



Structured Activities for Intelligent Learning

An elementary school resource book

by

RICHARD SKEMP

Expanded North American Edition

Edited and adapted

by

BRUCE and MARILYN HARRISON

Volume 2

for the later years

SAIL THROUGH MATHEMATICS: STRUCTURED ACTIVITIES FOR INTELLIGENT LEARNING

The author Richard Skemp is internationally recognized as a pioneer in the psychology of learning mathematics based on understanding rather than memorizing rules. A former mathematics teacher, he has lectured in twenty countries, is a former Professor of Educational Theory and Director of the Mathematics Education Research Centre at Warwick University, and a Past President of the International Group for the Psychology of Mathematics Education.

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SAIL THROUGH MATHEMATICS:

**STRUCTURED
ACTIVITIES FOR
INTELLIGENT
LEARNING**

Richard R. Skemp
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Volume 2

EEC Ltd.
Calgary

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And finally, but by no means least, yourselves, for allowing our efforts to find their hoped-for destination in your own classrooms.

NOTES

SAIL through Mathematics, Volume 1 is intended for Pre-grade Level 1 through Grade Level 2, and *Volume 2* for Grade Levels 3 through 6. Because of the wide range of children's abilities, this division can only be very approximate. In particular, some of the later activities in *Volume 1* will be found useful for children in Grade Level 3, especially if they come from schools where this approach has not been used. Every administrative grade placement brings together children currently operating at many different developmental levels in mathematics. The *SAIL* program is uniquely designed to accommodate a wide range of abilities.

Since the English language lacks a pronoun which means either he or she, I have used these alternately by topics.



SAIL Activities in the Classroom

A TEACHER'S GUIDE

to Structured Activities for Intelligent Learning

Drawing on ten years of experience using Richard Skemp's structured learning activities with her own classes and with many other teachers and their students, Marilyn Harrison has written the following teacher's guide and the accompanying sequencing guides to share insights, ideas, and suggestions.

Setting Sail

You are about to embark on an adventure in learning mathematics with understanding. The vehicle for this adventure consists of a set of structured and sequenced mathematics learning activities that have been designed in accordance with contemporary learning theory. Cooperative, schematic learning and prediction from mathematical patterns provide the basis for a powerful and unique approach to problem solving. Clusters of activities enable students to construct networks of mathematical ideas intelligently by building and testing through *direct experience, communication with others, and creative extrapolation*.

The learning activities in the *SAIL* volumes provide opportunities for cooperative group work, discussion, conjecturing, analyzing, justifying, writing, exploring, and applying mathematics. Where appropriate, memorization is encouraged for fluency and efficiency, but never in place of developing reasoning and understanding. The activities are designed for students to learn effectively by doing, saying, and recording. They explore a variety of appropriate methods. Prediction activities occur frequently, providing an excellent context for practicing estimation and reasonable conjecture skills while learning to reason mathematically.

The use of concrete materials in the *SAIL* program is extensive and carefully designed. Exemplary techniques are modelled for drawing the most out of the concrete and the more abstract learning situations.

Suggestions for Smooth Sailing

Classroom management

The activities are designed to engage students in pairs, in small groups of up to six students, or in whole class discussions. The optimal group size is indicated in the instructions for each activity. If students are not accustomed to working cooperatively in mathematics, their social learning skills need to be considered. They may have to learn such things as listening to each other, taking turns, discussing sensibly, and giving reasons rather than just arguing. The ways of learning mathematics which are embodied in *SAIL* both depend upon, and contribute to, social learning and clear speech. If these traits are already well established, you are off to a flying start.

Introducing the activities

It is the responsibility of a teacher to be familiar with each activity and its purpose before introducing it to students. To introduce an activity to the whole class, gather students in a circle and demonstrate the procedures by leading the activity with one or two students. Then have them do the activity in appropriate groups. Tell them to check that all the materials are on hand before they begin and that all the materials go back in their proper places at the end. It is worth the effort to establish well-defined routines from the beginning and most students appreciate the importance of keeping everything in place for those who will next do this activity. As the activity period concludes, it is important that the teacher and the students together reflect on and discuss their discoveries.

An alternative to introducing a new activity to the whole class would be to initiate it with a small group of students while the others are engaged in activities which they have already learned. This provides an opportunity to give 'high quality input' to small groups. One of the advantages of organizing your classroom this way is that the other students are learning on their own, doing their own thinking. This kind of teaching includes ways of managing students' learning experiences which are less direct, more sophisticated, and more powerful than traditional approaches.

You will find that the activities fall into two main groups: those which introduce new concepts, and those which consolidate them and provide a variety of applications. Activities in the first group always need to be introduced by a teacher to ensure that the right concepts are learned. Once they have understood the concepts, students can go on to do the consolidating activities together with relatively little supervision. These activities could be introduced by another adult helper. In some cases students, especially older students, can teach it to others with the help of the printed rules.

A set of 39 videoclips of the *SAIL* activities and of the theory which they embody has been produced. Each of the 3- to 11-minute activity videoclips models the introduction of a *SAIL* activity to a small group of children. Five of the videoclips demonstrate classroom management skills when the activities are used with a whole class. A 60-minute *Discussion Time with Richard Skemp* video provides a comprehensive overview of his theory of intelligent learning, illustrated with sample learning activities.

Charting the Course

Suggested sequencing

To help sequence the activities, especially for teachers new to the program, suggested grade-by-grade sequencing guides covering Grade Level 3 through Grade Level 6 have been included in the final pages of *SAIL Volume 2*. The Grade Level 3 sequencing includes concepts appropriate for most 8-year-olds; Grade Level 4, for most 9-year-olds; Grade Level 5, for most 10-year-olds; and Grade Level 6, for most 11-year-olds. There are many possible routes through the Networks. The suggested sequencing, though not unique, is offered as an aid for those wishing to be assured that the curricular expectations of each level are covered.

Materials

Many of the materials are found in most schools. The following list will enable you to quickly check which materials may be needed.

Collectables

Enlist the help of children and parents to provide the following:

- thin cord or yarn for set-loops
- little objects for sorting (natural objects – we do not want children to think that mathematics only involves plastic cubes): buttons, sea shells, beans, pebbles, bottle tops, keys, pasta, bread tags, nuts and bolts, seeds, crayons, screws (store in small containers, boxes or see-through plastic bags)
- sorting trays – trays with partitions; egg cartons; box lids
- catalogues and magazines for assembling pictures
- shaker for dice (though rolling the die without a shaker is fine)
- containers: egg cups (or 35 mm film containers); identical drinking glasses; an assortment of glasses, jugs, jars, varying as widely as possible in height and width; two identical containers of capacity equal to or greater than any of the preceding
- a collection of objects which approximate in shape to spheres, cuboids, cylinders, and cones; and others which are none of these

Purchased materials

- interlocking cubes – 1 cm (e.g., Centicubes)
- interlocking cubes – 2 cm (e.g., Multilink cubes) *Caution: Some commercially available linking cubes have dimensions slightly smaller than 2 cm and may cause problems when used with the 2 cm number tracks provided in the photomasters, especially when 'rods' are made and checked against a number track. If the dimensions of available cubes are not exactly 2 cm, alternate number tracks with appropriate dimensions could be drawn to avert unnecessary confusion for the children.*
- attribute blocks
- '1-6' dice
- '0-9' dice
- dice bearing only 1's and 2's (e.g., cover the faces with sticky paper and write '1' on half of them, '2' on the other half)
- 1¢, 5¢, 10¢, 25¢, and \$1.00 coins

Teacher's guide

- 2, 5, 10, 20, 50 dollar bills (play money)
- milk straws and popsicle sticks
- base ten material – ones, tens, hundreds, thousands
- thousands, hundreds, tens, and ones charts
- geometric models of spheres, cuboids, cubes, cylinders, cones, pyramids & prisms (a set of geometric solids)
- Plasticine
- non-permanent markers, i.e., water soluble pens such as those used with overhead projector acetate film
- coloured felt-tip pens
- squared paper
- rulers marked in centimetres, millimetres
- a metre-stick
- a 'skeleton' metre cube
- paper clips – small, medium, large (1 box of each)
- 6 wooden rods 20 cm long; 6 wooden rods 24 cm long
- 3 model trucks (could be made from *Lego* in various widths and heights)
- resealable plastic bags (e.g., *Ziploc* freezer bags)
- waxed string (e.g., *Wikki Stix*) (see Meas 2.2/3)
- balance scales (pan balance, spring balance)
- bolts (see Meas 4.4/1)
- analogue clock face with hour hand
- analogue clock face with hour and minute hands
- Celsius thermometers
- calendars
- calculators
- safety compasses
- protractors
- pocket- or purse-sized mirror or a MIRA
- set-square (i.e., a sturdy right-angled triangle, plastic or wooden)
- a magnetic compass
- a set of standard masses (varying from 10 gm to 1 kg)
- a set of plastic beakers (ranging in capacity from 10 mL to 2 L)

Preparing the activity cards

Two routes are available. You can duplicate the cards from the photomasters in Volumes 1a or 2a or purchase sets of prepared cards. To keep the card preparation manageable, it is recommended that you work on one Network or one Level at a time.

Perhaps the best way to keep all the materials for each activity together is to store them in a labelled plastic bag or a cardboard box. Plastic bags can be stored upright in a suitably sized open cardboard box. Alternatively, the bags can be suspended from pegboards with hooks. Some teachers find it useful to have with each activity a copy of the relevant 'Concept,' 'Ability,' 'Materials,' and 'What they do' sections of the activity instructions. These may be mounted on coloured card with the relevant 'discussion boxes' included on the back. The instruction activity cards could be colour coded by Network.

i) Preparing the materials from the Photomasters:

The preparation of materials from the Photomasters is described in *Volume 2a* on page x. Labelling the bags and underlining in red the materials (such as base 10 materials) which are not kept in these individual bags indicates what else has to be collected at the start of an activity.

ii) Sets of printed card are available either laminated or not.

Not laminated

Most of the cards are ready to be laminated but a few need to be taped together before laminating. These are clearly indicated. Laminate the cards on both sides to prevent curling.

Laminated card

Cut out individual cards, place elastic bands around decks of cards, and store them with the activity boards (if applicable) in plastic bags or boxes. Self-sticking printed labels are provided.

The prepared materials can be stored effectively in a central location or in individual classrooms. Even when school sets are available, teachers prefer to have copies of some of the consolidation activities in their classrooms. Extra elastic bands should be available in each classroom.

Parent involvement in the preparation of activities is discussed in detail on page 22.

It is worth remembering that, except for occasional replacement, the work of preparing the materials will not have to be repeated. The time spent is a capital investment, which will pay dividends in years to come. You are also contributing to children's long-term learning which makes all the hard work worth the effort.

Managing the Fleet

Data management

Many opportunities arise in classroom settings which spontaneously lead to data management activities.

Ask children to contribute collectable materials. As the materials are collected record the type of material contributed by each child on a class graph. This will likely encourage them to bring more.

As they complete the activities, ask students to graph their results. Here are some examples.

Renovating a house (Num 3.9/3)

Each 'couple' keeps a record on a graph of how many months they need to save to buy each item to renovate their house. Compare with other groups to find out which 'couple' is able to do their renovations in the shortest amount of time.

Products practice (Num 5.6/5)

Players record their time (after correction of errors) on a graph. Each student could keep an individual record over a period of time.

Evaluation

“In a book about teaching, the importance of assessment is that we must know how far children have reached in their understanding, to know what they are ready to learn next, or whether review and/or consolidation are needed before going on.”¹

i) Teacher Observation

Observe students as they do the activities. Even when introducing the activities to a whole class, a teacher has the opportunity to observe individual students as the procedures are demonstrated. This is a valuable opportunity to focus on a few children followed by more as all of the children work in small groups. Notes can be assembled for individual students in the group by making brief comments on 'post notes' which can later be filed.

ii) Evaluation checklists – keep a record of individual student progress using the Progress Record provided at the end of this book.

iii) Self/Peer Evaluation – discussions at the conclusion of each activity period will provide opportunities for individual and peer evaluation. Student journals can provide diagnostic assessment; e.g., “What I already know about division” before introducing the topic, followed by “What I learned about division” after completing the appropriate activities in the division network.

- iv) Portfolios of students' work – as students record their work, keep their sheets in a folder or provide a booklet. Keep a record of the problems they write, the patterns they make, etc.
- v) Teacher-Child Interview/Conferences – when others are engaged in consolidation activities, teachers can have conferences with individual students.
- vi) Specific assessment – assessment booklets for each network of activities would provide teachers with opportunities for written feedback.
- vii) Notebooks – as they do the activities, have the students record their written responses in a notebook. The amount of written work that they complete as they respond to the activities is often amazing!

Meeting individual needs

The concept maps on pages 27-63 provide a valuable vehicle for addressing individual needs by showing which concepts need to be understood before later ones can be acquired with understanding. The concept maps can be used diagnostically to provide activities at appropriate levels.

- i) Within each activity
Individual differences are accommodated by the very nature of the activities. For example, in **Crossing** (Num 3.2/4, *SAIL Volume 1*), students roll a die to move their counters up a 10-square number track. A student whose counter is on the ninth square, just needs to roll a '1' to finish. The other child is at square five. Each child rolls the die; the one at square five gets a '5' and wins the game. The first child says, "How did that happen? I was closer to the end." The teacher discusses how probability works when you throw dice. One child may be working on addition: $5 + 5 = 10$ whereas the other child is thinking about probability.
- ii) Individual differences can also be accommodated by introducing a more advanced activity to a small group while others continue to work on an earlier activity. For example, in **Making equal parts** (Num 7.1 Activities 1, 2, 4 and 5) one group of students can be making physical representations in Activities 1 and 2 while others move on to pictorial representations in Activity 4 and others can consolidate the concepts in a challenging game, Activity 5.
- iii) Because the activities are open ended, students are able to work at their own pace and at a suitable level within each activity. One group of students may complete 20 addition questions using the 'Start,' 'Action,' and 'Result,' cards, while another group, using counters, may only complete 5. Each feels successful.

Across the curriculum

By their very nature, the ways of learning mathematics embodied in this program both depend upon and contribute to social learning and clear speech.

Many opportunities arise which spontaneously lead to activities across the curriculum. For example, in **Feeding the animals** (Num 7.2/1) students may wish to expand their knowledge about nutrition for zoo animals. **Trainee keepers, qualified keepers** (Num 7.2/2) could lead to an examination of the different occupations at the zoo. **Cargo airships** (Num 6.8/4) might be used in conjunction with a study about the airport. **Treasure chest** (Num 5.10/2) could be used in conjunction with a unit about the sea. The real life situations provided by many of the activities can lead to many and diverse areas of study across the curriculum.

Problem solving

Skemp's theory-based approach to problem solving has students learning one new concept at a time in the context of activities that are interesting and engaging but low in mathematically irrelevant material. This facilitates the process of abstraction as progress is made through the relevant concept maps and the child's knowledge structure is developed. The carefully sequenced activities lead to the development of appropriate mathematical models which can be used to generate solutions to problems, which in turn are tested in the original problem situation. This approach contrasts strikingly with approaches that begin with high-noise problem situations and lead to a disorderly development of concepts and processes.

Personalized number stories are included in each Network: addition, pages 170-174 in *Volume 1*; subtraction, pages 212-215 in *Volume 1*; multiplication, pages 151-154 in *Volume 2*; and division, pages 260, 268 in *Volume 1*.

Students learn how to solve problems. They are deliberately led from verbal problems to physical representations of the objects, numbers, and actions described in the number story (*modelling*), and from the latter to the mathematical statement, not directly from words to mathematical symbols. They learn how to produce numerical models in physical materials corresponding to given number stories, to manipulate these appropriately, and to interpret the result in the context of the number story. Later, they do this by using written symbols only. (For an example, see **Number stories: abstracting number sentences**, Num 5.4, Activities 1, 2, 3.)

Students are encouraged to draw diagrams to show what they have done (concrete-pictorial-symbolic), e.g., **Different questions, same answer. Why?** (Num 6.3/1), to show the connection between grouping and sharing in division.

As they are engaged in the activities, students use problem solving skills. E.g., in **Treasure chest** (Num 5.10/2) they are applying problem solving skills as they predict the best strategies for choosing their treasure.

The students are helped to invent their own real-world problems, applying the concepts they have learned in practical situations. E.g., **Front window, rear window – make your own** (Num 4.9/4).

Meeting the NCTM Standards²

The close correlation between the NCTM *Standards* and the *SAIL* program written by Richard Skemp is undoubtedly the result of the convergence of two independent, thorough, and insightful explorations to the heart of what is needed for the intelligent learning of mathematics.

Accepting that the NCTM *Standards* have established a broad framework for guiding needed reform in school mathematics, an examination of some of the ways in which the Skemp materials fit that framework follows.

Mathematics as Problem Solving (Standard 1). The Skemp learning activities are, themselves, problem-solving tasks. The students are led to construct mathematical concepts and relationships from physical experiences designed to appeal to their imagination and to build on their real-world experiences. Students work cooperatively on well-designed, goal-directed tasks, making predictions, testing hypotheses and building relational understandings that facilitate routine and non-routine problem solving. (This is addressed at length above.)

Mathematics as Communication (Standard 2). The Skemp learning activities are designed to foster communication about mathematical concepts between students and between students and adults. A typical cooperative-group activity, **Number targets** (Num 2.8/1), engages the students in communicating about place value concepts with physical embodiments, spoken symbols, written symbols, oral and read symbols, all the while exploring the underlying mathematical meanings while using problem solving strategies to predict their best move.

Mathematics as Reasoning (Standard 3). Using patterns and relationships to make sense out of situations is an integral component of the Skemp learning activities. As mentioned previously, the activities frequently lead the students to explore, conjecture (make predictions), and test their conjectures. The program builds on relational understanding, and useful instrumental (habit) learning is promoted when appropriate.

Mathematical Connections (Standard 4). Skemp's detailed conceptual analysis of the elementary school mathematics curriculum has produced a set of well-defined concept maps or networks. The networks arrange the activities in optimal learning sequences and provide teachers with the framework to make relational connections within and across networks. Many of the activities require assembling previously learned concepts and processes to deal with the task at hand. An entire network (NuSp 1, **The number track and the number line**) is devoted to number tracks and number lines, which are of importance throughout mathematics, from kindergarten through university-level mathematics, and beyond. They provide a valuable support for our thinking about numbers in the form of a pictorial representation. Skemp's unerring notions about contexts that appeal to student imaginations have produced interesting lifelike settings in which the students learn. They compare possible outcomes of the moves they might make in **One tonne van drivers** (Num 3.10/3), and they are introduced to a budgeting activity in **Catalogue shopping** (Num 3.10/4).

In adult life, planning the use of money and other resources – e.g., time, labour – is one of the major uses of arithmetic. Because of interesting real life situations, connections to other curriculum areas are easily integrated. **Feeding the animals** (Num 7.2/1) and **Setting the table** (Num 4.5/2, *SAIL Volume 1*) are examples of activities which have spin-offs to art and health.

Estimation, Number Sense & Numeration, Numbers & Operations, Computation, Number Systems (Standards 5-7). One entire network of Skemp learning activities, Num 1, treats **Numbers and their properties** from ‘sorting dots’ to ‘square numbers’ and ‘relations between numbers.’ Another complete network, Num 2, **The naming of numbers**, carefully builds everything one needs to know about place value for numerals of any number of digits. A network is devoted to each of the basic arithmetic operations, **Addition, Subtraction, Multiplication, and Division**. In all of these networks there is emphasis on number sense, operations sense, reasonable estimates, making and testing predictions, mental computation, thinking strategies, and relationships between concepts and between operations . . . all recommended in Standards 5 through 8. Calculators are used when appropriate, as they would be in real-life situations. Activities which use calculators include: **One tonne van drivers** (Num 3.10/3), **Cargo boats** (Num 5.7/3), **Treasure chest** (Num 5.10/2), **Cargo airships** (Num 6.8/4), **Number targets: division by calculator** (Num 6.9/1), **Predict, then press** (Num 7.8/2), **“Are calculators clever?”** (Num 7.8/3), and **Number targets by calculator** (Num 7.8/4).

Geometry, Measurement (Standards 9, 10 [Grades K-4]; 12, 13 [Grades 5-8]). Space 1 and the new Space 2 network cover properties and components of two- and three-dimensional shapes, symmetry, and motion geometry. NuSp 1, **The number track and the number line** network, develops basic linear measurement skills while providing useful embodiments for the activities developing number concepts and arithmetic operations. The *SAIL* measurement networks (Meas 1, 2, 3, 4, 5, and 6) did not appear in the **Structured Activities for Primary Mathematics** volumes published in England by Routledge in 1989. Their treatment of **Length, Area, Volume & capacity, Mass & weight, Time, and Temperature** covers the concepts and relationships of Standards 9 & 10 for Grades K-4 and Standards 12 & 13 for Grades 5-8.

Statistics, Probability (Standards 11 [Grades K-4]; 10, 11 [Grades 5-8]). The prerequisite skills have been well developed in the ten original networks where students are introduced to probability from real-life situations. For example, as previously discussed (see page 7), in **Crossing** (Num 3.2/4, *SAIL Volume 1*), students roll a die to move their counters up a 10-square number track. A student whose counter is on the ninth square, just needs to roll a ‘1’ to finish. The other child is at square five. Each child rolls the die; the one at square five gets a ‘5’ and wins the game. The first child says, “How did that happen? I was closer to the end.” The teacher discusses how probability works when you throw a die. One child may be working on addition: $5 + 5 = 10$ whereas the other child is thinking about probability.

Fractions and Decimals (Standard 12 [Grades K-4]; Standards 5, 6 [Grades 5-8]).

The Num 7 network, **Fractions**, begins with the development of real-object concepts of 'equal parts,' 'denominators,' and 'numerators,' building to fully-symbolic treatments of fractions, decimal-fractions, and operations with decimals. The whole development of fraction 'number sense' is firmly grounded in the use of concrete and pictorial models. The relationships between fractions and decimal fractions are carefully developed.

The Standards and Professional Development

Inservice courses, based on Skemp's learning theory, have been developed to give teachers opportunities to learn about the theory, to do a selection of activities together, to make them, and to discuss them. Follow-up professional development support is available as teachers incorporate the activities in their classrooms. As mentioned previously, thirty-nine VHS videoclips of the *SAIL* activities and of the theory which they embody have been produced as part of that support. The *SAIL* program is well suited for addressing the *Standards* in a thorough, consistent, and well-organized manner.

Each of these activities embodies both a mathematical concept, and also one or more aspects of the theory. So by doing these with a group of children, both children and their teacher benefit. The children benefit by this approach to their learning of mathematics; and the teacher also has an opportunity to learn about the theory of intelligent learning by seeing it in action. Theoretical knowledge acquired in this way relates closely to classroom experience and to the needs of the classroom. It brings with it a bonus, since not only do the children benefit from this approach to mathematics, but it provides a good learning situation for teacher also. In this way we get 'two for the price of one', time-wise.³

*One ship sails East, another West
by the self-same winds that blow.
It isn't the gales, but the trim of their sails
that determines the way they go.
Traditional*

Notes

- ¹ Skemp, R. R. (1989). *Mathematics in the Primary School*. London: Routledge. p. 166.
- ² National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: The Council.
- ³ Skemp, 1989, op. cit., p. 111.



INTRODUCTION

1 Why this book was needed and what it provides

The *Curriculum and Evaluation Standards for School Mathematics*¹ document produced by the National Council of Teachers of Mathematics stresses the importance of a *developmentally appropriate* curriculum. There is now a wide consensus that practical work is essential for ensuring developmentally appropriate concept formation throughout the elementary school years, and not just for younger children. There are now a number of these activities available, and individually many of them are attractive and worthwhile. But collectively, they lack two essential requirements for long-term learning: structure, and clear stages of progression. *SAIL through Mathematics, Volumes 1 and 2*, provides a fully structured collection of more than four hundred seventy-five activities, covering a core curriculum for children aged from four to eleven years old and using practical work extensively at all stages.

This collection is not, however, confined to practical work. Mathematics is an abstract subject, and children will need in the future to be competent at written mathematics. Putting one's thoughts on paper can be a help in organizing them, as well as recording them for oneself and communicating them to others. What is important is that this should not come prematurely. It is their having had to memorize a collection of rules without understanding which has put so many generations of learners off mathematics for life, and destroyed their confidence in their ability to learn it. Practical, oral, and mental work can provide the foundation of understanding without which written work makes no sense. Starting with these, the present collection provides a careful transition from practical work to abstract thinking, and from oral to written work.

Activities for introducing new concepts often include teacher-led discussion. Many of the other activities take the form of games which children can play together without direct supervision, once they know how to play. These games give rise to discussion; and since the rules and strategies of the games are largely mathematical, this is a mathematical discussion. Children question each other's moves, and justify their own, thereby articulating and consolidating their own understanding. Often they explain things to each other, and when teaching I emphasize that "When we are learning it is good to help each other." Most of us have found that trying to explain something to someone else is one of the best ways to improve one's own understanding, and this works equally well for children.

This volume also provides the following:

- (a) A set of diagrams (concept maps, or networks) showing the overall mathematical structure, and how each topic and activity fits.
- (b) Clear statements of what is to be learned from each group of activities.
- (c) For each activity, a list of materials and step by step instructions. (In Volume 2a, photomasters are also provided to simplify the preparation of materials.)
- (d) For each topic, discussion of the mathematical concept(s) involved, and of the learning processes used.

The last of these will, it is hoped, be useful not only for classroom teachers, but also for support teams, mathematics advisers, those involved in the pre-service and in-service education of teachers, and possibly also those whose main interests are at the research level.

2 The invisible components and how to perceive them

The activities in this collection contain a number of important components which are invisible, and can only be perceived by those who know what to look for. These include (i) real mathematics, (ii) structure, and (iii) a powerful theory.

(i) *Real mathematics*. I contend that children can and should enjoy learning real mathematics. You might ask: “What do you mean by this? Is it just a puff?” I say “Begin,” because a fuller answer depends on personal experience. If someone asks “What is a kumquat?” I can tell them that it is a small citrus fruit, but two of the most important things for them to know are what it tastes like, and whether they like it or not. This knowledge they can only acquire by personally tasting a kumquat.

Real mathematics is a kind of knowledge. I can describe it, and I hope you will find this a useful start. But some of the most important things about mathematics people cannot know until they have some of this kind of knowledge in their own minds; and those who acquired real mathematics when they were at school are, regrettably, in a minority. A simple preliminary test is whether you enjoy mathematics, and feel that you understand it. If the answers are “No,” then I have good news for you: what you learned was probably not real mathematics. More good news: you can acquire real mathematics yourself while using these activities with your children. You will then begin to perceive it in the activities themselves: more accurately, in your own thinking, and that of your children, while doing these activities. And you will begin to discover whether or not what you yourself learned as a child was real mathematics.

Mathematics (hereafter I will use ‘mathematics’ by itself to mean real mathematics) is a kind of knowledge which is highly adaptable. In the adult world, this adaptability can be seen in the great variety of uses to which it is put. Mathematics is used to make predictions about physical events, and greatly increases our ability to achieve our goals. Our daily comfort and convenience, sometimes our lives, depend on the predictive use of mathematics by engineers, scientists, technicians, doctors and nurses. At an everyday level, we use mathematics for purposes such as predicting approximately how long we should allow for a journey. Highly sophisticated mathematics is required to project communication satellites into orbits whereby they hang stationary relative to the earth; and also in the design of the satellites themselves, whose electronic equipment allows us to watch on our television screens events many thousands of miles away.

Mathematics has also an important social function, since many of the complex ways in which we co-operate in modern society would not be possible without mathematics. Nuts could not be made to fit bolts, clothes to fit persons, without the measurement function of mathematics. Businesses could not function without the mathematics of accountancy. If the person in charge of this gets his calculations wrong, his firm may go out of business: that is to say, others will no longer cooperate by trading with them.

Another feature of mathematics is creativity — the use of one’s existing knowledge to create new knowledge. Can you say what are ninety-nine sevens? Probably not, but if you think “A hundred sevens make seven hundred, so ninety-nine sevens will be one seven less: six hundred and ninety three,” then you are using your own

mental creativity. Creating new mathematics which nobody ever knew before is creativity at the level of the professional mathematician; but anyone who has some real mathematics is capable of creating knowledge which is new to them, and this way of using one's mind can give a kind of pleasure which those who have not experienced it may find hard to understand.

These are some of the adult uses of mathematics, which make it so important in today's world of advanced science, technology, and international commerce. At school, most children still learn a look-alike which is called by the same name, but whose uses have little in common with the uses of real mathematics. School mathematics as it is experienced by children is mostly for getting check marks, pleasing teachers, avoiding reproofs and sometimes also the humiliation of being made to feel stupid. It is also used for passing exams, and thereafter quickly forgotten. Yet real mathematics can be taught and learned at school. For an example of mathematics used predictively, try **Missing stairs** (Org 1.5/1, *SAIL Volume 1*). Success in most of the games also depends largely on making good predictions. Mathematics is used socially in all children's work together in groups; and in some, e.g., **Renovating a house** (Num 3.9/3), a social use is embodied in the activity itself. I hope that you will find pleasure in discovering examples of creativity in the thinking of your own children when they are learning real mathematics in contexts like these.

(ii) *Structure*. This is an essential feature of real mathematics. It is this which makes possible all the features described in (i), so for emphasis I am giving it a section to itself.

By structure we mean the way in which parts fit together to make a whole. Often this whole has qualities which go far beyond the sum of the separate properties of the parts. Connect together a collection of transistors, condensers, resistors, and the like, most of which will do very little on their own, and you have a radio by which you can hear sounds broadcast from hundreds of miles away. That is, if the connections are right: and this is what we mean by structure in the present example.

In the case of mathematics, the components are mathematical concepts, and the structure is a mental structure. This makes it much harder to know whether it is there or not in a learner's mind. But the difference between a mathematical structure and a collection of isolated facts is as great as the difference between a radio and a box of bits. There is the same difference between a radio set and a wrongly connected assortment of components, but this is harder to tell by looking. The important test of the presence or absence of structure, i.e. of the right set of connections, may best be inferred from performance. This is also true for mathematics, and for its look-alike which goes by the same name. Of these two, only real mathematics performs powerfully, enjoyably, and in a wide variety of ways.

Each individual learner has to put together these structures in his own mind. No one can do it for him. But this mental activity can be greatly helped by good teaching, an important part of which is providing good learning situations.

The requirements of a good learning situation include full use of all of the three modes of building conceptual structures. Mode 1 is learning by the use of practical materials; Mode 2 is learning from exposition, and by discussion; and Mode 3 is expanding one's knowledge by creative thinking. These categories are expanded and discussed more fully in *Mathematics in the Primary School*.²

The activities in this book are intended to help teachers provide learning situations of the kind described. They are also fully structured, meaning that the concepts embodied in each fit together in ways which help learners to build good mathematical structures in their own minds. This also includes consolidation, and developing mathematical skills.

(iii) *A powerful theory.* In 1929, Dewey wrote “Theory is in the end . . . the most practical of all things”;³ and I have been saying the same for many years, even before I knew that Dewey had said it first. The activities in this book embody a new theory of intelligent learning. This had its origins in the present author’s researches into the psychology of learning of mathematics,⁴ and was subsequently expanded and generalized into a theory of intelligent learning which can be applied to the learning of all subjects.⁵ It is not essential to know the theory in order to use the activities. But readers who are professional teachers will want to know not only what to do but why. Mathematics advisers, or lecturers in mathematical education, will wish to satisfy themselves of the soundness of the underlying theory before recommending the activities.

This theoretical understanding is best acquired by a combination of first-hand experience, reading, and discussion. Each of the activities embodies some aspects of the general theory, so by doing the activities with children we can observe the theory in action. For school teachers, this is a very good way to begin, since the theoretical knowledge acquired in this way begins with classroom experience, and as it develops further will continue to relate to it. This also has the advantage that we get ‘two for the price of one,’ time-wise: what might be called a ‘happy hour’ in the classroom! These observations can then form the first part of the trio

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whose value for school-based inservice education will be mentioned again in Section 5.

Reading helps to organize our personal experience, and to extend our knowledge beyond what can be gathered first-hand. A companion book to the present volume is *Mathematics in the Primary School*,⁶ and this also offers suggestions for further reading.

3 How this book is organized and how to use it

The two *SAIL* volumes contain teaching materials for eight school years, together with explanations and discussions. This is a lot of information. Careful thought has therefore been given to its organization, to make it easy to find as much as is required at a given stage, and to avoid feeling overloaded with information. Mathematics is a highly concentrated kind of information, so it is wise to take one's time, and to go at a pace which allows comfortable time for assimilation. The amount eventually to be acquired in detail by a class teacher would be no more than one-eighth of the total, if all children in the class were of the same ability. In practice it will, of course, be more because of children's spread of ability.

The aim has been to provide first an overview; then a little more detail; and then a lot of detail, of which many readers will not need all, nor all at the same time. This has been done by organizing the subject matter at four levels, into THEMES, NETWORKS, TOPICS, and ACTIVITIES. The themes and networks are tabulated below.

THEMES	NETWORKS	CODES
Organizing	Set-based organization	Org 1
Number	Numbers and their properties	Num 1
	The naming and recording of numbers	Num 2
	Addition	Num 3
	Subtraction	Num 4
	Multiplication	Num 5
	Division	Num 6
	Fractions	Num 7
Space	Shape	Space 1
	Symmetry and motion geometry	Space 2
Synthesis of Number and Space	The number track and the number line	NuSp 1
Pattern	Patterns	Patt 1
Measurement	Length	Meas 1
	Area	Meas 2
	Volume and capacity	Meas 3
	Weight and mass	Meas 4
	Temperature	Meas 5
	Time	Meas 6

The five main themes run in parallel, not sequentially, though some will be started later than others. Within the theme of Number, there are seven networks. For some themes, there is at present just one network each; but I have kept to the same arrangement for consistency, and to allow for possible future expansion. By ‘network,’ I mean a structure of interrelated mathematical ideas. It can well be argued that all mathematical ideas are interrelated in some way or other, but the networks help to prevent information overload by letting us concentrate on one area at a time.

Greater detail for each network is provided by a concept map and a list of activities. These are on pages 27 through 63. It will be useful to look at a pair of these as illustrations for what follows. Each concept map shows how the ideas of that network relate to each other, and in particular, which ones need to be understood before later ones can be acquired with understanding. These interdependencies are shown by arrows. A suggested sequence through the network is shown by the numbers against each topic. Use of the concept map will help you to decide whether other sequences may successfully be followed, e.g., to take advantage of children’s current interests. The concept map is also useful diagnostically. Often a difficulty at a particular stage may be traced to a child’s not having properly understood one or more earlier concepts, in which case the concept map will help you to find out which these are. Another function of the concept maps is to help individual teachers to see where their own teaching fits into a long-term plan for children’s learning, throughout the primary years: what they are building on, and where it is leading.

Below each concept map is shown a list of the activities for each topic. (I call them ‘topics’ rather than ‘concepts’ because some topics do not introduce new concepts, but extend existing ones to larger areas of application.) Usually these activities should be used in the order shown. An alphabetical list of the activities is also given, at the end of the book.

To find activities for a particular topic, the best way is via the concept maps and the lists of activities opposite them. Suppose for example that you want activities for adding past ten. For this you naturally look at the concept map for addition, and find adding past ten as topic 7. On the adjacent page, for this topic, you will find seven activities. Not all topics have so many activities, but this indicates the importance of this stage in children’s learning of addition. If on the other hand you want to find a particular activity by name, then the alphabetical index at the end of the book will enable you to do so.

The codes for each topic and activity are for convenience of reference. They show where each fits into the whole. Thus Num 3.8/2 refers to Network Num 3, Topic 8, Activity 2. If the packet for each activity has its code on it, this will help to keep them all in the right order, and to replace each in the right place after use.

4 Getting started as a school

Since schools vary greatly, what follows in this section and the rest of this introduction is offered as no more than a collection of suggestions based on the experience of a number of the schools where the materials have already been introduced.

It has been found useful to proceed in two main stages: getting acquainted, and full implementation. Since the latter will be spread over one or two years at least, the first stage is important for getting the feel of the new approach, and to help in deciding that it will be worth the effort.

For getting acquainted, a good way is for each teacher to choose an activity, make it up, and learn it by doing it with one or more other teachers. (Different activities are for different numbers of persons.) Teachers then use these activities with their own children, and afterwards they discuss together what they have learned from observation of their own children doing the activities. It is well worth while trying to see some of the activities in use, if this can be arranged. Initially, this will convey the new approach more easily than the printed page.

When you are ready to move towards a full implementation, it will be necessary to decide the overall approach. One way is to introduce the activities fully into the first and second years, while other teachers gradually introduce them into later years as support for the work they are already doing. This has the advantage that the full implementation gradually moves up the school, children being used to this way of learning from the beginning. Alternatively, activities may be introduced gradually throughout the school, individual teachers choosing which activities they use alongside existing text-based materials while they gain confidence in the new approach. As another alternative, a school may wish to begin with one network (perhaps, for example, multiplication), arranging for all the teachers to meet together to do all of the activities in the network and then to implement them at the appropriate grade levels.

Arrangements for preparation of the materials need to be planned well in advance. This is discussed in greater detail in section 5. A detail which needs to be checked in good time is whether the commercial materials needed, such as Multilink or Unifix and base ten materials, are already in the school in the quantities needed.

In considering the approach to be used, it is important to realize that while benefit is likely to be gained from even a limited use of the activities, a major part of their value is in the underlying structure. The full benefit, which is considerable, will therefore only be gained from a full implementation.

5 Organization within the school

Overall, the organization of the new approach is very much a matter for the principal and staff of each individual school to work out for themselves; so, as has already been emphasized, what follows is offered only in the form of suggestions, based on what has been found successful in schools where this approach has been introduced.

Whatever organization is adopted, it is desirable to designate an organizer who will coordinate individual efforts, and keep things going. It is a great help if this teacher can have some free time for planning, organizing, advising, and supporting teachers as need arises.

One approach which has been found successful is as follows. The mathematics organizer holds regular meetings with the staff in each one- or two-year group, according to their number. Each teacher chooses an activity, makes it up if necessary, and teaches it to the other teachers in the group. They discuss the mathematics involved, and any difficulties. Subsequently they discuss their observations of their own children doing these activities, and what they have learned by reflecting on these. This combination may be summarized as

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and is an important contribution to school-based inservice education. (So much so, that several leading mathematics educators have said that it should be printed on every page. As a compromise, I have printed it at the end of every topic.)

It is good if the mathematics organizer is also in a position to help individual teachers, since it is only to be expected for them to sometimes feel insecure when teaching in a style which may be very different from that to which they are accustomed. Two useful ways to help are by looking after the rest of a teacher's class for a while, so that this teacher is free to concentrate entirely on working with a small group; and by demonstrating an activity with a small group while the class teacher observes, the rest of the class being otherwise occupied.

6 Getting started as an individual teacher

The most important thing is actually to do one or more activities with one's own children, as early on as possible. This is the best way to get the feel of what the new approach is about. After that, one has a much better idea of where one is going. If there are particular topics where the children need help, suitable activities may be found via the concept maps and their corresponding lists of activities. Alternatively, here is a list of activities which have been found useful as 'starters'. The stages correspond roughly to years at school.

	vol	page	
Stage 1 Lucky dip	Org 1.3/1	1	72
"Can I fool you?"	Org 1.3/2	1	73
Missing stairs	Org 1.5/1	1	76
Stage 2 Stepping stones	Num 3.2/3	1	162
Crossing Num 3.2/4	1	164	
Sequences on the number track	NuSp 1.2/1	1	309
Stage 3 The handkerchief game	Num 3.5/1	1	175
"Please may I have?" (complements)	Num 3.5/2	1	176
Number targets	Num 2.8/1	1	141
Stage 4 Slippery slope	Num 3.7/3	1	184
Slow bicycle race	NuSp 1.5/1	1	319
Doubles and halves rummy	Num 1.10/3	1	121
Stage 5 Place value bingo	Num 2.10/3	2	91
Renovating a house	Num 3.9/3	2	117
Constructing rectangular numbers	Num 6.4/1	2	194
The rectangular numbers game	Num 6.4/2	2	195
Stage 6 Cycle camping	Num 3.10/2	2	121
One tonne van drivers	Num 3.10/3	2	122
Multiples rummy	Num 5.6/6	2	168
Stage 7 Cargo boats	Num 5.7/3	2	172
Classifying polygons	Space 1.12/1	2	278
Match and mix: polygons	Space 1.12/3	2	279

When the children are doing an activity, think about the amount of mathematics which the children are doing, including the mental and oral activity as well as the written work, and compare it with the amount of mathematics they would do in the same time if they were doing written work out of a textbook.

So far, I have interpreted the heading of this section as meaning that the reader is an individual teacher within a school where most, or at least some, colleagues are also introducing the new approach. But what if this is not the case? When talking with teachers at conferences, I have met some who are the only ones in their school who are using this kind of approach to the teaching of mathematics.

This is a much more difficult situation to be in. We all need support and encouragement, especially when we are leaving behind methods with which we are familiar — even though they have not worked well for many children. We need to discuss difficulties, and to share ways we have found for overcoming them. So my suggestion here is that you try to find some colleague with whom you can do this. At the very least, you need someone with whom to do the activities before introducing them to the children; and further discussions may arise from this.

7 Parents

Parents are naturally interested in their children's progress at school. Written work is something they can see — what they cannot see is the lack of understanding which so often underlies children's performances of these 'rules without reasons.' Sometimes also they try to help children at home with their mathematics. Unfortunately, this often takes the form of drill-and-practice at multiplication tables, and pages of mechanical arithmetic. This is the way they were taught themselves, and some parents have been known to respond unfavourably when their children come home and say that they have spent their mathematics lessons playing games. As one teacher reported, "Games are for wet Friday afternoons. Mathematics is hard work. They aren't meant to enjoy it."

How you deal with problems of this kind will, of course, depend partly on the nature of existing parent-teacher relationships in your own school. When explaining to parents who may be critical of what you are doing, it also helps if you are confident in your own professional understanding, and if there is a good consensus within the school. These are areas in which sections 2 and 5 offer suggestions. Some approaches which have been used with success are described here. They may be used separately or together.

A parents' evening may be arranged, in which parents play some of the games together. Teachers help to bring out the amount of mathematics which children are using in order to decide what move they will make, or what card to play. To do this well, teachers need to be confident in their own knowledge of the underlying mathematics, and this can be built up by doing the activities together, and discussing with each other the mathematics involved.

A small group of parents may be invited to come regularly and help to make up activities. This needs careful organizing initially, but over a period it can be a great saving in teacher's time. When they have made up some activities, parents naturally want to do them in order to find out what they are for; so this combines well with the first approach described.

Some parents may also be invited to come into classrooms and supervise consolidation activities. (It has already been mentioned that activities which introduce new concepts should be supervised by a teacher.)

For parents who wish to help children at home with their mathematics, the games provide an ideal way to do this. Many of these can be played at varying levels of sophistication, which makes them suitable as family games; and there is usually also an element of chance, which means that it is not always the cleverest player who wins. None of the games depends entirely on chance, however. Good play consists in making the best of one's opportunities. Parents who help their children in this way will also have the benefit of knowing that what they are doing fits in with the ways in which their children are learning at school.

8 Some questions and answers

Q. How long will it take to introduce these materials?

A. The materials embody many ideas which are likely to be new to many teachers, in two areas: mathematics, and children's conceptual learning. So it is important to go at a pace with which one feels comfortable, and which gives time to assimilate these ideas into one's personal thinking and teaching. It took me twenty-five years to develop the underlying theory, three to find out how to embody it in ways of teaching primary mathematics, and another five to devise and test the integrated set of curriculum materials in this volume. If a school can have the scheme fully implemented and running well in about two years, I would regard this as good going. But of course, you don't have to wait that long to enjoy some of the rewards.

Q. Isn't it a lot of work?

A. Changing over to this new approach does require quite a lot of work, especially in the preparation of the materials. The initial and ongoing planning and organization are important, so that this work can go forward smoothly. As I see it, nearly all worthwhile enterprises, including teaching well, involve a lot of work. If you are making progress, this work is experienced as satisfying and worth the effort. As one teacher said, 'Once you get started, it creates its own momentum.' And once it is established, the fact that children are learning more efficiently makes teaching easier.

Q. How were the concept maps constructed?

A. First, by a careful analysis of the concepts themselves, and how they relate to each other in the accepted body of mathematical knowledge.⁷ This was then tested by using it as a basis for teaching. If an activity was unsuccessful in helping children to develop their understanding, this was discussed in detail at the next meeting with teachers. Sometimes we decided that the activity needed modification; but sometimes we decided that the children did not have certain other concepts which they needed for understanding the one we had been introducing – that is, the concept map itself needed revision. So the process of construction was a combination of mathematics, applied learning theory, and teaching experiment.

Q. Is it all right to use the activities in a different order?

A. Within a topic, the first activity is usually for introducing a new concept, and clearly this should stay first. Those which follow are sometimes for consolidation, in which case the order may not matter too much. Sometimes, however, they are for developing thinking at a more abstract level, in which case the order does matter. However, once the activities have been used in the order suggested, it is often good to return to earlier ones, for further consolidation and to develop connections in both directions: from concrete to abstract, and from abstract to concrete.

Whether it is wise to teach topics in a different order you can decide by looking at the concepts map itself. These, and the teaching experiments described, show that the order of topics is very important. They also show that for building up a given knowledge structure, there are several orders which are likely to be successful – and many which are likely to be unsuccessful.

Q. Is it all right to modify the activities?

A. Yes, when you are confident that you understand the purpose of the activity and where it fits into the long-term learning plan. The details of every activity have been tested and often rewritten several times, both from the point of view of the mathematics and to help them to go smoothly in the classroom; so I recommend that you begin by using them as written. When you have a good understanding of the mathematics underlying an activity, you will be in a position to use your own creativity to develop it further.

Sometimes children suggest their own variation of a game. My usual answer is that they should discuss this among themselves, and if they agree, try it together next time: but that rules should not be changed in the middle of a game. They may then discuss the advantages and disadvantages of the variation.

Q. Can I use the materials alongside an existing scheme? And what about written work, in general?

A. Especially with the older children, I would expect you to begin by introducing these activities alongside the scheme you are already using and familiar with. With younger children, the activities introduce as much written work as I think is necessary. Thereafter, existing textbook schemes can be put to good use for gradually introducing more written work of the conventional kind. But these should come after the activities, rather than before. Children will then get much more benefit from the written work because they come to it with greater understanding.

Q. Doesn't all this sound too good to be true?

A. Frankly, yes. Only personal experience will enable you to decide whether it can become true for you yourself, and in your school. Each school has its own microclimate, within which some kinds of learning can thrive and others not. Where the microclimate is favourable to this kind of learning, what I have been describing can become true, and I think you will find it professionally very rewarding. Where this is not at present the case, the problems lie beyond what can be discussed here. I have discussed them at length elsewhere.⁸

Q. What do you see as the most important points when implementing this approach?

A. Good organization; personal experience of using the activities; observation, reflection and discussion.

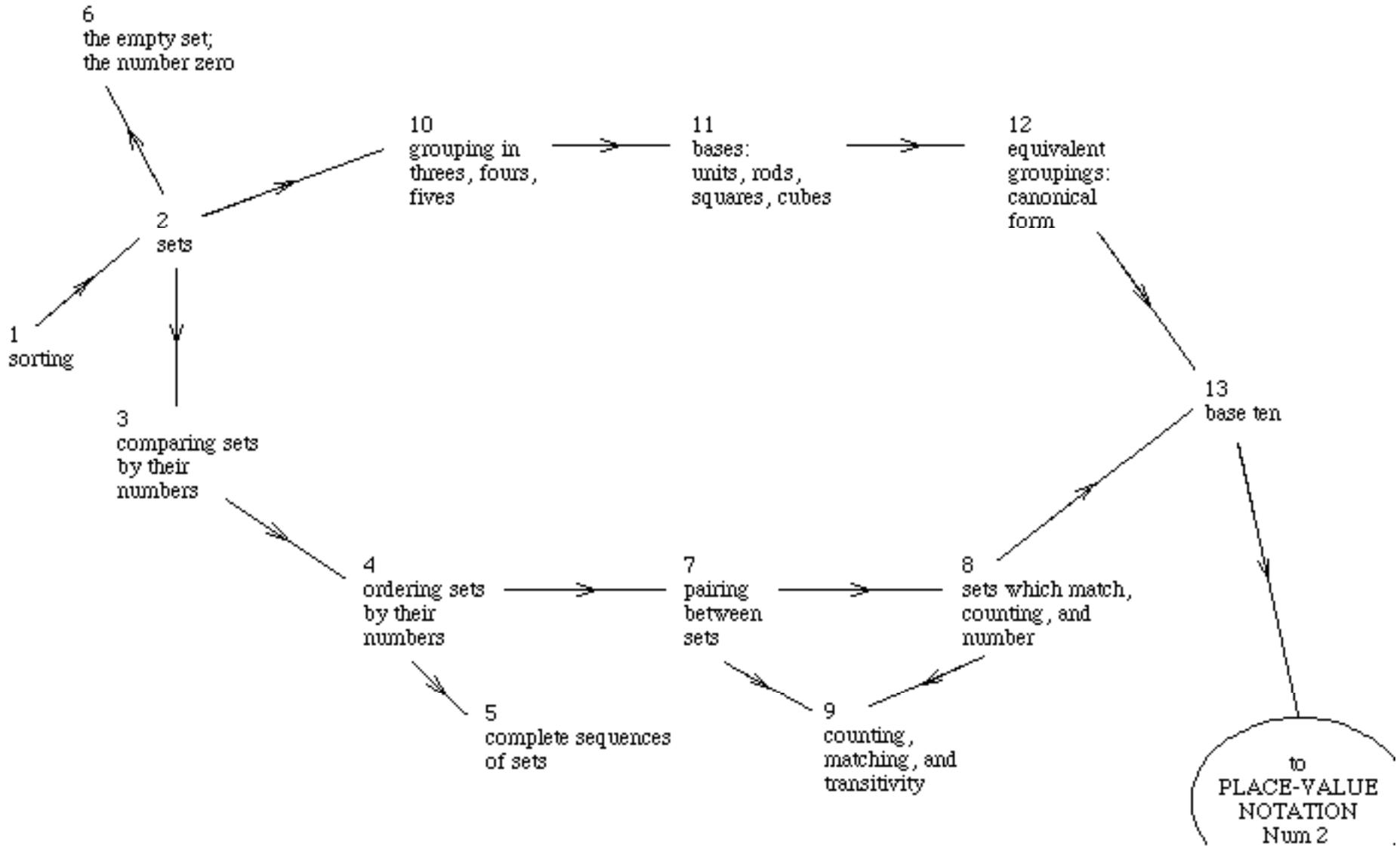
Notes

- ¹ National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: The Council.
- ² Skemp, R. R. (1989). *Mathematics in the Primary School*. London: Routledge.
- ³ Dewey, J. (1929). *Sources of a Science of Education*. New York: Liveright.
- ⁴ Skemp, R. R. (1971, 1986 2nd ed.). *The Psychology of Learning Mathematics*. Harmondsworth: Penguin.
- ⁵ Skemp, R. R. (1979). *Intelligence, Learning, and Action*. Chichester: Wiley.
- ⁶ Skemp, 1989, op. cit.
- ⁷ Skemp, 1971 and 1978, op. cit. See Chapter 2 of the first edition, or Chapter 1 of the second edition of Skemp, 1971.
- ⁸ Skemp, 1979, op. cit., Chapter 15.

CONCEPT MAPS and LISTS OF ACTIVITIES

Activities in **bold** are those found in this volume.

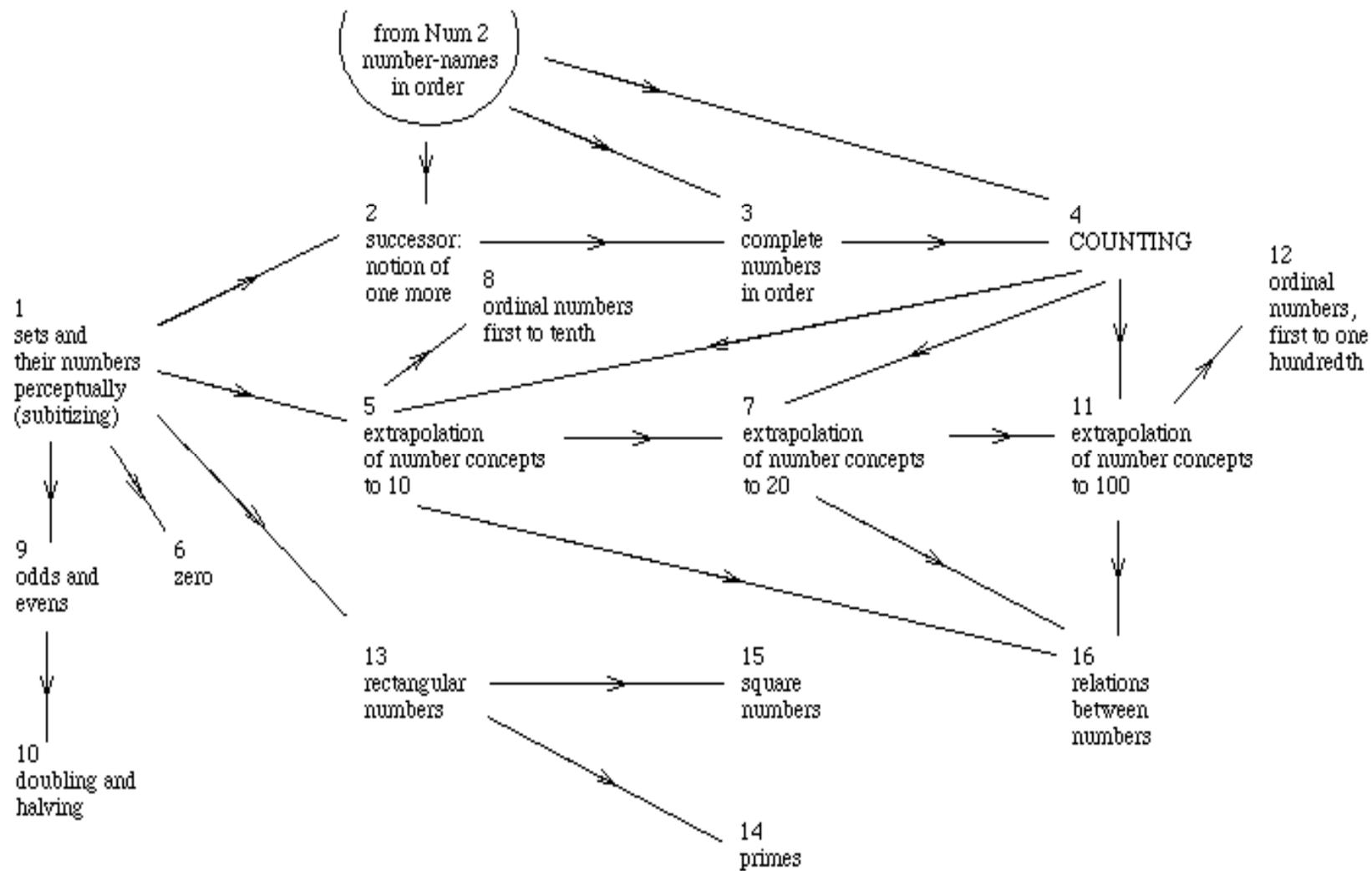
Org 1 Set-based organization



Org 1 Set-based organization

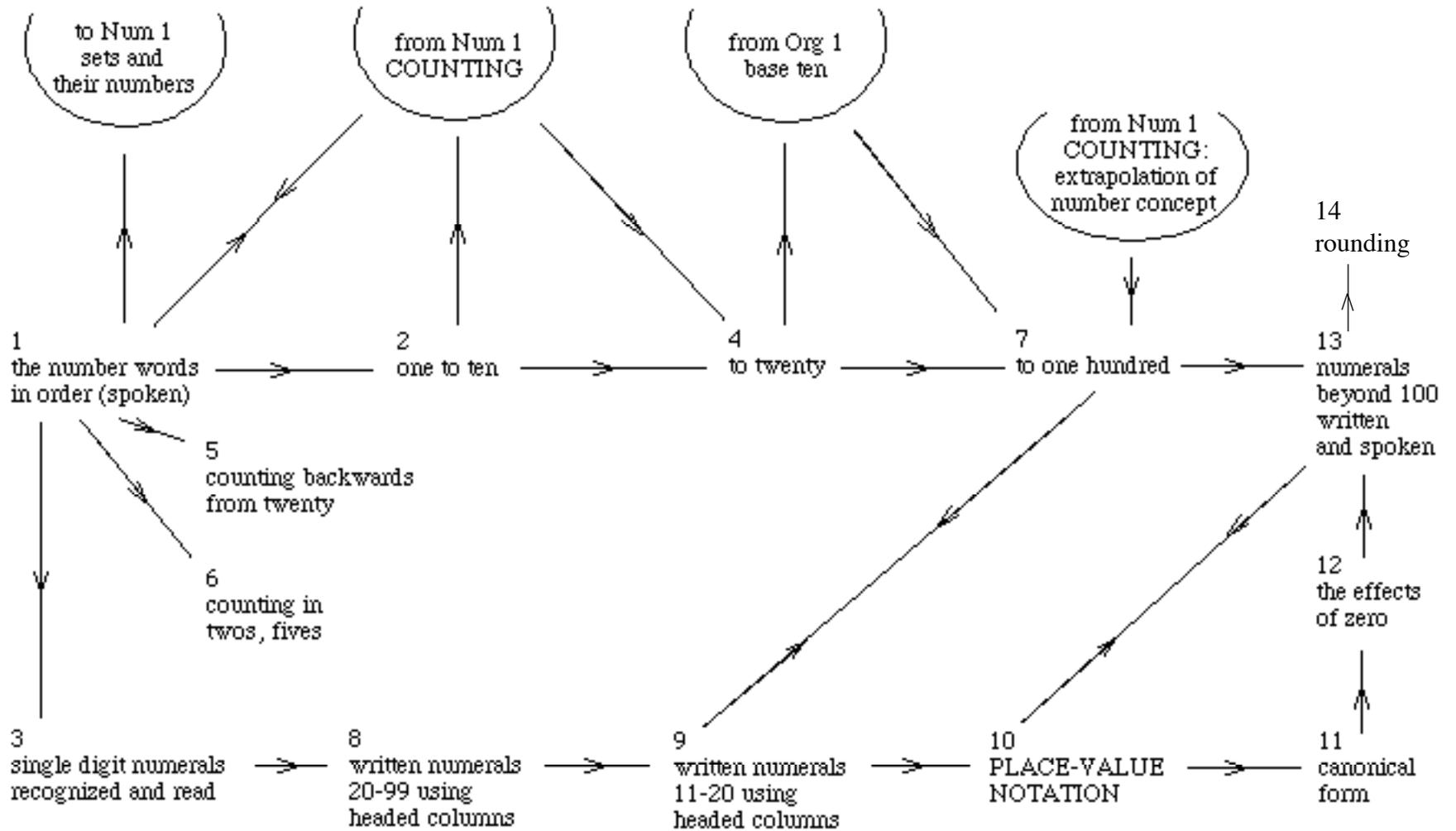
1	<i>Sorting</i>		
1/1	Perceptual matching of objects		
1/2	Matching pictures		
1/3	A picture matching game		
1/4	Dominoes		
1/5	Conceptual matching		
1/6	Conceptual matching		
1/7	Attribute cards		
2	<i>Sets</i>		
2/1	Introduction to Multilink or Unifix		
2/2	Making picture sets		
2/3	“Which set am I making?”		
2/4	“Which two sets am I making?”		
3	<i>Comparing sets by their numbers</i>		
3/1	Lucky dip		
3/2	“Can I fool you?”		
4	<i>Ordering sets by their numbers</i>		
4/1	Ordering several rods by their lengths		
4/2	Combining order of number, length, and position		
5	<i>Complete sequences of sets</i>		
5/1	Missing stairs		
6	<i>The empty set; the number zero</i>		
6/1	The empty set		
7	<i>Pairing between sets</i>		
7/1	Physical pairing		
7/2	Mentally pairing		
8	<i>Sets which match, counting, and number</i>		
8/1	Sets which match		
9	<i>Counting, matching, and transitivity</i>		
	No activity; but a note for teachers		
10	<i>Grouping in threes, fours, fives</i>		
10/1	Making sets in groups and ones		
10/2	Comparing larger sets		
10/3	Conservation of number		
11	<i>Bases: units, rods, squares and cubes</i>		
11/1	Units, rods and squares		
11/2	On to cubes		
12	<i>Equivalent groupings: canonical form</i>		
12/1	“Can I fool you?” (Canonical form)		
12/2	Exchanging small coins for larger		
13	<i>Base ten</i>		67
13/1	Tens and hundreds of cubes		67
13/2	Tens and hundreds of milk straws		68
13/3	Thousands		68

Num 1 Numbers and their properties



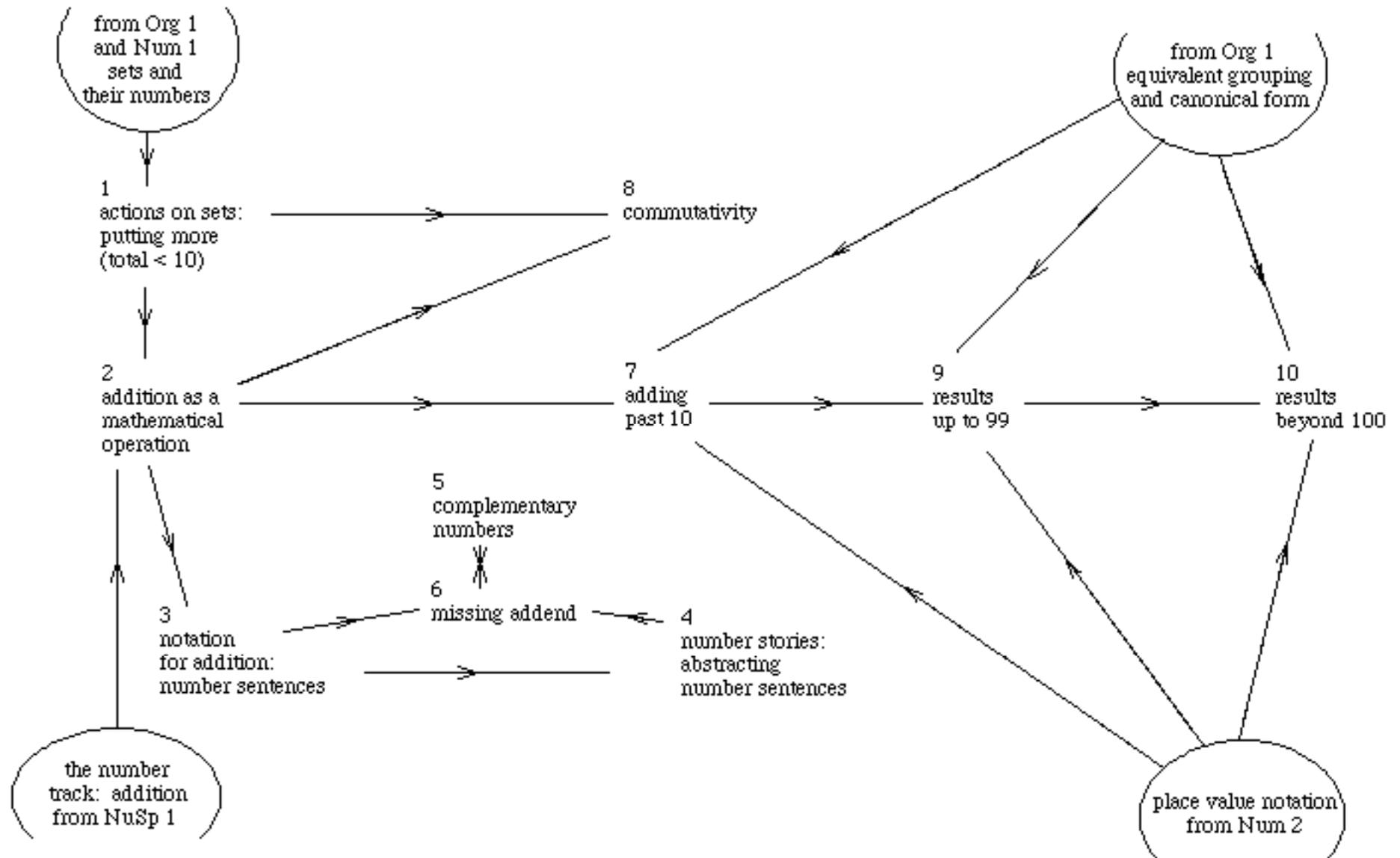
Num 1 Numbers and their properties

1	<i>Sets and their numbers perceptually (subitizing)</i>	
1/1	Sorting dot sets and picture sets	
1/2	Picture matching game using dot sets and picture sets	
2	<i>Successor: notion of one more</i>	
2/1	Making successive sets	
2/2	Putting one more	
3	<i>Complete numbers in order</i>	
3/1	“Which card is missing?”	
4	<i>Counting</i>	
4/1	Finger counting to 5	
4/2	Planting potatoes	
5	<i>Extrapolation of number concepts to 10</i>	
5/1	Finger counting to 10	
5/2	Missing stairs, 1 to 10	
5/3	“I predict-here” on the number track (Same as NuSp 1.1/1)	
5/4	Sequences on the number track (Same as NuSp 1.2/1)	
6	<i>Zero</i>	
6/1	“Which card is missing?” (Including zero)	
6/2	Finger counting from 5 to zero	
7	<i>Extrapolation of number concepts to 20</i>	
7/1	Finger counting to 20: “Ten in my head”	
8	<i>Ordinal numbers first to tenth</i>	
8/1	“There are . . . animals coming along the track”	
8/2	“I’m thinking of a word with this number of letters.”	
8/3	“I think that your word is . . .”	
9	<i>Odds and evens</i>	
9/1	“Yes or no?”	
9/2	“Can they all find partners?”	
9/3	“Odd or even?”	
10	<i>Doubling and halving</i>	
10/1	“Double this and what will we get?”	
10/2	“Break into halves, and what will we get?”	
10/3	Doubles and halves rummy	
11	<i>Extrapolation of number concepts to 100</i>	69
11/1	Throwing for a target	69
11/2	Putting and taking	71
12	<i>Ordinal numbers, first to one hundredth</i>	72
12/1	Continuing the pattern	72
12/2	Sorting proverbs	72
13	<i>Rectangular numbers</i>	74
13/1	Constructing rectangular numbers [Same as Num 6.4/1]	74
13/2	The rectangular numbers game [Same as Num 6.4/2]	75
14	<i>Primes</i>	76
14/1	Alias prime [Same as Num 6.5/3]	76
14/2	The sieve of Eratosthenes [Same as Num 6.5/4]	77
14/3	Sum of two primes	77
15	<i>Square numbers</i>	79
15/1	Square numbers	79
15/2	An odd property of square numbers	79
16	<i>Relations between numbers</i>	82
16/1	“Tell us something new.”	82
16/2	“How are these related?”	83



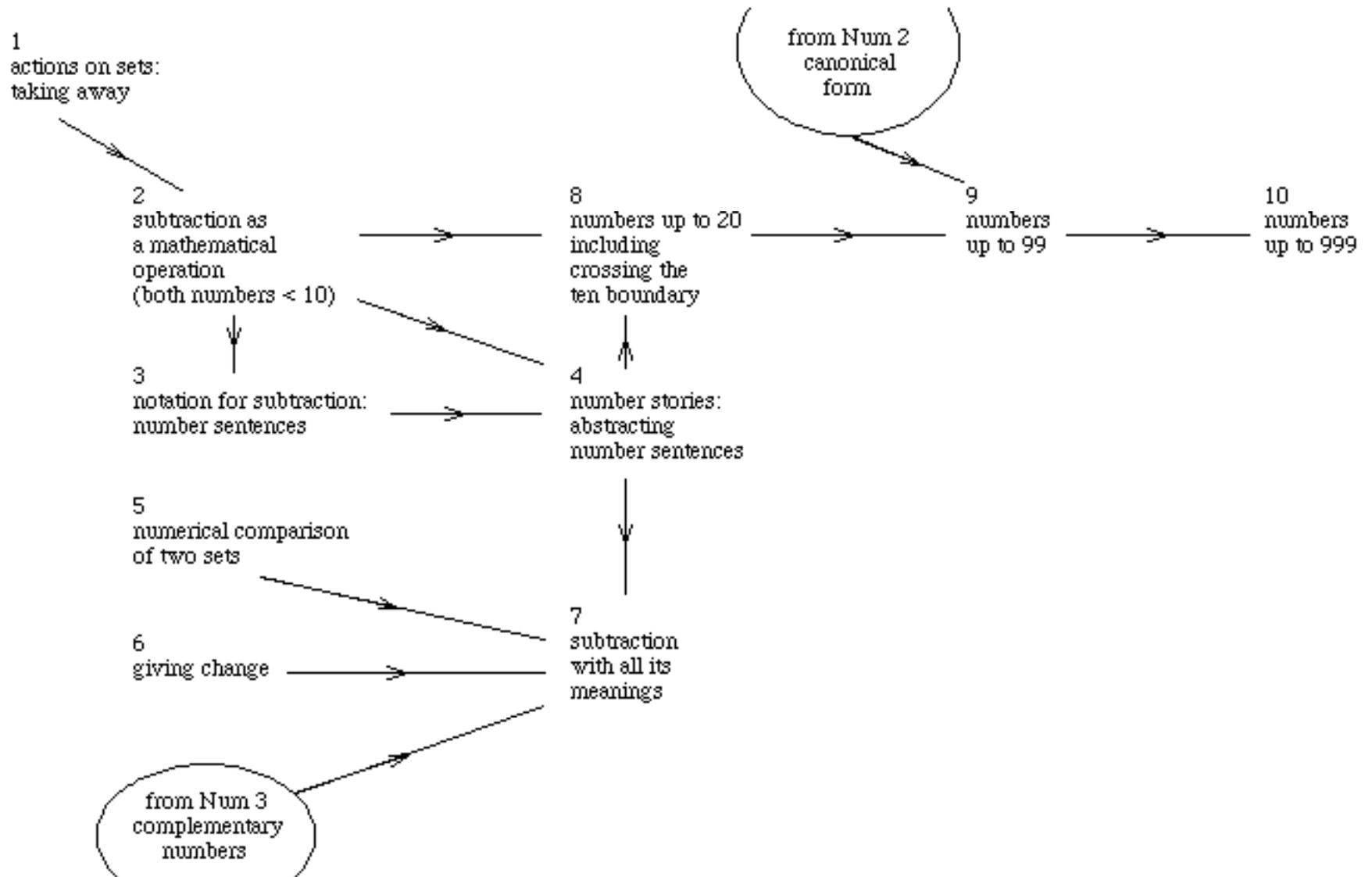
Num 2 The naming of numbers

1	<i>The number words in order (spoken)</i>		
1/1	Number rhymes		
2	<i>Number words from one to ten</i>		
2/1	Number rhymes to ten		
3	<i>Single-digit numerals recognized and read</i>		
3/1	Saying and pointing		
3/2	“Please may I have . . .?”		
3/3	Joining dots in order, to make pictures		
3/4	Sets with their numbers		
3/5	Sequencing numerals 1 to 10		
4	<i>Continuation of counting: 1 to 20</i>		
4/1	Number rhymes to twenty		
5	<i>Counting backwards from 20</i>		
5/1	Backward number rhymes		
5/2	Numbers backwards		
6	<i>Counting in twos, fives</i>		
6/1	Counting with hand clapping		
6/2	Counting 2-rods and 5-rods		
6/3	Counting money, nickels		
6/4	Counting sets in twos and fives		
7	<i>Extrapolation of counting pattern to one hundred</i>		
7/1	Counting in tens		
7/2	Counting two ways on a number square		
7/3	Tens and ones chart		
8	<i>Written numerals 20 to 99 using headed columns</i>		85
8/1	Number targets		86
8/2	Number targets beyond 100		87
9	<i>Written numerals from 11 to 20</i>		88
9/1	Seeing, speaking, writing 11-19		88
9/2	Number targets in the teens		89
10	<i>Place-value notation</i>		90
10/1	“We don’t need headings any more.”		90
10/2	Number targets using place-value notation		91
10/3	Place-value bingo		91
11	<i>Canonical form</i>		94
11/1	Cashier giving fewest coins		95
11/2	“How would you like it?”		97
12	<i>The effects of zero</i>		99
12/1	“Same number, or different?”		99
12/2	Less than, greater than		100
13	<i>Numerals beyond 100, written and spoken</i>		102
13/1	Big numbers		102
13/2	Naming big numbers		104
14	<i>Rounding (whole numbers)</i>		107
14/1	Run for shelter		107
14/2	Rounding to the nearest hundred or thousand		109
14/3	Rounding big numbers		111



Num 3 Addition

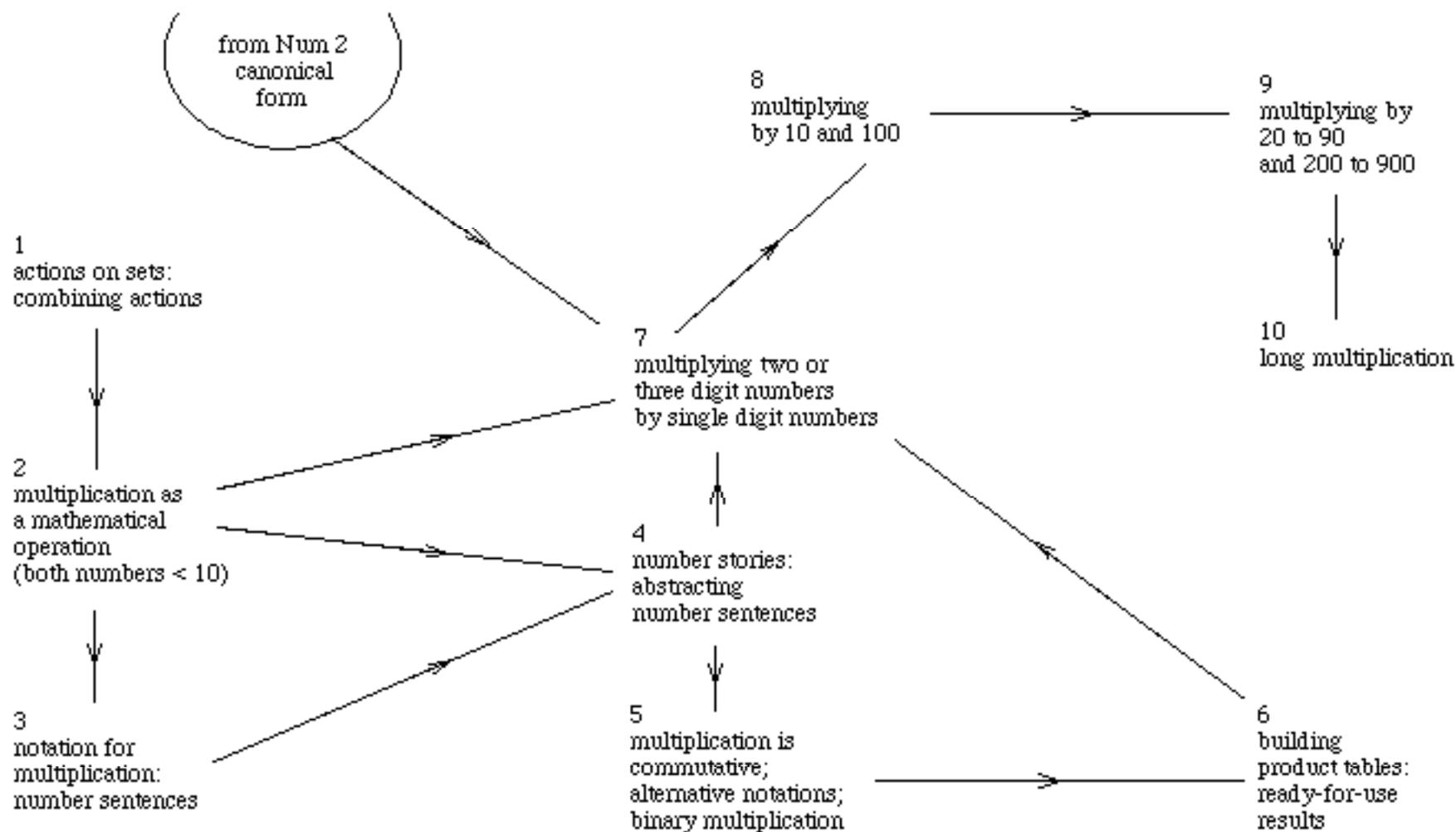
1	<i>Actions on sets: putting more (Total <10)</i>		
1/1	Start, Action, Result (do and say)		
1/2	Putting more on the number track (verbal) [NuSp 1.3/1]		
2	<i>Addition as a mathematical operation</i>		
2/1	Predicting the result (addition)		
2/2	“Where will it come?” (Same as NuSp 1.3/2)		
2/3	Stepping stones		
2/4	Crossing (Same as NuSp 1.3/3)		
3	<i>Notation for addition: number sentences</i>		
3/1	Writing number sentences for addition		
3/2	Write your prediction (addition)		
4	<i>Number stories: abstracting number sentences</i>		
4/1	Personalized number stories		
4/2	Abstracting number sentences		
4/3	Personalized number stories - predictive		
5	<i>Complementary numbers</i>		
5/1	The handkerchief game		
5/2	“Please may I have?” (complements)		
6	<i>Missing addend</i>		
6/1	“How many more must you put?”		
6/2	Secret adder		
6/3	Personalized number stories: what happened?		
7	<i>Adding past 10</i>		
7/1	Start, Action, Result over ten		
7/2	Adding past 10 on the number track (Same as NuSp 1.3/4)		
7/3	Slippery slope		
7/4	Addfacts practice		
7/5	Addfacts at speed		
7/6	Predictive number sentences past 10		
7/7	Explorers		
8	<i>Commutativity</i>		
8/1	Introducing commutativity		
8/2	Introducing non-commutativity		
8/3	Using commutativity for counting on		
8/4	Commutativity means less to remember		
9	<i>Adding, results up to 99</i>		113
9/1	Start, Action, Result up to 99		113
9/2	Odd sums for odd jobs		116
9/3	Renovating a house		117
9/4	Planning our purchases		116
9/5	Air freight		119
10	<i>Adding, results beyond 100</i>		121
10/1	Start, Action, Result beyond 100		121
10/2	Cycle camping		121
10/3	One tonne van drivers		122
10/4	Catalogue shopping		124



Num 4 Subtraction

1	<i>Actions on sets: taking away</i>	
1/1	Start, Action, Result (do and say)	
1/2	Taking away on the number track (do and say) [NuSp 1.4/1]	
2	<i>Subtraction as a mathematical operation</i>	
2/1	Predicting the result (subtraction)	
2/2	What will be left? (NuSp 1.4/2)	
2/3	Returning over the stepping stones	
2/4	Crossing back (NuSp 1.4/3)	
3	<i>Notation for subtraction: number sentences</i>	
3/1	Number sentences for subtraction	
3/2	Predicting from number sentences (subtraction)	
4	<i>Number stories: abstracting number sentences</i>	
4/1	Personalized number stories	
4/2	Abstracting number sentences	
4/3	Personalized number stories - predictive	
5	<i>Numerical comparison of two sets</i>	
5/1	Capture (NuSp 1.4/4)	
5/2	Setting the table	
5/3	Diver and winner	
5/4	Number comparison sentences	
5/5	Subtraction sentences for comparisons	
6	<i>Giving change</i>	125
6/1	Change by exchange	125
6/2	Change by counting on	126
6/3	Till receipts	126
7	<i>Subtraction with all its meanings</i>	128
7/1	Using set diagrams for taking away	128
7/2	Using set diagrams for comparison	129
7/3	Using set diagrams for finding complements	130
7/4	Using set diagrams for giving change	130
7/5	Unpacking the parcel (subtraction)	131
8	<i>Subtraction up to 20, including crossing the 10 boundary</i>	133
8/1	Subtracting from teens: choose your method	133
8/2	Subtracting from teens: "Check!"	135
8/3	Till receipts up to 20¢	135
8/4	Gift shop	136
9	<i>Subtraction up to 99</i>	137
9/1	"Can we subtract?"	138
9/2	Subtracting two-digit numbers	139
9/3	Front window, rear window	141
9/4	Front window, rear window - make your own	143
10	<i>Subtraction up to 999</i>	144
10/1	Race from 500 to 0	144
10/2	Subtracting three-digit numbers	145
10/3	Airliner	146
10/4	Candy store: selling and stocktaking	148

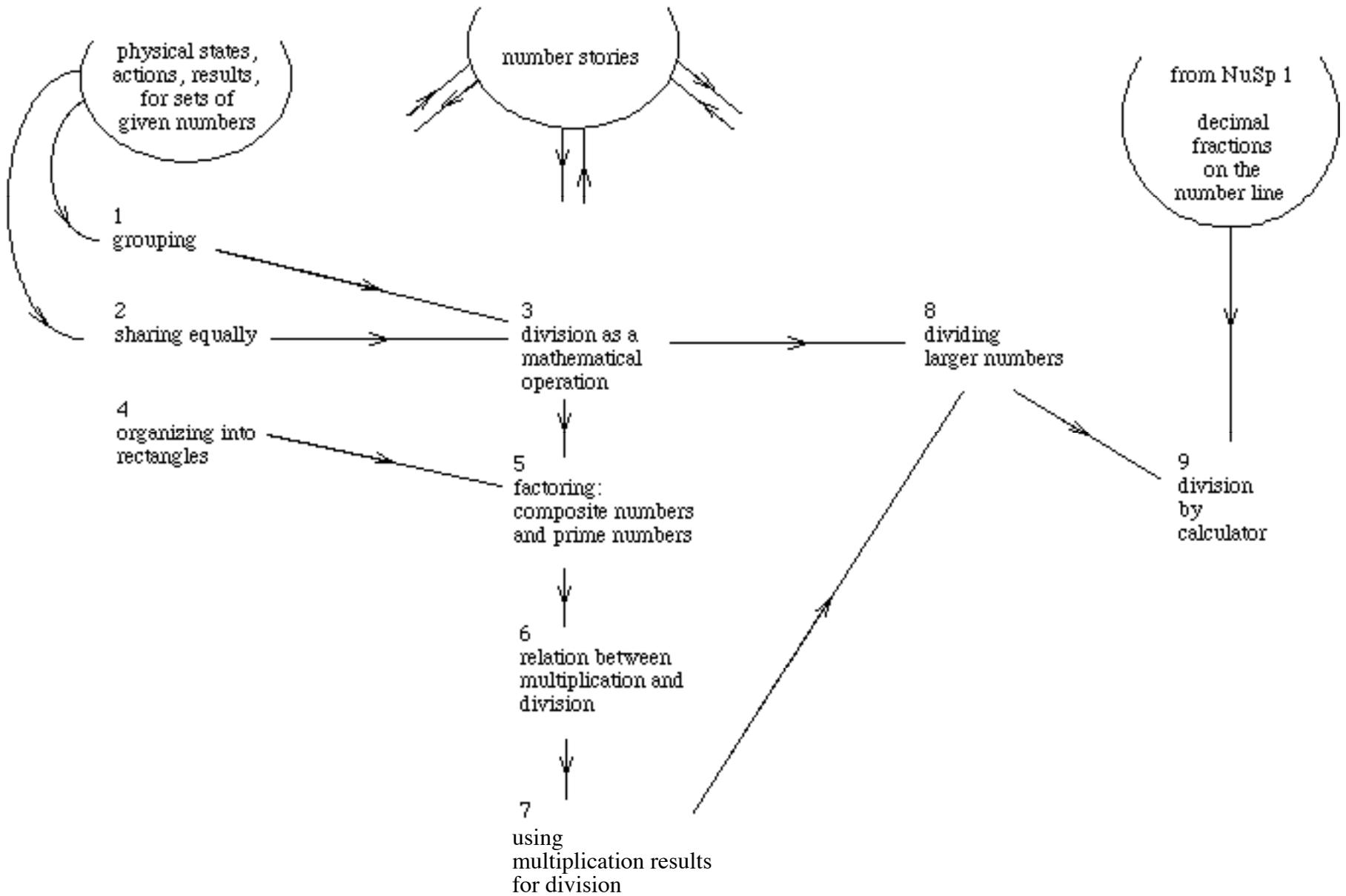
Num 5 Multiplication



Num 5 Multiplication

1	<i>Actions on sets: combining actions</i>		6	<i>Building product tables: ready-for-use results</i>	161
1/1	Make a set. Make others which match		6/1	Building sets of products	161
1/2	Multiplying on a number track		6/2	“I know another way”	163
1/3	Giant strides on a number track		6/3	Completing the products table	164
			6/4	Cards on the table	166
2	<i>Multiplication as a mathematical operation</i>		6/5	Products practice	166
2/1	“I predict - here” using rods		6/6	Multiples rummy	168
2/2	Sets under our hands				
3	<i>Notation for multiplication: number sentences</i>		7	<i>Multiplying 2- or 3-digit numbers by single-digit numbers</i>	170
3/1	Number sentences for multiplication		7/1	Using multiplication facts for larger numbers	171
3/2	Predicting from number sentences		7/2	Multiplying 3-digit numbers	172
			7/3	Cargo boats	172
4	<i>Number stories: abstracting number sentences</i>	151	8	<i>Multiplying by 10 and 100</i>	176
4/1	Number stories (multiplication)	151	8/1	Multiplying by 10 or 100	176
4/2	Abstracting number sentences	152	8/2	Explaining the shorthand	178
4/3	Number stories, and predicting from number sentences	153	8/3	Multiplying by hundreds and thousands	178
5	<i>Multiplication is commutative; alternative notations: binary multiplication</i>	155	9	<i>Multiplying by 20 to 90 and by 200 to 900</i>	179
5/1	Big Giant and Little Giant	156	9/1	“How many cubes in this brick?” (Alternative paths)	179
5/2	Little Giant explains why	158	9/2	Multiplying by n-ty and any hundred	180
5/3	Binary multiplication	159	10	<i>Long multiplication</i>	182
5/4	Unpacking the parcel (binary multiplication)		10/1	Long multiplication	182
	Alternative notations	159	10/2	Treasure chest	183

Num 6 Division

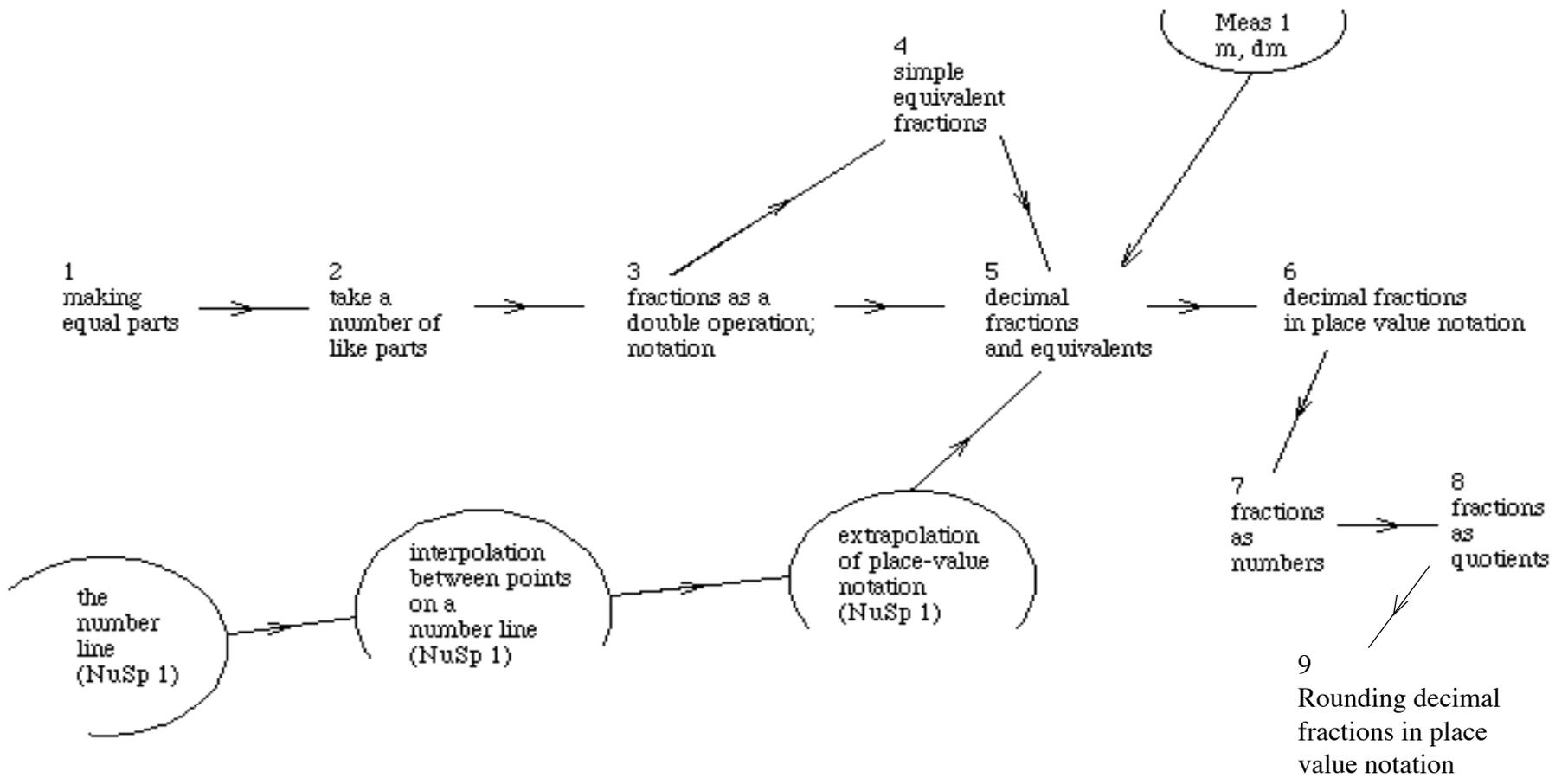


Num 6 Division

1	<i>Grouping</i>			
1/1	Start, Action, Result: grouping			
1/2	Predictive number sentences (grouping)			
1/3	Word problems (grouping)			
2	<i>Sharing equally</i>			
2/1	Sharing equally			
2/2	“My share is . . .”			
2/3	“My share is . . . and I also know the remainder, which is . . .”			
2/4	Word problems (sharing)			
3	<i>Division as a mathematical operation</i>	187		
3/1	Different questions, same answer. Why?	187		
3/2	Combining the number sentences	190		
3/3	Unpacking the parcel (division)	190		
3/4	Mr. Taylor’s game	191		
4	<i>Organizing into rectangles</i>	193		
4/1	Constructing rectangular numbers [Same as Num 1.13/1]	194		
4/2	The rectangular numbers game [Same as Num 1.13/2]	195		
5	<i>Factoring: composite numbers and prime numbers</i>		196	
5/1	Factors bingo		197	
5/2	Factors rummy		197	
5/3	Alias prime [Same as Num 1.14/1]		198	
5/4	The sieve of Eratosthenes [Same as Num 1.14/2]		199	
6	<i>Relation between multiplication and division</i>		201	
6/1	Parcels within parcels		201	
7	<i>Using multiplication results for division</i>		204	
7/1	A new use for the multiplication square		204	
7/2	Quotients and remainders		205	
7/3	Village Post Office		207	
8	<i>Dividing larger numbers</i>		209	
8/1	“I’m thinking in hundreds . . .”		209	
8/2	“I’ll take over your remainder”		210	
8/3	Q and R ladders		213	
8/4	Cargo Airships		214	
9	<i>Division by calculator</i>		217	
9/1	Number targets: division by calculator		217	

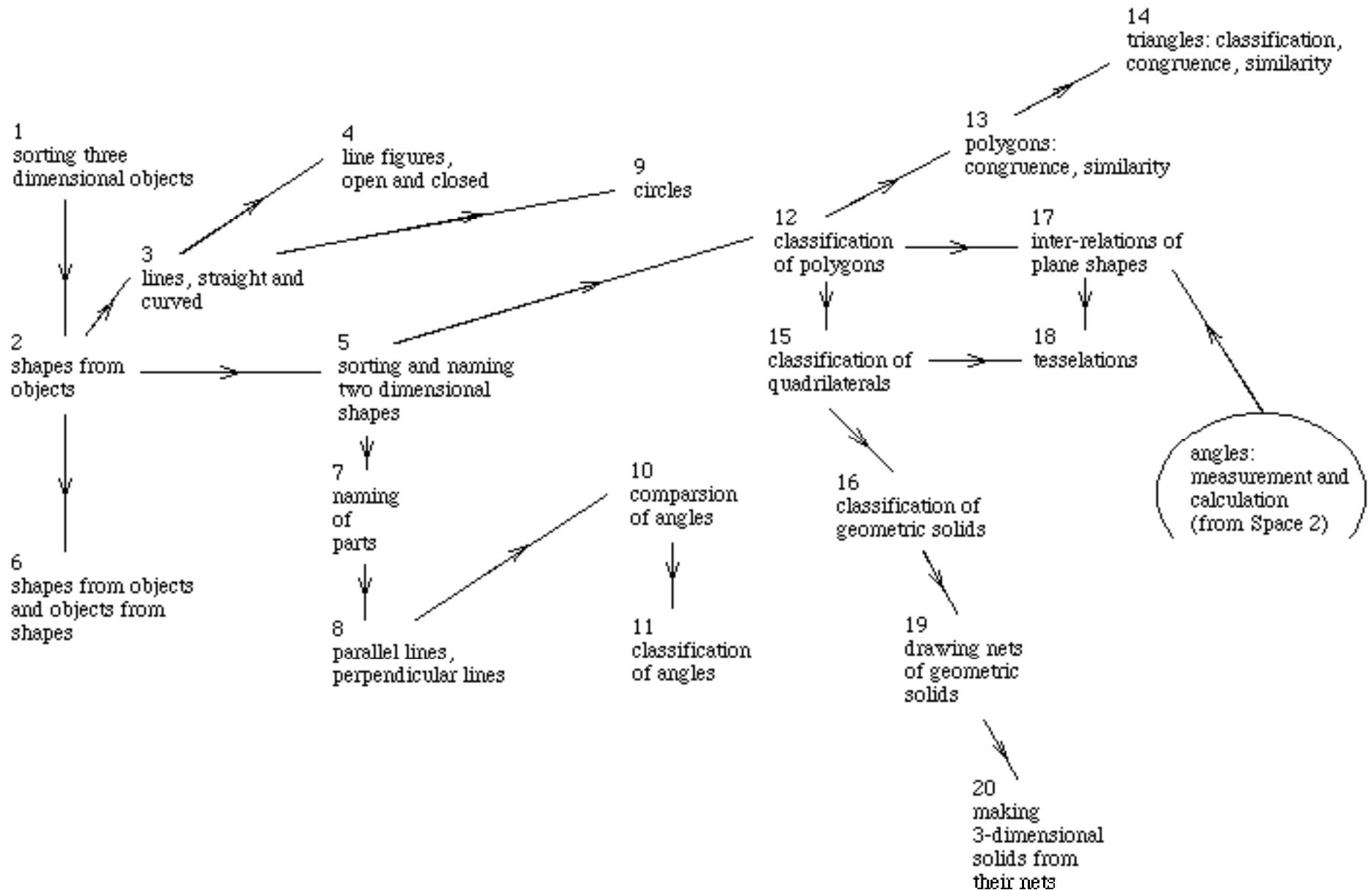
Num 7 Fractions

as a double operation
as numbers
as quotients



Num 7 Fractions

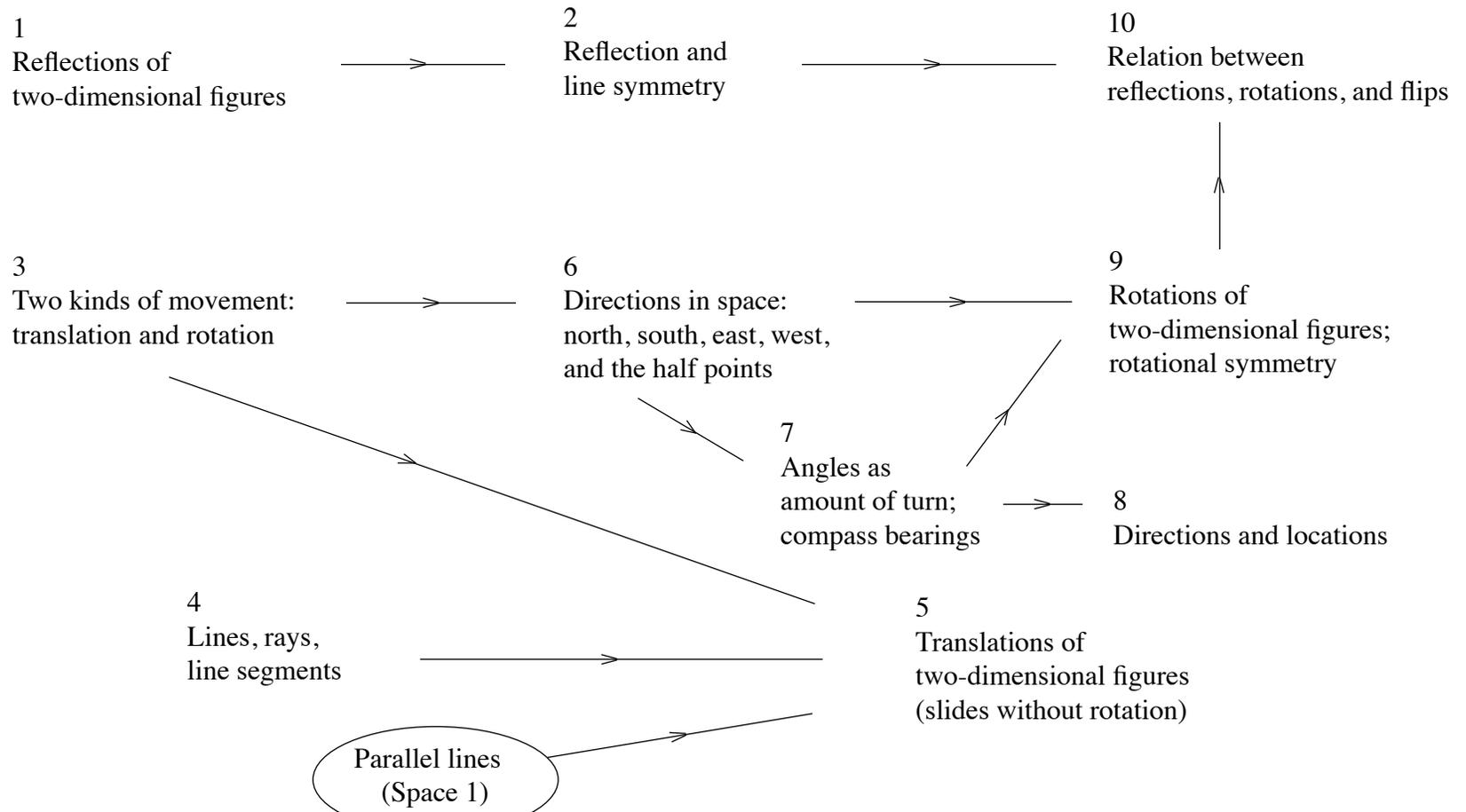
1	<i>Making equal parts</i>	219	5	<i>Decimal fractions and equivalents</i>	240
1/1	Making equal parts	219	5/1	Making jewellery to order	240
1/2	Same kind, different shapes	221	5/2	Equivalent fraction diagrams (decimal)	242
1/3	Parts and bits	223	5/3	Pair, and explain	243
1/4	Sorting parts	223	5/4	Match and mix: equivalent decimal fractions	244
1/5	Match and mix: parts	225			
2	<i>Take a number of like parts</i>	227	6	<i>Decimal fractions in place-value notation</i>	245
2/1	Feeding the animals	227	6/1	Reading headed columns in two ways	245
2/2	Trainee keepers, qualified keepers	229	6/2	Same number, or different?	247
2/3	Head keepers	230	6/3	Claiming and naming	248
3	<i>Fractions as a double operation; notation</i>	232	7	<i>Fractions as numbers. Addition of decimal fractions in place-value notation</i>	250
3/1	Expanding the diagram	233	7/1	Target, 1	251
3/2	“Please may I have?” (Diagrams and notation)	235	7/2	“How do we know that our method is still correct?”	252
4	<i>Simple equivalent fractions</i>	237	8	<i>Fractions as quotients</i>	254
4/1	“Will this do instead?”	237	8/1	Fractions for sharing	255
4/2	Sorting equivalent fractions	238	8/2	Predict, then press	257
4/3	Match and mix: equivalent fractions	239	8/3	“Are calculators clever?”	258
			8/4	Number targets by calculator	259
			9	<i>Rounding decimal fractions in place-value notation</i>	261
			9/1	Rounding decimal fractions	261
			9/2	Fractional number targets	262



Space 1 Shape

1	<i>Sorting three dimensional objects</i>			
1/1	Sorting by shape			
1/2	Do they roll? Will they stack?			
2	<i>Shapes from objects</i>			
2/1	Matching objects to outlines			
3	<i>Lines, straight and curved</i>			
3/1	Drawing pictures with straight and curved lines			
3/2	“I have a straight/curved line, like . . .”			
3/3	“Please may I have . . . ?” (Straight and curved lines)			
4	<i>Line figures, open and closed</i>			
4/1	“Can they meet?”			
4/2	Escaping pig			
4/3	Pig puzzle			
4/4	Inside and outside			
5	<i>Sorting and naming two dimensional shapes</i>			
5/1	Sorting and naming geometric shapes			
5/2	Sorting and naming two dimensional figures			
5/3	I spy (shapes)			
5/4	Claim and name (shapes)			
6	<i>Shapes from objects and objects from shapes</i>			
6/1	“This reminds me of . . .”			
7	<i>Naming of parts</i>			
7/1	“I am touching . . .” (three dimensions)			
7/2	“Everyone touch . . .” (three dimensions)			
7/3	“I am pointing to . . .” (two dimensions)			
7/4	“Everyone point to . . .” (two dimensions)			
7/5	“My pyramid has one square face . . .”			
7/6	Does its face fit?			
8	<i>Parallel lines, perpendicular lines</i>	263		
8/1	“My rods are parallel/perpendicular”	263		
8/2	“All put your rods parallel/perpendicular to the big rod”	264		
8/3	Colouring pictures	264		
9	<i>Circles</i>	266		
9/1	Circles in the environment	266		
9/2	Parts of a circle	266		
9/3	Circles and their parts in the environment	267		
9/4	Patterns with circles	268		
10	<i>Comparison of angles</i>	270		
10/1	“All make an angle like mine”	270		
10/2	“Which angle is bigger?”	271		
10/3	Largest angle takes all	271		
10/4	Angles in the environment	272		
11	<i>Classification of angles</i>			273
11/1	Right angles, acute angles, obtuse angles			273
11/2	Angle dominoes			274
11/3	“Mine is the different kind”			275
11/4	“Can’t cross, will fit, must cross”			275
12	<i>Classification of polygons</i>			278
12/1	Classifying polygons			278
12/2	Polygon dominoes			279
12/3	Match and mix: polygons			279
13	<i>Polygons: congruence and similarity</i>			282
13/1	Congruent and similar polygons			283
13/2	Sides of similar polygons			284
13/3	Calculating lengths from similarities			285
14	<i>Triangles: classification, congruence, similarity</i>			287
14/1	Classifying triangles			287
14/2	Triangle dominoes			288
14/3	Match and mix: triangles			288
14/4	Congruent and similar triangles			289
15	<i>Classification of quadrilaterals</i>			291
15/1	Classifying quadrilaterals			292
15/2	Relations between quadrilaterals			293
15/3	“And what else is this?”			294
15/4	“I think you mean . . .”			294
16	<i>Classification of geometric solids</i>			296
16/1	What must it have to be . . . ?			296
17	<i>Inter-relations of plane shapes</i>			298
17/1	Triangles and polygons			298
17/2	Circles and polygons			299
17/3	“I can see”			301
17/4	Triangles and larger shapes			302
18	<i>Tessellations</i>			304
18/1	Tessellating regular polygons			304
18/2	Tessellating other shapes			305
18/3	Inventing tessellations			306
18/4	Tessellating any quadrilateral			307
19	<i>Drawing nets of geometric solids</i>			309
19/1	Drawing nets of geometric solids			309
19/2	Recognizing solids from their nets			311
19/3	Geometric nets in the supermarket and elsewhere			312

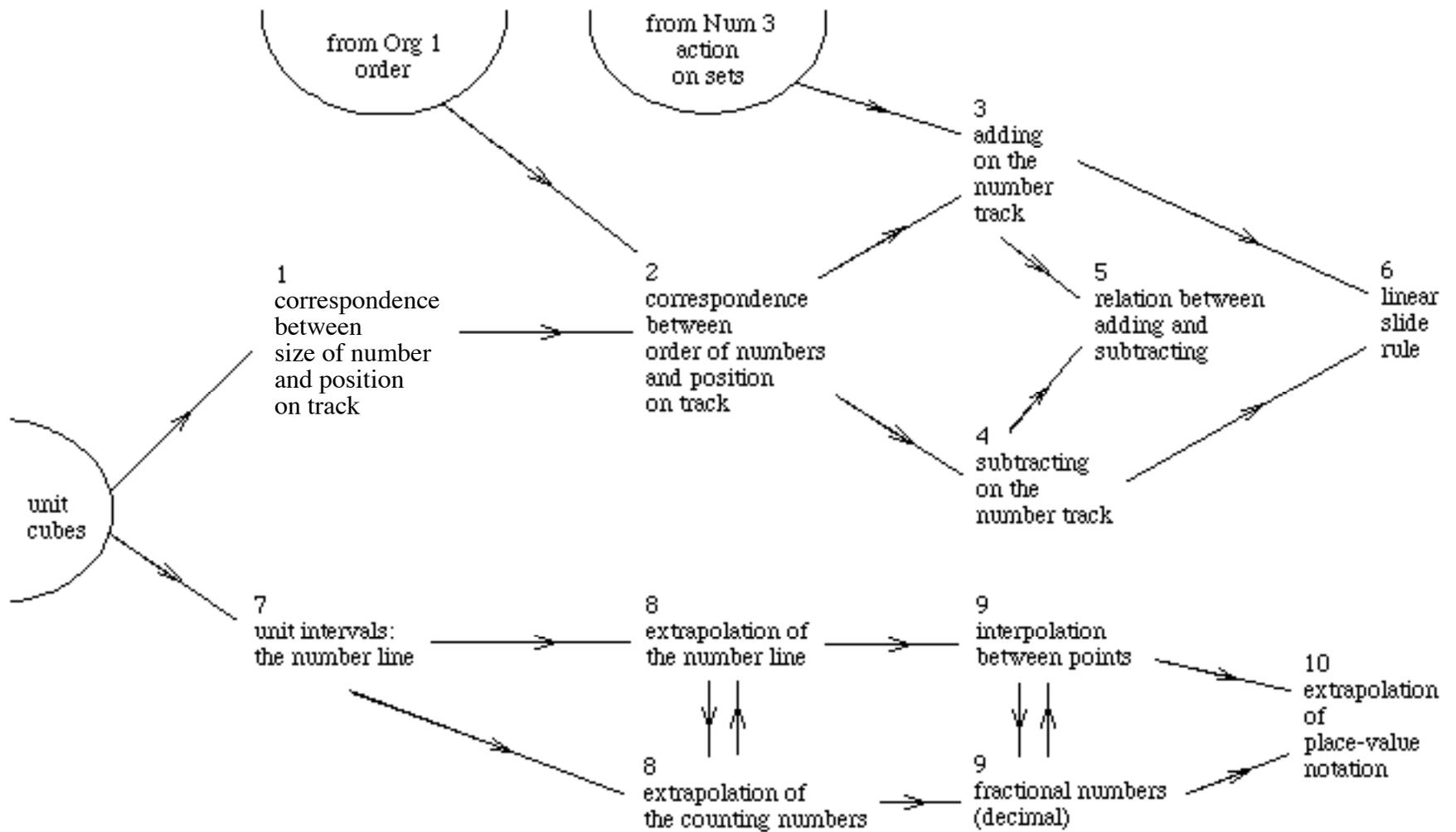
Space 2 Symmetry and motion geometry



Space 2 Symmetry and motion geometry

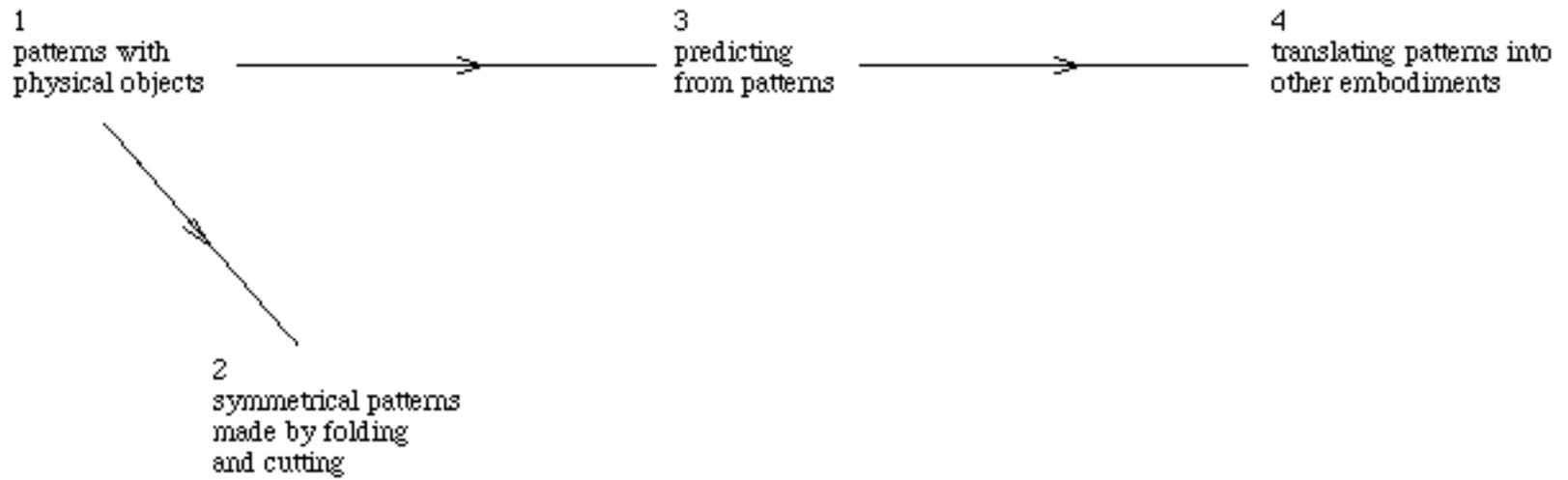
1	<i>Reflections of two-dimensional figures</i>	313		<i>Angles as amount of turn; compass bearings</i>	337
1/1	Animals through the looking glass	314	7/1	Directions and angles	338
1/2	Animals two by two	314	7/2	Different name, same angle	339
1/3	Drawing mirror images	315	7/3	Words from compass bearings	339
			7/4	Escape to freedom	341
2	<i>Reflection and line symmetry</i>	317	8	<i>Directions and locations</i>	344
2/1	Introduction to symmetry	317	8/1	Introduction to back bearings	344
2/2	Collecting symmetries	318	8/2	Get back safely	345
3	<i>Two kinds of movement: translation and rotation</i>	320	8/3	Where are we?	347
3/1	Walking to school	320	8/4	True north and magnetic north	350
4	<i>Lines, rays, line segments</i>	322	9	<i>Rotations of two-dimensional figures</i>	352
4/1	Talk like a Mathematician (lines, rays, line segments)	322	9/1	Centre and angle of rotation	352
4/2	True or false?	323	9/2	Walking the planks	353
5	<i>Translations of two-dimensional figures (slides without rotation)</i>	326	9/3	Introduction to rotational symmetry	355
5/1	Sliding home in Flatland	326	9/4	Match and mix: rotational symmetries	356
5/2	Constructing the results of slides	329	9/5	Match and mix (line symmetry)	357
5/3	Parallels by sliding	330	10	<i>Relation between reflections, rotations, and flips</i>	358
6	<i>Directions in space: north, south, east, west, and the half points</i>	332			
6/1	Compass directions	332			
6/2	Directions for words	333			
6/3	Island cruising	334			

NuSp 1 The number track and the number line



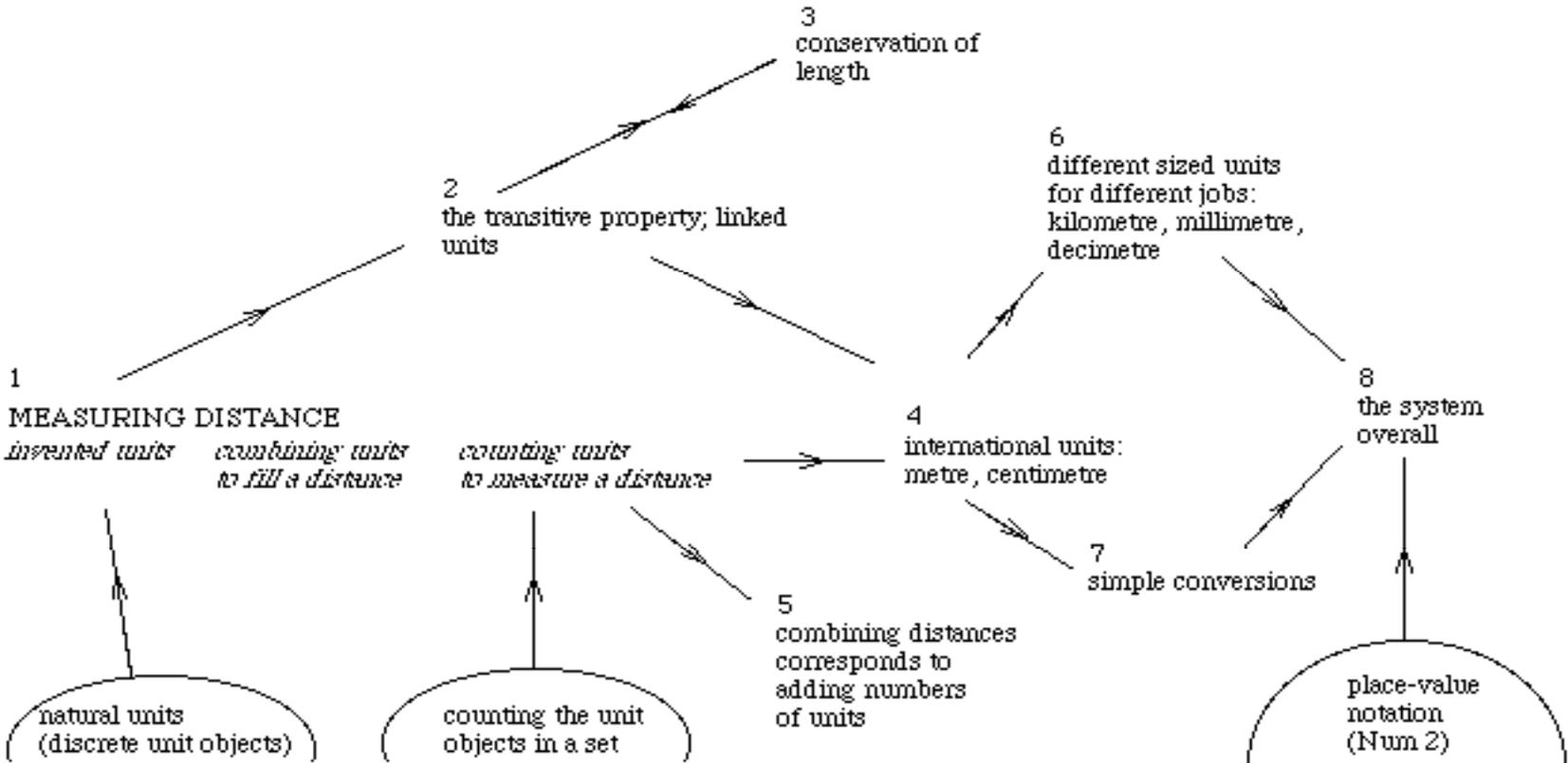
NuSp 1 The number track and the number line

1	<i>Correspondence between size of number and position on track</i>		
1/1	“I predict - here” on the number track		
2	<i>Correspondence between order of numbers and position on track</i>		
2/1	Sequences on the number track		
3	<i>Adding on the number track</i>		
3/1	Putting more on the number track (verbal) [Same as Num 3.1/2]		
3/2	“Where will it come?” (Same as Num 3.2/2)		
3/3	Crossing (Same as Num 3.2/4)		
3/4	“Where will it come?” (Through 10) (Same as Num 3.7/2)		
4	<i>Subtracting on the number track</i>		
4/1	Taking away on the number track (verbal) [Same as Num 4.1/2]		
4/2	What will be left? (Same as Num 4.2/2)		
4/3	Crossing back (Same as Num 4.4/4)		
4/4	Capture (Same as Num 4.5/1)		
5	<i>Relation between adding and subtracting</i>		
5/1	Slow bicycle race		
5/2	Ups and downs		
6	<i>Linear slide rule</i>		
6/1	Add and check		
6/2	Adding past 20		
7	<i>Unit intervals: the number line</i>		359
7/1	Drawing the number line		359
7/2	Sequences on the number line		360
7/3	Where must the frog land?		360
7/4	Hopping backwards		360
7/5	Taking		360
7/6	A race through a maze		361
8	<i>Extrapolation of the number line</i>		
	<i>Extrapolation of the counting numbers</i>		363
8/1	What number is this? (Single starter)		363
8/2	What number is this? (Double starter)		365
8/3	Is there a limit?		366
8/4	Can you think of . . . ?		366
9	<i>Interpolation between pairs</i>		
	<i>Fractional numbers (decimal)</i>		368
9/1	What number is this? (Decimal fractions)		369
9/2	Snail race		370
9/3	Snails and frogs		371
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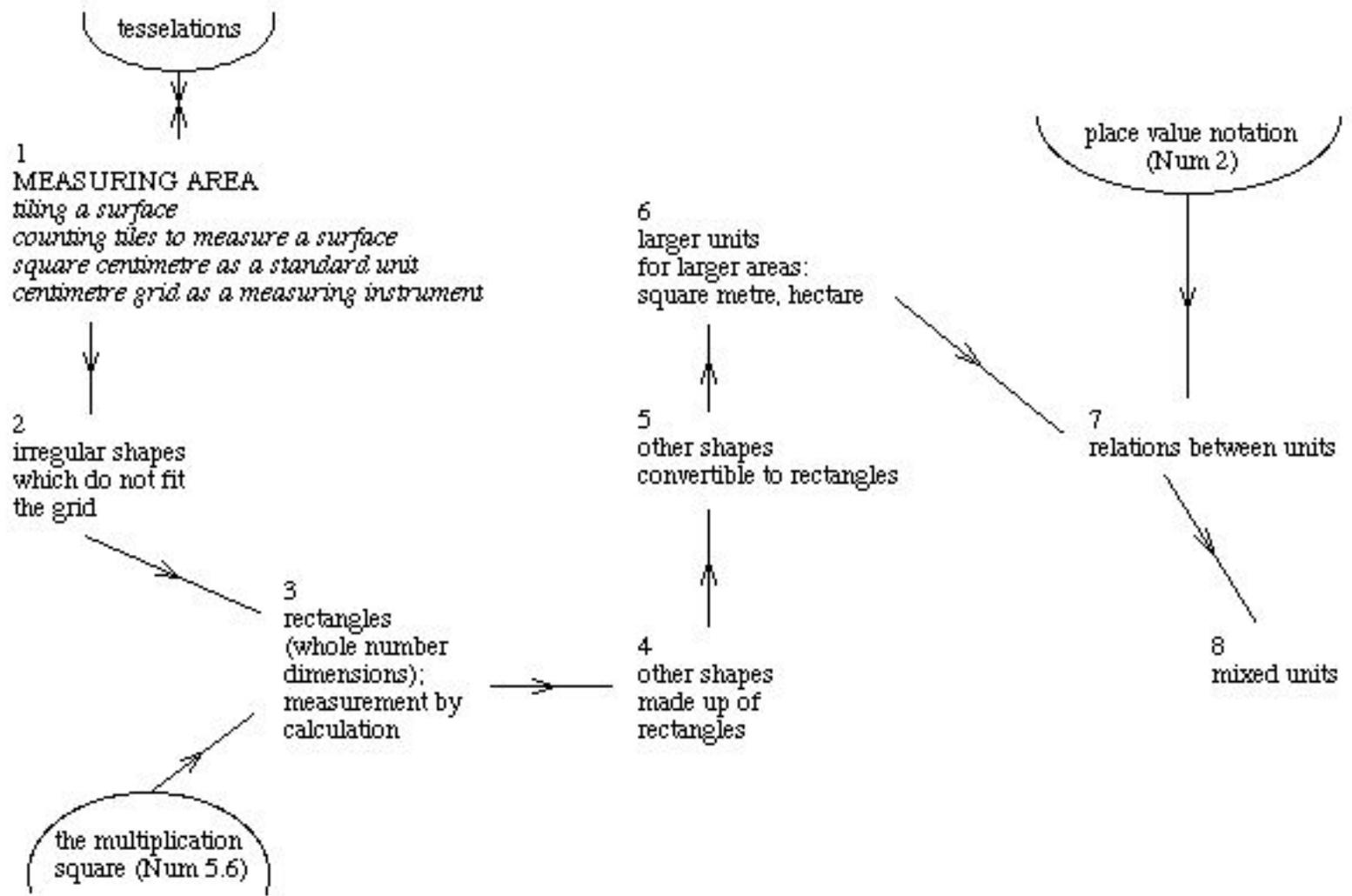
Patt 1 Patterns

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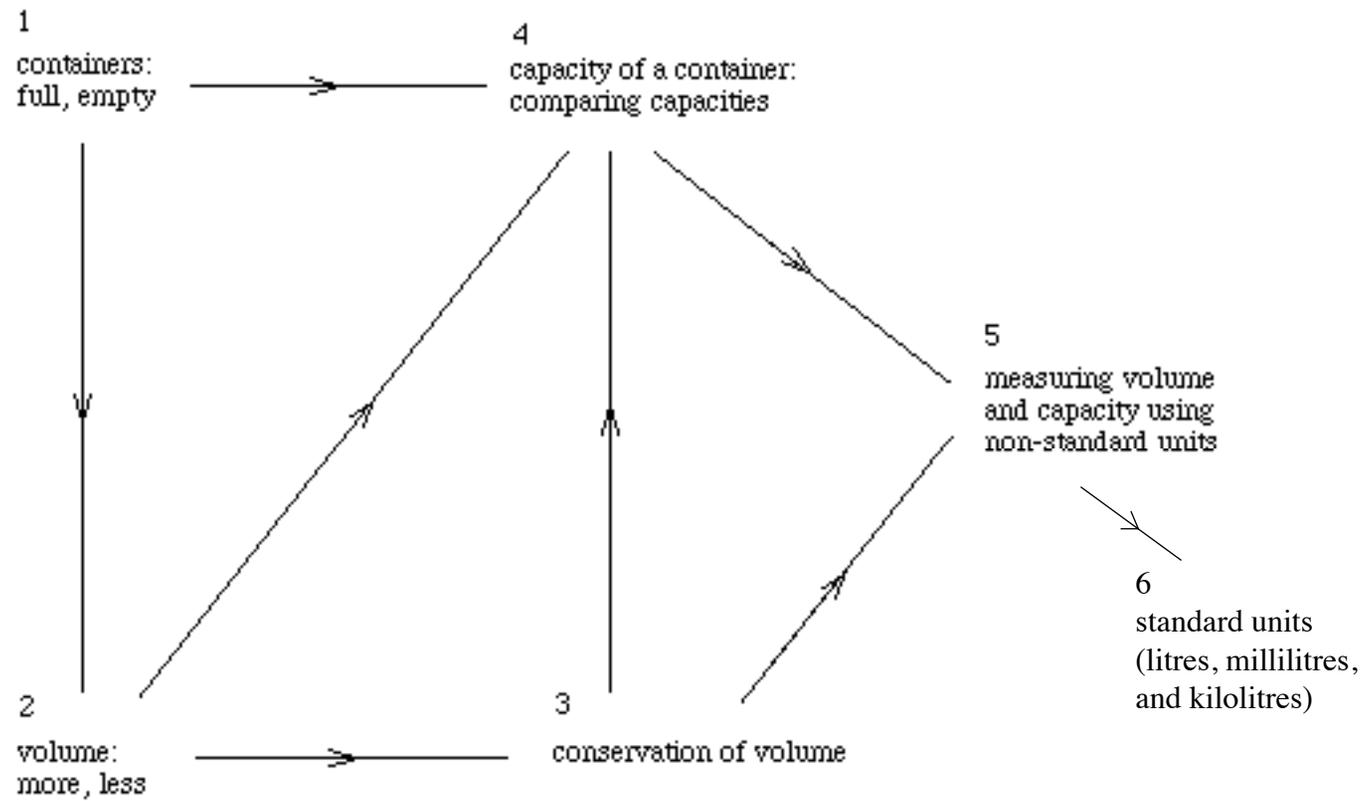
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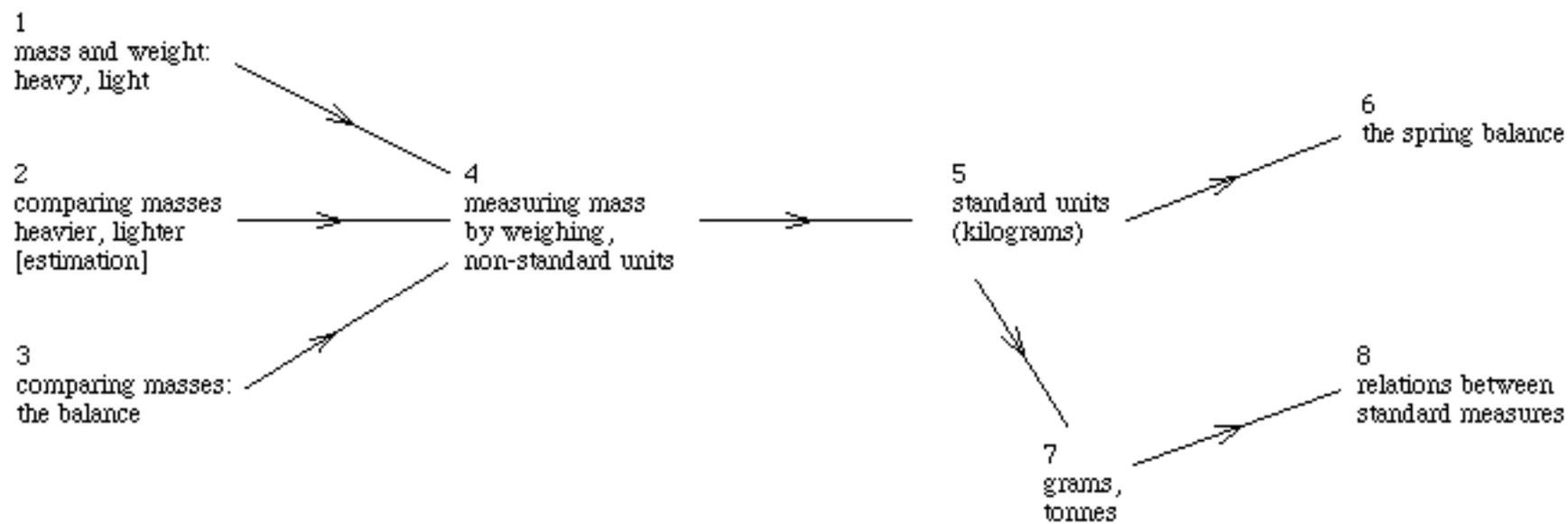
1	<i>Measuring area</i>				
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Meas 3 Volume and Capacity

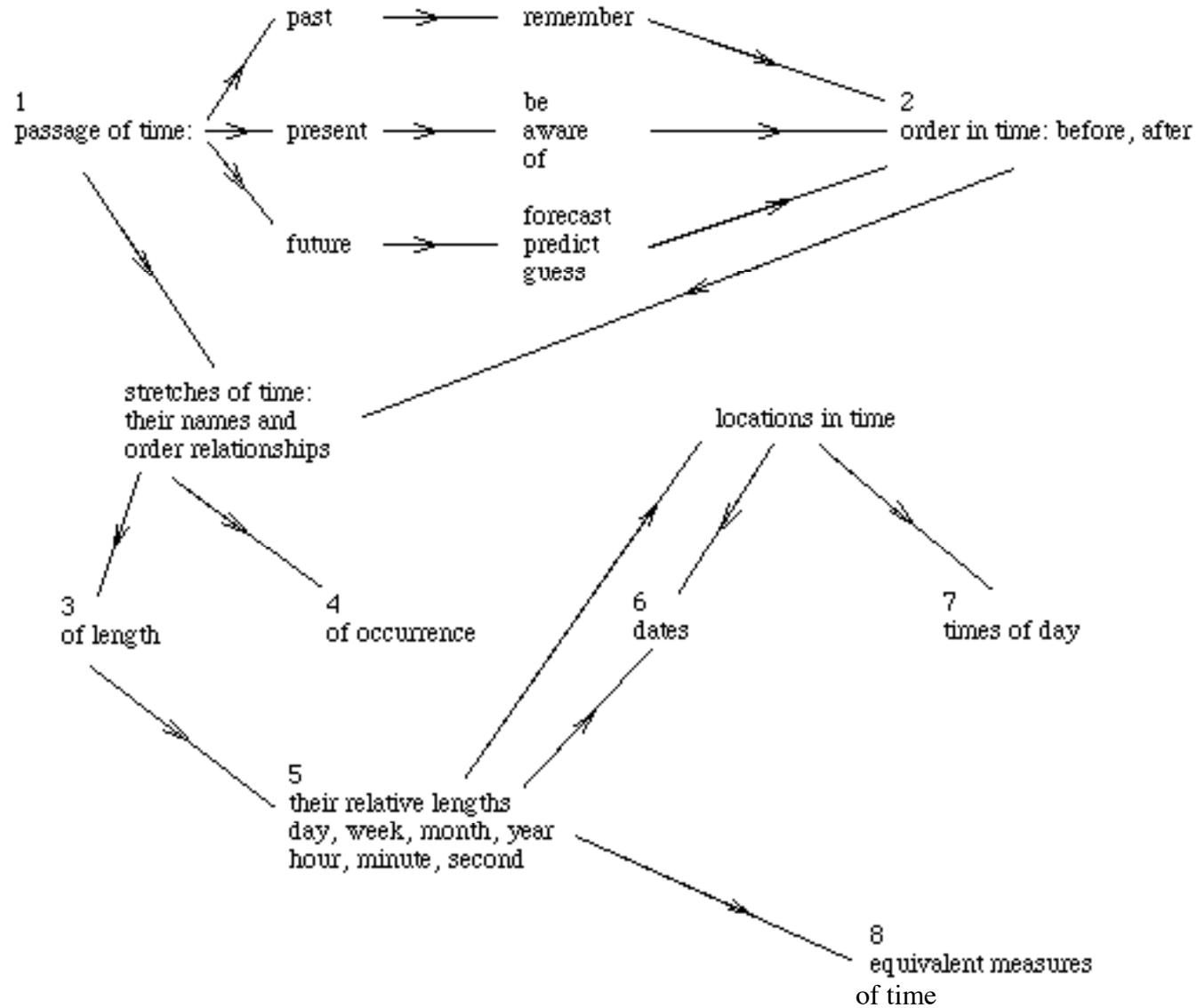
1	<i>Containers: full, empty</i>
1/1	Full or empty?
2	<i>Volume: more, less</i>
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3	<i>Conservation of volume</i>
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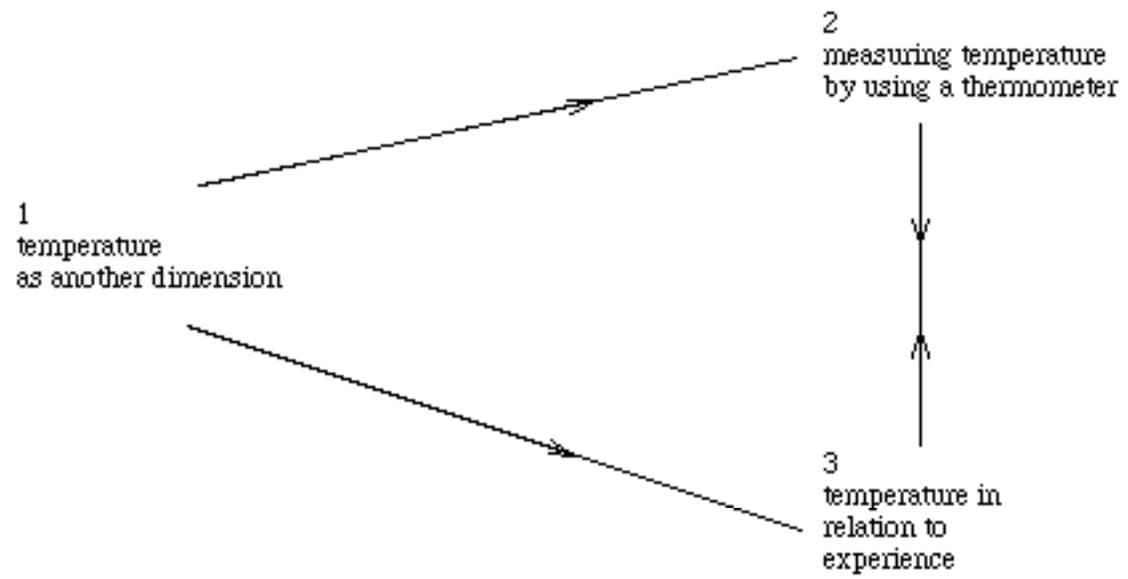
1	<i>Introductory discussion</i>		
	No activity; but a note for teachers		
2	<i>Comparing masses: heavier, lighter (estimation)</i>		
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THE NETWORKS AND ACTIVITIES



*'Slippery slope' [Num 3.7/3, SAIL Volume 1] **

* A frame electronically extracted from
NUMERATION, ADDITION AND SUBTRACTION,
a colour videocassette produced by the
Department of Communications Media,
The University of Calgary



Tens and hundreds of cubes [Org 1.13/1]

[Org 1] SET-BASED ORGANIZATION

Organizing in ways which lay foundations for concepts relating to number

Org 1.13 BASE TEN

Concept Tens, hundreds, thousands.

Abilities (i) To form and recognize sets of these numbers.
(ii) To recognize and match different physical representations of them, embodying different levels of abstraction.

Discussion of concepts Base 10 involves the same concepts as those used in bases 3, 4, 5. Practically, however, there are 3 major differences: (i) Our monetary system, and the measures used in commerce, technology and sciences, all use base 10; (ii) This base enables us to represent larger numbers with fewer figures; (iii) But 10 is too big to subitize. So manipulations which can be done perceptually for bases up to five depend on counting when working with base ten. Whether, using hindsight, it might have been better to choose some other base, may be for some an interesting subject for discussion. Many of us can remember the mixture of bases 4, 12, and 20 used in England's former monetary system. Our measurement of time still uses bases 60, 12, 24, 7, 30, 31, and 365! Computers work in base 2. This is not a good one for humans, who convert to base 16 (hexadecimal). But for most practical and theoretical purposes, base ten is the established one, and it is an important part of our job as teachers to help children acquire understanding, confidence, and fluency (in that order of importance) in working with base ten in decimal notation. It is with this aim that the foundations have been so carefully prepared in topics 1 to 12 of this network.

Activity 1 Tens and hundreds of cubes [Org 1.13/1]

A teacher-led activity for a small group. Its purpose is to help children to transfer to base ten the same thinking as they have developed for bases three, four, and five in earlier topics.

Materials • A large box of 1 cm cubes.

What they do

1. Introduce this activity by asking if they know what base is most used in everyday life. Relate this to the fact that we have ten fingers.
2. Ask them to make some ten-rods.
3. Ask them to join these up to make ten-squares.
4. Very likely they will run out of cubes: certainly they will not be able to make many ten-squares. This may surprise them! Tell them that the number of single cubes in a ten-square is called a hundred, which (as they have discovered) is quite a large number. Tell them that we're now on our way to big numbers.

Activity 2 Tens and hundreds of milk straws [Org 1.13/2]

An activity for a small group. Its purpose is to repeat grouping into tens and hundreds with a different embodiment.

- Materials*
- A large number of milk straws cut into halves.
 - Rubber bands.

- What they do*
1. Ask the children to find out how many hundreds, tens, and ones there are.
 2. Working together they group the straws first into tens with a rubber band around each ten.
 3. They then group the tens into bundles of ten tens, with a rubber band around each big bundle.

Activity 3 Thousands [Org 1.13/3]

A teacher-led activity for a small group. Its purpose is to combine their concept of a base cubed with that of base 10 to give them the concept of a thousand.

- What they do*
1. Repeat Activity 1, but ask them now to go on to see if they can make a ten-cube.
 2. They will not have enough cubes to do this. Explain that the number of single cubes in a ten-cube is called a thousand; it is a very large number.
 3. If we haven't enough single cubes to make a ten-cube, what can we do instead?
 4. One possibility is to make a hollow cube using a ten-square for base and ten-rods for the other eight edges. We then have to imagine a solid cube, made from 10 ten-squares on top of each other.
 5. Another possibility is to use the base 10 cubes from a multibase set. We then have to imagine all the cubes inside, indicated by the markings on the faces.

Discussion of activities In the preceding topics, children developed the concepts of a base, and of rods, squares, and cubes, using bases small enough for all the grouping to be done physically. With base ten, this becomes laborious; and when it comes to the cube of base ten, impractical.

However, provided that the earlier concepts have been well established, they can be combined with children's concept of the number ten in such a way that the concepts which were learned by using bases three, four, five expand to include base ten. This is what we have been doing in this topic. We have thus gradually been reducing children's dependence on physical objects for representing numbers and moving them towards representing them in other ways. We have also been extending children's concepts of numbers into the thousands. They have now come a long way, and should be allowed to return to the support of physical materials at any time when they feel the need. This will help to keep their concepts of numbers strong and active, and reduce the danger that when symbols become the main method for handling numbers, the concepts fade away.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Num 1] NUMBERS AND THEIR PROPERTIES

Numbers as mental objects which, like physical objects, have particular properties

Num 1.11 EXTRAPOLATION OF NUMBER CONCEPTS TO 100

Concepts The complete counting numbers in order to 100, grouped in tens.

Abilities (i) To state the number of a given set from 1 to 100.
(ii) To make a set of a given number from 1 to 100.

Discussion
of concept

Here we are concerned, not with a totally new concept (such as being odd or even), but with increasing the examples which a child has of his existing concept of number. Order and completeness provide a framework to ensure that these new examples fit the established pattern.

The key feature of the extrapolation is the idea that we can apply the process of counting, not only to single objects but to groups of objects, treating each group as an entity. So this topic links with all the topics in Org 1, 'Set-based organization,' which are shown in the upper part of the network as leading to topic 13 (Org 1.13, 'Base ten').

Activity 1 **Throwing for a target** [Num 1.11/1]

An activity for one player, two working together, or it may be played as a race between two players throwing alternately. Its purpose is to help children to extrapolate their number concepts up to 100, and to consolidate their use of grouping in tens.

Note Before this activity, children should have completed Org 1.

Materials

- A game board, see Figure 1.*
- For stage (a), base 10 material.
- For stage (b), a variety of other materials as described below.
- 2 dice.
- Slips of paper on which are written target numbers, e.g., 137, 285. (It is best not to go above 300 at most, or the game takes too long.)

* Provided in photomasters

What they do Stage (a)

1. The player throws the dice and adds the two numbers.
2. He puts down that number of ones on the game board, one per line and starting from the top.
3. Each time he reaches 10 ones he exchanges them for one 10, placing it on a line in the 'tens' column. Similarly, 10 tens are exchanged for one 100 to go in the 'hundreds' column.
4. He must finish with a throw of the exact number to reach the target number.
5. If the number is 6 or less he uses one die only.

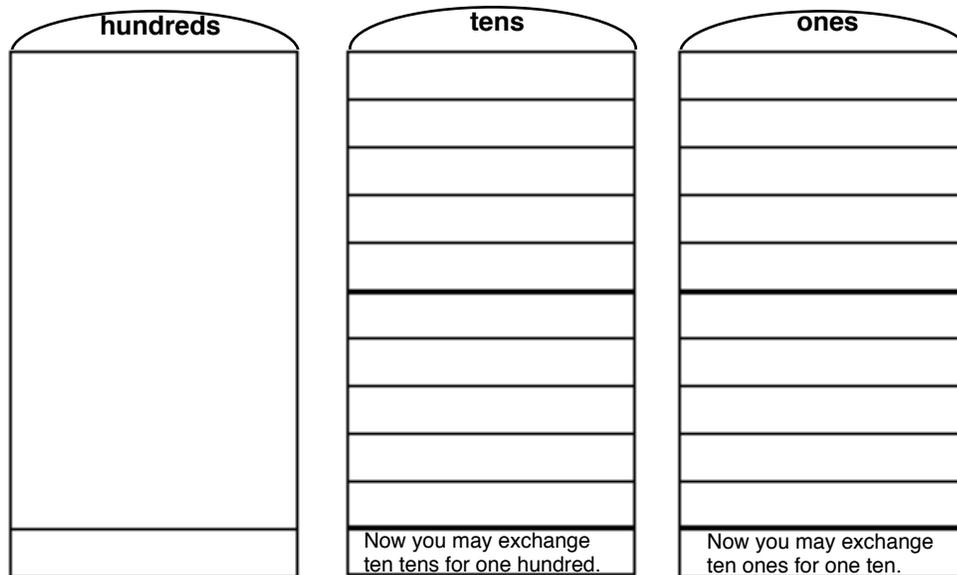


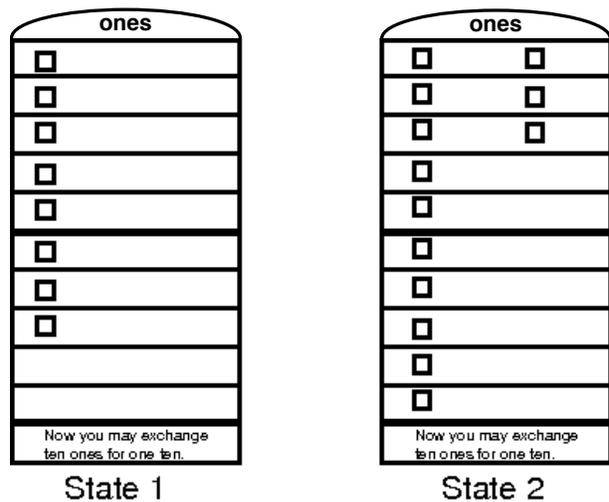
Figure 1 Throwing for a target

Stage (b)

To prevent children becoming too attached to a particular embodiment, this game should also be played with other suitable materials, such as popsicle sticks for ones, bundles of ten with a rubber band around, and ten bundles of ten. Drinking straws cut in half are good. Also: pennies, dimes, and \$1 coins.

Notes

- (i) At stage (a), children will often put more than ten ones in the 'ones' column, and these should be available. E.g., starting with the state on the left, and throwing 5, they put down 5 and get the state on the right. Then they remove the line of ten cubes, exchanging these for a ten-rod which they put in the ten column. This is a good way to begin.



- (ii) However, at stage (b) provide only 10 ones. This leads to a variety of strategies, which children should be given time to discover for themselves. It is important to restrain one's urge to tell them, or they may acquire the method but not its interiority. I recommend that you say something like this: "When you find short cuts, as soon as you are sure they are correct you may use them."

Activity 2 **Putting and taking** [Num 1.11/2]

A game for two players. It is a simple variant of Activity 1, with the same purpose.

- Materials*
- 50 (halved) milk straws in bundles of ten, for each player.
 - ‘Exchange pool’: 5 more tens and 20 ones.
 - 2 dice.
 - A box to put spares in.

- What they do*
1. Each player starts with 50 straws, in bundles of 10. They agree which will put, and which will take, each working separately with his own set of straws.
 2. They throw the dice alternately.
 3. The ‘putter’ begins and puts down at his place the number that is the total shown on the dice.
 4. The ‘taker’ plays next and takes away from the bundles at his place the total shown when he throws the dice.
 5. The ‘putter’ wins by reaching 100.
 6. The ‘taker’ wins by reaching 0.
 7. Exchanging of 1 ten for 10 ones will be necessary whenever they cross a ten boundary upwards or downwards.
 8. If the game is played with the same requirement as in Activity 1, that the exact number must be thrown to win, this could prolong the game unduly. It is therefore probably best to agree that a throw which would take the number past 100 or 0 is also acceptable.

Discussion
of activities

Activities 1 and 2 both use the now-familiar concepts of grouping in tens, and canonical form (see Org 1.12) to lead children on to 100. Although they are extrapolating their number concepts, which is schema building by Mode 3 (creativity), this extrapolation is also strongly based on physical experience (Mode 1 building). This is an excellent combination.

An interesting feature here is that the physical experience by itself would not be sufficient to lead to the formation of these new concepts. A suitable schema is also needed which can organize this experience, and contains a pattern ready to be extrapolated by this experience. This has never been put better than by Louis Pasteur, when he said: “Discoveries come to the prepared mind.”

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.12 ORDINAL NUMBERS, FIRST TO ONE HUNDREDTH

Concepts The ordinal numbers, first to one hundredth.

Abilities For any set of objects which have an order:

- (i) To identify the ordinal number of a given object;
- (ii) Given an ordinal number, to find the object to which it belongs.

Discussion of concepts The children should already have the concept of an ordinal number, with examples from first to tenth, so the present topic only requires the expansion of a pattern which they already know. This pattern is regular, apart from the teens; and even there, the derivation from the counting numbers is regular. The task is therefore a straightforward one, and it is likely that a number of children will already have learned it without having been specifically taught.

Activity 1 **Continuing the pattern** [Num 1.12/1]

A straightforward activity for a small group of children. Its purpose is to extend the pattern of ordinal numbers up to hundredth.

Materials None.

- What they do*
1. They begin by saying the ordinal numbers from first to tenth, in order clockwise around the table.
 2. “Now we go on in tens, like this. Tenth, twentieth, thirtieth, — can you go on?” The children continue in turn “fortieth, fiftieth, . . . hundredth.”
 3. If necessary, this is practised until the children can do it reliably.
 4. “Now we put in the numbers in between.” Starting at tenth, the children continue round the table, initially with prompting from yourself, “Eleventh, twelfth, thirteenth, . . . twentieth, twenty-first, twenty-second, . . . long enough to demonstrate that they have grasped the pattern.
 5. They should then be able to continue from any starting point, taking turns to begin with any ordinal number they like (below hundredth). The others then continue round the table. This continues until they can do it reliably, from any chosen starting point.

Activity 2 **Sorting proverbs** [Num 1.12/2]

An activity for children to do individually. Its purpose is to provide an application of the concepts introduced in Activity 1. This activity is not conceptually difficult, but requires concentration and accuracy in the use of ordinal numbers.

Materials For each child

- A copy of WINDING WORDS (Photomaster 6).*
- One of the four cards provided on Photomaster 7.*
- Pencil, paper, and eraser.

* Provided in *SAIL Volume 2a*.

- What they do*
1. Each card has on it four proverbs, in which the words have been replaced by their ordinal numbers in the winding assortment of words shown in WINDING WORDS. By using these numbers to find the corresponding words, each separate proverb can be sorted out from the mixture.
 2. To help organize the process, some of the words are printed in bold. The children may be left to find out for themselves what this indicates. They should also be told to write the words on a single line, like a sentence.
 3. What they must not do is to write anything on any of the cards. This would make things easier, but the object of this activity is to get lots of practice in using ordinal numbers. It would also spoil the cards for the next users.
 4. It is desirable that the children do not see each other's answers for the cards they have not yet done. One way to avoid this would be to have them all working on the same card, which would of course require extra copies.
 5. It may be necessary to call attention to the difference between pairs like 'thirtieth' and 'thirteenth.'
 6. Some of the results may be somewhat unexpected! I suggest that you indicate which words are incorrect, and leave it for the children to put them right.

Discussion
of activities

Activity 1 is simple, but it embodies a principle of great generality in mathematics: that of extending a known pattern. In Activity 2, I have taken the ordinals a little way beyond one hundredth, and have left this for the children to follow unassisted.

Having disentangled the proverbs, children may like to discuss their meanings, with your help. Paraphrasing is not always easy, since the essence of a proverb is that it says something in a way which is catchy and concise. The meaning may be more easily conveyed by examples. This discussion may also help to expand children's knowledge in other areas. For example, proverb 8 refers to migration; proverb 6, to good housekeeping (and it may also save losing a button). There may even be children who do not know how an omelette is made! I thought it more appropriate to leave proverb 14 as used in its country of origin, so an elementary knowledge of foreign currency is required.

A list of the proverbs is appended, for your own information.

1. There's many a slip twixt cup and lip.
2. He who laughs last laughs loudest.
3. Birds of a feather flock together.
4. Time and tide wait for no man.
5. All that glisters is not gold.
6. A stitch in time saves nine.
7. Don't look a gift horse in the mouth.
8. One swallow doesn't make a summer.
9. You can't make an omelette without breaking eggs.
10. Half a loaf is better than no bread.
11. Least said soonest mended.
12. Every cloud has a silver lining.
13. Darkest the night when dawn is nigh.
14. In for a penny in for a pound.
15. It's love that makes the world go round.
16. It's an ill wind that blows no one any good.

Num 1.13 RECTANGULAR NUMBERS

Concept Rectangular numbers, as the number of unit dots in a rectangular array. (See example below.)

Ability To recognize and construct rectangular numbers.

Discussion of concepts Here is another property which a number can have, or not have. The term ‘rectangular’ describes a geometrical shape, so when it is applied to a number it is being used metaphorically. Provided we know this, it is a useful metaphor, since the correspondences between geometry and arithmetic (also algebra) are of great importance in mathematics.

Rectangular numbers are closely connected with multiplication and with calculating areas. They provide a useful contribution to both of these concepts, and a connection between them.

Activity 1 **Constructing rectangular numbers** [Num 1.13/1]

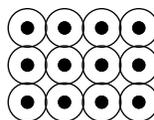
An activity for a small group, working in pairs. Its purpose is to build the concept of rectangular numbers.

Materials For each pair:

- 25 small counters with dots at their centres.
- Pencil and paper.

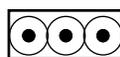
What they do 1. The activity is introduced along the following lines. Explain, “We think of these counters as dots which we can move around.”

2. Put out a rectangular array such as this one.



3. Ask, “What shape have we made?” (A rectangle.)
4. “How many counters?” (In this case, 12.)
5. “So we call 12 a rectangular number.”
6. Repeat, with other examples, until the children have grasped the concept. Note that the rectangles must be solid arrays like the one illustrated.
7. Next, let the children work in pairs. Give 25 counters to each pair, and ask them to find all the rectangular numbers up to 25.

8. If any of them think that a pattern like this might be a rectangle, remind them that the counters represent dots, draw these on paper, and ask: “Would you say this is a rectangle?”



9. The question always arises whether or not squares are to be included. The answer is “yes.” Both squares and oblongs are rectangles, in the same way as both girls and boys are children. This is discussed fully in Space 1.15.
10. Finally, let the children compare and check their lists with each other’s.

Activity 2 **The rectangular numbers game** [Num 1.13/2]

A game for two. It consolidates the concept of a rectangular number in a predictive situation. Children also discover prime numbers, though usually they do not yet know this name for them.

- Materials*
- 25 counters.
 - Pencil and paper for scoring.

- Rules of the game*
1. Each player in turn gives the other a number of counters.
 2. If the receiving player can make a rectangle with these, he scores a point. If not, the other scores a point.
 3. If when the receiving player has made a rectangle (and scored a point), the giving player can make a different rectangle with the same counters, he too scores a point. (E.g., 12, 16, 18).
 4. The same number may not be used twice. To keep track of this, the numbers 1 to 25 are written at the bottom of the score sheet and crossed out as used.
 5. The winner is the player who scores the most points.

Discussion of activities Activity 1 uses physical experience for building the concept of a rectangular number. Communication is also used to introduce the concept and to attach the accepted name. Schema building thus takes place by Modes 1 and 2 in combination.

Activity 2 uses rectangular numbers as a shared schema which forms the basis of a game. Success at this game depends on predicting whether or not a given number is rectangular. Each prediction is tested (Mode 1) as part of the game.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.14 Primes

- Concepts*
- (i) A prime number as one which is not a rectangular number.
 - (ii) A prime number as one which is not the multiple of any other number except 1 and itself.
 - (iii) A prime number as one which is not divisible by any other number except 1 and itself.

- Abilities*
- (i) To use these criteria to recognize prime numbers.
 - (ii) To give examples of prime numbers.
 - (iii) To be able to construct the set of primes less than 100.

Discussion of concepts This is a negative property, that of not having a given property. Children will already have formed this concept while playing the rectangular numbers game. In this topic we give the concept further meaning by relating it to other mathematical ideas.

Activity 1 **Alias prime** [Num 1.14/1]

A game for up to 6 players. Its purpose is to introduce children to the difference between composite and prime numbers, and give them practice in distinguishing between these two kinds of number.

Materials • Three counters for each player.

- Rules of the game*
1. Begin by explaining the meanings of ‘composite number’ and ‘prime.’ These concepts have been well prepared in earlier activities, and children have usually invented their own names for them.
 2. Explain that ‘alias’ means ‘another name for,’ often used to hide someone’s identity. In this game, all prime numbers use the alias ‘Prime’ instead of their usual name.
 3. Start by having the players say in turn “Eight,” “Nine,” “Ten,” . . . around the table.
 4. The game now begins. They say the numbers around the table as before, but when it is a player’s turn to say any prime number, they must not say its usual name, but say “Prime” instead.
 5. The next player must remember the number which wasn’t spoken, and say the next one. Thus the game would begin (assuming no mistake) “Eight,” “Nine,” “Ten,” “Prime,” “Twelve,” “Prime,” “Fourteen,” “Fifteen,” “Sixteen,” “Prime,” “Eighteen,” and so on.
 6. Any player who makes a mistake loses a life — i.e., one of her counters. Failing to say “Prime,” or saying the wrong composite number, are both mistakes.
 7. When a player has lost all her lives she is out of the game, and acts as an umpire.
 8. The winner is the last player to be left in the game.

Note When the players are experienced, they may begin counting at “One.” This gives rather a lot of primes for beginners!

Activity 2 The sieve of Eratosthenes [Num 1.14/2]

An activity for children working individually. As they get better at ‘Alias Prime,’ they will come to numbers about which they are doubtful whether these are prime or not. This is a method for constructing a set of primes by systematically eliminating multiples.

Materials

- For each child a 10 by 10 number square on which are written the numbers in order from 1 to 100. They can make these themselves, but two copies are included in the *Volume 2a* photomasters.

What they do

1. They begin by crossing off all the multiples of 2 except 2 itself.
2. Then they cross out all the multiples of 3 except 3 itself, and so on.
3. After 7, they will find that there are no new ones to cross out. Let them discover this. Why is it so? (This is quite a hard question.)
4. When finished, children check their results against each other’s.
5. They then make a separate list of the primes below 100.

Activity 3 **Sum of two primes** [Num 1.14/3]

A game for two. Its purpose is to give a further opportunity for thinking about prime numbers.

Materials

- Number cards 2 to 20.*

* Provided in the photomasters

What they do *Stage (a)*

1. The cards are shuffled and put on the table face down.
2. Player A turns over the top card and puts it down face up. (Say, 14.)
3. She now tries to express this as the sum of two primes, and if successful she scores a point. In this case she can: $13 + 1$.
4. If B can do so in a different way, she also scores a point. In this case she can: $3 + 11$.
5. B then turns over the next face-down card, and the game is continued as above.

Stage (b)

This is played as above, but using two dice to give larger numbers. These are of different colours, and one determines the tens, the other the ones.

Discussion
of activities

The concept of a prime number was implicit in Num 1.13/2, 'The rectangular numbers game,' when children encountered numbers which are not rectangular numbers. Here this negative property is made explicit and given a name. Activity 1 ('Alias prime') centres attention on prime numbers in a game based on this concept, and the concept is tested by Mode 2 (comparison with the schemas of others, leading sometimes to discussion).

Primes were initially conceptualized in relation to rectangular numbers, which is a physical and spatial metaphor. In Activity 2 they are thought of in a different way, that of not being multiples. They are therefore not divisible except by 1 and themselves. Multiplication and division are abstract mathematical operations, so the concept of a prime has now become independent of its physical/spatial beginnings. Independent does not however mean permanently detached from physical embodiments. While learning to climb a mountain higher and higher, we must always retain the ability to come down again. We also need to be able to put our mathematics to work, and this means relating it once again to physical embodiments.

Activity 3 is an abstract game with numbers, of a kind which only mathematicians enjoy. So if the children we are teaching do enjoy this game, we may congratulate both them and ourselves.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Square numbers [Num 1.15/1]

Num 1.15 Square numbers

Concept A square number as a particular kind of rectangular number, in which the corresponding rectangle has sides equal in length (i.e., is a square).

Ability To construct and recognize square numbers.

Discussion of concept Since a square is a particular kind of rectangle, we are now distinguishing a particular kind of rectangular number, using the term 'square' metaphorically as was the case for 'rectangle.' Note that 'a three-square' is a kind of square, whereas 'three squared' is a number: the number of dots in a three-square. The latter is the result of an operation on the number 3 (multiplied by itself); compare 3 doubled and 3 halved.

Activity 1 **Square numbers** [Num 1.15/1]

An activity for a small group. Its purpose is to introduce the concept of a square number.

Materials

- 50 counters.
- Pencil and paper for each child.

What they do

1. Ask the children to make the rectangles for the numbers 4, 9, 16.
2. Put these in the centre of the table, and ask the children what they notice.
3. Explain that since these rectangles are squares, their numbers also are called squares.
4. Ask the children each to make a list of all the square numbers up to 100. They may use counters if they wish, and if there are enough.
5. When they have finished, they compare lists. If there is any discrepancy, it is discussed.

Activity 2 **An odd property of square numbers** [Num 1.15/2]

This is an interesting investigation which can be done singly or in pairs. It relates the sequence of square numbers (present topic) and of odd numbers (Num 1.9 in *Volume 1*).

Materials

- Cubes: about 30 in each of several colours.
- Paper and pencil.

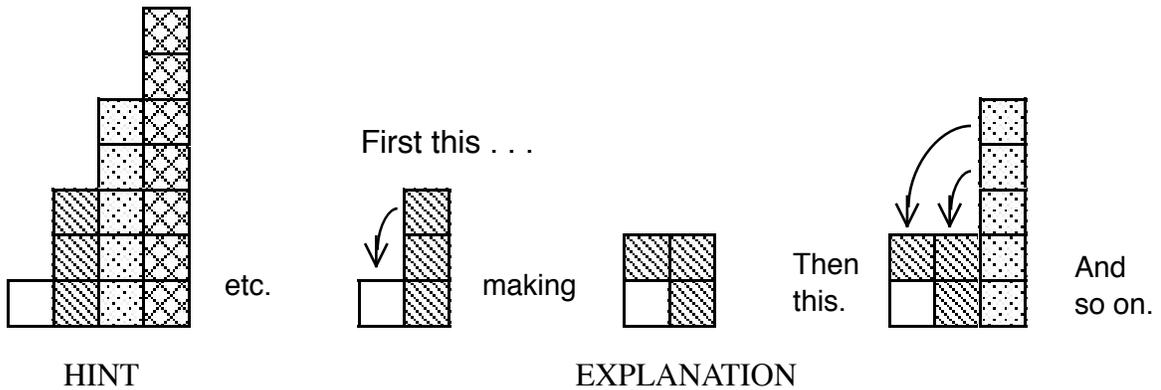
What they do

1. Ask the children to continue the following, each time adding the next odd number, until they have seen a pattern.

$$\begin{array}{rcl} 1 + 3 & = & 4 \\ 1 + 3 + 5 & = & 9 \\ 1 + 3 + 5 + 7 & = & 16 \end{array}$$

Num 1.15 Square numbers (cont.)

- When they have seen the pattern, ask them to try and find out why it is that adding the next odd number gives the next square number. This is difficult without the following hint: make a staircase of the odd numbers, using different colours for adjacent rods. The step from this to an explanation is shown in the diagram below; but if children can discover this for themselves, it is a pity to deprive them of this pleasure.



- A further step is to see that the sum of the first N odd numbers is equal to the square of N . This is difficult to express tidily in words without the use of N to stand for 'any number.'



An odd property of square numbers [Num 1.15/2]

Discussion
of activities

Activity 1 shows a familiar pattern. A new concept is introduced by examples of several physical embodiments. Children then use the concept to find more examples for themselves, first with and then without the help of physical embodiments. All three modes of schema building are thus brought into use.

Activity 2 introduces not just a property of certain numbers, but a relation between two patterns. One pattern is formed by the successive left-hand sides, the other by the successive right-hand sides of the equations.

This result can also be obtained starting from the other end, i.e. beginning with the explanation shown after step 2, and deriving the sequences of numbers on the left and right-hand sides. But I think that the way I have presented it makes it more intriguing, and increases the pleasure of discovering why.

For your
own
interest

Here is a question which has come up a number of times in discussion with teachers.

As explained in the discussion of the concept, 3 multiplied by itself is called 3 squared.

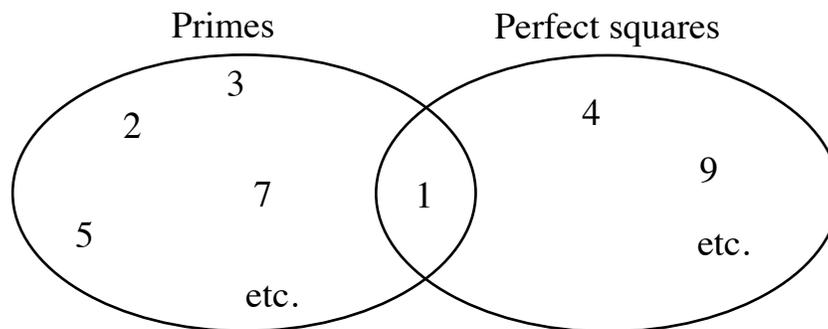
So 2 squared = 4
3 squared = 9
4 squared = 16 and so on.

The inverse relationship is called a square root.

So the square root of 4 = 2
the square root of 9 = 3
the square root of 16 = 4 and so on.

As long as we are talking about the counting numbers, only certain numbers have a square root. These are called perfect squares. So 4, 9, 16 . . . are perfect squares. They are also non-prime.

Are all perfect squares non-prime? In particular what about 1? You might like to think about this, and perhaps discuss it with others, before reading on. The discussion is probably more important than the result. Here are my own thoughts on the matter. 1 is not a square number, since a single dot can hardly be described as a square. However, 1 multiplied by itself is equal to 1, i.e., 1^2 is 1. So we now have the paradox that 1 is a perfect square arithmetically, but geometrically it is not a square number. By any definition, 1 is prime.



In this context, 1 is unique.

Num 1.16 Relations between numbers

Concept That all numbers are related, in many ways.

Abilities (i) To find several relationships between a given number and other numbers.
(ii) To find several relationships between two given numbers.

Discussion of concept From the second topic of this network onwards, there has been emphasis not only on individual numbers but on relationships between them. This is a particular case of one of the major emphases of this approach: that learning mathematics involves learning, not isolated facts, but a connected knowledge structure.

The relationships dealt with specifically in this topic are straightforward ones such as successor and predecessor, half and double. In other networks we meet (e.g.) sum and difference, multiple and factor. In this topic we go further, to bring forward a very general property of numbers: that every number is related to every other number, not just in one way but many. Indeed, the only limits to how many relationships we can find are those of time and patience.

Activity 1 “Tell us something new.” [Num 1.16/1]

A game for a small group. Its purpose is to give practice in thinking about properties of numbers, and their relationships with other numbers.

Materials

- A bowl of counters, say 3 per player.
- A pack of number cards (or any other way of generating assorted numbers). How high they should go depends on the ability of the players.

Rules of the game

1. The first card is turned over, e.g., 17.
2. Each player in turn has to say something new about this number, e.g.,
“17 is prime.”
“17 is $10 + 7$.”
“17 is $20 - 3$.”
“17 is half 34.”
“17 is four fours plus 1.” etc.
3. Each time a player makes a correct statement, the others say “Agree” and she takes a counter.
4. If it has been used already, the others say, “Tell us something new.”
5. If an incorrect statement is made, they say, “Tell us something true.”
6. When all the counters have been taken, many different properties and relationships have been stated about the same number.
7. The player with the most counters is the winner.
8. Another game may then be played with a different number.

Activity 2 **“How are these related?”** [Num 1.16/2]

A game for a small group. It is a variation of Activity 1, but harder.

Materials • As for Activity 1.

Rules of the game As for Activity 1, except that two numbers are used at a time. Players now have to say how these are related.

E.g., 25 and 7

“25 is more than 7.”

“25 is 8 more than 7.”

“3 sevens plus 4 makes 25.”

“Both 25 and 7 are odd numbers.”

“The sum of 25 and 7 is an even number.” etc.

Discussion of activities

These activities develop fluency and inventiveness in the handling of numbers. In Activity 1, children can choose to keep well within their domain (the mental region where they feel confident), or to stretch their abilities by devising more difficult answers and thereby to expand their domains. This is under each child’s control, so confidence is maintained and developed further. Activity 2 allows less choice, so it is more demanding.

They also increase the interconnections within children’s schemas. This makes much use of Mode 3 activity – the creative use of existing knowledge to find new relationships. Testing is by Mode 2, agreement and if necessary discussion, which in turn is based on Mode 3 – testing by consistency with what is already known.

OBSERVE AND LISTEN

REFLECT

DISCUSS



“How are these related?” [Num 1.16/2]



*'Number Targets' [Num 2.8/1] **

* A frame electronically extracted from
NUMERATION, ADDITION AND SUBTRACTION,
a colour videocassette produced by the
Department of Communications Media,
The University of Calgary

[Num 2] THE NAMING OF NUMBERS in ways which help us to organize them and use them effectively

Num 2.8 WRITTEN NUMERALS 20 TO 99 USING HEADED COLUMNS

Concept That a particular digit can represent a number of ones, tens, (and later hundreds . . .) according to where it is written.

- Abilities* (i) To match numerals of more than one digit with physical representations of ones, tens, (and later hundreds . . .).
(ii) To speak the corresponding number-words for numbers 20 to 99.

Discussion of concept

First, let us be clear about what a digit is. It is any of the single-figure numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (corresponding to the numbers we can count on our fingers). Just as we can have words of one letter (such as a), two letters (such as an), three letters (such as ant), and more, so also we can have written numerals of one digit (such as 7), two digits (such as 72), three digits (such as 702), and more.

The same numeral, say 3, can be used to represent 3 buttons, or shells, or cubes, or single objects of any kind. If we want to show which objects, we can do so in two ways. We can either write '3 buttons, 5 sea shells, and 8 cubes,' or we can tabulate:

buttons	sea shells	cubes
3	5	8

Likewise the same numeral, say 3, can be used to represent 3 single objects, or three groups of ten, or 3 groups of ten groups of ten (which we call hundreds for short). We could write '3 hundreds, 5 tens, and 8 ones'; or we could tabulate:

hundreds	tens	ones
3	5	8

We are so used to thinking about (e.g.) 3 hundreds that we tend not to realize what a major step has been taken in doing this. We are first regarding a group of ten objects as a single entity, so that if we have several of these we can count "One, two, three, four, five . . . *groups of ten*." Then we are regarding a *group of ten groups of ten* as another entity, which can likewise be counted "One, two, three . . ." And by the end of this network, we shall no longer be regarding these as groups of physical objects, but as abstract mental entities which we can arrange and rearrange. We shall also have introduced a condensed and abstract notation (place value).

These two steps need to be taken one at a time. While the first, described above, is being taken, we need to use a notation which states clearly and explicitly what is meant: i.e., headed column notation.

Also, because the correspondence between written numerals and number words only becomes regular from 20 onwards, we start children's thinking about written numerals here where the pattern is clear. The written numerals 11 - 19 are also regular, but their spoken words are not, so these are postponed until the next topic.

Num 2.8 Written numerals 20 to 99 using headed columns (cont.)

Activity 1 Number targets [Num 2.8/1]

A game for as many children as can sit so that they can all see the chart right way up; minimum 3. It follows on from ‘Tens and ones chart’ (Num 2.7/3 in *SAIL Volume 1*). Its purpose is to link the spoken number words, just learned, with the corresponding written numerals.

- Materials*
- Target cards.*
 - Tens and ones chart.*
 - Pencil and headed paper for each child.
 - Base 10 material, tens and ones.**

* Provided in the photomasters. See also Note (iii), following.

** This game should be played with a variety of base ten material such as milk straws or popsicle sticks in ones and bundles of ten; multibase material in base ten.

- What they do*
1. The target cards are shuffled and put face down.
 2. In turn, each child takes the top card from the pile. He looks at this, but does not let the others see it.
 3. Before play begins, 2 tens are put onto the chart. (This is to start the game at 20.)

Tens	Ones
4	7

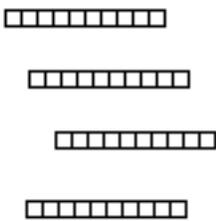
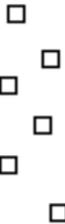
Tens	Ones
5	4

Target cards

Tens	Ones
------	------

Paper

4. The objective of each player is to have on the chart his target number of tens and ones.
5. Each player in turn may put in or take out a ten or a one.
6. Having done this, he writes on his paper the corresponding numerals and speaks them aloud in two ways. For example:

Tens	Ones
	

Chart

4	6
---	---

Writes

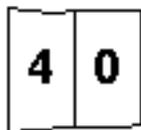
“four tens, six ones”;
“forty-six”

Speaks

Num 2.8 Written numerals 20 to 99 using headed columns (cont.)

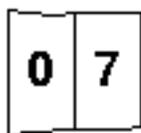
7. In the above example, if a player holding a 47 target card had the next turn, he would win by putting down one more one. He would then show his target card to confirm that he had achieved his target.
8. Since players do not know each others' targets, they may unknowingly achieve someone else's target for them. In this case the lucky player may immediately reveal his target card, whether it is his turn next or not.
9. When a player has achieved a target, he then takes a new target card from the top of the pile, and play continues.
10. The winner is the player who finishes with the most target cards.

Notes (i) If one side of the tray is empty, a corresponding zero must be written and spoken; e.g.,



“four tens, zero ones”;
“forty”

and also



“zero tens, seven ones”;
“seven”

- (ii) Players are only required to write the numbers they themselves make. It would be good practice for them to write every number, but we have found that children consider this to be cumbersome.
- (iii) Several sets of target cards should be prepared in which the numbers are reasonably close together, both the tens and the ones. If they are too far apart, the game may never end.

Variation It makes the game more interesting if, at step 5, a player is allowed two moves. For example, he may put 2 tens, or put 2 ones, or put 1 ten and take 1 one, etc. This may also be used if no one is able to reach his target.

Activity 2 Number targets beyond 100 [Num 2.8/2]

When children are familiar with numbers greater than 100, they can play this game with suitable modifications using targets in hundreds, tens and ones.*

* Provided in the photomasters

Discussion of activities

In preparation for place-value notation, it is important for children to have plenty of practice in associating the written symbols and their locations with visible embodiments of hundreds, tens, and ones . . . as well as associating both of these with the spoken words. In this topic ‘location’ means ‘headed column’; in Num 2.10 it will mean ‘relative position.’

This activity uses concept building by physical experience (Mode 1). The social context provided by a game links these concepts with communication (Mode 2) using both written and spoken symbols.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.9 WRITTEN NUMERALS FROM 11 TO 20

Concepts The written numerals 11 - 19 as having the same meanings as the number-words with which they are already familiar.

- Abilities*
- (i) To match the numerals 11 - 19 with the spoken number-words.
 - (ii) To match both with physical embodiments of these numbers.

Discussion of concepts The discussion of Num 2.8 applies equally here. However, the clear and regular correspondence which we find from 20 onwards, e.g.:

2 tens, 7 ones
twenty seven
27

does not apply from 11-19. E.g., although 1 ten, 2 ones is written (as we would expect) 12, it is spoken not as “onety two” but as “twelve.” And 1 ten, 7 ones is written (as we would expect) 17, but the spoken form is backwards, seventeen. So the present topic contains, implicitly, the notion of irregularity – departure from an expected pattern.

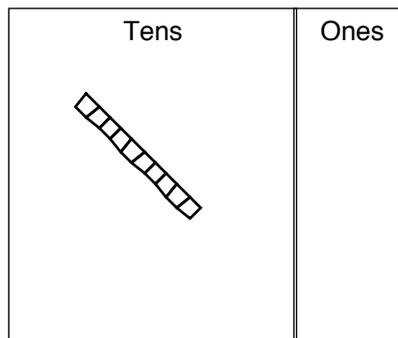
Activity 1 Seeing, speaking, writing 11-19 [Num 2.9/1]

A teacher-led activity for a small group. Its purpose is to relate the number-words spoken but not written down in Num 1.7/1 to the written numerals 11-19.

- Materials*
- Tens and ones chart.
 - Base 10 material.
 - Paper and pencil for yourself.

Suggested sequence for the discussion

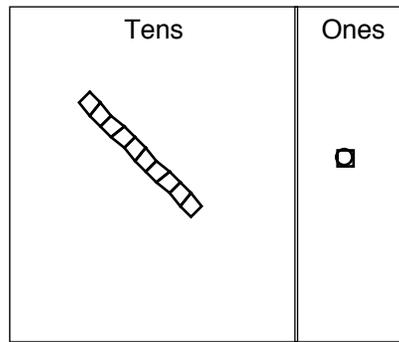
1. You put in



2. Write

Tens	Ones
1	0

3. Say: "One ten, zero ones. Ten."
4. Put in



5. Write



6. Say: "One ten, one one. Eleven."
7. Carry on through the teens. (So you don't put in another ten.)
8. Soon the children will join in. You might point out that if these followed the same pattern as 21, 31 . . . we would talk about onety-one, onety-two, etc., and explain that in olden times people hadn't thought about it carefully: so the 'teen' names came to be attached, and have 'stuck.'

Activity 2 Number targets in the teens [Num 2.9/2]

The same number targets game as in the topic just before this (Num 2.8/1) should now be played, starting with the tens and ones chart empty.

Materials As for (Num 2.8/1) except:

- Target cards now 11 to 19 (provided in the photomasters).

Discussion of activities The written numerals 10 - 19 follow the same pattern as those from 20 on, so these concepts are acquired by extrapolation (Mode 3 schema building). However, the spoken number-words do not follow this pattern, and do not correspond well to the numerals, even from 13 on. ("Thirteen," "fourteen" . . . are read from right to left.) However, children will already be familiar with the spoken number-words, and the activity now links these to the numbers in physical embodiments (tens and ones chart), and to the written numerals.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.10 PLACE-VALUE NOTATION

Concept That a particular digit can represent a number of ones, tens, hundreds, according to whether it comes first, second, third in order reading from right to left.

- Abilities*
- (i) To identify separately which digits represent ones, tens, hundreds, by their positions relative to each other.
 - (ii) To read aloud two and three digit numerals.
 - (iii) To match these with physical representations.

Note Hundreds are not included until the second time around, after Num 2.13, ‘Numerals beyond 100, written and spoken.’

Discussion of concept Provided that we have only one digit in each column (which may be a zero), we can leave out the headings and ruled columns and still know what each digit stands for. The result is a brilliantly simple notation, whose brilliance is easily overlooked just because it is so simple. The Greek mathematicians, excellent though they were in many ways, did not think of it; nor did the Romans, nor the earlier Egyptians nor the Babylonians. As a result, they made relatively slow progress in arithmetic and algebra.

It is also a condensed and abstract notation, and this is why it has been approached with such careful preparation in the preceding activities.

Activity 1 “We don’t need headings any more.” [Num 2.10/1]

A teacher-led activity for a small group. Its purpose is to introduce children to place-value notation, as described above.

Materials • Pencil and paper.

Suggested sequence for the discussion 1. Write some number between 21 and 99, with headed columns, as below.

Tens	Ones
4	7

- 2. Ask, “Can you say this number?” Accept either “forty-seven” or “four tens, seven ones.”
- 3. Then say, “Can you say it another way?” to get the alternative reading. (At this stage we want both, every time.)
- 4. Repeat with other numbers, including ones between 11 and 19. Numbers such as 30 should be read as “Three tens, zero ones; thirty.”

5. Fold the top of the paper under so that the headings for tens and ones do not show, and write another number. Ask the children if they can still say the numbers as before. Practise this until they are confident, and then do the same without the dividing line.
6. If the children are fully proficient in the new notation, say, "So we don't always need the headings now, though they are still useful sometimes," and continue to Activity 3. Children who have progressed through the earlier topics in this network should have no difficulty at this stage. For those who do, it would be best to go back to topics Num 2.8 and 2.9 to ensure that they are fully prepared. They should then do Activity 2 of this topic.

Activity 2 Number targets using place-value notation [Num 2.10/2]

The children should now play again the number targets game, exactly as in Num 2.8/1 and Num 2.9/1, but with plain number cards* and plain paper (no headings).

* Provided in the photomasters.

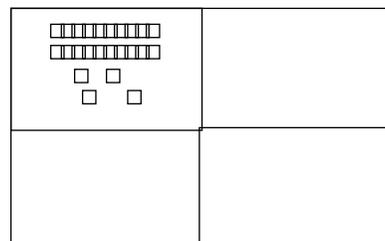
Activity 3 Place-value bingo [Num 2.10/3]

A game for 5 or 6 players. Its purpose is to consolidate children's understanding of the relationships between written numerals in place value notation, the same read aloud, and physical embodiments of the numbers.

- Materials*
- Number cards 0 to 59.*
 - Base 10 material, tens and ones.
 - Die 0 to 9.**
 - Die 0 to 5.**
 - Shaker.
 - For each player, pencil and letter-size paper.
- * Provided in the photomasters.
 ** Spinners may be used instead.

Rules of Stage (a) the game Preparation

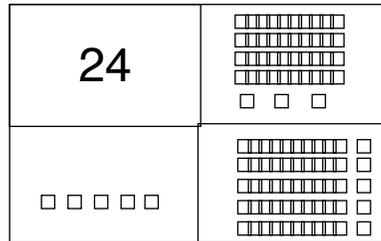
1. The players fold their papers once each way to make 4 rectangular spaces. These are used as bingo cards.
2. The first player throws the two dice. The 0 to 5 die gives the tens, the 0 to 9 die gives the ones.
3. Suppose that he gets 2 tens and 4 ones. He takes the corresponding ten-rods and ones from the box and puts these into the first space on his bingo card.
4. The other players do likewise in turn.
5. Steps 2, 3, 4 are repeated until each has filled all his spaces.



Num 2.10 Place-value notation (cont.)

The game

1. The pack of number cards is shuffled and put face downward on the table.
2. The players in turn turn the top card over, calling each two ways. E.g., “Three tens, five ones; thirty-five.” “Seven tens, zero ones; seventy.” “Zero tens,* four ones; four.”
3. It is important to speak each number in these two ways. After calling, the cards are put face down in another pile.
4. When a number is called corresponding to the ten-rods and units in one of a player’s spaces, he takes these off and writes the numeral for them.
5. The first player to replace all his ten-rods and units by numerals is the winner. He calls “Bingo.”
6. The game continues until all the players have done likewise.



Notes

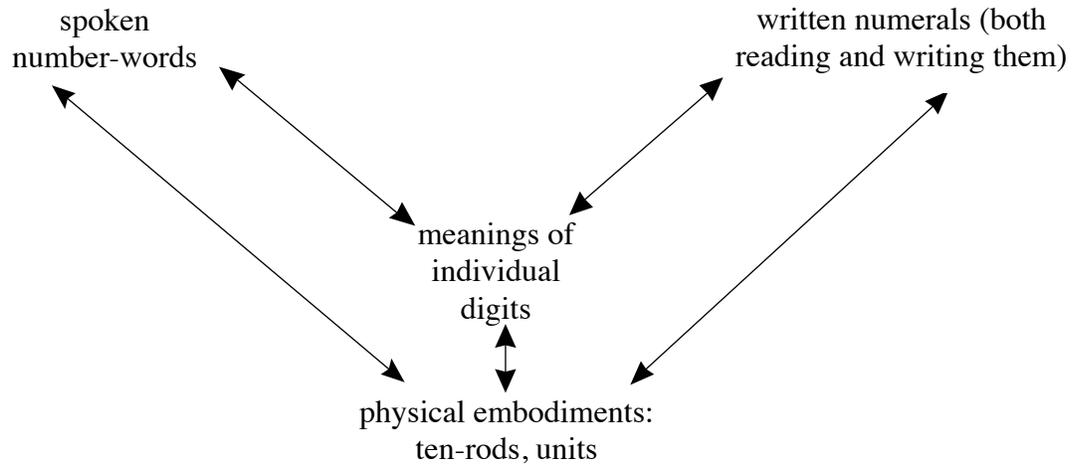
- * The question of whether zero may be omitted is considered later, in Num 2.12.
- (i) The game can be played with cards from 0 to 99, but this uses a lot of ten-rods and no new ideas are involved.
 - (ii) The paper may be folded to make 6 spaces if desired.

Stage (b)

This is played in the same way as Stage (a), except that instead of ten-rods and units, 10-cent and 1-cent coins (genuine or plastic) are used.

Discussion of activities Activity 1 introduces the final step into place-value notation, in which an important part of the meaning of each digit (namely whether it means that number of ones, tens, hundreds . . .) *is not written down at all*, but is implied by relative position. It is therefore very important that this meaning has been accurately and firmly established, which is the purpose of all the preparatory activities in earlier topics. Also, that the connection of the new notation with this meaning is established and maintained. Since the tens and ones no longer appear visibly, children now have to use their memory-images instead. These images are exercised and consolidated by having the children speak aloud (e.g., “four tens, seven ones”) every time. Activity 2 relates the new notation to the individual meanings of each digit both as expressed in words (five tens, three ones) and as embodied in physical materials (ten-rods and single cubes).

Activity 3 uses all these connections:



All three modes of schema building, and all three modes of testing, are brought into use for this final step into place-value notation.

- Mode 1 building* The new notation is related to physical embodiments.
- Mode 2 building* In Activity 1, the new notation is communicated, both verbally and in writing by a teacher. Activities 2 and 3 consolidate the shared meaning of this notation, by using it in a social context as basis for two games.
- Mode 3 building* The new notation is an extrapolation of the headed columns notation which they already know.
- Mode 1 testing* In Activity 1, children see column headings which confirm the accuracy of their readings in the ‘tens, ones’ form.
- Mode 2 testing* Activities 2 and 3 are games in which the players check the accuracy of each other’s actions.
- Mode 3 testing* This is implicit in the transition from headed columns to place value, since if the new notation were not consistent with the one they already know, children would not accept it so readily as they do.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.11 CANONICAL FORM

Concept Canonical form as being one of a variety of ways in which a number may be written.

- Abilities*
- (i) To recognize whether a number is or is not written in canonical form.
 - (ii) To rewrite a number into or out of canonical form.

Discussion of concepts The interchangeability of headed column notation and place-value notation depends on there being one digit only in each column, so that the first, second, third . . . columns reading from right to left always correspond one-to-one with the first, second, third . . . digits reading from right to left. However, it is one thing to note the advantages of having one digit only per column, and another to tell children that “We must not have more than one digit in any column,” or “We must not have numbers greater than nine in any column.” This is incorrect, because when adding (e.g.) 57 and 85, we shall (temporarily) have numbers greater than nine in both columns; and when calculating (e.g.) 53 - 16, children are told to take a ten from the tens to the ones column so that we can subtract 6.

The present approach recognizes that there are a variety of correct ways of writing numbers, one of these being called canonical form. This has only one digit per column (which may be a zero), and has the advantage that in this case, and not otherwise, headed column notation can be replaced by place-value notation. For this reason, canonical form is used unless there is a reason for using one of the alternatives – which we often need to do as a temporary measure.

Here are some examples of the same numbers written in non-canonical and canonical forms. For a given number there is only one canonical form, but many non-canonical.

<p>Canonical</p> <table border="1" style="border-collapse: collapse; width: 60px; height: 100px; margin: auto;"> <thead> <tr> <th style="padding: 2px 5px;">Tens</th> <th style="padding: 2px 5px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="text-align: center; padding: 10px;">4</td> <td style="text-align: center; padding: 10px;">2</td> </tr> </tbody> </table>	Tens	Ones	4	2	<p>← same number →</p>	<p>Non-canonical</p> <table border="1" style="border-collapse: collapse; width: 60px; height: 100px; margin: auto;"> <thead> <tr> <th style="padding: 2px 5px;">Tens</th> <th style="padding: 2px 5px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="text-align: center; padding: 10px;">3</td> <td style="text-align: center; padding: 10px;">12</td> </tr> <tr> <td style="text-align: center; padding: 10px;">2</td> <td style="text-align: center; padding: 10px;">22</td> </tr> <tr> <td style="text-align: center; padding: 10px;">1</td> <td style="text-align: center; padding: 10px;">32</td> </tr> </tbody> </table>	Tens	Ones	3	12	2	22	1	32
Tens	Ones													
4	2													
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1	32													
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Tens	Ones													
5	0													
Tens	Ones													
4	10													

Canonical		Non-canonical																		
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Hundreds	Tens	Ones																		
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6	4	65																		

Non-canonical forms like the above often arise during calculations. To give the final answers in place-value notation, facility in converting to canonical form is required. But place-value notation depends on there being only one digit per column, so I think that headed column notation should be used whenever changes to and from canonical form are involved, until the children are absolutely clear what they are doing. The conventional notation, involving little figures written diagonally above, is a further condensation which may be quicker to write but is not consistent with this requirement. And now that calculators are readily available, it is *understanding* that is most important in written calculations. For speed, calculators are better.

A further complication, usually ‘swept under the carpet,’ is that in non-canonical form, place-value notation makes an appearance within the headed columns. There is no mathematical inconsistency, but we are now using two notations in combination. I see no objection to this provided that we show both of them clearly. This is what I have tried to do in the activities which follow.

Activity 1 Cashier giving fewest coins [Num 2.11/1]

A game for up to five players. Its purpose is to present canonical form in a familiar embodiment. Children should first have done Org 1.12 (in *Volume 1*) and Org 13.

- Materials*
- Coins, 1-cent, 10-cent and, later, 100-cent (\$1).
 - Number cards 1 - 49, or 1 - 99 if preferred.*
 - Pieces of paper for cash slips.

* Provided in the photomasters

What they do Stage (a)

1. One child acts as cashier, and has a supply of 1-cent and 10-cent coins. (She should organize these to suit herself.)
2. The number pack is put face down on the table.

Num 2.11 Canonical form (cont.)

- Each player in turn, turns over the top card and records on her own cash slip what she gets, using headed columns.

Say she gets 47. This entitles her to 47 single pennies, or their equal in value. She first writes 47 in the single cent column.

Hundreds ¢	Tens ¢	Ones ¢
		47

- She hands it to the cashier.

The cashier replies, "I want to use the fewest coins," and returns the paper.

- The player changes her request to 4 ten-cent pieces and 7 single pennies, which the cashier accepts and pays. She records her agreement with a check mark, and the closing of the transaction with a line.
- If a player turns a card showing less than 10, conversion is of course not required.
- After three rounds, each player counts her money, and if she has more than nine coins of a kind, exchanges as appropriate with the cashier.
- The winner is the one who has most, and she acts as cashier for the next round.

Hundreds ¢	Tens ¢	Ones ¢
		47
	4	7✓

Extension

A bonus of 100¢ (\$1) may be earned by first adding the three underlined amounts, converting, and predicting the result before checking physically.

When the game is established, players may forestall the cashier's refusal by making the conversion before passing her their cash slip. Both forms should however always be recorded: first the figure on the number card, then the equivalent value in canonical form. The whole object of this game is to establish that these are two ways of writing the same number.

Stage (b)

(Return to this after Num 2.13, 'Numerals beyond 100, written and spoken.')

It is played exactly as for Stage (a) except that two number packs of different colours are used, one signifying dimes (10¢ coins) and the other pennies (1¢ coins).

The conversion to canonical form may now require several steps. Suppose a player turns up 15 tens, 12 ones.

First we make separate conversions.

Hundreds ¢	Tens ¢	Ones ¢
	<u>15</u>	<u>12</u>
	<u>1</u>	<u>2</u>
<u>1</u>	<u>5</u>	
1	6	2

Then we combine their results.

To begin with every step should be written, as illustrated. In this case it makes no difference whether the 12 singles or the 15 tens are converted first. With proficiency, however, the process of combining can be done mentally, and it is then easier to work from right to left.

A player who turns over (say) 46 tens, 71 singles will have further steps to take before conversion to canonical form is complete.

Hundreds ¢	Tens ¢	Ones ¢
	46	71
	— 7	— 1
— 4	— 6	
— 4	13	— 1
5	3	1

(One step has been done mentally here.)

Activity 2 “How would you like it?” [Num 2.11/2]

This is similar to Activity 1, except that the cashier behaves differently. Instead of paying out the amount in the smallest number of coins, she acts like a bank cashier who asks, “How would you like it?” when we cash a cheque.

Suppose, as before, a player turns over 47. She records this, as before, but she might then ask for it to be paid like this

or this

or even like this.

Hundreds ¢	Tens ¢	Ones ¢
		47
	3	17
	2	27
	0	47

Num 2.11 Canonical form (cont.)

So the cashier needs to keep plenty of piles of ten single pennies. If the cashier runs out of change before all have had three turns, she will have to ask the customers to help by giving her back some change in return for larger coins.

This game can be played at Stages (a) and (b), as in Activity 1. In both cases the final counting, and conversion to canonical form, will now be complicated, so Activity 2 should not be tackled until Activity 1 is well mastered. At any time when a player has made a mistake, the conversions should be done with coins to check the pencil-and-paper conversions.

Discussion of activities Children have already formed the concept of canonical form as it applies to the repeated grouping of physical materials, in Org 1.12 (*SAIL Volume 1*) and the topics leading up to it; and, for base 10, in Org 1.13. The present topic is concerned with the corresponding regroupings done mentally, and recorded using the mathematical notation shown in Activities 1 and 2.

This notation conveys, but more clearly, the same meaning as the makeshift devices which most of us learned at school, such as this:

31
~~47~~

Canonical form is concerned with different notations for representing the same numbers. The symbolic manipulations practised here all represent regrouping exchanges such as those done in Org 1 with physical materials. Children now need to acquire facility at a purely symbolic level, so the base 10 physical material is replaced by coins. These provide an excellent intermediate material for the present stage, since they represent number values in a way which is partly physical, partly symbolic.

This topic forms cross-links with the various calculations which cause non-canonical forms to arise, and practice with these will be found in the appropriate networks. But canonical form is a major concept in its own right, and other examples of it occur in later mathematics. It needs to be presented in a way which allows children to concentrate on learning this important concept by itself, without having at the same time to cope with other mathematical operations.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.12 THE EFFECTS OF ZERO

Concept Zero as a place-holder.

Ability In place-value notation, to recognize when zero is necessary for giving their correct values to other digits, and when it is not.

Discussion of concept In place-value notation, it is the position first, or second, or third . . . in order from right to left which determines the value of a digit. So the numerals 04 and 4 both mean the same: 4 is in the ones place in each, and there are zero tens – explicitly in the first, implicitly in the second. However, the numerals 40 and 4 do not mean the same, since 4 is in the tens place in the first numeral and in the ones place in the second. In the numeral 40, zero acts as a place-holder. By occupying the ones position, zero determines the meaning of 4.

Though the zero in 04 is not necessary, it is not incorrect. Its use is becoming increasingly common, e.g., in digital watches, and in figures on cheques and elsewhere for computer processing.

Activity 1 “Same number, or different?” [Num 2.12/1]

A game for two or more players. They all need to see the cards the same way up. Its purpose is for children to learn when the presence or absence of a zero changes the meaning of a numeral, and when it does not.

- Materials*
- A double pack of single headed number cards from 0 to 9.*
 - A mixed pack of cards on which are the words ‘same number’ or ‘different number,’ about 5 of each.*
 - An extra zero, of a different colour.*

* Provided in photomasters

Rules of the game Stage (a)

1. Both packs of cards are shuffled and put face down on the table.
2. The player whose turn it is has the extra zero.
3. He turns over one card from each pile.

Suppose he gets these.

He now has to put down a zero so that what

is on the table still means the same number as before. (It does NOT tell him to “add a zero.” This would mean something quite different.)

4. If he does this correctly, the others award him a point.

5. The circulating zero now goes to the next player, who might turn over these cards.

7

same number

0	7
---	---

same number

3

different number

Num 2.12 The effects of zero (cont.)

6. This would be his correct response.

3	0
---	---

different number

7. A player who turns over 0 from the pile, together with 'different number,' still gets a point for saying, "It can't be done," and explaining why.

8. It is good for the players to speak aloud the numbers before and after putting down the zero. This makes it sound different if the numbers are different. E.g., in steps 3 and 4: "Seven ones." "Zero tens, seven ones." And in steps 5 and 6: "Three ones." "Three tens, zero ones."

Stage (b)

(Return to this after Num 2.13, 'Numerals beyond 100, written and spoken.') This is played as in Stage (a), except that they turn over *two* number cards from the pile and put them side by side.

Suppose a player gets this:

4	2
---	---

different number

In this case (though not always) there are two correct responses.

4	2	0
---	---	---

different number

If he gives both, and speaks both correctly, he is awarded 2 points.

4	0	2
---	---	---

different number

Activity 2 Less than, greater than [Num 2.12/2]

Stage (a)

A game for two players. Its purpose is to provide further examples of the effects of zero.

Materials For each player:

- A pack of single headed number cards 1-9.*

Shared between two:

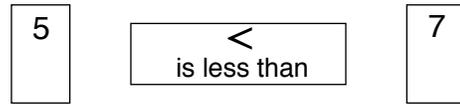
- A single zero card.*
- An 'is less than' card.*
- An 'is greater than' card, both as illustrated on the following page.*

* Provided in the photomasters

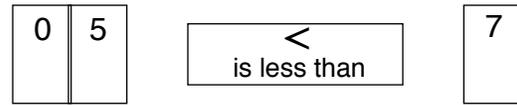
What they do

1. The 'is less than' card is placed between the two players. Also on the table is the zero card.
2. Each player holds his own pack face down. Each takes the top card from his own pack and lays it on his side of the 'is less than' card.
3. The first player to have a turn then picks up the zero and must place it next to his own card to make a true statement.

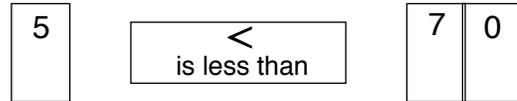
Suppose that here it is the left-hand player's turn.



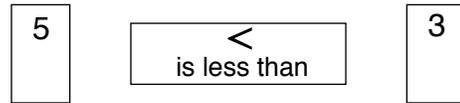
His correct response is:
For this he gets one point.



Had it been the right-hand player's turn, one correct response would have been:
(What is the other?)



In this case, only the right hand player can give a correct response.



If it was the left-hand player's turn, he could if he wished take a chance with an incorrect response. In that case, the other may say "Challenge," and if the challenge is upheld the challenger gets two points.

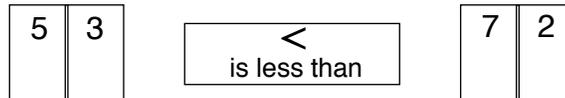
When all cards have been used, the game is repeated using the 'is greater than' sign.

Stage (b)

(Return to this after Num 2.13, 'Numerals beyond 100, written and spoken.')

As with Activity 1, this can be played with larger numbers. Each player now begins by putting down two cards. Several correct responses may then be possible, and players get a point for each which is both tabled and verbalized.

For example,
in this situation



the right-hand player has three correct responses: 072, 702, and 720.

Discussion of activities

In this topic, children are now working at a purely symbolic level, without any support from physical materials. The activities involved are quite sophisticated, since they involve changing meanings for the same symbols. Agreement about these changing meanings depends on a shared schema for assigning values to symbols: this schema being (as we have noted) that of a condensed and sophisticated notation, in which much of the meaning is not explicitly written down at all, but is implied by relative positions.

Any difficulties will usually be best dealt with by relating these activities to the more explicit headed columns notation; and, if necessary, by providing further backup in the form of base ten physical materials. These can be used to make very clear what a great difference in meaning can result from quite small changes in the positions of symbols.

Num 2.13 NUMERALS BEYOND 100, WRITTEN AND SPOKEN

- Concepts*
- (i) That a particular digit can also represent a number of hundreds, thousands, ten-thousands, hundred-thousands, millions, according to whether it comes third, fourth, fifth, sixth, seventh in order, reading from right to left.
 - (ii) That each new position in this order (together with the associated word) represents groups of ten of the group just before.

- Abilities*
- (i) To identify separately which digits represent hundreds, thousands, ten-thousands, hundred-thousands, millions.
 - (ii) To write numbers in expanded notation.
 - (iii) To read aloud numerals having up to seven digits.
 - (iv) To match these with physical representations.

Discussion of concepts Here we are extrapolating the place-value concept from its limited application to tens and ones successively to hundreds, thousands, The essence of this is a repetition, each time we move one place further from right to left, of the grouping process, each new larger group containing ten of the group just before.

This extrapolation can and should be begun physically. Since it is not likely that a million units are available, we are led naturally to an intermediate representation which is partly physical, partly symbolic.

Activity 1 Big numbers [Num 2.13/1]

This is a teacher-led activity for as many as can see the chart properly. Its purpose is to extrapolate children's use of place-value notation up to the seventh place, representing millions.

- Materials*
- A chart prepared as in the illustration below.
 - Base ten material.
 - A skeleton metre cube (representing a million) would be very helpful if possible. This would have to be filled with units to represent a million physically.

millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	ones
----------	-------------------	---------------	-----------	----------	------	------



hinged here

Stage (a)

1. Place the chart where the children can see it with the hinged section folded under so that only the right-hand section is visible (hundreds, tens, ones).
2. Point to the ones section, and (with the base-ten material on view) ask, “Which do we put here?”
3. The answer is “units” [or “ones”], and you put a specimen unit in the space, saying “Like these.”
4. Similarly for the tens and hundreds sections, the specimens in these cases being a ten-rod and a ten-square respectively. They will already know that the number of units in a ten-square is called a hundred.

Now go straight on to Stage (a) of Activity 2, ‘Naming big numbers.’

Stage (b)

5. Now open the chart out to show all the sections. Put a specimen from the base-ten material in the ones, tens, hundreds.

millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	ones
				□	—	•

6. Continue, pointing to the thousands space, “What do we put here?” The answer is “A ten-*cube*” and you put a specimen in the thousands space. The number of units in a ten-cube is called a thousand.
7. For the next space ten-thousands, a problem arises. Even if there are ten ten-cubes available, they would make a tall and unstable column. Ask the children whether, instead, they can *imagine* a ten-rod made up of thousands in this space.
8. Next we come to the hundred-thousand space. Ask “Now, what should we imagine in *this* space?” (pointing). The answer is a ten *square* of thousands; and for the next (millions) space, a ten-*cube* of thousands.
9. It would be helpful to support their imaginations with a skeleton metre cube (representing a million).



Big numbers [Num 2.13/1]

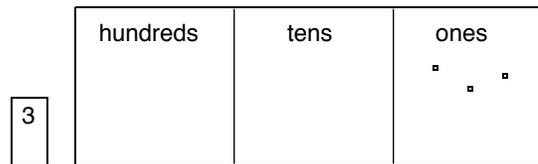
Activity 2 Naming big numbers [Num 2.13/2]

For as many players as can see the material the same way up. (Otherwise the order gets reversed.)

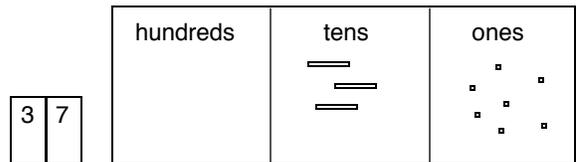
- Materials*
- A chart as used in Activity 1.
 - A box of base ten material.
 - A pack of single headed number cards 0 to 9.*
 - Pencil and paper for writing numbers in expanded notation.
- * Provided in the photomasters

What they do Stage (a)

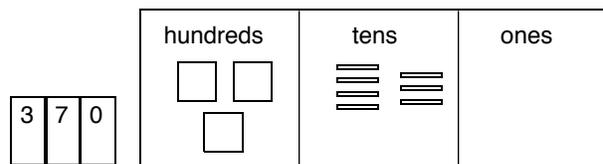
1. The pack is shuffled and put face down on the table.
2. The first player picks up the top card and puts it face upward beside the chart, and says (in this example) “Three ones: three,” and puts the three units into the correct space.



3. The next player picks up the next card, and puts it *on the right* of the one already there. He now says (in this example) “Three tens, seven ones: thirty-seven,” puts down material accordingly, and writes the number in expanded notation and in place value notation: ‘ $30 + 7 = 37$.’



4. The next player picks up the next card, and again puts it on the right of the one already there. She now says (in this example) “Three hundreds, seven tens, zero ones: three hundred seventy,” puts down material accordingly, and writes: ‘ $300 + 70 + 0 = 370$.’



5. The chart is now cleared of material, and the cards returned randomly to the pack.
6. Steps 2 to 4 are repeated until the children have mastered this stage.
7. Now please read the directions at the end of Stage (b), on the next page.

Stage (b)

Materials

- As in Stage (a) to begin with.
- Pencil and paper for writing expanded notation and for scoring.

What they do

This is played a little differently. The base ten material is no longer used. For the first few rounds the chart from ones to millions is left on display as a reminder, after which it is put away and the game continued mentally.

1. The 0 - 9 pack is put face down on the table.
2. The players in turn pick up the top card and put it down on the right of those already there, as in stage (a).

3. The first player says
“Five ones: five.”

5

4. The next player says
“Five tens, seven ones:
fifty-seven,” and writes ‘ $50 + 7 = 57$.’

5	7
---	---

5. The next player says
“Five hundreds, seven tens,
one one: five hundred, seventy-one”
and writes ‘ $500 + 70 + 1 = 571$.’

5	7	1
---	---	---

and so on.

6. This player should say “Five
ten-thousands, seven thousands,
one hundred, six tens, two ones:
fifty-seven thousand, one hundred,
sixty two,” writing
‘ $50\ 000 + 7\ 000 + 100 + 60 + 2 = 57\ 162$.’

5	7	1	6	2
---	---	---	---	---

7. Scoring for a correct answer is 1 point for each correct digit and 1 point for correctly expressing the number in expanded form.
8. If no one makes a mistake, the round continues to the millions, and the player to whom this falls will score eight points.
9. In this case, the following round starts with the next player. This ensures (provided that the number of players is not seven) that the millions turn will come to a different player each time.

As soon as children are confident with Activity 1, ‘Big numbers,’ Stage (a), they should continue straight on to Stage (a) of Activity 2, ‘Naming big numbers.’ After that, there are two paths open to them.

- (i) They may return to Num 2.11 and Num 2.12 doing these activities at the more difficult stages involving hundreds, tens, ones.
- (ii) They may continue to Stages (b) of Activities 1 and 2 (‘Big numbers,’ and ‘Naming big numbers’), and then follow path (i).

These two paths may also be followed alternately.

Discussion of activities This topic makes considerable use of Mode 3 schema building: extrapolation of an existing pattern, and combining ideas which they already have.

The pattern is that of repeatedly putting together the same number of groups to make the next group larger. This pattern was built up using physical materials (Mode 1) in Org 1, topics 10 to 12 in *SAIL Volume 1* and topic 13 in this volume. To start with it was based on numbers small enough to be subitized (perceived without counting). Then it was extended to base 10, using prefabricated physical materials so that the process of putting together physical materials to form larger groups was replaced by its mental equivalent.

In Activity 1, Stage (a), this pattern is picked up again, using physical materials by way of review. In Stage (b), it is continued much farther. For this, we let go of the physical materials and we use our imagination. The children will in these activities be combining this extended pattern with words they have probably already heard: “hundred,” “thousand,” possibly even “million.”

Note how in Activity 2, every digit changes its place value at every turn, thereby emphasizing how the meaning depends both on the digit itself and on its location. Expanded notation is introduced explicitly at this point, at first with the support of base ten material and oral interpretation. You may wish to postpone the introduction of expanded notation for some students. Others may be ready to make use of exponential notation, as in:

$$57\ 162 = 5 \times 10^4 + 7 \times 10^3 + 1 \times 10^2 + 6 \times 10 + 2$$

OBSERVE AND LISTEN

REFLECT

DISCUSS



Naming big numbers [Num 2.13/2]

Num 2.14 ROUNDING (WHOLE NUMBERS)

Concept That of rounding a given whole number to the nearest ten, hundred, thousand,

Ability To round any given number as above.

Discussion of concept Sometimes too much information is as unhelpful as too little. For example, if we want to compare the populations of several large cities, this can be grasped more easily if they are not shown to the last person. To the nearest hundred would be accurate enough, or even perhaps the nearest thousand. Rounding is thus a method of giving a suitable amount of information for a particular purpose.

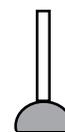
Activity 1 Run for shelter [Num 2.14/1]

A game for up to four players. Its purpose is to introduce the concept of rounding in the context of a number line. (In this case, it is a number line with embellishments!)

- Materials*
- Game board (see Figure 3).*
 - Die, marked 1, 2, 4, 5, R, R.
 - Shaker (or the die may be rolled from the hands.)
 - Markers, one for each player.†

* Provided in the photomasters

† These need to be quite small. I use short pieces of coloured milk straws with a small blob of Blu-tak (or Plasticine). [See illustration.]



- What they do*
1. The board represents a winding track with a shelter at every tenth location. The game is of the familiar kind, in which players roll the die and move forward that number of places. More than one player may be at the same location.
 2. There is, however, this difference: if the die shows R, this means *Rain*. It also means *Run to the nearest shelter*.
 3. Which shelter this is depends on the present location of the player. If she is at 1, 2, 3, 4 after a ten, she has to return to the ten just passed. If she is at 6, 7, 8, 9 after a ten, she runs forward to the next shelter ahead. If she is on a 5, she is half way between two shelters, so naturally chooses to run to the one ahead.
 4. The winner is the first to reach the camp at end of the track. For this either the exact number, or an R, is required. The others may continue if they wish.
 5. Explain that what they have been learning is called *rounding to the nearest ten*. Now the R also means rounding.

Note If this game is found too long, it may be played finishing at (say) 50 or 60.

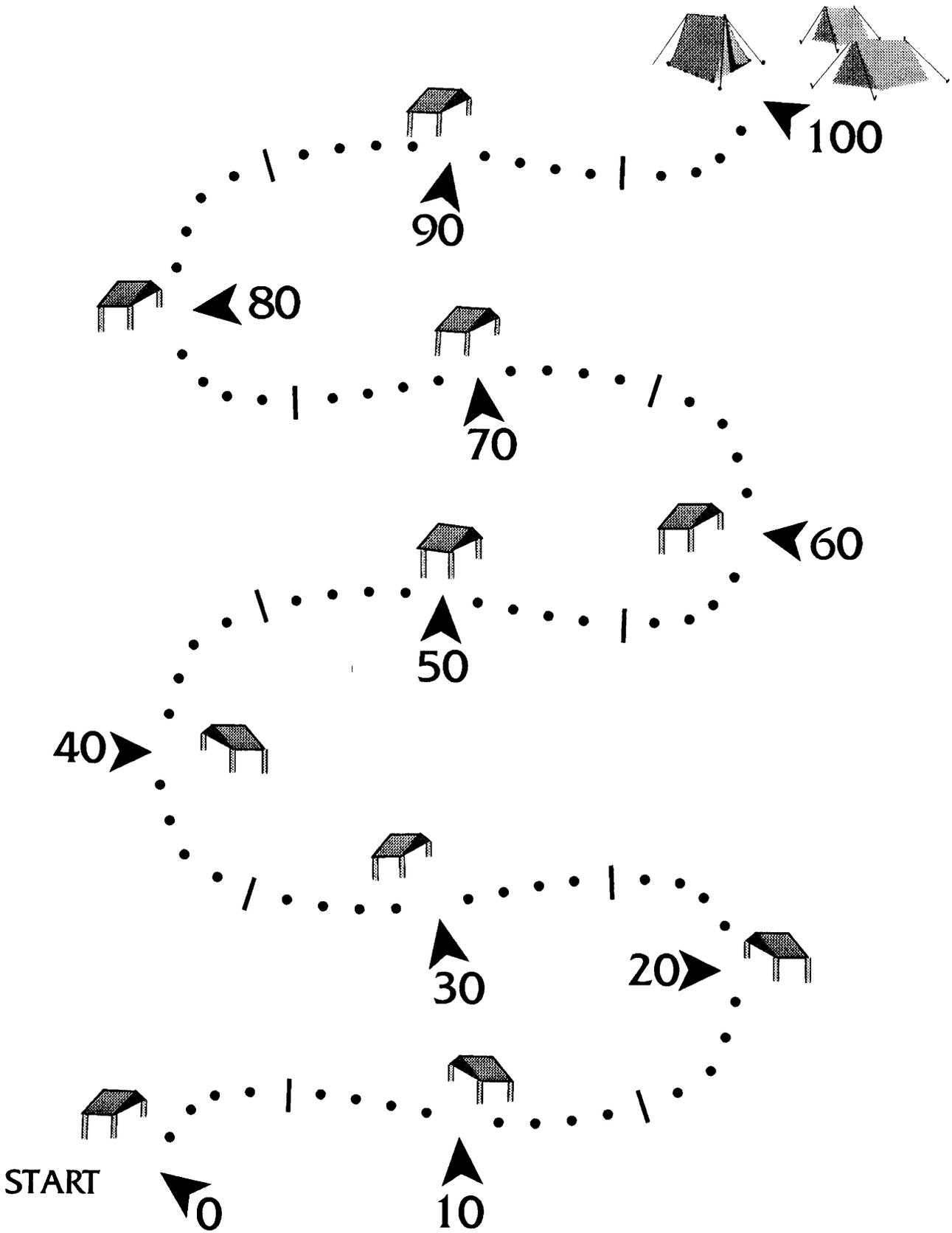


Figure 3 'Run for shelter' game board

Activity 2 Rounding to the nearest hundred or thousand [Num 2.14/2]

An activity for as many children as can see the materials the right way up. (Otherwise the direction gets reversed.) Its purpose is to make explicit and extend the concept of rounding, and to help the ability become independent of the number track.

Materials For the group:

- A card with number lines, as illustrated in Figure 4.*
- A pack of single headed number cards, 0 to 9. *
- Something to point with which does not make a mark.
- A plain card on which to lay out the number cards is also useful.

* Provided in the photomasters

Stage (a) Teacher-led introduction

*Suggested
sequence
for the
introduction*

1. Show the number line card to the children, and explain that this is part of a long number line: too long to show the whole of it. Ask them to identify the unmarked points.
 2. Put out three number cards to show a number between 200 and 300:
E.g.:
- | | | |
|---|---|---|
| 2 | 7 | 3 |
|---|---|---|
3. Ask one of them to point to its approximate position on the number line. In this case, this is somewhere between the points for 270 and 280.
 4. “So if we go to the nearest hundred, where would this be?” Ask them to say the number as well as pointing. (300.) “This is called ‘Rounding to the nearest hundred.’”
 5. Ask them to put into words what decides which way they go. Something like “If it’s less than half way, you go back. If it’s at or above half way, you go on.”
 6. Put out other numbers between 200 and 300, some nearer 200 and some nearer 300. And ask the children in turn first to point to it on the number line, and then say the result of rounding to the nearest hundred.
 7. Now lay out another number which does not begin with 2, e.g.:
- | | | |
|---|---|---|
| 6 | 1 | 4 |
|---|---|---|
8. “For this we need another part of the number line.” (Pointing to the next number line down, and indicating the ‘hundreds’ markers:)
“What would these two points have to be? (600 and 700.) So where would this be, approximately? (Between 610 and 620).”
 9. Can you round this to the nearest hundred? (600.)
 10. Repeat as long as seems necessary.

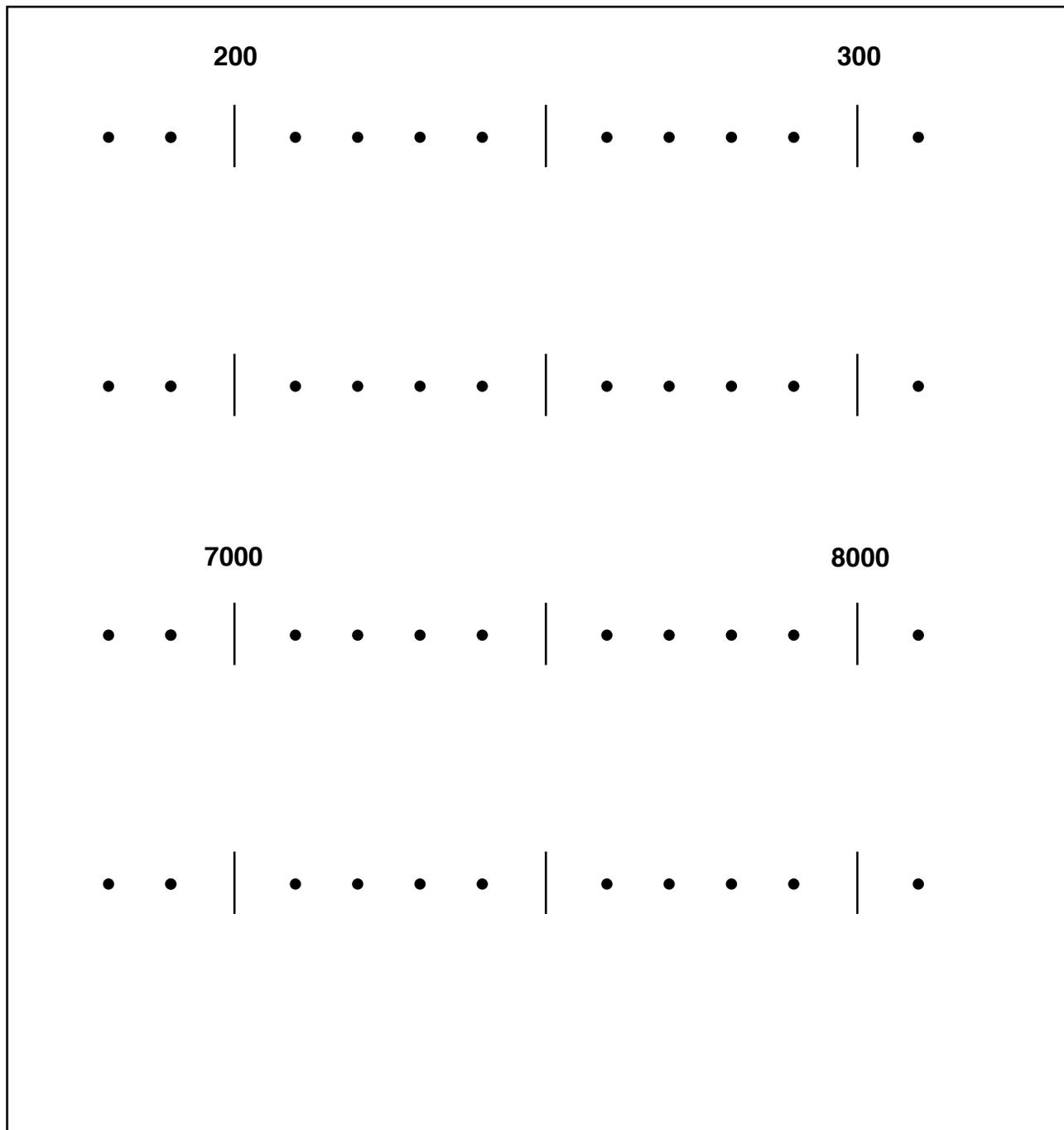


Figure 4 Number lines card for 'Rounding to the nearest 100 or 1000'

Stage (b) (for them to continue on their own)

1. The pack is shuffled and put face down.
2. The child whose turn it is to begin turns over the top three cards and puts them in a row, as in Stage (a).
3. All write down what this is when rounded to the nearest hundred.
4. The child whose turn it is reads out her result. Others say whether they agree or not. Any disagreements are resolved as described in step 6.
5. The next child puts the top card from the pack face up on the right of those already there, and removes the left hand card which she puts in a separate face down pack.
6. Steps 3 and 4 are then repeated. Each time, the child whose turn it is reads out her result. If there is disagreement, it is for her to explain, with the aid of the number line, why she thinks that her answer is the right one.
7. This continues until the children are fluent at rounding to the nearest hundred. When the pack is used up, it is shuffled and replaced for a fresh start.

Stage (c) As for Stage (b), but using four cards, and rounding to the nearest thousand. Thousands number lines are available, if required, on the same card as the hundreds number line already used.

Activity 3 Rounding big numbers [Num 2.14/3]

A game for as many children as can see the materials the right way up. It follows Activity 2 closely, and its purpose is to extend the concept and ability of rounding numbers up to the nearest million, and beyond if they choose. It requires good concentration, and offers a challenge to the more able children.

- Materials*
- Single headed number cards, as for Activity 2. For this game, a double pack may be preferred.
 - Pencil and paper for scoring.

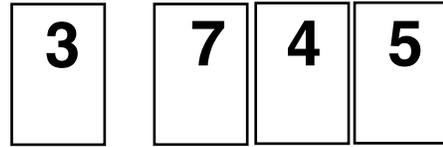
- Rules of the game*
1. One player is appointed score-keeper. The pack is shuffled and put face down on the table.
 2. This game is similar to Stage (b) of the previous activity, except that two cards are turned up by the first player and no card is removed by subsequent players. The number of cards showing thus increases by one for each successive player.
 3. The first player picks up the top two cards, puts them face up side by side, and says (in this example) “Thirty-seven, rounded to the nearest ten, is forty.” For this she scores 1 point.

3	7
---	---
 4. The next player picks up the top card, puts it face up on the right of those already on the table, and says (in this example) “Three hundred seventy-four, rounded to the nearest ten, is three hundred seventy. Rounded to the nearest hundred, is four hundred.” Score, 2 points.

3	7	4
---	---	---

Num 2.14 Rounding (whole numbers) (cont.)

5. The next player picks up the top card, puts it face up on the right of those already on the table, and says (in this example) “Three thousand seven hundred forty-five, rounded to the nearest ten, is three thousand seven hundred fifty. Rounded to the nearest hundred, it is three thousand seven hundred.” (Not three thousand eight hundred, a mistake which is sometimes made. A number line will make this clear if necessary.) Rounded to the nearest thousand, it is four thousand.” Score, 3 points.



6. If no one makes a mistake, the game continues to the millions, i.e., with seven cards face up, and the player to whom this falls will score 6 points.
7. The next player shuffles the pack and starts again, with two cards as in step 3. They then continue with steps 4 to 7 as before.
8. If a player makes a mistake, she scores the points she has earned, but the turn then passes to the next player.
9. This game requires much concentration. Therefore, if anyone makes any kind of interruption, except “Yes” or “No,” while a player is having her turn, this player is allowed a penalty-free mistake.

Discussion of activities Once again the number line is used as a powerful support to our thinking. Activity 1 embodies it in a game which introduces rounding in its simplest form. Activity 2 extrapolates the concept to include rounding to the nearest hundred or thousand. Activity 3 requires mental concentration, which needs peace and quiet: and it was during field testing that I proposed the rule in step 9 of the last activity. All five children unanimously agreed that it was a good rule, although most of them had been interrupting! So in this activity, the children are learning more than the mathematics itself.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Num 3] ADDITION

A mathematical operation which corresponds with a variety of physical actions and events

Num 3.9 ADDING, RESULTS UP TO 99

Concept Adding when the results are between 20 and 99.

Ability To add 2-digit numbers, the results still being 2-digit numbers.

Discussion of concept What is new here is not the concept of addition, but extension of the ability to do this with larger numbers.

Activity 1 Start, Action, Result up to 99 [Num 3.9/1]

This is a continuation of Num 3.7/1 (in *Volume 1*), 'Start, Action, Result over 10,' with larger numbers.

- Materials*
- SAR board as illustrated below in step 1.*
 - Start and Action cards with assorted numbers from 1 to 49.*
 - Base ten material, tens and ones.
 - Paper and pencil for each child.
- * Provided in the photomasters

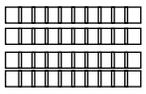
What they do *Stage (a) Headed columns and materials*
(The steps for this are the same as in Num 3.7/1, and are repeated here for convenience.)

- What they do*
1. The Start and Action cards are shuffled and put in position as usual, and the top two cards turned over.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 10px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 5px;">S</div> <div style="margin: 0 5px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 5px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 5px;">A</div> <div style="margin: 0 5px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 5px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	<table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <th colspan="2">Do and say</th> </tr> <tr> <th style="width: 50%;">Tens</th> <th style="width: 50%;">Ones</th> </tr> <tr> <td style="height: 100px;"> </td> <td> </td> </tr> </table>	Do and say		Tens	Ones		
Do and say							
Tens	Ones						

Num 3.9 Adding, results up to 99 (cont.)

2. Tens and single cubes are put down as shown on these cards by one of the children, who describes what she is doing and what it corresponds to on the SAR board. E.g., “The Start card means ‘Put down four tens and six ones.’” “The Action card means ‘Put three tens and eight ones more.’” Each child records this on her own paper, which she first rules into headed columns. The board, and records, will now appear as below.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 20px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">S</div> <div style="margin-right: 10px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">A</div> <div style="margin-right: 10px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say
	<div style="width: 45%; text-align: center;"> Tens  </div> <div style="width: 45%; text-align: center;"> Ones  </div>

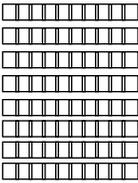
Tens	Ones
4	6
+ 3	8

3. The next stage is shown below.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 20px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">S</div> <div style="margin-right: 10px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">A</div> <div style="margin-right: 10px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say
	<div style="width: 45%; text-align: center;"> Tens </div> <div style="width: 45%; text-align: center;"> Ones </div>

Tens	Ones
4	6
+ 3	8
7	14

4. Finally, if there are (as in this case) more than ten ones, ten of them are exchanged for a ten-rod. This is transferred to the tens column. The children again record this individually (as illustrated on the following page).

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 20px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">S</div> <div style="margin-right: 10px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">A</div> <div style="margin-right: 10px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say										
	Tens	Ones									
		<table border="1" style="margin: auto; border-collapse: collapse;"> <thead> <tr> <th style="padding: 2px 5px;">Tens</th> <th style="padding: 2px 5px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="text-align: center; padding: 2px 5px;">4</td> <td style="text-align: center; padding: 2px 5px;">6</td> </tr> <tr> <td style="text-align: center; padding: 2px 5px;">+ 3</td> <td style="text-align: center; padding: 2px 5px;">8</td> </tr> <tr style="border-top: 1px solid black;"> <td style="text-align: center; padding: 2px 5px;">7</td> <td style="text-align: center; padding: 2px 5px;">14</td> </tr> <tr> <td style="text-align: center; padding: 2px 5px;">8</td> <td style="text-align: center; padding: 2px 5px;">4</td> </tr> </tbody> </table>	Tens	Ones	4	6	+ 3	8	7	14	8
Tens	Ones										
4	6										
+ 3	8										
7	14										
8	4										

5. Children compare their final results. There should be no difference, but if there is, they will need to repeat the process and check each step together.
6. The board is cleared, and steps 1 to 5 are repeated with different numbers.

Stage (b) Headed columns, no materials

1. The SAR board and base ten materials are no longer used.
2. Start and Action cards are turned over as before, and all the stages of addition are written in headed column notation as in stage (a).

Stage (c) Place-value notation

1. Start and Action cards are turned as before. If the addition would result in 10 or more in the units position, a streamlined form of headed column notation is used.

2. Here is a simple example.

First the addition
to be done
is written.

$$\begin{array}{r} 3 \quad 7 \\ + 2 \quad 5 \\ \hline \end{array}$$

3. Adding 7 and 5 gives a result in 2 digits, so we need headed columns.

$$\begin{array}{r|l} \text{T} & \text{Ones} \\ 3 & 7 \\ + 2 & 5 \\ \hline 5 & 12 \end{array}$$

4. Regroup.

$$\begin{array}{r|l} \text{T} & \text{Ones} \\ 3 & 7 \\ + 2 & 5 \\ \hline \cancel{5} & \cancel{12} \\ 6 & 2 \end{array}$$

Num 3.9 Adding, results up to 99 (cont.)

5. Finally, write the result in place value notation.

$$\begin{array}{r} 37 \\ + 25 \\ \hline 62 \end{array}$$

6. The whole may be set out like this.

$$\begin{array}{r} 37 \\ + 25 \\ \hline 62 \end{array}$$


T	Ones
3	7
+ 2	5
5	12
6	2

This result is written last.

Activity 2 Odd sums for odd jobs [Num 3.9/2]

A game for about four children. Its purpose is to use the skills learned in Activity 1.

- Materials*
- 1 die bearing only 1s and 2s [for tens].
 - 1 normal 1-6 die (or 0-9, if you have one) [for ones].
 - 1 set of ‘odd job’ cards.*
 - 1 set of ‘target’ cards.**
 - Plastic or real money: 10¢, 5¢, 1¢ coins.
 - Money boxes with a slit in the lid for all but one of the players.
 - Paper and pencil for each player.

*Written on these are jobs they might do such as “Wash car,” “Clean all downstairs windows.” (provided in the photomasters)

**Written on these are the kind of things they might be working for, such as “Swimming, 60¢ for two,” “Present for sister, 78¢.” (provided in the photomasters)

- What they do*
1. One child acts as ‘Mom’ or ‘Dad.’
 2. Each child picks a target from the target cards which are held in a fan or spread out face downwards on the table.
 3. Then, starting with the child on the left of ‘Mom’ or ‘Dad,’ the first ‘does a job’ by turning over a card, and since there is no fixed rate, throws the dice to see what she is paid.
 4. The player acting as ‘Mom’ gives her the money in coins.
 5. She puts these in her money box, which she may not open without permission.
 6. Each takes her turn, putting the money in her box each time.
 7. To know when she has enough money, each child keeps a record, adding on her earnings each time.
 8. The first who thinks she has enough asks permission to open her money box, and her recorded total is checked by her ‘parent’ against the money in the box.
 9. If correct she gets to be ‘Mom’ next time.
 10. If not she has to go on doing jobs!

Activity 3 Renovating a house [Num 3.9/3]

A co-operative game for 4 children. Its purpose is to practise the skills developed in Activities 1 and 2 in another play situation.

- Materials*
- Gameboard, see Figure 5.*
 - On cards:*
 - house
 - chimney
 - windows
 - doors
 - Tens die, numbered 1 & 2 only.
 - Ones die, numbered 0 to 9.
 - Play money: \$10 bills, \$5 bills, and \$1 coins.
 - Slips of paper, pencils.

* Provided in the photomasters

It is a good idea to introduce the game with costs in round numbers, e.g., window \$30, door \$50, and progress to harder numbers as shown. For this, two houses will be needed or adhesive labels used, after laminating, on which there are various sets of prices.

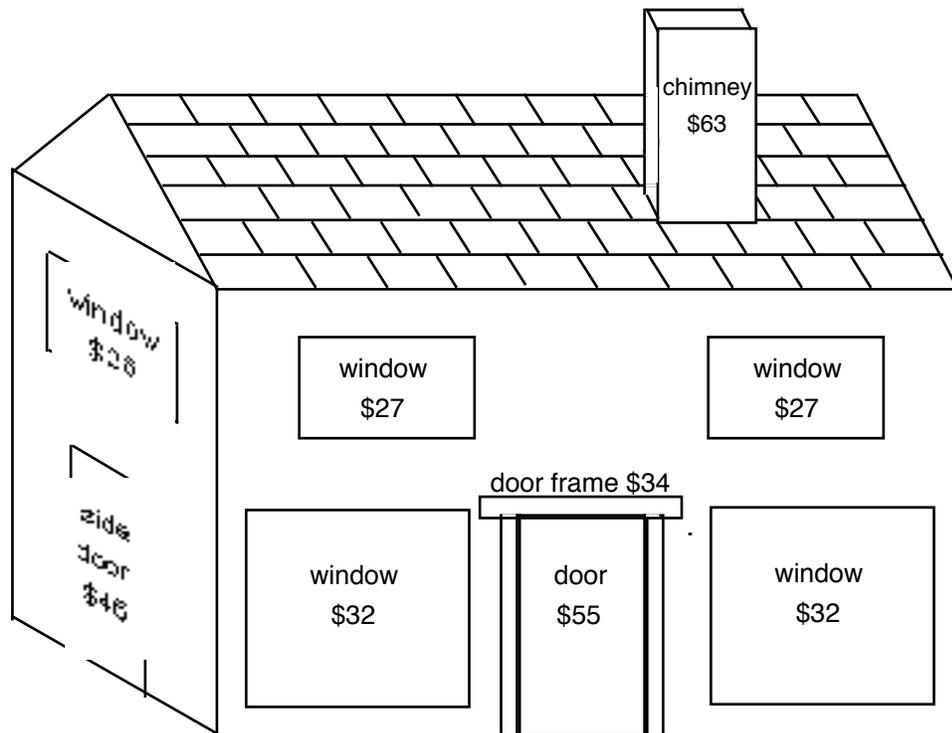


Figure 5 Renovating a house

Num 3.9 Adding, results up to 99 (cont.)

- What they do*
1. One child acts as banker, one as building supplies dealer, and two as a young couple who are saving what they can each week and putting the money towards parts for their house. This is an old house which they have bought cheaply and are renovating.
 2. Each of the couple throws the dice to determine how much they have saved from their earnings that month.
 3. They record these amounts on a slip of paper, and add them together.
 4. They take their slip to the banker, who checks their total and gives them cash in exchange, keeping the slip.
 5. When they have enough, they go to the building supplies dealer and buy a door, window, or chimney. The banker may be asked to exchange larger bills for smaller.
 6. When the house is built, they may play another game, exchanging roles.

$$\begin{array}{r} 24 \\ + 17 \\ \hline 41 \end{array}$$

Activity 4 Planning our purchases [Num 3.9/4]

For a small group. This activity uses the same mathematics for a situation which is the reverse of Activity 2. Instead of accumulating money for a predetermined single target (which may be exceeded), they have a given amount of money which they plan to spend on a variety of purchases, keeping to a total cost within the given amount.

- Materials*
- A variety of labelled articles for the shop.*
 - Slips of paper or card.**
 - Plastic money.
 - Pieces of paper and pencils for each child.

*Varied prices in the range of 48¢, 19¢, 7¢ and a few penny objects.

**On these are varying sums from 50¢ to 90¢, in tens.

- What they do*
1. One child first acts as banker, and has charge of the money; then as shopkeeper.
 2. The other players draw slips of paper to discover how much they have to spend.
 3. Each child takes her slip to the banker to get cash.
 4. She then makes a shopping list which she totals.
 5. When ready, she goes to the shop with her money and shopping list.
 6. The shopkeeper then sells her the goods, naturally making sure that she receives the correct payment.

Shoppers can use penny objects to make up an exact amount, thereby avoiding the necessity for giving change. Children could also ask the banker to change a 10¢ coin into 5¢ and/or single pennies, which is good practice for mentally changing between tens and singles. Receiving change is however something the children will already have experienced outside school, so you may prefer to leave out the penny objects and let the shopkeeper give change instead.

Activity 5 Air freight [Num 3.9/5]

A more difficult addition activity, for children playing in pairs.

- Materials*
- Cards with pictures of objects together with their weights.*
 - Cards representing containers.

* Suitable pictures might be bags, suitcases, small items of furniture, domestic items like tape recorders, televisions, etc.

It is good to have several assorted sets of these. For example, a set having 19 cards, each with a picture from a mail order catalogue, marked with the following weights:

64 kg,	54 kg,	52 kg,	48 kg,	46 kg,	42 kg,	
39 kg,	38 kg,	35 kg,	31 kg,	28 kg,	22 kg,	
20 kg,	19 kg,	17 kg,	15 kg,	12 kg,	9 kg,	8 kg

This makes a total of 599 kg, which it is possible to pack into 6 containers (5 holding 100 kg each, and 1 holding 99 kg). You could use other combinations of weights. It is probably easiest to choose them by breaking down 100 kg's initially,

e.g.,	62,	27,	11.	
	53,	17,	22,	8,
	49,	33,	18,	
	35,	31,	19,	10, 5 etc.



Air freight [Num 3.9/5]

Num 3.9 Adding, results up to 99 (cont.)

- What they do*
1. The object is to pack all the objects into the smallest possible number of containers. No single container may weigh more than 100 kilograms.
 2. They may work however they choose. One way is for one to act as packer, and the other as checker who makes sure that no container exceeds 100 kg.
 3. If there are several pairs doing this activity at the same time, they may compare results to see which pair is the best at packing the containers.

Discussion of activities These involve the extrapolation of techniques already learned to larger numbers, and the combining of concepts already formed. So the schema building involved in this topic is a good example of Mode 3: creativity. Grouping and regrouping in tens from Org 1, adding past ten from the topic just before this, and notations for tens and singles from Num 2, are the chief concepts to be synthesized.

In Activity 1, stage (a) renews the connection between written addition and its meaning as embodied in base-ten materials. Stage (b) is transitional to addition using place-value notation in stage (c). In all these stages, place-value notation sometimes occurs in combination with headed columns. This hybrid was only adopted after careful analysis and discussion with teachers. It is implicit in the conventional notation, when small ‘carrying’ figures are used; so is it not better to show clearly what we are doing, especially since it represents such an important part of the calculation? The layout suggested is very little slower than the traditional one – and if speed is the main object, then calculators are the best means to achieve it.

Activity 2 has been devised in the form shown to introduce a predictive element – checking the total on paper against coins in the money box. This you will recognize as an example of Mode 1 testing.

Activities 2, 4 and 5 introduce multiple addends. In Activity 2, this is done by stages, each total being recorded before the next one is added. This is how we add mentally, so it is a good preparation for Activity 4 where the running totals need not be recorded unless children find it helpful. Activity 3, and the more difficult Activity 5, have been included to give a further choice of activities in this section, since it is good for children to get plenty of assorted practice at addition with regrouping.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.10 ADDING, RESULTS BEYOND 100

Concept Adding when the results are from 100 to 999.

Ability To add 2 or 3 digit numbers, results in the above range.

Discussion of concept As in Num 3.9, what we are concerned with in this topic is the further extension of their existing abilities in addition.

Activity 1 Start, Action, Result beyond 100 [Num 3.10/1]

This repeats Activity 1 of the last topic with hundreds, tens and ones. Its familiarity should give children confidence in tackling these larger numbers before they use them in the new activities which follow.

- Material*
- SAR board as for Num 3.9 but with an extra column for hundreds.*
 - Start and Action cards with assorted numbers between 1 and 499.*
 - Base ten material, hundreds, tens and units.
 - Pencil and paper for each child.
- * Provided in the photomasters

Stages (a), (b), (c) These are the same as for Num 3.9/1 except for the larger numbers.

Activity 2 Cycle camping [Num 3.10/2]

A board game for 2 or more. Its purpose is to exercise children's skills in addition, relating this to what they have learned on the number line.

- Materials*
- Board (see photomaster in Volume 2a).
 - Markers in the form of tents.
 - Die marked 1 - 6.
 - Die marked 0 - 9.
 - Shaker.

- What they do*
1. The board represents a long and winding road through beautiful countryside.
 2. The players begin at 0, and in turn throw the dice to determine the distance to be travelled for the day.
 3. The 1 - 6 die gives the tens, and the 0 - 9 die gives the ones. The result is a variable daily distance from 10 to 69 km. Children may use their imaginations to explain these.
 4. Adding is done by 'counting on' separately in tens and in ones, a decade at a time for the tens.
 5. Before moving, a cyclist points to where he thinks the day's kilometres will take him.
 6. The others check and say 'Agree' or 'Disagree'. If the cyclist's prediction is incorrect, he has a puncture that day and cannot move.
 7. To finish, in this game the exact number need not be thrown. Any number of kilometres equal to or greater than the remaining distance will do.

Activity 3 One tonne van drivers [Num 3.10/3]

A board game for up to four players. Their roles as van drivers give them plenty of practice in addition of numbers with sums up to one thousand.

- Materials*
- Game board, see Figure 6.*
 - Cards representing vans, one for each player.*
 - Cards representing loads.**
 - A calculator.
 - Pencil and paper for each player.
 - Paper clips, one for each player.

* Provided in the photomasters

** The choice of loads affects both the length and the difficulty of the game. As a start I suggest you try the following.

6 large: 509, 576, 624, 697, 718, 745.

8 medium: 132, 194, 246, 287, 355, 369, 424, 431.

12 small: 12, 17, 21, 25, 34, 39, 40, 46, 65, 78, 89, 96.

- What they do*
1. One player acts as supervisor, the others are van drivers.
 2. The supervisor has a calculator, the drivers do not.
 3. The load cards are piled face down at the depots, which have respectively large, medium, and small packages.
 4. The drivers have cards representing vans of 1 tonne (1000 kg) capacity. They drive these to the depots, and bring back a load to the warehouse. The load cards are kept on the vans by paper clips.
 5. Players start at the warehouse, and move in turn. In a single move they may do any one of the following.
 - (a) Drive to any depot and take what is offered (the top card).
 - (b) Stay where they are and take another item.
 - (c) Return to the warehouse.
 - (d) Unload at the warehouse and have the weight of their load checked by the supervisor.
 6. If, when his load is checked, it is found to be overweight (more than 1 tonne), the driver must wait for 2 turns at the warehouse while his van is overhauled and tested. The excess weight is not credited.
 7. The drivers use pencil and paper to add up their loads as they go along. If offered a package which would make them overweight, they should refuse it. This uses up their turn. The package is replaced at the bottom of the pile.
 8. The game ends when all the goods have been collected and unloaded at the warehouse. The winner is the driver who has brought back most. He becomes the supervisor for the next game.

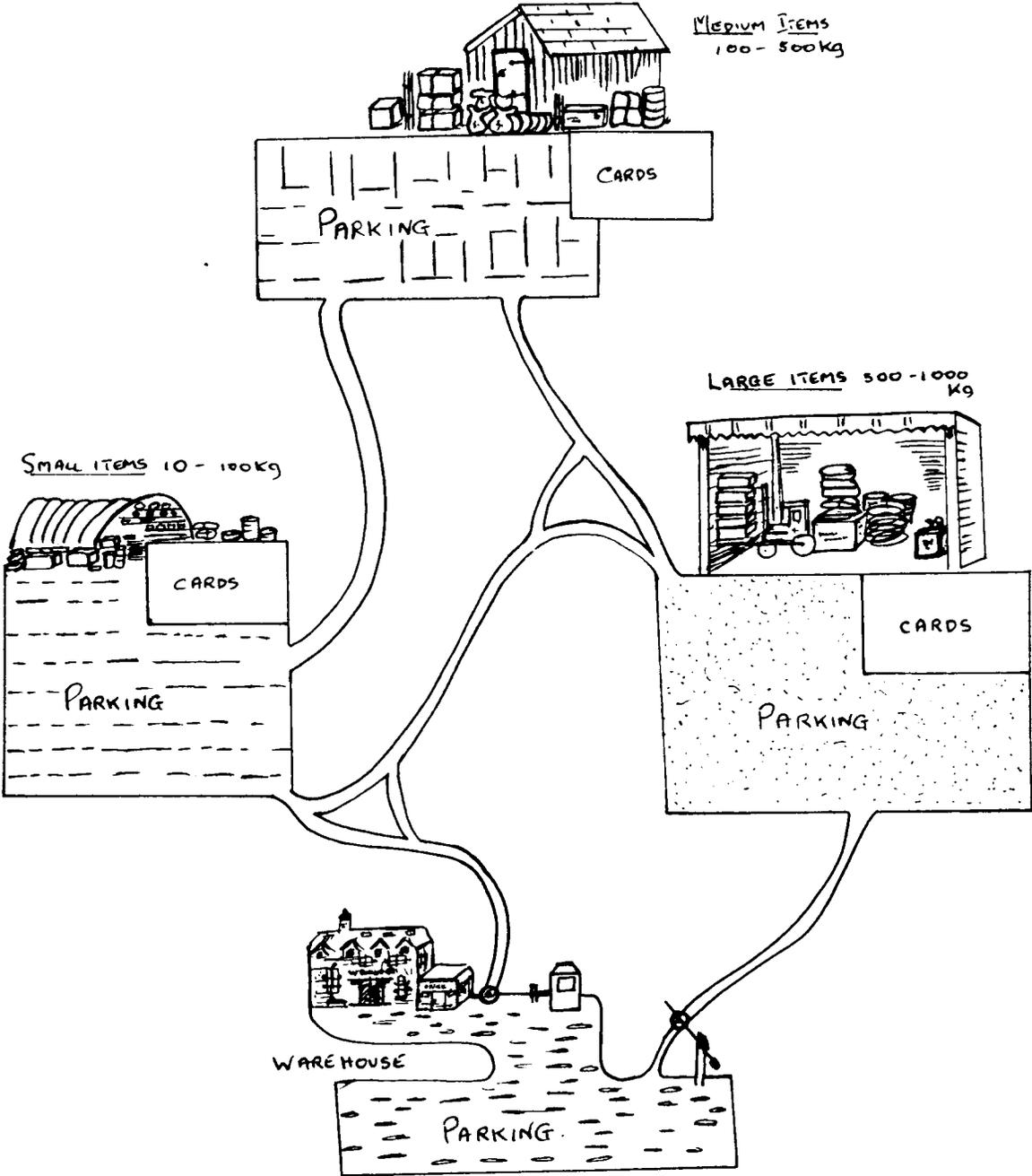


Figure 6 One tonne van drivers

Activity 4 Catalogue shopping* [Num 3.10/4]

An activity for a small group. This gives further uses for their addition skills. Since 1 dollar = 100 cents, working in dollars and cents is equivalent to working in hundreds, tens and units.

- Materials*
- Pages from a mail order catalogue, pasted on card and headed, e.g., ‘\$100 to spend.’ or ‘\$25 to spend.’
 - Pencil and paper for each child.

- What they do*
1. The children are given a catalogue card along with pencil and some pieces of paper.
 2. They are asked to prepare a list of items to be ordered, within the total amount stated at the top of the card. They should first make a rough list and then write out a tidy order.

3. Initially, headed column notation should be used to make the correspondence obvious.

dollars	dimes	cents
100¢	10¢	1¢
5	2	7
527¢		

4. Later this may be written as . . . using place-value notation.
5. At the present stage we are still using cents as our units, and a dollar is just another name for 100 of these. Later, when the dollar is taken as the unit, the ‘period’ becomes a ‘decimal point’ and the next two digits represent tenths and hundredths. Decimal fractions are still in the future, but this transitional notation is excellent preparation.

\$5.27

* For this activity I am indebted to Mrs. A. Cole, Headteacher of Loveston County Primary School, Dyfed.

Discussion of activities We begin with the familiar ‘Start, Action, Result’ in the same three stages as before. This picks up the group of related concepts which are by now well established, and which we want them to extrapolate further. ‘Cycle camping’ (Activity 2) uses a number line, in the form of a game, as another way of adding. Though the results go past 100, the new numbers to be added on do not. With this grounding in the expanded technique, children should be ready for Activities 3, ‘One tonne van drivers,’ and 4, ‘Catalogue shopping.’ The first of these is a little like ‘Air Freight’ from the previous topic, but it also involves comparing possible outcomes of the moves they might make. The other is a budgeting activity. In adult life, planning the use of money and other resources – time, labour – is one of the major uses of arithmetic. This is one of the reasons I have introduced games of this kind.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Num 4] **SUBTRACTION**

Taking away, Comparison, Complement, Giving change

Num 4.6 GIVING CHANGE

Concept Paying a required amount by giving more and getting change.

- Abilities*
- (i) To give the correct change.
 - (ii) To check that one received the correct change.

Discussion of concepts This is another contributor to the comparison aspect of subtraction. In this case, the larger number is the amount tendered, the smaller number is the cost of the purchase, and the difference is the change.

Activity 1 Change by exchange [Num 4.6/1]

An activity for 3 or 4 children (not more). Its purpose is to ‘spell out’ with the coins themselves what is happening when we give or receive change.

- Materials*
- Play money: 30¢ for each customer made up as in step 2, following; 1¢ and 5¢ coins for shopkeeper.
 - A ‘till’ (tray with partitions).
 - Pictures on cards representing objects for sale, with prices marked, all less than 10¢.

- What they do*
1. One child acts as shopkeeper, the rest as customers.
 2. The customers start with 30¢, made up of two 10¢ coins, one 5¢ coin, and five 1¢ coins. The shopkeeper has a plentiful supply of 1¢ and 5¢ coins.
 3. The shopkeeper sets out her wares. If there is not enough table space for all the goods, some may be kept ‘in the stock room’ and put out later.
 4. The customers in turn make their purchases one at a time.
 5. To start with, they pay with exact money. When they no longer have the exact money for their purchases, they pay by giving more and getting change.
 6. Suppose that a customer asks for a 6¢ apple, and hands a 10¢ coin to the shopkeeper.
 7. The shopkeeper says “I have to take 6¢ out of this, so I need to exchange it.” She puts the 10¢ coin into her till and takes out 10 pennies. (With experience, a combination of 5¢ and 1¢ coins will be used.)
 8. Spreading these smaller coins out, she then says “I’m taking 6¢ for your apple” and does so. “The rest is your change: 4 cents.”
 9. The shopkeeper gives the apple and the 4¢ change to the customer, who checks that she has received the right change.

Num 4.6 Giving change (cont.)

Activity 2 Change by counting on [Num 4.6/2]

A continuation from Activity 1, for 3 to 6 children. (May be included or by-passed, at your discretion.) It is placed between Activities 1 and 3 to relate to its mathematical meaning the method of giving change often used by cashiers.

Materials • The same as for Activity 1.

What they do 1 - 4. are the same as in Activity 1.

5. The method of giving change is now different. Assume as before that the customer has handed a 10¢ coin to the cashier for a 6¢ apple. The cashier goes to the till and picks up coins to make the total up to 10¢, saying to herself “7, 8, 9, 10.”

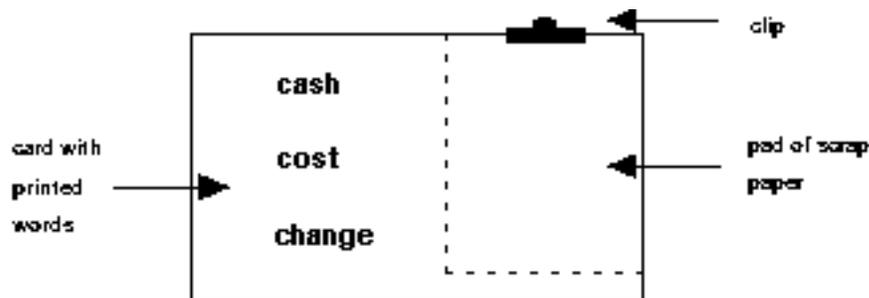
6. She says to the customer “6¢ for your apple,” and then, while counting the change into the customer’s hand, “7, 8, 9, 10.”

Note that with this method, the amount of the change is not explicitly stated or written.

Activity 3 Till receipts [Num 4.6/3]

A continuation from Activity 2, for 3 to 6 children. Its purpose is to relate the kind of subtraction involved in giving change (comparison) to the conventional notation for subtraction.

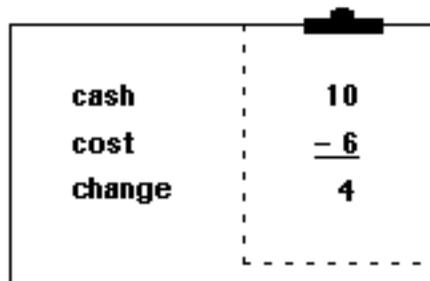
Materials • The same as for Activities 1 and 2, and also
• A pad of till receipts. (See illustration below.)



What they do 1 - 4. are as in Activity 1.

5. Having arrived at the right change by any means she likes, the shopkeeper then writes a till receipt for the customer.

6. She shows it to the customer like this, before removing it from the pad and handing it to the customer.



7. This is what the customer receives, together with her purchase and change.

$$\begin{array}{r} 10 \\ - 6 \\ \hline 4 \end{array}$$

Discussion of activities Once again, there is more here than meets the eye. The counting on method for giving change, as usually practised in shops, produces the correct change and allows the customer to check. But it does not say in advance what amount this will be, nor does it lead to subtraction on paper.

So in Activities 1, 2, 3, we have a sequence. In Activity 1, the emphasis is on the concept itself of giving change, using the simplest possible way of arriving at the amount. Note that the customers begin with assorted coins, so that the activity does not begin with giving change, but with paying in the direct way. Giving more and receiving change is then seen as another way of paying the correct amount. We found that when this approach was not used, some children continued to give change even when the customer, having collected the right coins by receiving change, had then paid the exact amount! This shows how easily habit learning can creep in instead of understanding, and also how important it is to get the details right in these activities. Activity 2 uses counting on as a method for first producing the correct change, and then allowing the customer to check. Finally, Activity 3 transfers this to paper and makes explicit the amount of change which the customer should receive. It also relates this new aspect of subtraction to the notation with which children are already familiar. This helps to relate it to the overall concept of subtraction.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 4.7 SUBTRACTION WITH ALL ITS MEANINGS

Concept Subtraction as a single mathematical operation with 4 different aspects.

Ability To relate the overall concept of subtraction to any of its embodiments.

Discussion of concepts In this topic we are concerned with recapitulating the 4 earlier aspects of subtraction: taking away, comparison, complement, and change. Finally, in Activity 5, these are fused together into a concept of subtraction from which can be extracted all of these particular varieties.

Activity 1 Using set diagrams for taking away [Num 4.7/1]

A teacher-led discussion for a small group. Its purpose is to relate the take-away aspect of subtraction to set diagrams.

Materials • Pencil and paper for all.

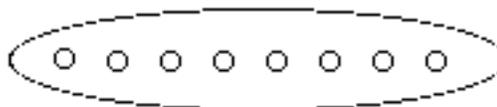
Suggested sequence for the discussion

1. Write on the left of the paper:

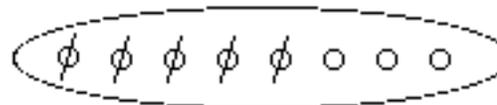
$$\begin{array}{r} \\ \underline{} \\ \end{array} \quad \begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$

2. Say and draw (pointing first to the 8, then the 5):

“This says we start with 8.”

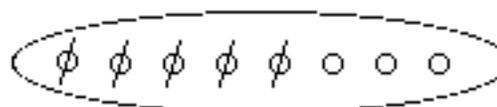


“This says we take away 5.”
Cross out the 5 to be taken away.



3. Write the result, 3.
4. Review the correspondences between the number sentence and the starting set, the action (crossing out), and the result.

$$\begin{array}{r} 8 \\ - 5 \\ \hline 3 \end{array}$$



5. Let the children repeat steps 1 to 4 with another example. Use vertical notation, as above.
6. Give further practice if needed.

Activity 2 Using set diagrams for comparison [Num 4.7/2]

A teacher-led discussion for a small group. Its purpose is to relate the comparison aspect of subtraction to set diagrams, and thereby to what they have just done.

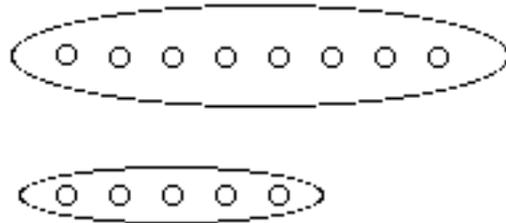
Materials • Pencil and paper for all.

Suggested sequence for the discussion

1. Write as before.
$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$

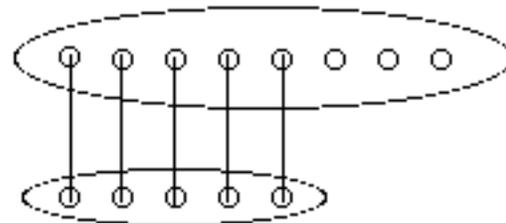
2. Say, "This subtraction can have another meaning, besides taking away." Here we have two numbers, the larger one above.

3. Draw these on the right of the subtraction sentence.



4. Ask (pointing): "How many more are there in this set, than this?"

5. If they answer correctly, say "Let's check." Draw lines like this and say (pointing) "These 5 lines show where the sets are alike, so these 3 without lines show where they are different."

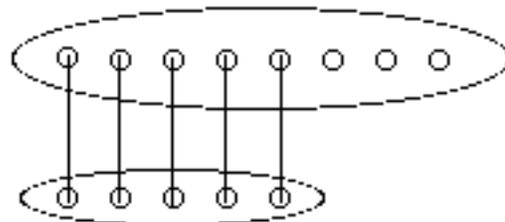


6. If they do not answer correctly, use step 5 to show how they can find the result.

7. Either way, write the result in the subtraction sentence.

8. Review the correspondences between the number sentence, the two sets, the action (comparison), and the result.

$$\begin{array}{r} 8 \\ - 5 \\ \hline 3 \end{array}$$



9. Explain: "We haven't been taking away, so we shouldn't read the number sentence as 'take away.' We say "8, subtract 5, result 3."

10. Give further practice, as required. Use vertical notation only.

11. Say, "Now we have 2 meanings for this subtraction sentence." Review these.

Num 4.7 Subtraction with all its meanings (cont.)

Activity 3 Using set diagrams for finding complements [Num 4.7/3]

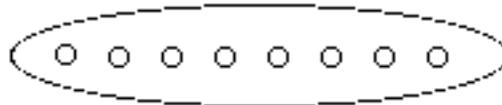
A teacher-led discussion for a small group. Its purpose is to relate the complement aspect of subtraction to set diagrams, and thereby to what they did in Activities 1 and 2. You may wish to review Num 3.5, ‘Complementary numbers,’ Activities 1 and 2, in the Addition network in *SAIL Volume 1*.

- Materials*
- Pencil and paper for all.
 - Red and blue felt tips.

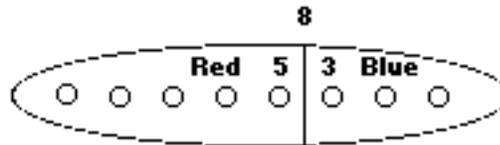
Suggested sequence for the discussion

1. Write, and draw in pencil.

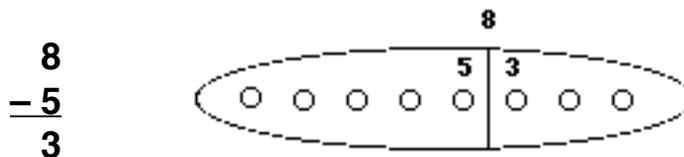
$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$



- Say, “Here is another meaning. We’re told that these are all to be coloured red or blue. If 5 are coloured red, how many blue?”
- If (as we hope) they say “Three,” check by colouring. If not, demonstrate.
- Say, “If we didn’t have red and blue felt tips, what could we do instead?”
- Accept any sensible answers, and contribute the suggestion below. Tell them that they may continue to use their own way if they like.



- Invite other meanings, e.g., 8 children, 5 girls, how many boys?
- Review the correspondences between the number sentence, the whole set, and the two parts of the set.



$$\begin{array}{r} 8 \\ - 5 \\ \hline 3 \end{array}$$

- Remind them that the larger number has to be above. This time the larger number is the whole set, the next number is one part, and the last number is the other part.
- Give further practice, as required. Use vertical notation only.
- They now have 3 meanings for the subtraction sentence. Review these.

Activity 4 Using set diagrams for giving change [Num 4.7/4]

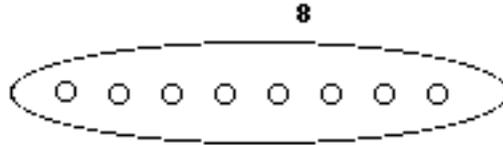
A teacher-led discussion for a small group. Its purpose is to relate the ‘cash, cost, change’ aspect of subtraction to set diagrams, and thereby to what they did in Activities 1, 2, and 3.

- Materials*
- Pencil and paper for all.

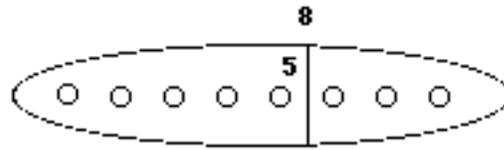
Suggested sequence for the discussion

1. Write, on the left of the paper:

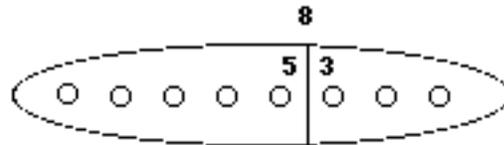
$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$
2. Say, "There's just one more meaning we can give this. Suppose you are a shop-keeper, and a customer gives you 8¢ for an apple. But the apple only costs 5¢. What money will you give back to him?"
3. Assuming that they answer correctly, say, "Yes. Now let's check."
4. Draw.



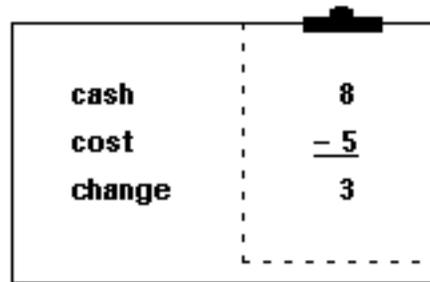
5. Say, "These are the 5 pennies for the apple," and draw the partition line. Write the 5 inside.



6. Continue: "And so these are the pennies you give back to the customer." Point to the right-hand subset and write the 3 inside.



7. Relate the foregoing to 'Cash, cost, change.' (Num 4.6/3).



8. Give further practice, as required.
9. Review all 4 of the meanings they now have for subtraction.

Activity 5 Unpacking the parcel (subtraction) [Num 4.7/5]

A game for up to 6 children. Its purpose is to consolidate children's understanding of the 4 different aspects of subtraction.

- Materials*
- 'Parcel' cards, of two kinds, as illustrated in Stage (a) and Stage (b).*
 - A bowl of counters.

* Provided in the photomasters

Num 4.7 Subtraction with all its meanings (cont.)

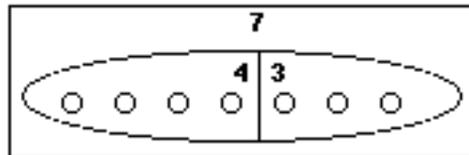
Rules of Stage (a)

- the game*
1. The first set of parcel cards is put face down, and the top one turned over. (Reminder: this is read as “7, subtract 3, result 4,” NOT as “7, take away 3 . . .”)
 2. Explain that this has a number of different meanings which can be ‘taken out,’ one at a time, like unpacking a parcel.
 3. The children take turns to give one meaning. If the others agree, the child whose turn it is takes a counter.
 4. There are four different mathematical meanings, as in Activities 1, 2, 3, and 4.
 5. An unlimited number of situational meanings can also be found, and these can become repetitive. E.g., if someone says, “7 boxes, 3 empty, so 4 have something in them,” and someone else then says, “7 cups, 3 empty, so 4 have something in them,” this is so little different as to be hardly worth saying. If the rest of the group unanimously think that an example is of this kind, they might reject it even though correct. This might lead to discussion as to what is acceptable as a genuinely different meaning.

$$\begin{array}{r} 7 \\ - 3 \\ \hline 4 \end{array}$$

Stage (b)

1. This is played in the same way as Stage (a), except that the second set of parcel cards is used, like this one:



These have even more possible meanings, and the children should write these down before expanding them further. E.g.:

$$\begin{array}{r} 4 \\ + 3 \\ \hline 7 \end{array}$$

$$\begin{array}{r} 7 \\ - 4 \\ \hline 3 \end{array}$$

$$\begin{array}{r} 7 \\ - 3 \\ \hline 4 \end{array}$$

2. We thus have ‘parcels within parcels.’ This activity involves much concentration of mathematical meaning, and children should return to it at intervals until all 4 aspects are mastered. Children find the part-whole relationships harder than the take-away and comparison aspects of subtraction.

Discussion of activities

In this topic we have a good example of the highly abstract and concentrated nature of mathematical ideas. Hence its power, but hence also the need for very careful teaching.

In the topics which lead up to this one, the 4 different aspects of subtraction are introduced separately, with the use of materials to provide a less abstract approach. In the present topic these are brought together by using set diagrams, which again provide a less abstract symbolism than the purely numerical symbols whose use, with full understanding, is the final learning goal.

In Activity 5, the children are learning explicitly something about the nature of mathematics, namely its concentration of information. We ourselves have been taking notice of this from the beginning.

Num 4.8 SUBTRACTION OF NUMBERS UP TO 20, INCLUDING CROSSING THE TEN BOUNDARY

Concept Expansion of the subtraction concept to include larger numbers.

Ability To subtract numbers up to 20, including examples which involve crossing the 10 boundary.

Discussion of concepts ‘Crossing the ten boundary’ means calculations like $12 - 3$, $14 - 6$, $17 - 9$. All subsequent examples which involve regrouping, such as $52 - 4$, $82 - 36$, $318 - 189$, depend on this.

Over the years there has been much discussion whether children should be taught to subtract by decomposition or complementary addition. The argument for decomposition has been that it can be demonstrated with physical materials, and so is better for teaching with understanding. For complementary addition, it has been claimed that it is easier to do, and makes for faster and more accurate calculations. The latter can also be justified sensibly, though it seldom is: ‘borrowing’ and ‘paying back’ is a nonsensical explanation.

The present approach is based on the physical regroupings which the children have learned in Org 1, and the concept of canonical form which comes in both Org 1 and Num 2. It has several advantages:

(i) It is mathematically sound.

(ii) It allows children to use whichever technique they find easier: counting back, corresponding to the take-away aspect of subtraction, or counting on, corresponding to complementation. Both of these they have already experienced with physical materials. Taking away across the ten boundary involves decomposing one larger group into 10 smaller groups; complementing involves putting together 10 smaller groups into one larger group. Either way, regrouping and canonical form are the key concepts.

(iii) The same technique (changing out of or into canonical form) is good for adding, multiplying, and dividing.

The present topic is preparatory to the full technique, which follows in Num 4.9. It teaches what we do after changing out of canonical form.

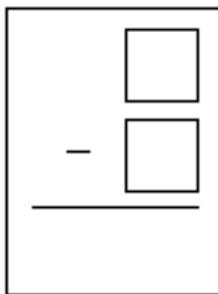
Activity 1 Subtracting from teens: choose your method [Num 4.8/1]

A teacher-led activity for up to 6 children. Its purpose is to show them two ways of subtracting across the tens boundary, help them to see that these are equivalent, and choose which method they prefer.

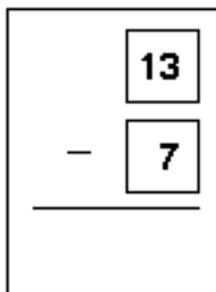
- Materials*
- Two sets of number cards, in different colours. One set is from 10-19, the other is from 0 to 9.*
 - Subtraction board, as illustrated.*

* Provided in the photomasters

Num 4.8 Subtraction of numbers up to 20 (cont.)



- What they do*
1. Before starting, they should review finger counting, including ‘Ten in my head.’ (See Num 1.4/1, Num 1.5/1 and Num 1.7/1 in *SAIL Volume 1*.)
 2. The subtraction board is put where all the children can see it the same way up. Both sets of cards are shuffled and put face down, near the board, with the teens set on the left.
 3. The top card from each pack is turned over, and put one in each space on the board to give (e.g.)



4. Demonstrate the two ways of doing this using finger counting.
 - (a) Counting back. “We start at 13 and count back to 7, putting down one finger each time. 12 (one finger down), 11, 10, 9, 8, 7 (six fingers down). The difference is 6.” They all do this.
 - (b) Counting on. “We start at 7 and count on to 13, putting down one finger each time. 8 (one finger down), 9, 10, 11, 12, 13 (six fingers down). The difference is 6.” They all do this.
5. Note that
 - (i) We use the word ‘difference’ in both cases to link with this aspect of subtraction.
 - (ii) We do *not* put down a finger for the starting number.
 - (iii) This method gives (as intended) a mixture of examples in which some do and some do not involve crossing the tens boundary.
6. With another pair of numbers, half the children arrive at the difference by counting back, and half by counting on. They should all have down the same number of fingers.
7. Step 5 is repeated until the children are proficient.
8. Tell them that having tried both, they may use whichever method they prefer from now on. They may like to discuss the reasons for their preference.

Activity 2 Subtracting from teens: “Check!” [Num 4.8/2]

A game for 4 or 6 children, playing in teams of 2. Its purpose is to give them fluency in subtracting across the tens boundary.

- Materials*
- Number cards 10-19 and 0-9.*
 - Subtraction boards.*
 - A bowl of counters.

*The same as for Activity 1.

- What they do*
1. A subtraction board is put where all can see it the same way up. Both sets of cards are shuffled and put near the board, with the ‘teens’ set on the left.
 2. In the first team, each player turns over the top card from one of the packs and puts it on the board, placing teens cards in the upper space.
 3. The two players then do the subtraction independently by any method they like.
 4. Another player says, “Ready? Check!”
 5. On the word “Check,” both players immediately put fingers on the table to show their results. No alteration is allowed.
 6. In some cases the result will be over 10. Example: $15 - 2$. Both players should now put down 3 fingers, saying “10 in my head.” (See Num 1.7/1 in *Volume 1*.)
 7. The other players check, and if both have the same number of fingers on the table (and the others agree that this is the correct answer), this team takes a counter.
 8. Steps 2 to 6 are repeated by the next team.
 9. The game continues as long as desired, the number cards being shuffled and replaced when necessary. All teams should have the same number of tries.
 10. The winners are those with the most counters.

Activity 3 Till receipts up to 20¢ [Num 4.8/3]

A continuation from Num 4.6/3, for 3 to 6 children. Its purpose is to consolidate their new skill in a familiar activity.

- Materials*
- Play money. The customers each have four dimes as well as one nickel, and three pennies. The shopkeeper has a good assortment of all coins.
 - A tray with partitions, used as a till.
 - Pictures on cards representing objects for sale with prices marked, ranging from (say) 3¢ to 19¢.
 - Base 10 material, units and ten-rods.

- What they do*
- 1-7. The same as in the earlier version of ‘Till receipts’ (Num 4.6/3), except that the prices range all the way up to 19¢.
 8. Customers may now purchase several objects at a time, provided only that the total is below 20¢.
 9. At any time when children have difficulty in writing the till receipts or checking them, they should help themselves by using base 10 material. They may also use the counting on method of Num 4.6/2.

Num 4.8 Subtraction of numbers up to 20 (cont.)

Activity 4 Gift shop [Num 4.8/4]

A game, continuing on from the previous activity, for 3 to 6 children.

Materials The same as for Activity 3, together with a ‘notice’* as illustrated in step 9. Its purpose is further to consolidate their new skills, and extend these to subtraction other than from multiples of 10.

* Provided in the photomasters

Rules of play 1 - 8. The same as in Activity 3.

9. However, the shopkeeper also displays a notice:

**YOUR PURCHASE FREE
if I give the wrong change.**

10. If a customer thinks he has received incorrect change, the other customers also check. If it is agreed that the change was incorrect, the shopkeeper must give the customer his purchase free. The cash is returned to the customer, and the change to the shopkeeper.
11. If this happens 5 times the shopkeeper goes broke, and someone else takes over the shop. (The number of mistakes allowed to the shopkeeper should be adjusted to the children’s ability.)
12. To make things more difficult for the shopkeeper, customers may pay with whatever amounts they like. E.g., they could hand over 17¢ for an object costing 9¢. (This rule should not be introduced until the children have learned the rest of the game.)

Discussion of activities It will be noticed that the activities of this topic do not begin with the use of physical materials, in spite of the importance which in general we attach to these. Base ten material is good for teaching the exchange of 1 ten for 10 ones, but they already have plenty of experience of this. It is also good for teaching conversion into and out of canonical form, and this too is done in earlier contributors to the present topic. It lends itself well to teaching subtraction in its ‘take-away’ form, but not nearly so easily to the comparison and complementation forms. The latter are easier to do mentally, since counting forward is easier than counting back. So the approach in this topic relies on the foundations laid by Mode 1 schema-building in earlier topics, and uses finger counting as a transitional technique which applies equally well to either aspect of subtraction. This will fall into disuse as children gradually learn, and use for subtraction, their addition facts.

These are followed by the application of these new techniques in familiar activities. The last activity, ‘Gift shop,’ introduces a penalty for the shopkeeper if he makes too many mistakes, and a reward for the customer who detects a mistake. It is not only in this game that a shopkeeper who cannot do his arithmetic finds himself in difficulties!

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 4.9 SUBTRACTION UP TO 99

Concept Their existing concept of subtraction, expanded to include subtraction of two-digit numbers.

Abilities (i) To subtract two-digit numbers.
(ii) To apply this to a variety of situations.

Discussion of concept So far as the concept itself is concerned, all that is new is the size of the numbers to which the operation is applied. However this requires the introduction of new techniques, and it is important that the manipulations of symbols which children learn at this stage should be meaningful in terms of the underlying mathematics.

The method we recommend has been reached by much thought, discussion, and field trials with children. It begins in Num 4.8, and continues here. However, rather than split the discussion, the whole of it was given at the beginning of Num 4.8. It would therefore be useful to reread this.

Before embarking on this,
we need to guard against the
common error of subtracting
the wrong way round, e.g.,

$$\begin{array}{r} 32 \\ - 17 \\ \hline 25 \end{array}$$

This gives the wrong result because subtraction is non-commutative. This contrasts with addition, which is commutative. The result of these two additions is the same if we interchange the numbers:

$$\begin{array}{r} 7 \\ + 2 \\ \hline 9 \end{array} \qquad \begin{array}{r} 2 \\ + 7 \\ \hline 9 \end{array}$$

Not so for these two subtractions:

$$\begin{array}{r} 7 \\ - 2 \\ \hline 5 \end{array} \qquad \begin{array}{r} 2 \\ - 7 \\ \hline \end{array}$$

cannot be done, with the numbers they know about so far

I leave it to your own judgment relative to the children you teach, whether or not to introduce the term 'non-commutative' at this stage. The important practical result is what Activity 1 is about.

Activity 1 “Can we subtract?” [Num 4.9/1]

A teacher-led discussion for a small group, or for the class as a whole. Its purpose is to emphasize that one can only subtract when the first number of the pair is greater than or equal to the second. In vertical notation, the upper number must be greater than or equal to the lower. (At this stage we are not concerned with negative numbers, but see the note at the end of this activity.)

- Materials*
- Pencil and paper, or
 - Chalk and chalkboard.

Suggested sequence for the discussion

1. Write a subtraction in vertical notation, e.g.,

$$\begin{array}{r} 6 \\ - 2 \\ \hline \end{array}$$

2. Ask for the result.
 3. Remind them of the first of the meanings for subtraction, taking away. (See Num 4.7, ‘Subtraction with all its meanings’)

$$\begin{array}{r} 6 \\ - 2 \\ \hline 4 \end{array}$$



4. Reverse the numbers, and ask “Can we subtract this way round?”

$$\begin{array}{r} 2 \\ - 6 \\ \hline \end{array}$$

5. Depending on their responses, let them see that this would mean making a set with number 2 and crossing out 6.



6. If there is still any doubt, let them try to do it with physical objects.
 7. Repeat steps 1 to 6 with other examples. Include equal numbers, and also cases where one number is zero. Start sometimes with the not-possible case.
 8. Continue, “How about one of the other meanings of subtraction? Let’s try it with cash, cost, change.”

9. Clearly in this case you get change.

cash	cents
cost	9
change	<u>- 5</u>
	4

10. But in this case, the shopkeeper would say, in effect, “Can’t be done.”

cash	cents
cost	5
change	<u>- 9</u>

11. Repeat steps 8 to 10 with one or more further examples.
12. Ask “What have we learned?” and obtain agreement on a suitable formulation, in their own words. It should mean the same as the learning goal described in the heading for this activity.

Note Some children may say that they can subtract a larger number from a smaller (e.g., referring to digging a hole, or temperatures below zero, . . .). In this case you could tell them that they are quite right, but they are talking about a different kind of number system, called integers, which they haven’t come to yet. When they do, they will find that it still uses a method for subtraction like the one they are now learning.

Activity 2 Subtracting two-digit numbers [Num 4.9/2]

A teacher-led activity for a small group. Its purpose is to teach subtraction of two-digit numbers, including cases which involve crossing the tens boundary. This now extends its meaning to include that between teens and twenties, twenties and thirties, etc.

- Materials*
- Subtraction board.*
 - Two sets of number cards, in different colours. One set contains twenty cards bearing assorted numbers between 50 and 99, the other contains twenty cards with assorted numbers all less than 50 (provided in the photomasters).**

* Like that for Num 4.8/1

** Suitable assortments are:

- Under 50 – 8, 9, 12, 14, 17, 18, 21, 23, 26, 29, 30, 33, 36, 39, 41, 42, 44, 45, 47, 48.
- 50 and over – 50, 53, 56, 59, 61, 67, 68, 72, 74, 75, 78, 79, 80, 82, 84, 87, 91, 93, 96, 99.

Suggested sequence for the discussion

1. Put out the subtraction board and number cards as in Num 4.8/1. Explain that this is like the earlier activity, but they are going to learn how to do subtraction of larger numbers.

2. Begin with an example such as this, in which both the upper digits are larger than the digits below them.

$$\begin{array}{r} 68 \\ - 23 \\ \hline \end{array}$$

3. Explain: “We subtract a column at a time, ones from ones, tens from tens.”

$$\begin{array}{r} 68 \\ - 23 \\ \hline \end{array}$$

4. Let them practice a few examples of this kind.
5. Next, introduce an example for which this is not so.

$$\begin{array}{r} 74 \\ - 26 \\ \hline \end{array}$$

6. “Can we do ‘4 subtract 6’?” (No.) “So this is what we do. First, we write it in headed columns.”

Tens	Ones
7	4
- 2	6

Num 4.9 Subtraction up to 99 (cont.)

7. “Next, we write the upper number differently.”
(This is changing it into a non-canonical form.)

Tens	Ones
6 7	4 14
- 2	6

8. “Now we can subtract a column at a time, as before.”

Tens	Ones
6	14
- 2	6
4	8

9. This is the answer in place-value notation.

$$\begin{array}{r} 74 \\ - 26 \\ \hline 48 \end{array}$$

10. The whole process may be set out as below. (The sign \Leftrightarrow means ‘is equivalent to.’ Its use is optional.)

$$\begin{array}{r} 74 \\ - 26 \\ \hline \end{array} \Leftrightarrow \begin{array}{c|c} \text{Tens} & \text{Ones} \\ \hline 7 & 4 \\ - 2 & 6 \\ \hline \end{array} \Leftrightarrow \begin{array}{c|c} \text{Tens} & \text{Ones} \\ \hline 6 \cancel{7} & 4 \cancel{14} \\ - 2 & 6 \\ \hline 4 & 8 \end{array} \Leftrightarrow \begin{array}{r} 74 \\ - 26 \\ \hline 48 \end{array}$$

This way they can see all the steps. At present the goal is understanding before speed.

11. The two middle steps may soon be combined mentally.

$$\begin{array}{r} 74 \\ - 26 \\ \hline \end{array} \Leftrightarrow \begin{array}{c|c} \text{Tens} & \text{Ones} \\ \hline 6 & 14 \\ - 2 & 6 \\ \hline 4 & 8 \end{array} \Leftrightarrow \begin{array}{r} 74 \\ - 26 \\ \hline 48 \end{array}$$

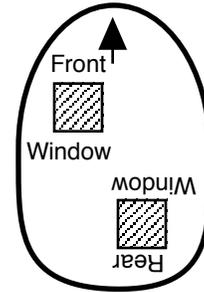
12. Now let the children use the subtraction board as in Num 4.8/1, to give assorted examples. That is, the two piles of number cards are shuffled and put face down, and the top card from each pile is turned over and put on the board. The children copy what is there onto their papers, calculate the results, and check.

- Notes*
- (i) If step 7 is not understood, use base 10 material to demonstrate the exchange of 1 ten-rod for 10 cubes. The rest of the calculation is more easily dealt with symbolically, with the help of finger counting if necessary. This allows either the use of counting forward from the smaller number (the complement aspect of subtraction), or counting back from the larger number (the take-away aspect of subtraction): see Num 4.8/1, ‘Subtracting from teens: choose your method.’ Many children find complementation the easier method.
 - (ii) Eventually, the intermediate steps may be done mentally, but this should not be done until children have had a lot of practice in the written form.

Activity 3 Front window, rear window [Num 4.9/3]

A game for two. Its purpose is to practise subtraction of two-digit numbers.

- Materials*
- ‘Front window, rear window’ game board.*
 - ‘Car,’ as illustrated.*
 - Pencil and paper for each player.
- * Provided in the photomasters



The ‘car,’ in coloured cardboard, has windows (shaded areas) which are cut out as holes. The arrow shows the direction of travel.

- What they do*
1. This game is based on the fact that the road signs we see looking backward tell us distances to places we have left behind.
 2. The players sit opposite each other with the board between them.
 3. The car is put at the starting town (bottom left), with the arrows pointing in the direction of movement. The players find out which window they look through from the writing they see right way up.
 4. The car moves to the first road sign and each writes the number he sees (say, 87 [km] through the front window and 27 [km] through the rear window).
 5. The car moves on to the next road sign and each again writes the number he sees (say, 68 [km] through the front window and 46 [km] through the rear window).
 6. Each has now recorded two distances. By subtraction, each finds the distance they have travelled between the road signs. Though the numbers are different, the distances are of course the same, so each passenger should get the same result. For the first two signs in this example, the subtractions are:

$$\begin{array}{r} 87 \\ - 68 \\ \hline 19 \end{array} \quad \text{and} \quad \begin{array}{r} 46 \\ - 27 \\ \hline 19 \end{array}$$

7. The car moves on to the next road sign and steps 5 and 6 are repeated with the two latest numbers.
8. This continues to the end of the journey. At the intermediate towns the signs now relate to the next town ahead and the town just left. The car now begins to travel in the opposite direction, so the board needs to be turned around so that the passengers continue to face the same way relative to the car.
9. By changing seats on their next trip, the passengers could get a different set of calculations.
10. Further practice may be given by using other figures, either by making other boards or by putting stickers on the existing board.

Num 4.9 Subtraction up to 99 (cont.)



Front window, rear window [Num 4.9/3]

Another suitable set of figures for the 'Front window, rear window' board:

Road section	Road length					
1	113	Front window	100	79	52	33
		Rear window	13	34	61	80
2	125	Front window	99	76	51	36
		Rear window	26	49	74	89
3	110	Front window	81	55	25	16
		Rear window	29	55	85	94
4	100	Front window	78	57	38	19
		Rear window	22	43	62	81

Activity 4 Front window, rear window – make your own [Num 4.9/4]

An activity and game for two. It is an extension of Activity 3, 'Front window, rear window,' in which the children work out their own road signs. This is appreciably harder than Activity 3.

- Materials*
- A board similar to that for Activity 3, without the numbers written in, but with the lines showing locations of the road signs. This board must be covered in transparent film or laminated.
 - 'Car,' as for Activity 2.
 - 2 dice, one marked 0-2 for tens, the other 1-6 for ones.
 - A washable non-permanent marker, and a damp rag.

- What they do*
1. They first decide on the distance between the first two towns. This must be between 105 and 130 kilometres. They both write this number down.
 2. They throw both dice, to obtain the distance travelled from the start.
 3. One player marks in this distance for the 'rear window,' the other subtracts from the total distance to get the 'front window' number and marks this on the board.
 4. They throw the dice again for the next distance travelled.
 5. They each work out their own number, and mark it on the board.
 6. Steps 4 and 5 are then repeated twice more to complete the first stage of the journey.
 7. They then repeat steps 1 to 6 for the remaining stages of the journey, choosing a different distance between each pair of towns.
 8. They should then play as in Activity 3 to check their calculations, or swap boards with another pair to do this. At first they might prefer to check at the end of each stage of the journey.

Discussion of activities Activity 1 is intended to prevent the error of subtracting the wrong way around, discussed first in the 'Discussion of concepts' for this topic.

Activity 2 then shows children how to subtract two-digit numbers. Since children should already be familiar with moving in and out of canonical form from the addition network, the suggestion is that they work at this level rather than go back to physical embodiments in base ten material. This avoids going right back to the take-away aspect of subtraction, whereas the concept now includes other components. Help from the latter may however be used to demonstrate the change from canonical form, also (correctly) called decomposition and (incorrectly) called borrowing, without detriment to the foregoing.

Activities 3 and 4 are more sophisticated applications of the concept of subtraction than they have encountered so far, using the new technique which they have learned.

OBSERVE AND LISTEN**REFLECT****DISCUSS**

Num 4.10 SUBTRACTION UP TO 999

Concept The existing concept of subtraction, extrapolated to numbers up to 999.

Abilities (i) To subtract numbers up to 999.
(ii) To apply this to situations involving any of the four aspects of subtraction already encountered.

Discussion of concept Although all that is new here is (as in the previous topic) an expansion of the size of numbers to which the operation of subtraction is applied, this results in a substantial increase in the amount of information to be handled. This makes necessary careful organizing, with the help of symbols on paper.

Activity 1 Race from 500 to 0 [Num 4.10/1]

An activity for 2 children. Its purpose is to make sure that the relation between larger numbers and physical materials, and the process of exchanging, are kept active before the children embark on the symbolic work of Activity 2.

Materials

- Base 10 materials in box:
 - 5 ten-squares
 - at least 20 ten-rods
 - at least 20 unit cubes
- 2 dice 1-6
- Two hundreds, tens, ones boards, as illustrated below, one for each player.

Hundreds	Tens	Ones

Rules of the game

1. Each player starts with 5 ten-squares, representing 500. These are put in the hundreds space on his board.
2. They throw the two dice alternately, and these determine the number to be subtracted. If the numbers thrown are (e.g.) 2 and 6, the player may choose to subtract either 2 tens and 6 units, or 6 tens and 2 units.

3. They physically take away that number in the base ten material, exchanging with the material in the box where necessary.
4. It is important to return material not in use to the box, or confusion results.
5. The player who first reaches zero is the winner.
6. Zero must be reached exactly. To do this, players may add or subtract the numbers thrown. Both must be used. E.g., a player throwing 5 and 2 could subtract either 7 or 3, but not 5 or 2.
7. Either player may delay taking his turn in order to check what his opponent is doing.

Activity 2 Subtracting three-digit numbers [Num 4.10/2]

A teacher-led activity for a small group. It follows on from Num 4.8/1, and its purpose is to extend the children’s ability to subtract to three digit numbers.

- Materials*
- Subtraction board.*
 - Two sets of number cards, in different colours. One set contains twenty cards bearing assorted numbers between 500 and 999, the other contains twenty cards bearing numbers all less than 500.**

* Like that for Num 4.8/1

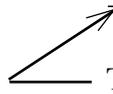
** Suitable assortments provided in the photomasters:

Under five hundred: 36, 54, 60, 71, 115, 142, 179, 187, 200, 241, 254, 298, 323, 332, 387, 399, 406, 415, 423, 468.

Over five hundred: 514, 533, 576, 587, 606, 641, 652, 688, 751, 727, 765, 790, 825, 843, 879, 890, 900, 914, 939, 968.

- Suggested sequence for the discussion*
1. Put out the subtraction board and number cards as in Num 4.8/1 and 4.9/2. Explain that they are now going to continue their learning of subtraction to hundreds, tens and ones.
 2. As before, the top card from each pile is turned over and put on the subtraction board. The children copy onto their paper.
 3. The method is a direct continuation of that already learned, so you may invite their own suggestions as to how to deal with these new examples.
 4. Changing out of non-canonical form may now involve one column, or two. Example: one column involved.

$\begin{array}{r} 825 \\ - 241 \\ \hline 584 \end{array}$	<table style="border-collapse: collapse;"> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">H</td> <td style="border-right: 1px solid black; padding: 0 5px;">T</td> <td style="padding: 0 5px;">Ones</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">8</td> <td style="border-right: 1px solid black; padding: 0 5px;">2</td> <td style="padding: 0 5px;">5</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">- 2</td> <td style="border-right: 1px solid black; padding: 0 5px;">4</td> <td style="padding: 0 5px;">1</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">5</td> <td style="border-right: 1px solid black; padding: 0 5px;">8</td> <td style="padding: 0 5px;">4</td> </tr> </table>	H	T	Ones	8	2	5	- 2	4	1	5	8	4	<table style="border-collapse: collapse;"> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">H</td> <td style="border-right: 1px solid black; padding: 0 5px;">T</td> <td style="padding: 0 5px;">Ones</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">7</td> <td style="border-right: 1px solid black; padding: 0 5px;">12</td> <td style="padding: 0 5px;">5</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">- 2</td> <td style="border-right: 1px solid black; padding: 0 5px;">4</td> <td style="padding: 0 5px;">1</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 0 5px;">5</td> <td style="border-right: 1px solid black; padding: 0 5px;">8</td> <td style="padding: 0 5px;">4</td> </tr> </table>	H	T	Ones	7	12	5	- 2	4	1	5	8	4
H	T	Ones																								
8	2	5																								
- 2	4	1																								
5	8	4																								
H	T	Ones																								
7	12	5																								
- 2	4	1																								
5	8	4																								

 This 584 is written last.

Num 4.10 Subtraction up to 999 (cont.)

Example: two columns involved.

9	1	4
-	2	9
<hr/>		
6	1	6

H	T	Ones
9	1	4
-	2	9
<hr/>		

H	T	Ones
9	0	14
-	2	9
<hr/>		

H	T	Ones
8	10	14
-	2	9
<hr/>		
6	1	6

This 616 is written last.

Not all examples will be as hard as the second; but even with this kind, and the one below, this technique keeps thinking under control. It reduces the amount which has to be dealt with to one step at a time, and still allows children to do these mentally when they feel ready.

Example: two zeros.

9	0	0
-	1	4
<hr/>		
7	5	8

H	T	Ones
9	0	0
-	1	4
<hr/>		

H	T	Ones
8	10	0
-	1	4
<hr/>		

H	T	Ones
8	9	10
-	1	4
<hr/>		
7	5	8

This 758 is written last.

5. If one of the harder kinds comes up before the children are able to deal with it, it may be by-passed so that children build up confidence with easier examples.
6. If children still have difficulty, they probably need more practice changing from canonical form with the use of physical materials. (See Num 2.11.)
7. Steps 2 to 4 are now repeated. Each child copies from the board onto his paper, and does the subtraction. They then compare results with those sitting next to them. They should continue practicing on their own until all are proficient.

Activity 3 Airliner [Num 4.10/3]

A game for two children. Its purpose is to illustrate the practical use of subtraction, and give practice in the subtraction of three-digit numbers.

- Materials*
- Airliner board, see Figure 7.*
 - Three packs of 0-9 number cards.*
 - Pencil and paper for each player.

* Provided in the photomasters

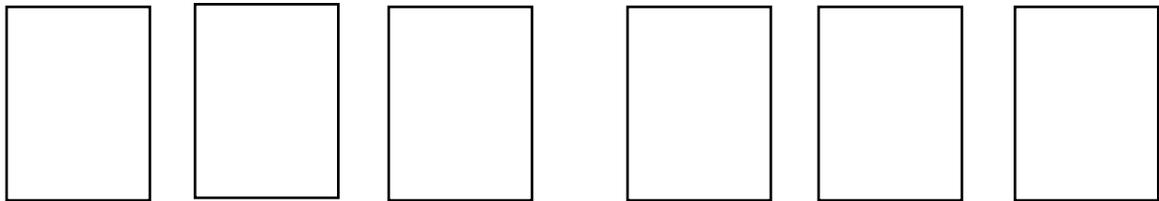
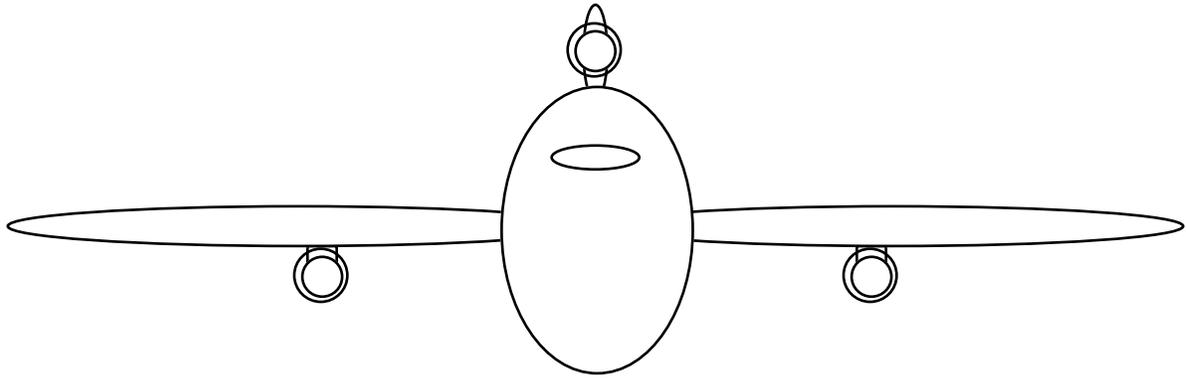
- What they do*
1. One player acts as flight engineer, the other as fuel operator.
 2. They sit opposite each other with the board in between, facing the flight engineer. Some kind of screen is arranged so that only the flight engineer can see the aircraft's fuel gauges (as would be the case in the actual situation).
 3. The number packs are put together and shuffled.
 4. The 6 top cards are put in the 6 spaces. This gives a simulation of the readings on the fuel gauges when the aircraft lands.

- The flight engineer wants to take on board the exact amount of fuel to equalize her port and starboard tanks. Suppose that these are the readings.

PORT TANK
724

STARBOARD TANK
419

She calculates that she wants 305 gallons in her starboard tank, and asks the fuel operator to put this in.



Starboard Fuel Tank

Port Fuel Tank

Figure 7 Airliner

- The fuel operator asks, "What is your present reading, please?" The flight engineer tells her.
- The fuel operator calculates what the reading should be when he has pumped in 305 gallons (in the present case). This of course is 724, the reading on the port tank gauge which he cannot see.
- The fuel operator says, "Check, please. I have put in 305 gallons according to my own gauge, so yours should now read 724."
- If both calculations are correct, this figure will be the same as the reading on the port tank gauge (which the fuel operator cannot see). If not they look for the mistakes in their calculations.
- Steps 4 to 9 are repeated, simulating another flight.
- After a few flights, the players change roles.

Activity 4 Candy store: selling and stocktaking [Num 4.10/4]

An activity for 4 to 6 children. Its purpose is to consolidate their new abilities in a game which links it back again with physical embodiments.

- Materials*
- ‘Candies.’ Whatever candy-sized objects you can obtain.
 - Small self-sealing plastic bags.
 - Cardboard boxes to hold ten bags.
 - A pack of 30 spending cards on which are written sums of money varying between 4¢ and 25¢.*
 - Play money.
 - Pencil and notebook for each child.
 - Shallow money containers, one for each player (useful but not essential).
- * Provided in the photomasters

- What they do*
1. One acts as shop owner, one as shop assistant, and the rest as customers.
 2. Before opening the shop, the shop owner and shop assistant pack the candies into bags of 10. They also pack boxes containing 10 bags.
 3. The players begin with about 100¢ each, and the shop has about 200¢ (not these exact amounts, however).
 4. Before the shop opens, the shop owner checks and records his stock and cash: e.g., stock 347 candies, cash 185¢.
 5. Likewise each customer notes how much money he starts with.
 6. Shopping now begins. Each candy costs 1¢. The customers find it hard to decide how much to spend on candies, so each in turn takes the top card from the pile of spending cards which is put face down on the table.
 7. Having taken a card, the customer spends that amount on candies, keeping the card as a record of what he has spent.
 8. The shop assistant does the selling. She breaks bulk as may be necessary to give customers the number of candies they ask for. She also gives change if required.
 9. This continues until each customer has made (say) 3 purchases. With 4 customers, this will make 12 transactions.
 10. At closing time, the shop owner checks stock and cash. The total value of the candies still in stock, added to the money taken, should be equal to the stock at the beginning of the day.
 11. Meanwhile the customers also check. The total of their spending cards would be equal to the value of their candies. Also, the money they have spent subtracted from what they started with should be equal to the amount they have left.
 12. Another pair then takes over as shop owner and shop assistant. Everyone helps in restoring all the materials to their starting positions, and another day’s shopping can begin.

Note When children are proficient at this, the price per candy may be varied. This complicates matters considerably. It might be a good activity for use with calculators.

Discussion of activities Activity 1 returns to the use of Mode 1 schema building, to make sure that the symbolic manipulations do not lose their connections with their underlying concepts. Base ten material embodies grouping and exchanging particularly well, so we use it once again here.

Activity 2 extends the pencil-and-paper technique for subtraction, already learned for numbers up to 99, into the domain of three-digit numbers. This is a simple form of Mode 3 building - extrapolating ideas and skills they already have into new situations. It is simple, because no new concepts are involved.

Activity 3 is an easy game in which the players have a good reason for checking each other's calculations.

Activity 4 is a fairly ambitious one which brings together all the aspects of subtraction described earlier in this network. There is taking away, since the candies sold are taken from those in the shop. At any time, the value of the candies in stock and the money taken are complementary, the total being the starting value of the stock before any is sold and before any money is taken. This is also true for the candies in stock and the money in the till. Giving change is also involved, and the change due is the difference between the cost of the goods and the cash tendered. The predictions are made and tested, in the checking of stock and takings, money spent and candies acquired. And it is likely that at some stage of the activity children will decide to use recording to help them in what they are doing. There is a lot in this package, so it will be worth while letting children repeat it for as many times as they still enjoy it, so that they may have time to consolidate the many relationships which are embodied.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Candy store: selling and stocktaking [Num 4.10/4]



Number stories (multiplication) [Num 5.4/1]

[Num 5] **MULTIPLICATION** Combining two operations

Num 5.4 NUMBER STORIES: ABSTRACTING NUMBER SENTENCES

Concept Numbers and numerical operations as models for actual happenings, or for verbal descriptions of these.

- Abilities*
- (i) To produce numerical models in physical materials corresponding to given number stories, to manipulate these appropriately, and to interpret the result in the context of the number story: first verbally, then recording in the form of a number sentence.
 - (ii) To use number sentences predictively to solve verbally given problems.

Discussion of concepts The concept of abstracting number sentences is that already discussed in *SAIL Volume 1* in Num 3.4, and it will be worth reading this again. In Num 4.4 it was expanded to include subtraction, and here we expand it further to include multiplication.

In some applications of multiplication, we need to place less emphasis on the first action and the second action, and more on their results: namely a small set (of single objects) and a large set (a set of these sets). The use of small set ovals and a large set loop from the beginning provides continuity here.

Activity 1 Number stories (multiplication) [Num 5.4/1]

An activity for 2 to 6 children. Its purpose is to connect simple verbal problems with physical events, linked with the idea that we can use objects to represent other objects.

- Materials*
- Number stories on cards, of the kind in the example overleaf. Some of these should be personalized, as in Num 3.4/1 in *Volume 1*, but now there should also be some which do not relate to the children, more like the kind they will meet in textbooks. Also, about half of these should have the number corresponding to the small set coming first, and about half the other way about.**
 - Name cards for use with the personalized number stories.
 - Number cards 2 to 6.**
 - 30 small objects to manipulate: e.g., bottle tops, shells, counters, . . .
 - 6 small set ovals. *
 - A large set loop.*
 - Slips of blank paper, pencils.

* As for Num 5.1/1 in *SAIL Volume 1*.

** Provided in the photomasters

Num 5.4 Number stories: abstracting number sentences (cont.)

What they do (apportioned according to how many children there are)

1. A number story is chosen. The name, cards and number cards are shuffled and put face down.
2. If it is a personalized number story; the top name card is turned over and put in the number story. Otherwise, explain “Some of these stories are about you, and some are about imaginary people.”
3. The top number card is turned over and put in the first blank space on the story card.
4. The next number card is turned over and put in the second blank space.
5. The number story now looks like this.

There are children in each rowing boat
and rowing boats on the lake. So
altogether children are boating on the lake.

6. Using their small objects (e.g., shells) to represent children, the oval cards for boats, and the set loop for the lake, the children together make a physical representation of the number story. If it is a personalized number story, this should be done by the named child.
7. The total number of shells is counted, and the result written on a slip of paper. This is put in the space on the card to complete the story.
8. While this is being done, one of the children then says aloud what they are doing. E.g., “We haven’t any boats, so we’ll pretend these cards are boats, and put shells on them for children. We need 3 children in each boat, and 4 boats inside this loop which we’re using for the lake. Counting the shells, we have 12 children boating on the lake.”
9. The materials are restored to their starting positions, and steps 1 to 8 are repeated.

Activity 2 Abstracting number sentences [Num 5.4/2]

An extension to Activity 1 which may be included fairly soon. Its purpose is to teach children to abstract a number sentence from a verbal description.

- Materials*
- As for Activity 1, and also:
 - Pencil and paper for each child.

What they do As for Activity 1, up to step 8.

9. Each child then writes a number sentence, as in step 6 of Num 5.3/1. They read their number sentences aloud.
10. The materials are restored to their starting positions, and steps 1 to 8 are repeated.

Activity 3 Number stories, and predicting from number sentences [Num 5.4/3]

An activity for 2 to 6 children. It combines Activities 1 and 2, but in this case completing the number sentence is used to make a prediction as in Num 5.3/2.

- Materials*
- Number stories asking for predictions.**
 - Name cards for use with the personalized number stories.*
 - Two sets of number cards 2 to 6.**
 - 30 objects to manipulate. *
 - 6 oval small-set cards. *
 - A set loop. *
 - Pencil and paper for each child. *

*As for Activity 1.

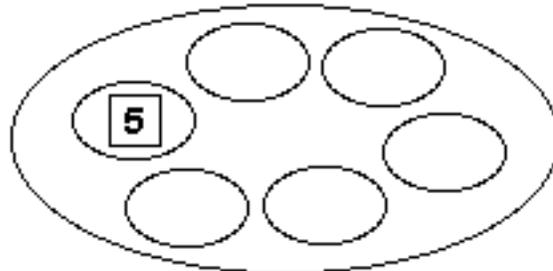
** Provided in the photomasters

What they do (apportioned according to how many children there are)

1. A number story is chosen. The name cards and number cards are shuffled and put face down.
2. If it is a personalized number story, the top name card is turned over and put in the number story. Otherwise, explain “Some of these stories are about you, and some are about imaginary people.”
3. Two more children turn over the top two number cards, and put these in the first two spaces on the story card.
4. The number story now looks something like this:

Giles is collecting fir cones in a wood.
 He gets **5** cones into each pocket and he has **6** pockets.
 When he gets home and empties his pockets, how many fir cones will he have?

5. The named child (if there is one) then puts out an appropriate number of small-set cards, and a number card from the second pack, to represent the situation. (This corresponds to step 6 of Activity 1.) For the present example, this is what he should put.



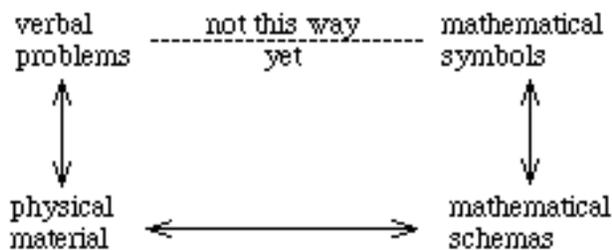
Num 5.4 Number stories: abstracting number sentences (cont.)

6. He explains, using his own words and pointing: “These 6 ovals represent my 6 pockets, and this 5 is the number of fir cones I have in each.” (This verbalization is an important part of the activity.)
7. All the children then write and complete number sentences, as in Activity 2. For the present example, these would be either $6(5) = 30$
“six fives make 30”
or $5 \times 6 = 30$
“five, six times, equals 30”
They read these aloud.
8. Another child then tests their predictions by putting the appropriate number of objects in each small-set oval. The number sentences are corrected if necessary.
9. They then write their answers to the question as complete sentences.
E.g. (in this case), “Giles will have 30 fir cones.”
10. The materials are returned to their starting positions, and steps 1 to 9 are repeated using a different number story and numbers.

Discussion of activities

The activities in this topic parallel those in Num 3.4 (addition) and Num 4.4 (subtraction), and in view of their importance it will be worth rereading the discussion of these.

Solving verbally stated problems is one of the things which children find most difficult, as usually taught. This is because they try to go directly from the words to the mathematical symbols. In the early stages, it is of great importance that the connection is made via the use of physical materials, since these correspond well both to the imaginary events in the verbally-stated problem, and to the mathematical schemas required to solve the problem. The route shown below may look longer, but the connections are much easier to make at the present stage of learning.



Since these physical materials have already been used in the earlier stages for schema building, they lead naturally to the appropriate mathematical operations. Side by side with this, they learn the mathematical symbolism which will in due course take the place of the physical materials.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 5.5 MULTIPLICATION IS COMMUTATIVE; ALTERNATIVE NOTATIONS; BINARY MULTIPLICATION

- Concepts*
- (i) The commutative property of multiplication.
 - (ii) Alternative notations for multiplication.
 - (iii) Multiplication as a binary operation.

- Abilities*
- (i) To understand why the result is still the same if the two numbers in a multiplication sentence are interchanged.
 - (ii) To recognise and write multiplication statements in different notations with the same meanings.
 - (iii) To multiply a pair of numbers.

Discussion of concepts (i) Particularly when considered in a physical embodiment, the commutative property of multiplication is interesting, and surprising, if we come to it with fresh eyes. Why should 5 cars with 3 passengers in each convey the same number of persons as 3 cars with 5 passengers in each? And likewise whatever the numbers? If you think that the answers are obvious, try to explain the second of these before reading further. I have tried to introduce this element of surprise in the first two activities.

(ii) So far the children may have used only one notation for multiplication but there are several in common use.

5(3)	which may be read as “Five times three” or “Five threes.”
3 x5	read as “Three, five times.”
3 x 5	read as “Three multiplied by five” (note the equal spacing) which may be taken either to mean the same as the one above, or as representing binary multiplication.

There is also this	3
vertical notation	<u>x 5</u>

which will be needed later for calculations like this	473 <u>x 5</u>
---	-------------------

We are more likely to read the lower one as “five threes . . . five sevens [seventies]. . . five fours [four-hundreds] . . .” than as “three, five times . . . seven [seventy], five times . . . four [four hundred], five times.” So here is an inconsistency. This inconsistency is neatly removed by using the notation for binary multiplication, which is explained in the next section.

(iii) Consider three different multiplications:

5(3)
6(3)
7(3)

The operand in every case is 3, but there are three different operations. These are, unary multiplication by 5, by 6, and by 7.

In contrast, binary multiplication has a *pair* of numbers as operand; and there is just one operation, multiply, for all pairs of numbers.

All the foregoing can be taken care of at a single stroke, by introducing computer notation for multiplication.

$$3 * 5 = 15 \quad \text{read as "Three star five equals fifteen"}$$

means the binary product of 3 and 5. Because multiplication is commutative (see Activity 2), it includes and replaces all the following:

$$5(3) = 15 \qquad 3 \times 5 = 15$$

$$3(5) = 15 \qquad 5 \times 3 = 15$$

$\begin{array}{r} 3 \\ \times 5 \\ \hline 15 \end{array}$	$\begin{array}{r} 5 \\ \times 3 \\ \hline 15 \end{array}$
---	---

This seems good value to me, particularly since children need to learn computer notation anyway. The computer books do not specifically mention binary multiplication: I am including this meaning as a bonus.

All the above will become clearer as you work through the activities in this topic.

Activity 1 Big Giant and Little Giant [Num 5.5/1]

An activity for up to six children. It is a sequel to ‘Giant strides on a number track’ (Num 5.1/3 in *SAIL Volume 1*), and its purpose is to introduce the commutative property of multiplication in a way which does not make it seem obvious (which it is not).

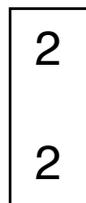
- Materials*
- Activity board, see Figure 8.*
 - Two sets of number cards 6 to 9 (single).*
 - One set of number cards 2 to 5 (double).*
 - Double width number track 1 to 50.*
 - Blu-tak or Plasticine.

* Provided in the photomasters

The number cards need to fit the spaces in the activity board. See illustration following.

The squares on the number track need to be accurate in size, 1 cm by 1 cm, since the track will also be used with 1 cm cubes in Activity 2.

Number cards



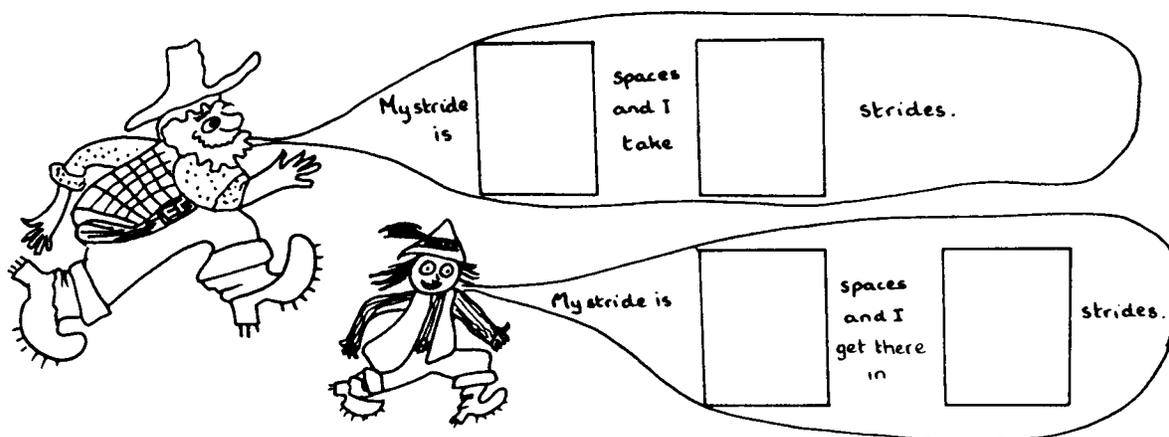


Figure 8 Big Giant and Little Giant

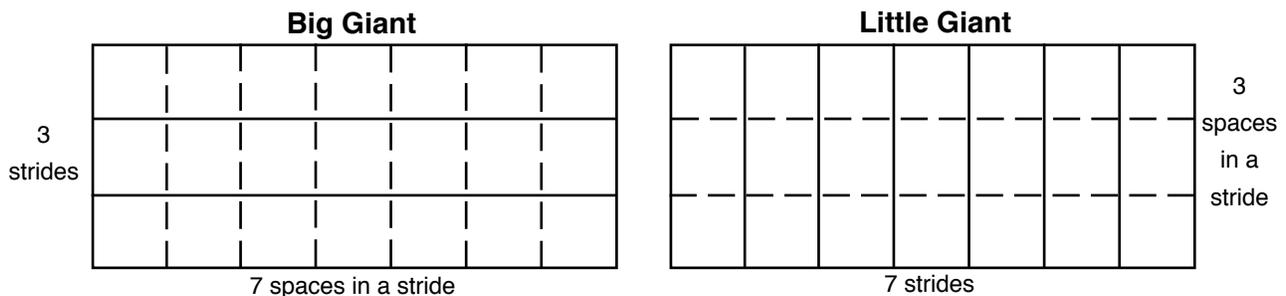
- What they do*
1. Two children act the parts of Big Giant and Little Giant, respectively.
 2. Big Giant has the set of double number cards and one of the sets of single number cards. Little Giant has the other set of single cards.
 3. Big Giant turns over the top cards in each of his packs, and puts these face up in their respective spaces on the activity board.
 4. Big Giant puts into action the statement which he has completed. He does this by putting footprints (small blobs of Plasticine) on the number track, one for each stride of the given length. He uses the upper part of the number track.
 5. Little Giant's stride is of a different length. He has to find out how many of these strides he must take to arrive at the same place as Big Giant. He does this in the same way as Big Giant, using the lower part of the number track.
 6. He then puts the appropriate number card in his second space to complete his own statement.
 7. The children then read aloud from the board what they have done.
 8. Steps 3 to 6 are now repeated, with other children taking the parts of Big Giant and Little Giant. Say to Little Giant: "If you think you know how many strides you need, to get to the same place as Big Giant, put the number in the space first and then see if you were right. If you don't know, find out first and then put in the number."
 9. Eventually one of the children acting as Little Giant will realize that his own numbers are always the same as Big Giant's the other way around. He will then be able to predict successfully every time. He keeps this discovery to himself for the time being.
 10. This continues, with children taking turns at Little Giant, until all have discovered how to predict, though probably not yet why.
 11. When this stage is reached, ask: "But can you explain why your method always works?"
 12. If any Little Giant can give a good explanation before having done Activity 2, then he is really clever!

Activity 2 Little Giant explains why [Num 5.5/2]

A teacher-led discussion for up to 6 children. Its purpose is to provide the explanation asked for in step 11 of Activity 1. For this, it may be necessary for you to take the part of Little Giant.

- Materials*
- The same as for Activity 1, and also
 - Squared paper and pencil for each child.
 - 100 1 cm cubes, if possible 20 each in 5 different colours.

- What they do*
1. Start with a set of cards in position on the activity board, as in steps 3, 4, 5 of Activity 1. This time, Little Giant’s card should also be face up. E.g.:
Big Giant: “My stride is 7 spaces and I shall take 3 strides.”
Little Giant: “My stride is 3 spaces and I shall get there in 7 strides.”
 2. Instead of footmarks on the number track, the giants represent their strides by rods. So in this example, Big Giant uses a 7-rod to represent one of his strides, and puts together 3 of these on the number track. Little Giant uses a 3-rod for one of his strides, and joins together 7 of these alongside those of Big Giant. Adjacent strides should be of different colours, to keep them distinguishable.
 3. At this stage it is still not obvious why both are of the same length, since the two journeys look different.
 4. “Now,” says Little Giant, “we arrange the rods like this.” The long rods are then separated again into single strides, and arranged in rectangles as shown below.



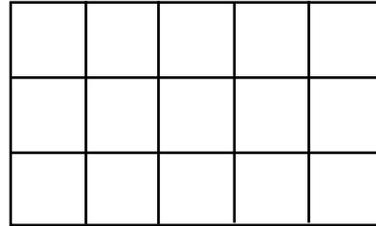
5. From this it can be seen why the two journeys are of the same length. 3 strides, each of 7 spaces makes the same rectangle as 7 strides, each of 3 spaces. Both rectangles have the same number of cubes.
6. Little Giant asks, “Will it always happen like this, whatever the numbers?”
7. The group as a whole discusses this. One or two other examples might be done using the cubes.
8. This should now be continued as a pencil and paper activity, using squared paper, as follows.
9. Two more numbers are shown on the activity board: e.g., 6 and 4.
10. All the children draw rectangles 6 squares long and 4 squares wide.
11. The children make these into diagrams for the two different ways of getting to the same place, as shown in step 4. About half the children make diagrams representing Big Giant’s journey (6 spaces in a stride, 4 strides), and the rest make diagrams representing Little Giant’s journey (4 spaces in a stride, 6 strides).
12. They interchange, and check each other’s diagrams.
13. Finally, Little Giant tells the others: “There is a word for what we have just learned. Multiplication is *commutative*. It means that if we change the two numbers around, we still get the same result. Always.”

Activity 3 Binary multiplication [Num 5.5/3]

Continuing the teacher-led discussion in Activity 2.

Materials • Pencil and squared paper.

Suggested sequence for the discussion 1. Draw a rectangle, say 5 squares by 3 squares



2. Ask the children how many different multiplication sentences they can write based on this diagram. Collect these in two groups, those meaning 5 sets of 3 and those meaning 3 sets of 5. (See ‘Discussion of concepts.’)
3. Now that they have learned that multiplication is commutative, all the notations in both groups are different ways of writing and thinking about the same thing. So it would be sensible to have just one way of writing all these.
4. Computer notation provides just what we need.

$$3 * 5$$

combines all the above in just one notation. So we shall use this from now on, unless there is a particular reason for using one of the others.

5. (This part of the discussion is optional) It follows that $5 * 3$ and $3 * 5$ mean the same thing as each other. It would be nice to have a notation in which neither number was written before the other, but it is hard to think how this could be done. The nearest representation we can get to this is the rectangle 3 squares by 5 squares, shown in step 1. Here, there is no distinction between rows and columns; i.e., no distinction between the 3 and the 5.

Activity 4 Unpacking the parcel (binary multiplication)
Alternative notations [Num 5.5/4]

A game for up to 6 children. Its purpose is to consolidate the connections between the notation just learned, and its various possible meanings. Also, to show what a great amount of information is contained in a single statement.

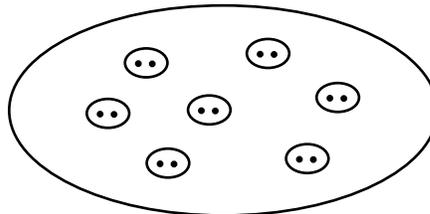
- Materials* • About 10 cards on which are written an assortment of number sentences like $3 * 5 = 15, 6 * 4 = 24$.†
- Pencil, plain and squared paper for each child.
 - Number track (e.g., the double width number track from Num 5.5/1).
 - A bowl of counters.

† Provided in the photomasters

Num 5.5 Multiplication is commutative; alternative notations; binary multiplication (cont.)

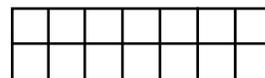
What they do

1. The cards are shuffled and put face down.
2. The top card is turned over: e.g., $2 * 7 = 14$.
3. The players in turn ‘unpack the parcel,’ by giving some of the various meanings contained in this number sentence. They may use drawings, or appropriate number sentences.
E.g., (drawing)
“Make a set of 2.
Make it 7 times.
Combined result, 14”



writes: $2 \times 7 = 14$

4. The next player might draw this, and say
“Rectangle 2 by 7,
14 squares. The notation for
this is the one on the card.”

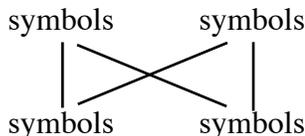


5. The next players might use the same rectangle, first as 7 2-rods, then as 2 7-rods, writing these respectively as $7(2) = 14$ and $2(7) = 14$.
6. Points may also be scored by writing a number sentence in a different notation, which has the same meaning as that which another player has just given. This may be done only once for each example.
7. Others might use the number track, in the ways already learned.
8. Each time a player gives a meaning, the others say “Agree” if they think he is correct, and he takes a counter. If they do not agree, they explain why not and if necessary discuss.
9. When all have made a statement about this card, or when players run out of ideas, the next card is turned, and steps 2 to 6 are repeated.
10. The player with the most counters at the end of play is the winner.

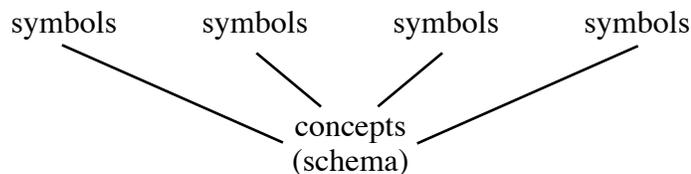
Discussion of activities

In this topic, the children are introduced to some more ways of symbolizing multiplication. For their future work they will need to know all of these. To minimize the likelihood of confusion and maximize the advantages described, the important thing is to continue to strengthen the connections between the symbols and the concepts, rather than between symbols and each other.

Not this:



but this:



The best way to do this is by frequent use of physical materials and drawings.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 5.6 BUILDING PRODUCT TABLES: READY-FOR-USE RESULTS

- Concepts*
- (i) Product tables, as an organized collection of ready-for-use results.
 - (ii) The complete set of products, up to $10 * 10$.

- Abilities*
- (i) To recall easily and accurately whatever results are needed for a particular job.
 - (ii) To build new results from those which are already known.

Discussion of concepts In view of the strong emphasis throughout this book on concepts, schemas, and understanding, it may come as a surprise that I believe firmly in the importance of children 'knowing their tables,' even in these days of inexpensive calculators. But there is no inconsistency.

To put our knowledge to good use, which includes using it to extend its own boundaries, we must have readily available all the results which we need for frequent use. Just as our writing would be very slow if we had not memorized the correct spelling of a large number of words, so our arithmetic would be very slow if we had not learned our addition facts and product tables.

However, we would hardly think it sensible to make children memorize the spelling of words which to them were meaningless. Yet until recently, and perhaps even now, children are required to memorize multiplication tables while their concept of multiplication is so weak that they still have to ask, "Please, Teacher, is it an add or a multiply?"

Understanding of the relationship between multiplication and physical events, both with actual objects and events as described in number stories, has been carefully developed in preceding topics. In the present topic, although the activities are intended to help children acquire a repertoire of ready-for-use results, you will see that this is done in such a way that many of these are built up from results already known. So children are learning, not a collection of isolated facts but a system of interrelated results.

Activity 1 also uses an important property of multiplication, called by mathematicians the distributive property: "multiplication is distributive over addition." E.g.,

$$4 * 8 = 4 * (5 + 3) = 4 * 5 + 4 * 3$$

The use of the product patterns shows this clearly, and at this stage of children's learning this intuitive and pictorial understanding is enough. A statement like the above would only confuse.

Later, we shall use this same property to multiply by numbers greater than 10. E.g., to multiply by 58, we multiply by 50 and by 8 and add the results.

Still later, this property will be much used in algebra.

$$a * (y + z) = a * y + a * z$$

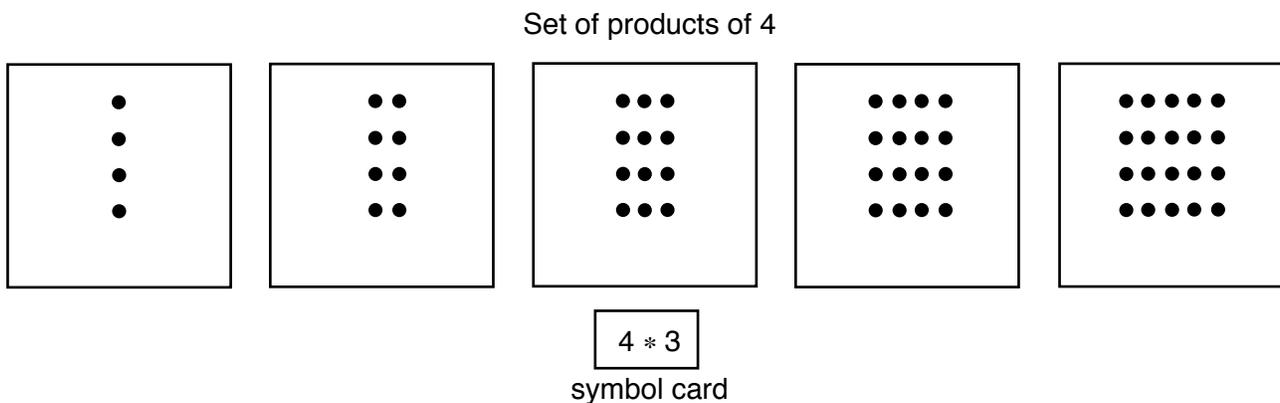
Activity 1 Building sets of products [Num 5.6/1]

An activity for children in pairs. Its purpose is to help children build sets of product results; and to know these, ready for use. Stages (a) and (b) also give practice in mental addition.

- Materials*
- 4 sets of product patterns, for the products of 2, 3, 4, 5 respectively. *
 - 4 corresponding sets of symbol cards. *
- * Provided in the photomasters. See examples in step 3 below.

What they do Stage (a) Products to 5

1. Each pair uses one set of product patterns, and the first half (* 1 to * 5) of the corresponding set of symbol cards. We will suppose that they are using the sets of products of 4, so the symbol cards will be those from $4 * 1$ to $4 * 5$ inclusive.
2. The product patterns are laid face up in order, and the symbol cards are shuffled and put face down.
3. Each child in turn turns over the top symbol card, puts it below the corresponding product pattern, and says what this product is. (That is, the total number of dots in the rectangle.) E.g., she picks up $4 * 3$, puts it as shown below,

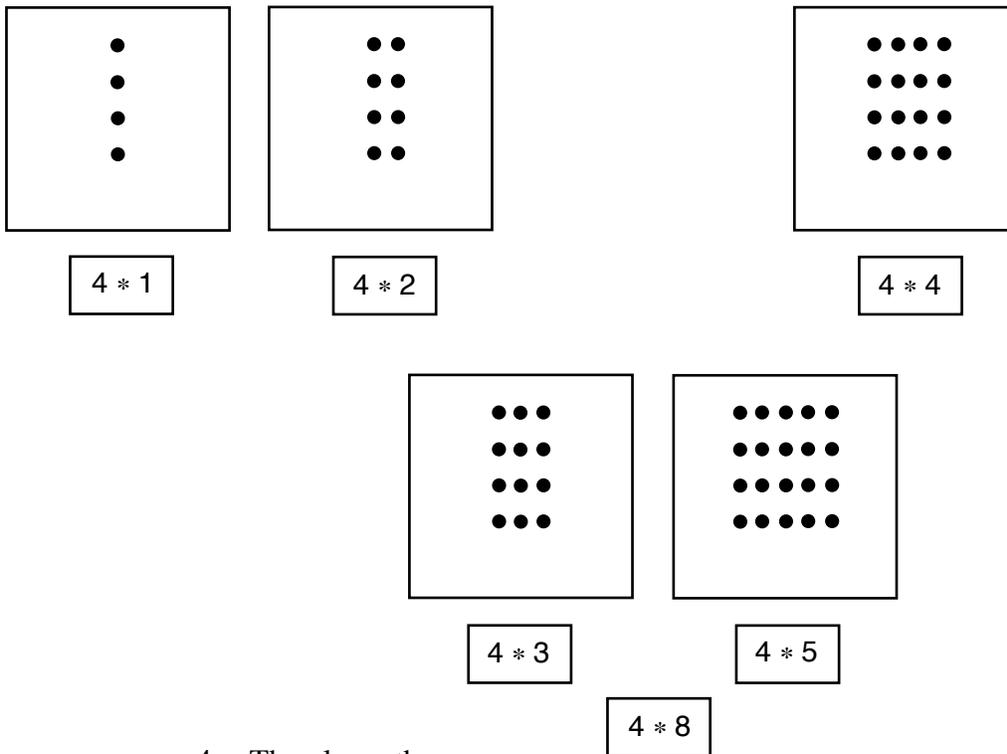


and says “4 star 3 is 12.”

4. She may obtain this result either by counting the dots, or by recalling it from memory.
5. The other player then says either “Agree” or “Disagree.” If “Disagree,” they check by counting the dots.
6. The symbol card is then replaced in the pack, face down as before.
7. It is important that steps 3 to 6 are repeated until the children can recall all the results from memory before they continue to Stage (b).

Stage (b) Products to 10

1. A set of product patterns is put in a row, each with its symbol card just below. (See illustration in step 3, above.)
2. The other half of the symbol cards are now used. As before they are shuffled and put face down, and the top one is turned over. Suppose that it is $4 * 8$.
3. To deal with this, the product patterns for $4 * 3$ and $4 * 5$ are put together, each with its symbol card, as shown on the next page.



4. The player then says:

$$\begin{array}{r}
 4 * 8 \text{ is } 4 * 3 \text{ plus } 4 * 5 \\
 \qquad \qquad \qquad 12 \text{ plus } 20 \\
 4 * 8 \text{ is } \qquad \qquad 32
 \end{array}$$

5. The symbol card is put on one side, and the product rectangles put back in line.
6. This is continued until all the symbol cards have been used. This activity is repeated until the children are fluent.

Notes (i) Stage (b) is best introduced with even sets of products, $2 *$ and $4 *$. These give multiples of 10 for the products $2 * 5$ and $4 * 5$.
(ii) The products with 10 should come late in the sequence. Let children devise their own responses. Some double the product with 5, some put together 3 product patterns, some already know the result without a product pattern.

Activity 2 “I know another way.” [Num 5.6/2]

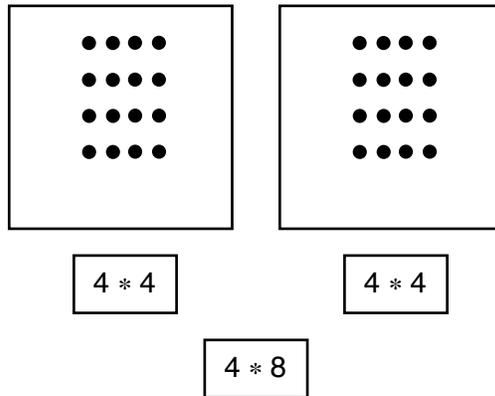
An activity for children in pairs. Its purpose is to consolidate the results already known, to make further connections between these, and to extend their use of the distributive property of multiplication over addition. (This activity often arises spontaneously during Activity 1, Stage (b).)

- Materials*
- A double set of product patterns.*
 - Symbol cards, full sets.*
- *As used in Activity 1, Stage (b).

Num 5.6 Building product tables: ready-for-use results (cont.)

Rules of the game

- 1, 2, 3. As in steps 1, 2, 3 of Activity 1, Stage (b). In this case there are two similar rows of product rectangles. Note that in Step 2, the beginning player may now use either a single product rectangle if available (e.g., for $4 * 3$) or two of these (e.g., $4 * 2 + 4 * 1$).
4. If the other player agrees, she says “Agree.”
5. She then says “I know another way.” Using again the example given in Activity 1, Stage (b), Step 3, and following on from there, she might put together product rectangles like this



and says

“ $4 * 8$ is $4 * 4$ plus $4 * 4$
 16 plus 16
 $4 * 8$ is 32
 as before.”

6. If the first player agrees, she says “Agree.”
7. The cards are replaced, and steps 1 to 6 are repeated.
8. As the players progress, they may use 3 product rectangles if they wish.

Activity 3 Completing the product table [Num 5.6/3]

An investigative activity for up to 6 children. The amount of help needed from their teacher will depend on the ability of the children.

Materials For each child:

- A partly-completed product square, from 1 to 50.*
- An L-card, see illustration on next page.*
- Pencil.

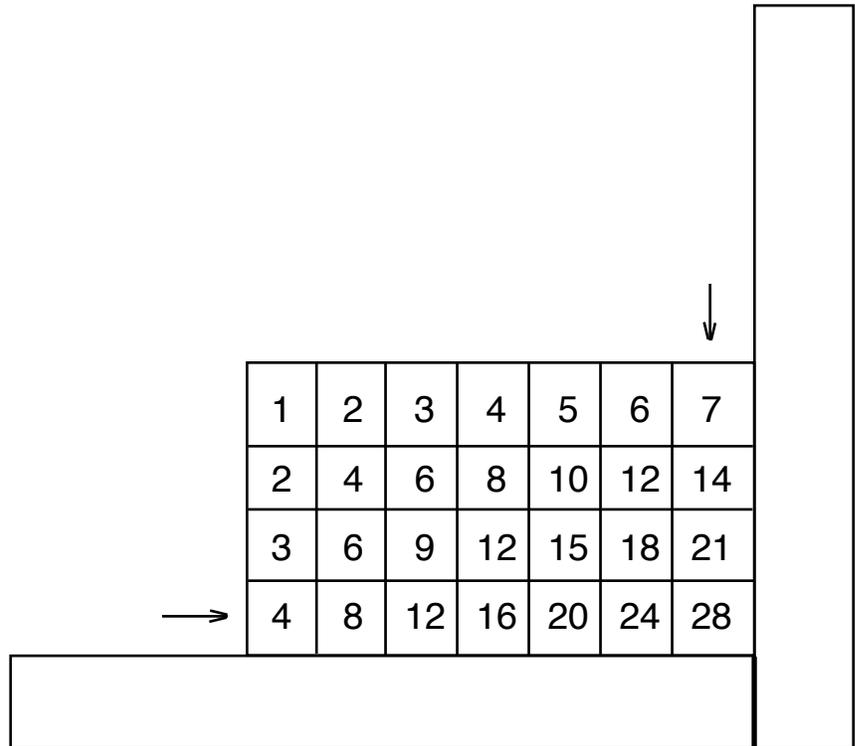
* A full-size product square is provided in the photomasters, *SAIL Volume 2a*.

What they are asked to find out Stage (a) What is the connection between the numbers in the squares, and the product patterns and results they have learned so far? (The answer is shown, in brief, in the illustration.)

The lower right-hand number gives the number of squares in the rectangle, which is the same as the number of dots in the corresponding product pattern.

Here we have:

$$4 * 7 = 28$$



Stage (b)

How can they complete the product table by filling in the blank squares? (The left hand column, of course, continues from 5 to 10). We then want all the products of 5, of 6, of 7, of 8, of 9, and of 10. One way would be by laying down the L shape for each product, and counting squares. This is not as bad as it sounds, since it only involves counting the squares additional to those already numbered. But there are better ways.

- (i) The first column, of course, continues to 10.
- (ii) The next three columns can be completed by using their knowledge that multiplication is commutative. So $6 * 2 = 2 * 6$ which is already known, and so on.
- (iii) Row 5 can be completed by counting by 5's. [5, 10, 15, 20, . . .]
- (iv) Row 6 can now be completed by adding 6 each time. This is in fact another use of the distributive property.

$$\begin{aligned} 6 * 6 &= 6 * 5 + 6 * 1 \\ &= 30 + 6 \\ &= 36 \end{aligned}$$

And so on.

- (iv) The children should write all their new results lightly in pencil, until all their figures are checked with each others'. Then they should complete their product table as neatly as possible in ink (e.g., ballpoint, thin fibre-tip) for their own future use. Tell them to keep these carefully.

Activity 4 Cards on the table [Num 5.6/4]

An activity, for children to play in pairs, as many as you have materials for. They may with advantage make their own, and practice in odd times which would otherwise be wasted. Its purpose is to practice the recall of all their product results.

- Materials*
- 9 sets of symbol cards, each with 10 cards in each set, from $2 * 1$ to $10 * 10$.†
 - One product table and L-card for each pair.†
- † Provided in the photomasters

- What they do*
1. In each pair, one child has in her hand a single pack of cards, shuffled and face down. The other has on the table her multiplication table and L-card.
 2. Child A looks at the top card, say $3 * 8$, and tries to recall this result. Child B then checks by using her multiplication square and L-card.
 3. If A's answer was correct, this card is put on the table. If incorrect, it is put at the bottom of the pile in her hand so that it will appear again later.
 4. A continues until all the cards are on the table. This method gives extra practice with the cards she got wrong.
 5. Steps 1 to 4 are repeated until A makes no mistake, and her hand is empty.
 6. The children then change roles, and repeat steps 1 to 5.
 7. Steps 1 to 5 are then, possibly at some other time, repeated with a different pack until all the packs are known.
 8. The final stage is to mix all the packs together. Each child then takes from these a pack of 10 mixed cards, and repeats steps 1 to 5 with this pack.
 9. Step 7 is then repeated with a different pack.
 10. This activity should be continued over quite a long period: say, one new pack a week, with review of earlier packs, including mixed packs.

Activity 5 Products practice [Num 5.6/5]

A game for up to 6 children. Its purpose is further to consolidate children's recall of multiplication results. This game may be introduced for variety before children have completed Activity 4, using the packs which they have learned so far.

- Materials*
- Multiplication cards: all the packs which they have learned, mixed together.*
 - A multiplication table and L-card.*
 - Products board, see Figure 9.*
- * Provided in the photomasters

- Rules of play*
1. All, or nearly all, the cards are dealt to the players. Each should have the same number, so when the remaining cards are not enough for a complete round, they are put aside and not used for the game. The products board is put on the table between them.
 2. The players hold their cards face down. In turn they look at their top card (e.g., $7*8$) and put it in the appropriate space on the products board (in this case 56). It does not matter if there is a card in that space already – the new card is then put on top.
 3. The others check. If it is wrong, they tell her the correct answer and she replaces the card at the bottom of the pack.

Num 5.6 Building product tables: ready-for-use results (cont.)

4. If she does not know, she asks and someone tells her. She then replaces the card at the bottom of the pack.
5. Any disagreements are settled by using the multiplication board.
6. Play continues until all have put down all their cards. If there are no mistakes, all will finish in the same round. Those who do make mistakes, or do not know, will be left with cards in their hands to put down in subsequent rounds.
7. If one player finishes a clear round ahead of the others, she is the winner.

Variation If a stopwatch is available, this game may also be played as a race. To make a fair race, each player needs to be using the same pack. This suggests various forms, e.g.,

Form (a) A single pack, of a table to be consolidated or reviewed.

Form (b) Several packs mixed.

Form (c) (For advanced players.) All packs from twos to tens, making 90 cards in all.

The rules for all forms are the same:

1. One player acts as starter and timekeeper.
2. The others in turn see how quickly and accurately they can put down all their cards.
3. Those not otherwise involved check for accuracy, *after* all the cards have been put down.
4. For each incorrect result, 5 seconds are added to the time. (This figure may be varied according to the skill of the players.)
5. The winner is the player with the fastest time after correction for errors.

Products Practice	= 2	= 3	= 4	= 5	= 6
= 7	= 8	= 9	= 10	= 12	= 14
= 15	= 16	= 18	= 20	= 21	= 24
= 25	= 27	= 28	= 30	= 32	= 35
= 36	= 40	= 42	= 45	= 48	= 49
= 50	= 54	= 56	= 60	= 63	= 64
= 70	= 72	= 80	= 81	= 90	= 100

Figure 9 Products board

Activity 6 Multiples rummy [Num 5.6/6]

A popular game of the rummy family. Best for 2 to 5 players; 6 is possible. Its purpose is to consolidate their knowledge of multiplication facts.

- Materials*
- Two packs of double-headed number cards, a beginners' pack and an advanced pack.* The beginners' pack containing two each of all the numbers from 2 to 30, excluding primes greater than 20 (54 cards). The advanced pack contains one each of all the numbers from 2 to 100, excluding primes greater than 40 and even numbers greater than 20 (43 cards).

* Provided in the photomasters

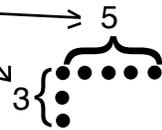
- Rules of the game*
1. The packs are put together and shuffled. Five cards are dealt to each player.
 2. The rest of the pack is put face down on the table, with the top card turned over to start a face upwards pile.
 3. The object is to get rid of one's cards by putting down sets of 3 multiples of the same number; e.g., 3, 15, 21; 2, 18, 20.
 4. Players begin by looking at their cards and putting down any trios they can. They check each others' trios.
 5. The first player then picks up a card from either the face down or the face up pile, whichever she prefers. If she now has a trio, she puts it down. Finally she discards one of her cards onto the face-up pile.
 6. In turn the other players pick up, put down if they can, discard. (This sequence should be memorized.)
 7. The winner is the first to put down all her cards. Play then ceases.
 8. The others score the total of the remaining cards in their hands. The lower the score the better, the winner scoring zero.
 9. Another round may then be played, and the scores added to those of the previous round. The overall winner is the one with the lowest total score.



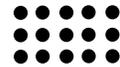
Multiples rummy [Num 5.6/6]

Discussion of activities The emphasis of this topic is to combine easy and accurate recall of multiplication results (products) with knowledge of many of their interrelationships. This means that although there is memorizing to be done, the facts which are thus learned are related both to each other, and to their underlying numerical concepts. It also means that if a particular result is temporarily forgotten, the children know ways of reconstructing it for themselves.

Thus in Activity 1, the products are related to patterns of dots which show both the numbers to be multiplied



and their product



Numerals are not written on the product pattern cards, so that children do not just match symbols to symbols in this activity. The number of rows and columns are however both small enough to be subitized – perceived without counting.

In Stage (b), known products are used to construct new ones. This is Mode 3 concept building, creativity. Here they are also building up the interconnections mentioned above. Practice in mental addition is also provided – though you may allow them to use pencil-and-paper while gaining confidence.

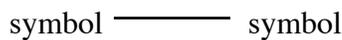
Activity 2 further builds up the interconnections, and consolidates use of the distributive property.

Activity 3 is another constructive, extrapolative activity: Mode 3 again.

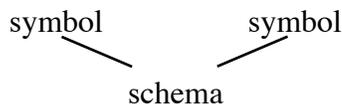
Activities 4 and 5 are for developing effortless recall – ‘facts at their fingertips.’ But with such a strong relational beginning in Activities 1, 2, and 3, this should not become just rote learning. What we want children to acquire is fluency: easy recall without loss of meaning.

In Activity 4, it is tempting to write the results on the back of the cards; e.g., to write 18 on the back of $6 * 3$. I suggest that it is better not to do this, since it is a step in the direction of rote memory.

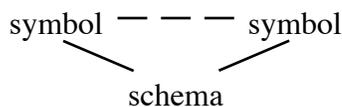
It strengthens this link



whereas we want this



Eventually we want all three links



and Activity 5 promotes learning of the link shown in dashes. But the other two links, shown in continuous lines, are the most important.

Activity 6, multiples rummy, is a popular game which uses recall of relevant products to make the best decisions.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 5.7 MULTIPLYING 2- OR 3-DIGIT NUMBERS BY SINGLE DIGIT NUMBERS

Concept Expansion of the multiplication concept to include examples in which the larger number has 2 or 3 digits, the smaller still having only 1 digit.

Ability To calculate any result of the kind described.

Discussion of concept

It is no small thing that once we know the multiplication facts from $1 * 1$ to $9 * 9$, we can use these to multiply numbers of any size, such as $5283 * 364$. Now that calculators are so widely available, it is no longer necessary for children to achieve a high degree of proficiency in doing calculations of this kind, and there are some teachers who think that these should now be omitted altogether. My personal view is that this ability to make use of a relatively small number of multiplication facts (55, including products with 1 and 10) to multiply any two numbers we choose, without learning any more multiplication tables, is a fine example of extending our knowledge from within. Also, it depends on certain properties of multiplication and addition which, together with the idea of a variable, form the main foundations of algebra. So I continue here along the path towards long multiplication, in the belief that there are interesting things to be learned on the way.

There are five of these properties altogether, of which two are important for this expansion of multiplication. They have been given technical names by mathematicians.

(i) Multiplication is distributive over addition.

$$\begin{aligned}\text{For example,} \quad 47 * 6 &= (40 + 7) * 6 \\ &= (40 * 6) + (7 * 6)\end{aligned}$$

If we think of 40 as 4 tens, we can calculate this because we know $4 * 6$. We already know $7 * 6$. The distributive property says we can get the correct result by adding these two products.

(ii) Multiplication is associative.

E.g, $2 * 3 * 4$ will give the same result whether the first two, or the second two, are multiplied first:

$$\begin{array}{lcl}(2 * 3) * 4 = 6 * 4 & & 2 * (3 * 4) = 2 * 12 \\ = 24 & & = 24\end{array}$$

You can easily check that this is not true for division. Its importance is that it allows us to multiply by one factor at a time. E.g.:

$$\begin{aligned}38 * 40 &= 38 * (10 * 4) \\ &= (38 * 10) * 4 \\ &= 380 * 4\end{aligned}$$

which we can calculate using the distributive property.

(Note that we have also used a shortcut for multiplying by 10, which needs to be justified.) It is not easy to decide how much of this should be explained to the children. A full explanation is probably too much for most, but to teach just the methods is to teach instrumentally something which calculators can do much better.

So what I have done is to present the distributive property intuitively, as embodied in base 10 material; and the associative property explicitly, followed by a demonstration of its usefulness. In both cases I have replaced the technical names by simpler descriptions, but some of your children may like to know the former.

Activity 1 Using multiplication facts for larger numbers [Num 5.7/1]

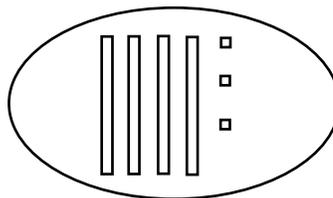
An activity for any number of children working in pairs. Its purpose is to take the first steps in extending children's ability to multiply. Before they do this activity, it may be desirable to review canonical form (Num 2.11).

- Materials*
- Base 10 material, tens and units.
 - 6 small set ovals.*
 - Large set loop.
 - Two 1-to-6 dice for each pair.
 - Two 1-to-9 dice for each pair.
 - Pencil and paper for each child.
- * Two are provided in the photomasters

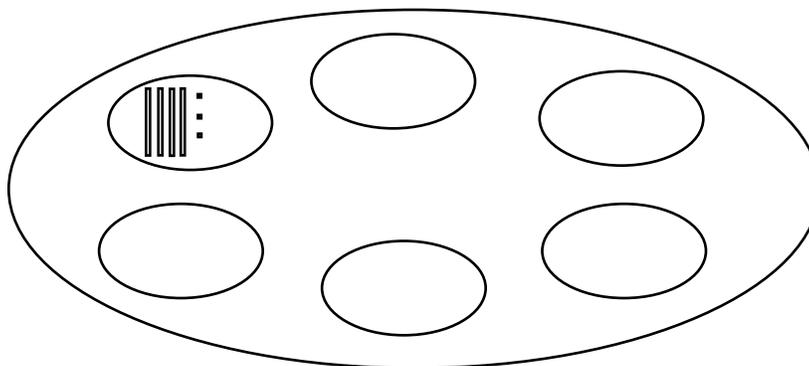
- What they do*
1. The 1-to-6 dice are used to begin with. First two are thrown together, to give a two digit number. Then a single die is thrown to give a single digit number.
 2. These are then written by each pair as a multiplication, in vertical notation, using headed columns. For example:

$$\begin{array}{r|l}
 \text{H} & \text{T} \text{ Ones} \\
 \hline
 & 4 \quad 3 \\
 * & \quad 6 \\
 \hline
 \end{array}$$

3. Base 10 material is put in one of the small set ovals to represent 43.



4. They then put more small set ovals to make six in all, and a set loop around them. (Base 10 material is only used in one of the small set ovals.)



5. They think and write as follows. "We have to make 6 sets like the one shown. 6 sets of 3 units is 18 units. 6 sets of 4 tens is 24 tens.

$$\begin{array}{r|l}
 \text{H} & \text{T} \text{ Ones} \\
 \hline
 & 4 \quad 3 \\
 * & \quad 6 \\
 \hline
 24 & 18
 \end{array}$$

Num 5.7 Multiplying 2- or 3-digit numbers by single digit numbers (cont.)

6. Rearrange in canonical form (working mentally from right to left), and there is our answer.

H	T	Ones
4	3	
*		6
2	5	8

$$43 * 6 = 258$$

7. They compare their final answers. If these are different, they discuss.
 8. Steps 1 to 6 are repeated.
 9. The use of the base 10 material may be discontinued when you think that the children have well established the concept that what they are doing on paper represents making many sets like the one in the set loop.
 10. When the children are proficient with these smaller numbers, they may change to the two 1-to-9 dice.
 11. The notation shown in steps 5 - 6 leads naturally to the conventional notation, in which step 5 is omitted and the rearrangement into canonical form is done as we go along, with help from small 'carrying' figures. This conventional notation is quick and convenient, but it is also very condensed. The notation in step 5 shows much more clearly what we are doing, so I recommend using this to start with. *Note that the condensed notation will be needed by the time they reach long multiplication in Num 5.10.*

Activity 2 Multiplying 3-digit numbers [Num 5.7/2]

An activity for any number of children working in pairs. It follows directly on from Activity 1, but the larger number now goes up to 999.

Materials As for Activity 1, but with base 10 materials in hundreds, tens, and ones, and 3 dice for each pair of children.

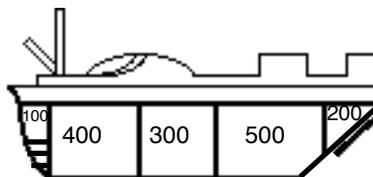
What they do This is so like Activity 1 that it does not need to be described in detail. The method already learned is easily extended to these larger numbers.

Activity 3 Cargo boats [Num 5.7/3]

This game is best for 2 or 3 players, though more could play. Its purpose is to give plenty of practice in multiplication.

- Materials*
- Game board, as in the photomasters and illustrated in Figure 10.
 - Die 1 - 6 (for easier game), 1 - 9 (for harder game).
 - Cargo boat for each player as in the photomasters.
 - Loading list for each player, as in the photomasters.
 - Calculator.

The cargo boats are cut out of cardboard and look like this.



Num 5.7 Multiplying 2- or 3-digit numbers by single digit numbers (cont.)

Rules of the game

1. Each player has a cargo boat, which he sails to each of 5 islands picking up cargo.
2. On each island there are packages of varying weight.
3. The first player sails his boat to the first island, and throws the die. This tells him the number of packages which he must take.
4. Although the number of packages is given by the roll of the die, he may choose the kind of packets he takes. All packets of the same kind must go into the same hold. The capacity of these is shown both on the boat and on the loading list (see next step).

5. Each player makes a loading list, on which he records what he takes on board. If he threw a five at the first island, his loading list might then look like this.

Compartment	Packages		Load
	Number	Weight	
100			
200			
300	5	* 50	250
400			
500			
Total Cargo			

6. The first player then sails his boat to the second island and repeats the process. Meanwhile the next player sails to the first island and starts loading; and so on. This means that by the third turn, all three players are occupied in making their calculations.
7. The winner is the player who returns to the home port with the heaviest cargo.
8. Players use a calculator to check each other's calculations.

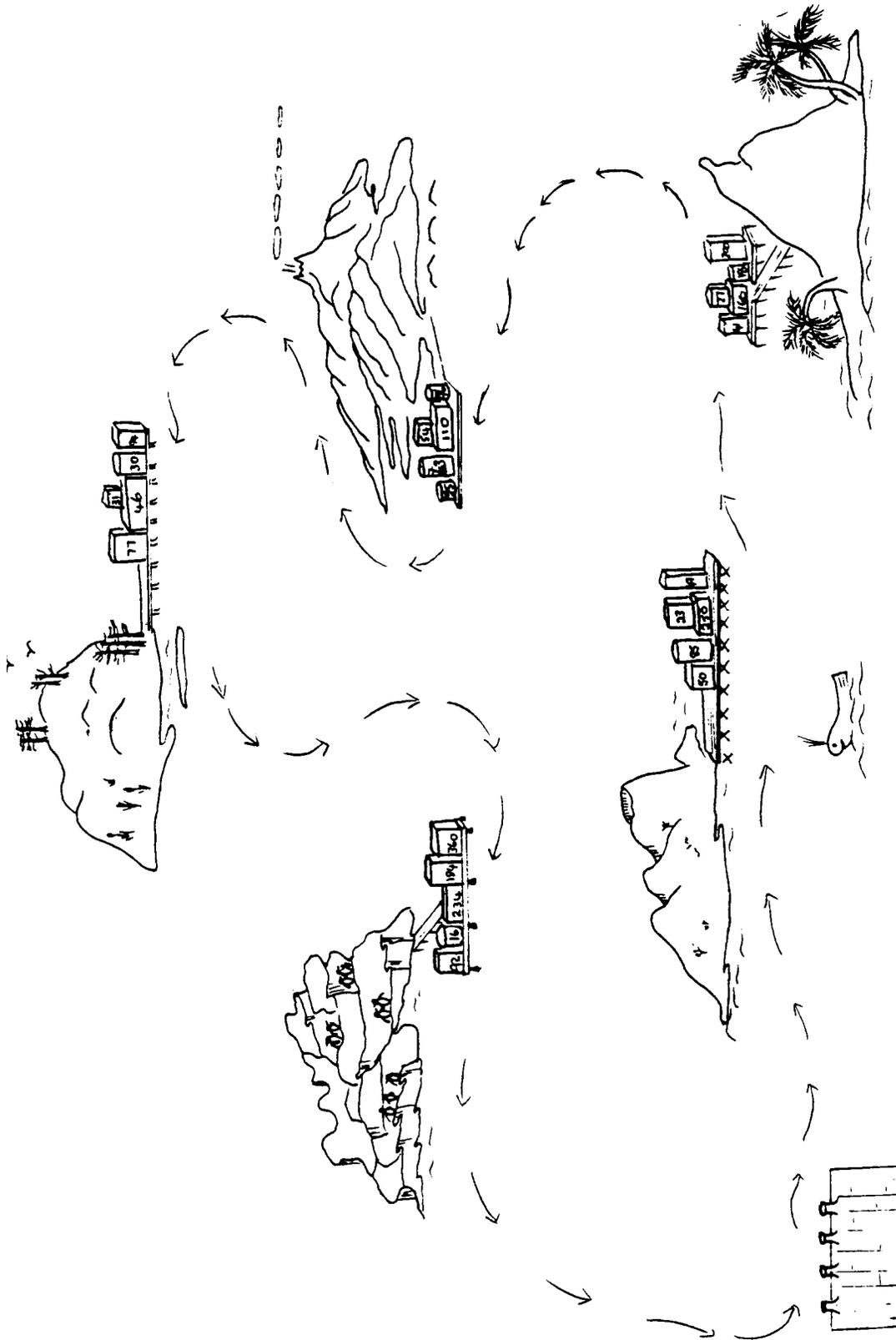


Figure 10 Cargo boats

Discussion of activities

Activity 1 is a good example of Mode 3 schema building: the extrapolation of schema and technique to include larger numbers. In doing this, we want the extrapolation to be based on expanding the concept itself, and not just repeating a way of manipulating symbols in a different situation. The latter takes place too, but the justification for doing it is provided by the base-10 material. This takes us back to the concept of multiplication: start with a set, make it many times. Now our set has tens and units, but the way to make many of it is visually obvious. And it is this which leads to, and justifies, the written method.

Here as elsewhere, headed column notation combined with the concept of canonical form is an important way of representing what we are doing symbolically. Since the vertical lines are there to act as separators, there is no loss of meaning if we leave out the outside lines, left and right. And although I have still included the headings Th, H, T, Ones, I would see no objection to letting these gradually become, as it were, 'unwritten headings.'

The extension to hundreds, tens, ones is so straightforward that I saw no reason for postponing it. Hence Activity 2.

Activity 3 is another game like 'Air freight' and 'One tonne van drivers,' for practising calculations in an interesting situation. You will easily be able to devise others. Catalogue shopping also lends itself easily to purchasing several of a given article, and hence to price calculations which require multiplication.

I think that we should at this stage allow children to use a calculator for checking their work. In the present situation I see calculators as labour-saving devices, rather than as completely removing the need for children to learn to do these calculations themselves.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Cargo boats [Num 5.7/3]

Num 5.8 MULTIPLYING BY 10 AND 100

Concept The mathematics behind the well-known shorthand.

Abilities (i) To multiply by 10 or 100, using the shorthand discussed below.
(ii) To explain it in terms of place-value.

Discussion of concept If asked “How do you multiply by 10?”, many children will say “You add a zero.” This of course is not true. Adding zero to any number leaves it unchanged. Writing a zero after the last digit is a useful shortcut, but try using it to multiply 2.5 by 10 and it will give a wrong answer. Short-cuts can lead you astray if you don’t know how they work.

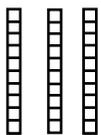
Activity 1 Multiplying by 10 or 100 [Num 5.8/1]

A teacher-led discussion for up to 6 children. Its purpose is to relate the ‘shorthand’ method of multiplying by 10 or 100 to its meaning.

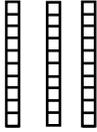
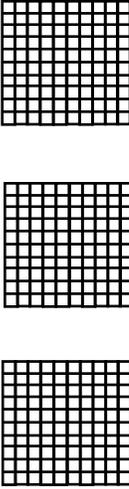
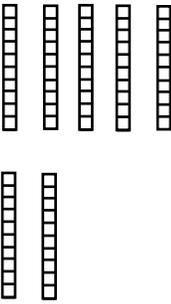
- Materials*
- Two 1-to-9 dice.
 - Base10 material.
 - Base 10 Th H T Ones board.*
 - Pencil and paper for each child.
- * As illustrated below in step 3. This needs to be fairly large.

Suggested sequence for the discussion

1. Throw the two dice to give a random 2-digit number; say, 37.
2. Have them put out base 10 material on the board to represent this. In the present example:

Th	H	T	Ones
			

3. Ask them to show what this would look like if multiplied by 10, using additional base 10 material in the space below.

	Th	H	T	Ones
				
* 10				

4. Ask them to record the result.

$$37 * 10 = 370$$

- Repeat steps 1, 2, 3 using different numbers.
- Repeat step 1, and ask if they can predict the result. If anyone talks about “adding a zero,” help them to see that this is incorrect by asking them what they get by adding (say) $3 + \text{zero}$.
- What we are really doing is moving every figure one place to the left, so that the units become the same number of tens, the tens become the same number of hundreds, and so on.

Activity 2 Explaining the shorthand [Num 5.8/2]

This is a teacher-led discussion for up to 6 children. It replaces Activity 1 for children who already know the shorthand, but do not understand what is behind it.

Materials • As for Activity 1.

- Suggested sequence for the discussion*
1. Use the dice to give a random 2 digit number as before, and let them use their shorthand to multiply by 10.
 2. Ask them to explain why this works, using the same explanation (that they are not adding a zero) as in Activity 1 step 6.
 3. If they give an explanation like that in Activity 1, step 7, or any other which is mathematically correct, good. Otherwise produce the base 10 Th H T Ones board, and take them through steps 2, 3, and 7 of Activity 1. Repeat, if necessary, until you are satisfied that they understand what they are doing.

Activity 3 Multiplying by hundreds and thousands [Num 5.8/3]

A continuation of Activity 1 or Activity 2.

First review Num 2.13/2 ‘Naming big numbers’ in the original form. Then repeat it using zeros to the right of a 2 digit number. Then, using zeros to the right of a 3 digit number.

Discussion of activities This is a topic in which the use of the short cut without understanding is widespread. Activities 1 and 2 therefore use a strong combination of headed columns and base 10 materials to provide a sound conceptual foundation for what is undoubtedly a useful shortcut. Activity 3 parallels these at the symbolic level, emphasizing the changes in meaning of each digit which result from their different positions.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 5.9 MULTIPLYING BY 20 TO 90 AND BY 200 TO 900

Concept The combination of their new knowledge of how to multiply by 10 and by 100 with their existing multiplication tables.

Abilities To multiply any two or three digit number (later, more) by 20 to 90 and by 200 to 900.

Discussion of concept The next step is easy. To multiply, say, 47 by 30 we multiply by 10 and by 3 in either order. What makes this true is the associative property, mentioned in Num 5.7. This tells us that

$$47 * 30 = 47 * (10 * 3) = (47 * 10) * 3 = 470 * 3$$

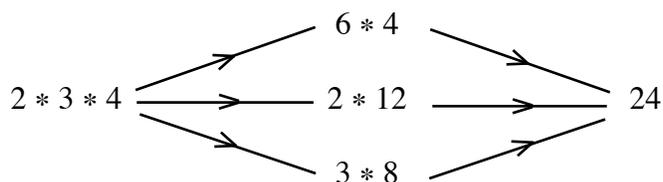
and for the last we only need to know our 3 times table.

Activity 1 “How many cubes in this brick?” (Alternative paths) [Num 5.9/1]

An activity for up to 6 children. Its purpose is to introduce children to the associative property.

- Materials*
- About 100 2 cm cubes.
 - Pencil and paper for each child.

- What they do*
1. Some of the children make a solid 2 by 3 by 4 brick. (Call it a cuboid if you prefer.) Others make bricks of different dimensions, according to how many children and how many cubes there are.
 2. Using the first brick, ask how many small cubes (or units) are there, and how they arrived at this result.
 3. There are 3 possible paths, depending on which layer they calculate first.



4. Elicit all three of these, and record them as above.
5. Discuss what this shows, namely that when multiplying three numbers together the result is the same whichever pair we combine (associate) first.
6. Help them to interpret these three results in relation to the brick. E.g. for the top path, $2 * 3$ was calculated first. This corresponds to a layer, 2 cubes by 3 cubes, having 6 cubes in all. $6 * 4$ means that there are 4 such layers.
7. Ask whether they think that this is always true, whatever the numbers. (It is: the number of cubes in a brick of any size can be calculated in all of these ways.)

Activity 2 Multiplying by n-ty and any hundred [Num 5.9/2]

An activity for up to 6 children. Its purpose is to combine the shorthand just learned with the associative property, in order to multiply by 20, 30, . . . , n-ty (where n is any number from 2 to 9); and likewise for 200, 300, . . . , any hundred.

- Materials*
- Two 1-to-9 dice.
 - Pencil and paper for each child.

What they do Stage (a)

1. Begin with numbers which do not give rise to special difficulties, such as extra zeros. E.g., $87 * 40$.
2. Explain that by writing this as:

$$87 * 40 = 87 * 10 * 4$$

we can get the result without knowing our 40 times table, or our 87 times table. We just need to know how to multiply by 10, and then by 4. (Activity 1 tells us that we will get the same result this way. Think of a brick 87 by 10 by 4.)

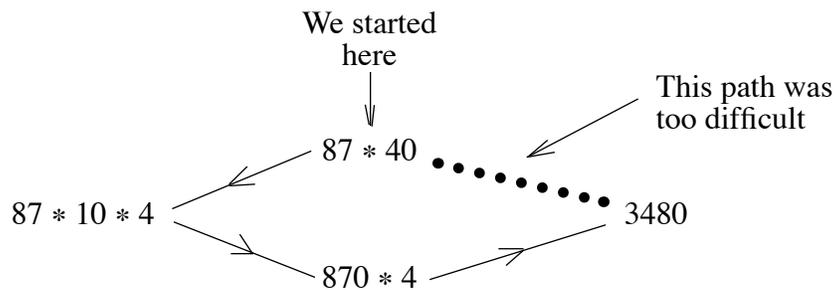
3. The work may be set out as below.

$$87 * 40 = 87 * 10 * 4$$

	Th	H	T	Ones	
			8	7	
* 10 =		8	7	0	
* 4 =		32	28	0	
=	3	4	8	0	[Canonical form]

$87 * 40 = 3480$

4. Remind them that when converting into canonical form we work from right to left. When the new method is understood, the conventional notation may be reintroduced. (See Num 5.7/1, step 11.)
5. This is what we have now done.



So we went the other way around, which was easier.

6. Even though this path is easier, there is a lot going on here, and it is important at this stage to consolidate these new ideas by discussion and by doing plenty of examples.
7. Repeat with other numbers.
8. When the children are confident at multiplying by numbers like 30 or 80, let them extend the method to multiplying by numbers such as 300 or 800.

Stage (b)

In preparation for long multiplication, they multiply simultaneously by 10 and by 4, as below. This may be taken in three steps.

1. $87 * 40 = \underline{\hspace{2cm}}$

	Th	H	T	Ones
			8	7
$* 40 =$		32	28	0
$=$	3	4	8	0

2. $87 * 40 = 3480$

	Th	H	T	Ones
			8	7
$* 40 =$	3	4	8	0

3.
$$\begin{array}{r} 87 \\ * 40 \\ \hline 3480 \end{array}$$

Discussion of activities We have now taken the first step into long multiplication, and this topic offers yet another example of how much is condensed into what on the surface looks like a simple process. I would not expect children necessarily to remember in detail all the reasons which they have been shown. What I think is important is for them to remember that there are reasons which they understood as they worked through them. They know that they did not learn rules without reasons, and that if necessary – possibly with a little help – they could remember why the new methods are correct.

Although the children have already learned to change mentally into canonical form in topic 5.7/1, step 11, we back-track a little into the headed column notation because this notation is less condensed, and shows the meaning more explicitly. In other words, we are refreshing the connections between the methods and the underlying concepts.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 5.10 LONG MULTIPLICATION

Concept The use of just 55 known results (from $2 * 1$ to $10 * 10$) to calculate any desired product.

Ability To multiply two numbers in which the smaller has 2 or 3 digits, with understanding and accuracy but not necessarily with speed.

Discussion of concept In the past, the main emphasis has usually been on the technique of long multiplication. The wide availability of inexpensive calculators has so changed the situation that if all that mattered was getting the answer, there would no longer be any good reason to teach this technique. This may well be the case for less able children. And I think that calculators should be freely used by all children in cases where the result is what is important, or when the figures are difficult.

However, the validity of the technique of long multiplication depends on several important mathematical principles, which have already been mentioned and which continue to be important in later mathematics. Two of these have already been specifically dealt with, in topic 5 (multiplication is commutative) and topic 9 (multiplication is associative). A third has been well used, and made explicit for teachers but not for pupils, in topic 6 (multiplication is distributive over addition). There are two more which perhaps I should mention for completeness, but you do not need to think about them unless you want to. These are, that addition also is both commutative and associative.

These properties make it possible to use known results to construct new knowledge, and thus to exercise the creative function of intelligence. This seems to me a good reason for continuing to teach long multiplication to those children who can grasp the principles on which it is based; and perhaps it is the only good reason.

The contributory concepts and abilities have been carefully prepared in earlier topics, and it is now simply a matter of putting them together.

Activity 1 Long multiplication [Num 5.10/1]

For any number of children. Its purpose is to show them how methods and results which they already know can be combined to multiply any two numbers.

Materials

- Pencil and paper for each child.
- Any source of numbers to be multiplied, such as dice, number cards, text book.

A suggested teaching approach

1. Explain that having learned 55 multiplication results – called products for short – they can now multiply any two numbers they want, without having to memorize any new product tables. Discuss this in contrast to other learning situations. (Having learned the names of 55 people does not enable us to know the right name for every other person we will ever meet.)

2. Recall how they worked out new products from those they already knew.

E.g.,

$$\begin{aligned} 4 * 7 &= 4 * 5 + 4 * 2 \\ &= 20 + 8 \\ &= 28 \end{aligned}$$

3. Show a suitable introductory example, such as $87 * 43$, and invite suggestions.
 4. What do they already know how to work out? Accept and list all suggestions which might be useful.
 5. Depending on your own judgement of the ability of the children, you may wish to leave them time for their own investigations, or you may decide that a direct exposition is appropriate at this stage.
 6. There are a number of ways of setting out the work. Here are two which show clearly the principles involved.

(a) $87 * 43 = 87 * 40 + 87 * 3$

$$\begin{array}{r} 87 \qquad 87 \qquad 3480 \\ * 40 \quad * 3 \quad + 261 \\ \hline 3480 \quad 261 \quad 3741 \\ \hline 87 * 43 = 3741 \end{array}$$

←

(b) $87 * 40 = 3480$
 $87 * 3 = \underline{261}$
 $87 * 43 = 3741$

7. I would regard the transition to the conventional, condensed layout as optional. It saves little time, and does not show so clearly the mathematical principles used.
 8. Give the children further practice in these until the method is well established.
 9. You could explain that calculators are a quick and convenient way of doing calculations like these; but it is good to be independent of calculators, and also to understand the mathematics behind the result.

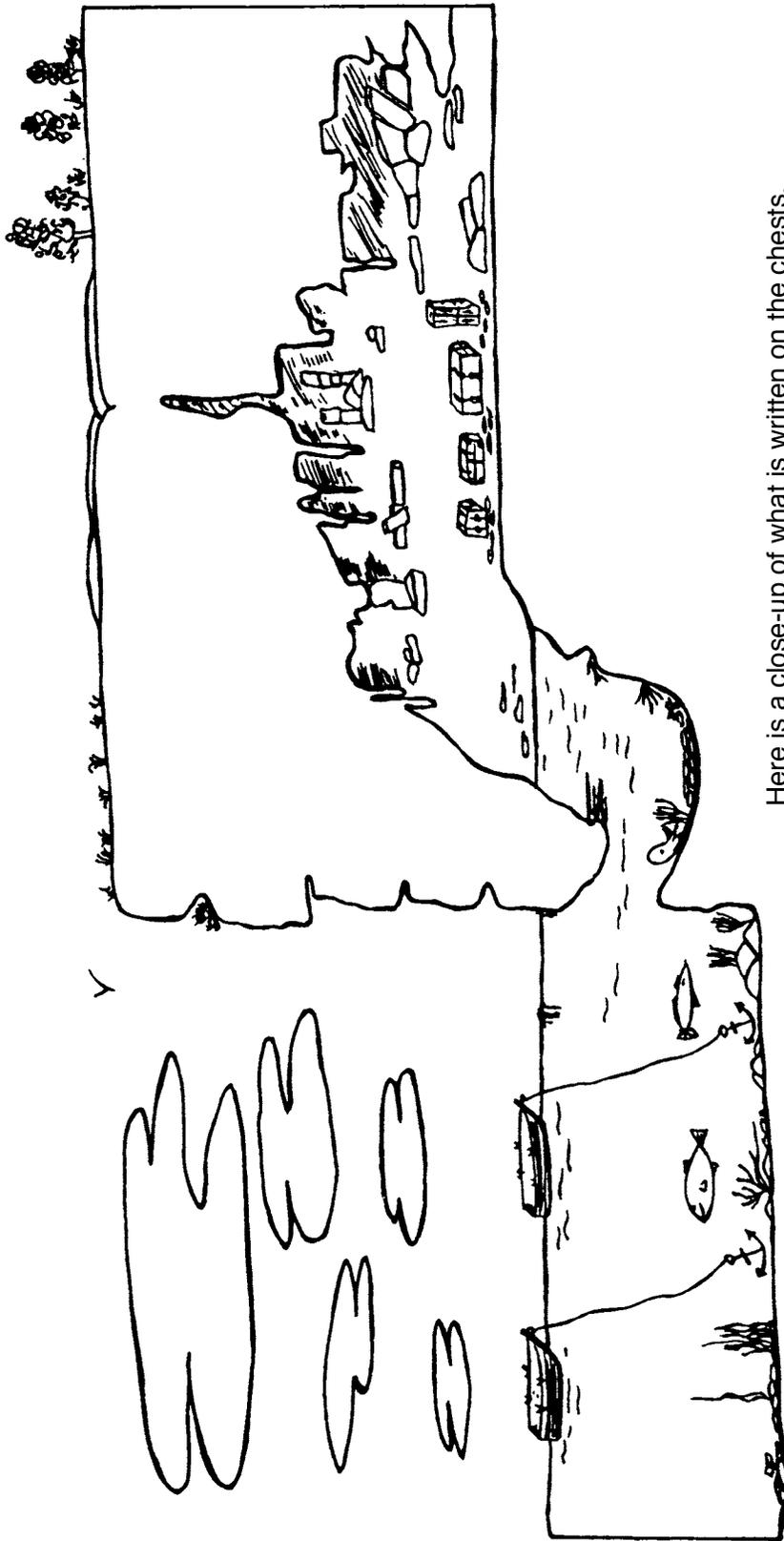
Activity 2 Treasure chest [Num 5.10/2]

A game for two crews, each of up to 3 children. Its purpose is to give practice in using long multiplication to make the best decisions.

- Materials*
- ‘Treasure chest’ game board, see Figure 11.*
 - Sets of treasure cards, 4 to a set.*†
 - Pencil and paper for each child.

† One of these is illustrated. Six more sets of suitable numbers are given at the top of the second following page. If you prefer, you may vary the objects according to your own imagination. These figures have been calculated to give close results, for which estimation is not sufficient. This makes it necessary to use long multiplication.

* Provided in the photomasters



Here is a close-up of what is written on the chests.

32 garnets worth \$ 74 each	45 amber brooches worth \$ 53 each	38 jet beads worth \$ 64 each	36 amethysts worth \$ 67 each
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Figure 11 Treasure chest

Suitable numbers for treasure cards:

First set	Amber 43 & 75,	Jasper beads 59 & 56,	Garnets 93 & 35,	Jet 39 & 64
Second set	Amethysts 82 & 67,	Topaz 86 & 64,	Gold 47 & 117,	Rubies 24 & 230
Third set	Silver goblets 24 & 166,	Emeralds 33 & 139,	Garnets 56 & 79,	Doubloons 31 & 145
Fourth set	Amber 96 & 78,	Amethysts 56 & 134,	Pearls 36 & 208,	Diamonds 26 & 278
Fifth set	Diamonds 26 & 249,	Jet 225 & 29,	Sapphires 44 & 148,	Garnets 94 & 69
Sixth set	Jasper 175 & 54,	Diamonds 28 & 338,	Pearls 37 & 256,	Amber 263 & 36

- Rules of the game*
1. The two crews of treasure seekers have swum under water to the cave, where they find four small sealed chests.
 2. The contents are painted on the outside.
 3. To avoid conflict, they reach the following agreement.
 - (i) Each crew will take back two chests.
 - (ii) One crew will have first and fourth choice, the other will have second and third choice. (Why is this the fairest way?)
 4. They determine which crew will have which pair of choices by guessing in which hand a pebble is held.
 5. Since they have no calculator with them, they have to use long multiplication to work out the values of the chests.
 6. It should be left to them to realize that it is better for the crew which guesses correctly in step 4 to delay saying whether they will choose first until after they have worked out the values of the chests.
 7. In turn, as described in step 3, they take their chests and swim back to the boat.
 8. On arriving on board, they may check with a calculator to find out which team has done best.
 9. The game may be repeated with other sets of chests.

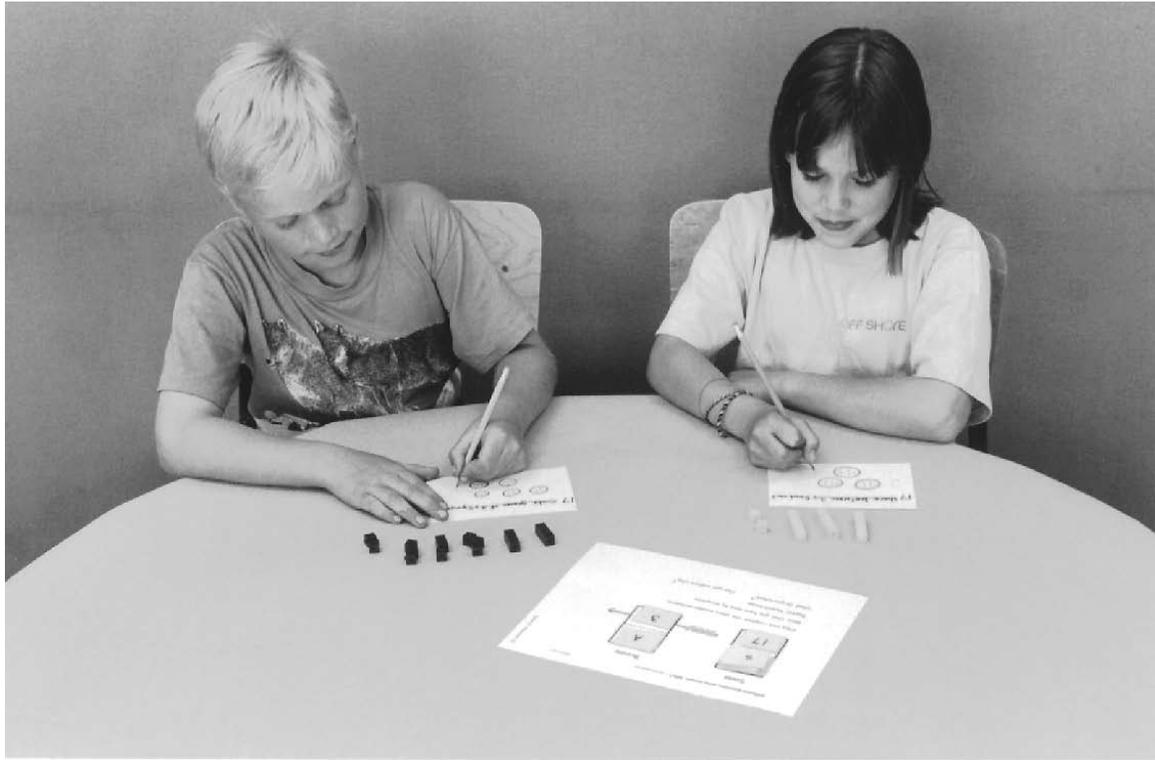
Discussion of activities Activity 1 involves using a combination of concepts and methods already learned to extend their ability to multiply into a much larger domain. When this method has been learned, Activity 2 provides a game in which to consolidate it.

There is something further to be said about calculators. What these manipulate with such speed, convenience, and accuracy are not numbers but numerals – not mathematical concepts, but symbols for these. It is their users who attach meanings to the symbols. So the more we use calculators, the more important it is to keep sight of the meaning of what we are doing.

OBSERVE AND LISTEN

REFLECT

DISCUSS



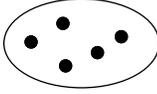
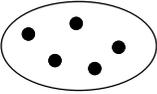
Different questions, same answer. Why? [Num 6.3/1]

[Num 6] **DIVISION**
Sharing equally, grouping, factoring

Num 6.3 DIVISION AS A MATHEMATICAL OPERATION

Concept The connection between grouping and sharing.

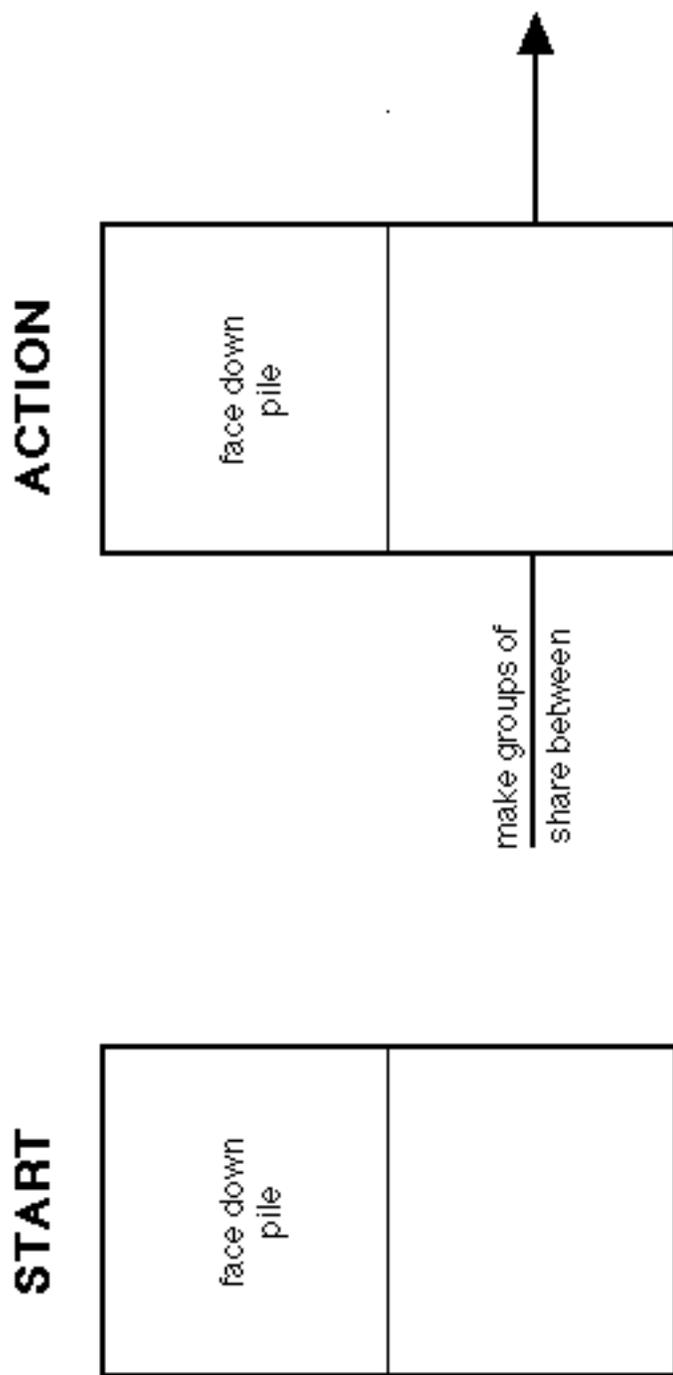
Ability To explain this connection. (This is most easily done with the help of physical materials.)

Discussion of concept	Physically, grouping and sharing look quite different.	
	Start with 15.	
	Make groups of 3.	
	Resulting number of groups is 5.	
	Start with 15.	
	Share among 3.	
Number in each share is 5.		
It is only at the level of thought that we can see that these are, in a certain way, alike. When children have grasped the connection between grouping and sharing, they have the higher order concept of division.		

Activity 1 Different questions, same answer. Why? [Num 6.3/1]

A problem for children to work at in small groups. (I suggest twos or threes.) The purpose is for them to discover for themselves the connection between grouping and sharing.

- Materials*
- Two-question board, see Figure 12.*
 - Start cards 10 to 25.*
 - Action cards 2 to 5.*
 - 50 (or more) small objects.
 - Pencil and paper for each child.
- * Provided in the photomasters



Copy and complete the above number sentences.

Show what you have done by diagrams.

Repeat several times.

What do you notice? Can you explain why?

Figure 12 Different questions, same answer. Why?

Introducing the problem

1. The start and action cards are shuffled and put face down in the upper spaces on the two question board.
2. The top card of each pile is turned and put face up in the lower space.
3. By reading above and below the line, there are now two unfinished number sentences. E.g.,

$$\boxed{17} \xrightarrow[\text{share between}]{\text{make groups of}} \boxed{3}$$

4. One (or more) in each group copies down the upper sentence, and one (or more) the lower sentence. They then complete whichever sentence they have written, using physical objects if they like. E.g.,

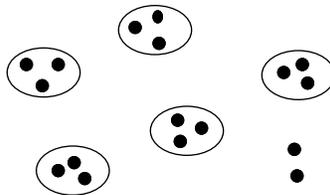
$$17 \xrightarrow{\text{make groups of } 3} 5 \text{ groups rem } 2$$

$$17 \xrightarrow{\text{share between } 3} 5 \text{ each rem } 2$$

N.B. They should write these neatly, and keep them for later use in Activity 2.

5. They draw diagrams to show what they have done. These diagrams could be used in step 4, instead of physical objects. E.g.,

Grouping



Sharing



6. They compare the two results.
7. Steps 2, 3, 4 are repeated. Step 5 need not be repeated every time: its purpose is to emphasize that these are two different questions.
8. Ask, "Will the two results always be the same? If so, why?"
9. Leave them to discuss this, and to arrive at a clear explanation. (One suggestion will be found in the 'Discussion of activities'.)
10. Return and hear their explanation, discussing it if necessary.

Num 6.3 Division as a mathematical operation (cont.)

Activity 2 Combining the number sentences [Num 6.3/2]

An activity for a small group. Its purpose is to teach the notation for the mathematical operation of division.

- Materials*
- The number sentences which they have written in Activity 1.
 - Pencil and paper for each child.

What they do 1. Have them compare the first number sentence of each kind. E.g.,

$$17 \xrightarrow{\text{make groups of 3}} 5 \text{ groups rem } 2$$

$$17 \xrightarrow{\text{share between 3}} 5 \text{ each rem } 2$$

2. Tell them that these two meanings may be combined in one number sentence. In this case, it would be

$$17 \div 3 = 5 \text{ rem } 2$$

This is read as

“17, divide by 3, result 5 remainder 2.”

or as

“17 divided by 3 equals 5 remainder 2.”

3. They then repeat steps 1 and 2 for the other number sentences.

Activity 3 Unpacking the parcel (division) [Num 6.3/3]

A continuation of Activity 2, for a small group. Its purpose is to remind children of the two possible meanings of a number sentence for division.

- Materials*
- Pencil and paper for each child.

What they do 1. They all write a division number sentence on their own paper. The first number should not be greater than 20.

$$11 \div 4 = 2 \text{ rem } 3$$

2. The first child shows his sentence to the others.
3. The next two on his left give the grouping and sharing meanings. In this case, “Start with 11, make groups of 4, result 2 groups remainder 3” followed by “Start with 11, share among 4, 2 in each share (or, each gets 2), remainder 3.” Note that “. . . groups,” or “. . . in each share,” are important parts of the expanded meaning.
4. Steps 2 and 3 are repeated until all have had their sentences ‘unpacked.’

Activity 4 Mr. Taylor's Game [Num 6.3/4]

This game for 2 players was invented by Mr. Stephen Taylor, of Dorridge Junior School, and I am grateful to him for permission to include it here. Its purpose is to bring together addition, subtraction, multiplication, and division, in a simple game.

- Materials*
- Number cards: 1 set 0 to 25, 3 sets 0 to 9.*
 - Game board, see Figure 13.*
 - Counters of a different colour for each player.
- * Provided in the photomasters

- What they do*
1. The object is to get 3 counters together in a line. They must be in the same row, column, or diagonal.
 2. The number cards are shuffled and put face down, the 0 to 25 cards in one pile and the 3 sets of 0 to 9 cards in another.
 3. The first player turns over the top card of each pile. He may choose to add, subtract, multiply, or divide the numbers shown. Division must, however, be exact.
 4. He puts one of his counters on the square corresponding to the result of the operation chosen.
 5. The other player turns over the next cards on the two piles and carries out steps 3 and 4.
 6. Play continues until one player has 3 in a row.
 7. Another round may now be played. The loser begins.

Discussion of activities Activity 1 poses a problem, for children to solve by the activity of their own intelligence. When they have seen the connection between grouping and sharing, they have the higher order mathematical concept of division.

The easiest path to seeing the correspondence is, I think, a physical one. If we have 15 objects to share among 3 persons, a natural way to do this is to begin by giving one object to each person. This takes 3 objects, a single 'round,' which we may think of as a group of 3. The next round may be thought of as another group of 3, and so on. Each round gives one object to each share, so the number of rounds is the number in each share.

This verbal description by itself is harder to follow than a physical demonstration accompanied by explanation. This I see as yet another demonstration of the advantage of combining Modes 1 and 2.

Activity 2 provides a notation for these two aspects of division. Note that the first number is the operand, that on which the operation is done. The operation is the division sign together with the second number, e.g., $\div 3$.

Activity 3 is another example of 'Unpacking the parcel.' There is much less in this one than in the subtraction parcel, but is still a useful reminder of the two physical meanings combined in a single mathematical notation.

After all this, they deserve a game. Mr. Taylor's game fits in nicely at this stage.

OBSERVE AND LISTEN**REFLECT****DISCUSS**

5	20	19	25	1
21	7	14	6	16
3	23	11	18	24
9	17	8	13	2
12	4	10	22	15

Figure 13 Mr. Taylor's Game.

© Stephen Taylor, 1981

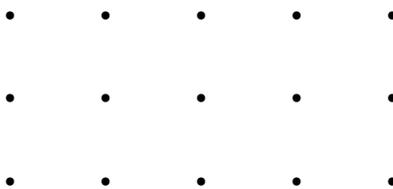
Num 6.4 ORGANIZING INTO RECTANGLES

Concept A rectangular number.

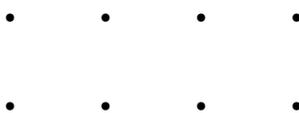
Ability To recognize and construct rectangular numbers.

Discussion of concept A rectangular number is one which can be represented as the number of dots in a rectangular array. Here are some examples.

15

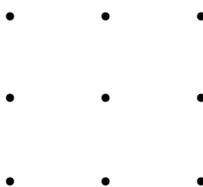


8



How about
this one?

9



The term 'rectangle' is currently used with two meanings: any shape which has a right angle at every corner, which includes both squares and oblongs; and oblongs only. For myself, I agree with the teacher who said during a discussion of this point, "It makes no more sense to me to talk about squares and rectangles, than about girls and children." So I hope that we may agree on the first meaning, i.e., that oblongs and squares are two kinds of rectangle.

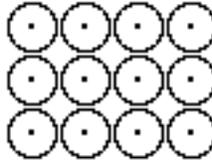
A rectangular number is of course the same as a composite number, but in this aspect it provides a very good link between multiplication and division, and an introduction to factoring.

Activity 1 Constructing rectangular numbers [Num 6.4/1]

An activity for a small group, working in pairs. Its purpose is to build the concept of rectangular numbers.

- Materials* For each pair:
- 25 small counters.
 - Pencil and paper.

- What they do*
1. The activity is introduced along the following lines. Explain, “We think of these counters as dots which we can move around.”
 2. Put out a rectangular array such as this one.



3. Ask, “What shape have we made?” (A rectangle.)
4. “How many counters?” (In this case, 12.)
5. “So we call 12 a rectangular number.”
6. Repeat, with other examples, until the children have grasped the concept. Note that the rectangles must be solid arrays like the one illustrated.
7. Next, let the children work in pairs. Give 25 counters to each pair, and ask them to find all the rectangular numbers up to 25.
8. If any of them think that a pattern like this might be a rectangle, remind them that the counters represent dots, and ask: “Would you say this makes a rectangle?”
9. The question nearly always arises whether or not squares are to be included. I suggest that you wait until someone raises this point, and then answer along the lines given in ‘Discussion of concept.’
10. Finally, let the children compare and check their lists with one another.



Activity 2 The rectangular numbers game [Num 6.4/2]

A game for two. It consolidates the concept of a rectangular number in a predictive situation. Children also discover prime numbers, though usually they do not yet know this name for them.

- Materials*
- 25 counters.
 - Pencil and paper for scoring.

- Rules of the game*
1. Each player in turn gives the other a number of counters.
 2. If the receiving player can make a rectangle with these, she scores a point. If not, the other scores a point.
 3. If when the receiving player has made a rectangle (and scored a point), the giving player can make a different rectangle with the same counters, she too scores a point. (E.g., 12, 16, 18).
 4. The same number may not be used twice. To keep track of this, the numbers 1 to 25 are written at the bottom of the score sheet and crossed out as used.
 5. The winner is the player who scores the most points.

Discussion of activities In Activity 1, the children first form the concept of a rectangular number from physical examples (Mode 1 building), and then construct further examples for themselves (Mode 3 building, Mode 1 testing).

Activity 2 uses the new concept in a predictive situation (Mode 1 testing).

OBSERVE AND LISTEN

REFLECT

DISCUSS



The rectangular numbers game [Num 6.4/2]

Num 6.5 FACTORING: COMPOSITE NUMBERS AND PRIME NUMBERS

- Concepts* (i) Factoring.
(ii) Composite numbers and prime numbers.

- Abilities* (i) To factor a given composite number.
(ii) To distinguish between composite and prime numbers.

Discussion of concepts A composite number is one which can be written as the product of two (or more) numbers, other than itself and 1. E.g.,

$$15 = 3 \times 5$$

$$22 = 11 \times 2$$

Often this may be done in more than one way. E.g.,

$$24 = 2 \times 12$$

$$24 = 3 \times 8$$

$$24 = 4 \times 6$$

To write a number as the product of other numbers is called factoring, and these other numbers are called factors of the original number. E.g., 3 and 8 are factors of 24. The process may often be continued, e.g.

$$\begin{aligned} 140 &= 14 \times 10 \\ &= 2 \times 7 \times 2 \times 5 \end{aligned}$$

which may be rearranged as

$$140 = 2 \times 2 \times 5 \times 7$$

In this topic we shall only deal with 2 factors, since we are here seeing it largely as a preliminary to the use of multiplication facts for dividing larger numbers.

1 is a factor of every number. A number which has no factors other than 1 and itself is called prime. So the terms 'rectangular number' and 'composite number' are interchangeable; and numbers which are not rectangular numbers are prime.

Activity 1 Factors bingo [Num 6.5/1]

A game for up to 6 players. Its purpose is to introduce the concept of factoring at a level where the calculations are still easy. It only requires knowledge of the multiplication tables up to 6×6 .

- Materials*
- 6 ‘factors’ bingo cards.*
 - Die 1-6 and shaker.
 - A non-permanent marking pen for each player.
 - Damp rag.

* Provided in the photomasters. See example in step 5 below. These should be covered with plastic film, so that they can be cleaned and re-used.

- Rules of the game*
1. Begin by explaining the meaning of ‘factor.’
 2. Each player has a Factors Bingo card and a non-permanent pen.
 3. Players in turn, throw the die, and call out the number which comes up. This number is available for use by everyone.
 4. They fill their cards by writing the given number in any space where it is a factor of the number on the left. This may only be done once for each throw.
 5. A partly completed card might look like this, assuming that the numbers 2, 6, 4, 5, 3 have been thrown so far. (This player was not able to use 5.)
 6. The winner is the first to fill his card.
 7. This game is not quite as simple as it might first appear. If a player writes (say) 2 as one of the factors of 28, he is left with 14 as the other factor, and will not be able to fill his card. A similar problem arises with 1.

4	=	2	x	x
6	=	x	x	x
24	=	6	x	4
18	=	3	x	x
12	=	x	x	x

Activity 2 Factors rummy [Num 6.5/2]

A game for up to 6 players. Its purpose is to develop mental agility in factoring.

- Materials*
- Product pack: one card each of the numbers 10, 12, 14, 15, 16, 18, 20, 21, 24, 28, 30, 36.*
 - Factors pack: six cards each of the numbers 2, 3, 4, 5, 6, 7, 8, 9.*
- * Provided in the photomasters. These two packs should be double headed (to facilitate holding a ‘hand.’) and of different colours.

- Rules of the game*
1. The product pack is put face down on the table. The dealer deals 5 cards from the factors pack to each player, and puts the rest of the factors pack face down on the table.
 2. He then turns over the top card of each pack, and puts it face up beside the pile.
 3. The object is to get rid of one’s cards by putting down pairs of cards which, when multiplied together, make the product shown on the face-up product card.

Num 6.5 Factoring: composite numbers and prime numbers (cont.)

4. Players begin by looking at their cards, and putting down any pairs they can.
5. The first player then picks up a factor card from either the face up or the face down pile, whichever he prefers. If he now has a pair, he puts it down. Finally he discards one of his cards onto the face up pile.
6. In turn the other players pick up, put down a pair if they can, discard.
7. At the end of each round, the dealer turns over the top product card from the face down pile, and puts it face up beside (not on top of) the product card already showing. In this way, the number of possible products increases for every new round.
8. After step 7, players put down any pairs now made possible, before the next round is played.
9. Steps 5 to 8 are repeated, until a player puts down all his cards.
10. Play then stops, and each player scores the total of the cards still in his hand. The player who went 'out' scores zero.
11. In this game, it is possible for two players to go 'out' simultaneously when a new product card is turned. This does not matter: they both score zero.
12. The scores are recorded, and another round may then be played.
13. The winner is the player with the lowest total score.

Activity 3 Alias prime [Num 6.5/3]

A game for up to 6 players. Its purpose is to introduce children to the difference between composite and prime numbers, and give them practice in distinguishing between these two kinds of number.

Materials • Three counters for each player.

- Rules of the game*
1. Begin by explaining the meanings of 'composite number' and 'prime.' These concepts have been well prepared in Activities 1 and 2, and in topic 4.
 2. Explain that 'alias' means 'another name for,' often used to hide someone's identity. In this game, all prime numbers use the alias 'Prime' instead of their usual name.
 3. Start by having the players say in turn "Eight," "Nine," "Ten," . . . round the table.
 4. The game now begins. They say the numbers around the table as before, but when it is a player's turn to say any prime number, its usual name must not be said but "Prime" is said, instead.
 5. The next player must remember the number which wasn't spoken, and say the next one. Thus the game would begin (assuming no mistake):
"Eight," "Nine," "Ten," "Prime," "Twelve," "Prime," "Fourteen," "Fifteen," "Sixteen," "Prime," "Eighteen," and so on.
 6. Any player who makes a mistake loses a life – i.e., one of his counters. Failing to say "Prime," or saying the wrong composite number, are both mistakes.
 7. When a player has lost all his lives he is out of the game, and acts as an umpire.
 8. The winner is the last player to be left in the game.

Note When the players are experienced, they may begin counting at "One." This gives rather a lot of primes for beginners!

Activity 4 The sieve of Eratosthenes [Num 6.5/4]

An activity for children to play individually, or in small groups. Its purpose is to teach them a classical piece of mathematics for identifying primes.

- Materials*
- Number squares 1-100, see Figure 14.*
 - Pencil for each child.
- * One each, or shared. Provided in the photomasters.

- What they do*
1. In the previous game, when players are good enough to reach largish numbers, there is sometimes doubt whether a number is prime or not. Explain that this is a method which gets rid of all the composite numbers and leaves only primes.
 2. Starting with 2 (the smallest prime other than 1), they cross out every multiple of 2 except 2 itself.
 3. The next prime is 3, so they cross out every multiple of 3 except 3 itself.
 4. Likewise for each successive prime.
 5. The numbers remaining when the process is complete are all primes.
 6. After a while, the children will notice that no new numbers are being crossed out. With numbers up to 100, the process is complete when all the multiples of 7 have been crossed out.
 7. The intriguing question is, Why? When we come to cross out the multiples of 11, 13, etc., why do we find that these have already been done? While steps 1 to 6 are straightforward, this requires careful thought.
 8. A simpler question is, “Why do we only cross out multiples of primes?”
 9. This method is quite general, and may be extended beyond 100 if desired.

Discussion of activities In this topic we are mainly concerned with introducing the concept of factoring as a preparation for its use in division, so that children will later be able to use their knowledge of the multiplication tables for dividing. This saves learning a whole new body of facts. For this reason the numbers involved are kept small. Activities 3 and 4 develop an offshoot of factoring, namely prime numbers. We do not develop the study of primes any further than this, but they have long intrigued mathematicians, and children should at least know what they are.

Using their knowledge of multiplication results to arrive at division results is, of course, a strong example of Mode 3 schema building. Doing things backwards is usually harder than forwards, and it certainly is in this case. So division, both conceptually and as a skill, is prepared and developed carefully over a number of topics.

OBSERVE AND LISTEN

REFLECT

DISCUSS

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

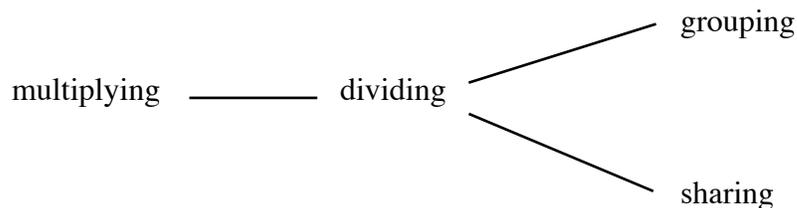
Figure 14 The sieve of Eratosthenes

Num 6.6 RELATION BETWEEN MULTIPLICATION AND DIVISION

Concept The relation between multiplication and division.

Ability To express the same relationship in a variety of different ways: as multiplying, grouping, sharing, dividing.

Discussion of concepts The new concept to be formed in this topic is an awareness of how all of the following are connected.



This is also a convenient place to introduce vertical notation for division, in preparation for dividing larger numbers in the next topic.

Activity 1 Parcels within parcels [Num 6.6/1]

A game for up to 6 children. Its purpose is to make explicit the relationships between multiplication and the various aspects of division.

- Materials*
- 6 or more parcel cards.*
 - Pencil and paper for each child.
 - Bowl of counters.

* Provided in the photomasters

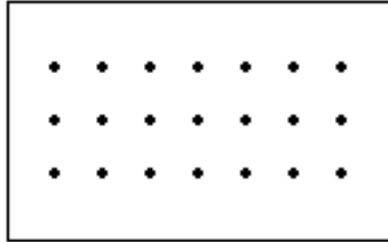
Rules of the game Stage (a)

1. The parcel cards are shuffled and put face down.
2. The first player turns over the top parcel card.
3. She then starts to “unpack the parcel” by first writing a number sentence which she shows to the others, and then reading it aloud in as many ways as she can. (See the example parcel card at the top of the page, overleaf.)
4. If the others agree, they say “Agree.” The first player takes a counter for each agreed statement.
5. If they disagree, the first player has to justify his statement by reference to the parcel card. E.g., for $3(7) = 21$, she would point out that there are 3 rows of 7, making 21.
6. Steps 3,4,5 are repeated by the other players in turn until the parcel has been fully unpacked.
7. The next player turns over the top card, and steps 1-6 are repeated.
8. The winner is the player with the most counters at the end.

Num 6.6 Relation between multiplication and division (cont.)

Example

Parcel card turned over:



Contents

- A. Writes: $3(7) = 21$
Reads: “3 sets of 7 make a set of 21” or “3 sevens are/make/equal 21” (1 point)
- B. Likewise for $7(3)$. (1 point)
- C. Writes: $3 * 7 = 21$.
Reads: “3 star 7 equals 21.”
“3 sevens are 21.”
“7 threes are 21.” (3 points)
- D. Writes: $21 \div 3 = 7$
Reads: “21 divided by 3 equals 7”
“21, make groups of 3, result 7 groups.”
“21, share between 3, result 7 in each share.” (3 points)
- E. Writes: $21 \xrightarrow{\text{make groups of 3}} 7$
Reads: “21, make groups of 3, result 7 groups.” (1 point)
- F. Writes: $21 \xrightarrow{\text{share between 3}} 7$
Reads: “21, share between 3, result 7 each.” (1 point.)
- G,H,I. Likewise for $21 \div 7 = 3$.

As may be seen from this example, it is quite a sophisticated game. C, D and G are clearly the ones to choose first. But there are still points to be collected when these have been used.

Stage (b)

1. When they have a good grasp of the foregoing, introduce the following notation. Explain that this is another way of writing a division sentence, which is needed for larger numbers.

$$\begin{array}{r} 3 \\ \hline 5 \overline{) 15} \end{array}$$

- This is read as: “How many 5’s in 15? Answer, 3.”
2. It should *not* be read as “five into fifteen . . .”, as one sometimes hears. We are certainly not dividing 5 into 15 shares.
 3. Continue as in Stage (a), including this notation.

Discussion of activity

Stage (a) is conceptually quite complex. This is a necessary consequence of the great variety of information which can be extracted from one 'parcel.' All these ways of thinking are needed for different purposes; and it is much more efficient, and economical of memory, to learn them all as different ways of thinking about the same number-relationship. This relationship is represented much more clearly by the rectangular array than it can be by any notation, each of which shows only a particular aspect. All these aspects are concepts already learned. What the children are now doing is becoming more aware of their interrelationships.

Stage (b) is simpler, since it only involves learning a new notation for what they already know.

Note that in both stages of the activity, the main links which are being strengthened are between notations and their various meanings, not between one notation and another. Their various meanings are all implicit in the rectangular arrays – the parcel cards.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Parcels within parcels [Num 6.6/1]

Num 6.7 USING MULTIPLICATION RESULTS FOR DIVISION

Concept The use of multiplication results in reverse, to obtain division results.

Ability To arrive at a division result mentally by recalling an appropriate multiplication result.

Discussion of concept The close relationship between the two statements

“3 fives are 15”

and

“How many fives in 15? Answer 3.”

has already been learned, using small numbers and visual representations. In this topic the concept is expanded to include all products up to and including 10×10 , and made independent of visual representations.

Activity 1 A new use for the multiplication square [Num 6.7/1]

An activity for children to do in pairs. Its purpose is to develop the ability described above.

- Materials*
- Multiplication square for each child.*
 - 1 to 9 die.
 - Pencil and paper for each child.

* They should still have the multiplication squares they made for themselves in Num 5.6/3. A photomaster is also provided in *Volume 2a*.

- What they do*
1. Player A has his multiplication square face up. Player B has his square face down.
 2. Player A throws the die. Suppose it shows 7. He therefore takes the row starting with 7, and writes

$$7$$

3. He then chooses any number in this row, say 35. He writes

$$7 \overline{) 35}$$

and passes the paper to Player B.

4. Player B first reads this aloud, as “How many sevens in 35?”
5. Next, if he can, he recalls from his multiplication tables that 5 sevens are 35. He says this aloud, and completes the written division.

$$5$$
$$7 \overline{) 35}$$

6. If he cannot do this, he turns over his multiplication table, finds the row starting 7, and then locates 35 in this row. He uses this to say aloud, “5 sevens are 35,” and then continues as in step 6.
7. In either case, player A checks B’s result and says “Agree.”
8. The players then interchange roles, and steps 2 to 6 are repeated.
9. Players should be encouraged to give each other practice with all the products in a row, and not just the harder ones. For this reason, no scoring system is included in this activity, which might cause players to give each other only the more difficult examples.

Activity 2 Quotients and Remainders [Num 6.7/2]

An activity for up to 4 children. Its purpose is to give practice in division, and to introduce the terms ‘quotient’ and ‘remainder’ using easy numbers.

- Materials*
- Q and R board, see Figure 15.*
 - Start cards 9 to 30.
 - Die 1-6.
 - For each child, 5 counters of 1 colour.
 - Pencil and paper for each child (optional).
- * Provided in the photomasters

- Rules of the game*
1. The start cards are shuffled and put face down.
 2. The first player throws the die, and then turns over the top start card. This gives a number to divide by, and a starting number; e.g., 3 and 23.

$$\begin{array}{r} 7 \text{ rem } 2 \\ \underline{3 } \\ 3 \end{array}$$

3. He says, “Quotient 7, remainder 2,” so that the others can check.
4. He may then place one of his counters on a 7 and one on a 2, the aim being to get 3 counters in a row. This row may be across, down, or diagonal.
5. The next player repeats steps 2 to 4, and so on in turn until either a player has won by getting 3 in a row, or this has become impossible.
6. A player who throws 1 may have another turn. If the quotient is a number not on the board, only the remainder is used.
7. A new game may then be played.
8. (Optional) Pencil and paper may be used for written calculation.

Note This board has been designed to give a quick result. For a longer game, play continues until each player has put down all his counters. For the last counter, he chooses either the quotient or the remainder. The winner is then the player with the longest row.



Quotients and Remainders [Num 6.7/2]

0	6	2	5	1	7	0
3	1	8	2	4	3	1
9	8	4	6	5	4	10
3	1	5	0	10	2	4
9	7	2	5	1	3	10
4	3	5	4	3	5	2
0	9	7	2	6	8	0

Figure 15 Q and R board

Activity 3 Village Post Office [Num 6.7/3]

A game for 2 to 6 children. Its purpose is to use the relationship between division and multiplication in situations where division involves a remainder.

- Materials*
- Multiplication square.*
 - Notice.*†
 - Stamp cards, 3 to 10.*
 - Parcel cards.**
 - Pencil and paper for each child.

* Provided in the photomasters

† See illustration in step 3 below

** As many as you like. I suggest that children begin with an easy pack, with numbers up to 30¢; and progress to a harder pack, up to 70¢. Suitable numbers are:

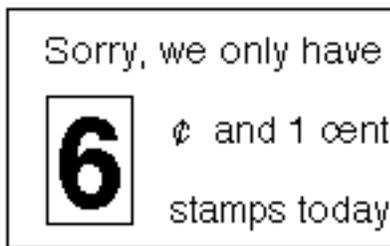
Easy 11, 14, 17, 19, 23, 25, 26, 28, 29, 30.

Hard 31, 33, 35, 36, 37, 40, 41, 43, 46, 53, 55, 57, 62, 65, 67, 69.

With the harder set of parcel cards, only stamp cards 7-10 should be used.

Rules of the game For 2 children

1. One acts as postal clerk, the other as customer.
2. The postal clerk shuffles the stamp cards, the customer shuffles the parcel cards, and these are then put face down in separate piles.
3. The postal clerk opens the shop by turning over the top stamp card and putting it in the space in the notice. It now reads (e.g.),



4. The customer then takes the top parcel card and turns it over. Suppose it reads 26¢. He calculates $26 \div 6 = 4 \text{ rem } 2$ and should therefore ask for 4 six-cent and 2 one-cent stamps. Note that the postal clerk likes customers to ask for the least number of stamps which gives the required total. (He doesn't take kindly to requests for 26 one-cent stamps.)
5. The postal clerk checks: $4(6) + 2 = 26$.
6. If correct, he says "Correct" and accepts the parcel. If the results do not agree, the multiplication square may be used to discover where their mistake lies.
7. The next parcel card is then taken by the customer, and steps 4 to 6 are repeated.

For up to 6 children

In addition to the postal clerk, there are now 1 or 2 assistants. The remaining children are customers, and are served by whichever clerk is free. Otherwise the game is played as above.

Discussion of activities In the earlier topics, learning has been directed towards the establishment of the concept of division in its two aspects, grouping and sharing; and with relating it to multiplication. In the present topic, the purpose is to develop fluency in calculation.

The basis of this is the use of already-known multiplication facts for division (Activity 1). The notation which children will need for dividing larger numbers is also used here. Activity 2 gives further practice at these in the form of a game, still using fairly small numbers. Activity 3 is important for linking these calculation skills back to the concepts involved.

The terms 'quotient' and 'remainder' are useful ones for later work. I leave it for you to decide whether to include 'divisor' and 'dividend' also.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Village Post Office [Num 6.7/3]

Num 6.8 DIVIDING LARGER NUMBERS

Concept The concept of division, as already learned.

Ability To divide a 2, 3, or 4-digit number by a single digit number, first without regrouping and then with regrouping.

Discussion of concepts The concept of division has already been learned. What we are here concerned with is extending the ability to larger numbers. The method for doing this uses non-canonical form, which has already been used in addition, subtraction, and multiplication.

Activity 1 “I’m thinking in hundreds” [Num 6.8/1]

An activity for children in groups of 3. Its purpose is to teach them the first steps in the ability described above.

Materials For each group:

- HTOnes division board, laminated.*
- Non-permanent pen and wiper.
- Examples card.**

* See illustration in step 2 below. A division board is provided in the photomasters, from which expendable sheets can be reproduced.

** These are the examples:

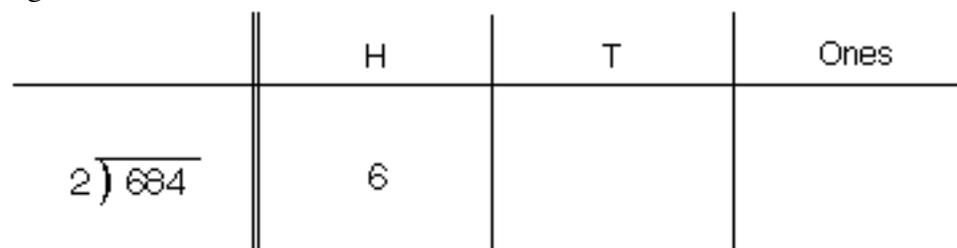
$$\begin{array}{cccccc} 684 \div 2 & 936 \div 3 & 848 \div 4 & 408 \div 2 & 630 \div 3 & \\ 880 \div 4 & 690 \div 3 & 906 \div 3 & 208 \div 2 & 550 \div 5 & \end{array}$$

- What they do*
1. The children need to sit facing the same way.
 2. The first of the examples is copied in the left hand space of the HTOnes division board by the child sitting in the middle, using the notation shown.



She reads this aloud: “How many 2’s in 684?”, and passes the board and non-permanent pen to the child on her left.

3. The left hand child says, “I’m thinking in hundreds,” and writes the hundreds figure in the hundreds column.



Num 6.8 Dividing larger numbers (cont.)

4. Next she writes, reading aloud as she does so: “How many 2’s in 6 hundreds? Answer, 3 hundred.” (N.B. 3 hundred, not 3.)

	H	T	Ones
$2 \overline{) 684}$	$2 \overline{) 6} \begin{array}{r} 3 \\ \hline \end{array}$		

5. Steps 3 and 4 are repeated for the tens figure by the middle child. She begins by saying, “I’m thinking in tens.”
 6. Steps 3 and 4 are repeated for the ones figure by the right hand child. She begins, “I’m thinking in ones.”
 7. The lower part of the board should now look like this.

	H	T	Ones
$2 \overline{) 684}$	$2 \overline{) 6} \begin{array}{r} 3 \\ \hline \end{array}$	$2 \overline{) 8} \begin{array}{r} 4 \\ \hline \end{array}$	$2 \overline{) 4} \begin{array}{r} 2 \\ \hline \end{array}$

8. The board is then passed back to the middle child, who completes the number sentence on the left.

	H	T	Ones
$2 \overline{) 684} \begin{array}{r} 342 \\ \hline \end{array}$	$2 \overline{) 6} \begin{array}{r} 3 \\ \hline \end{array}$	$2 \overline{) 8} \begin{array}{r} 4 \\ \hline \end{array}$	$2 \overline{) 4} \begin{array}{r} 2 \\ \hline \end{array}$

She reads it aloud. “How many two’s in six hundred eighty four? Answer, three hundred forty-two.”

9. The board is wiped clean, and steps 2 to 8 are repeated with the next example on the card.
 10. The order in which the children sit should be varied.

Activity 2 “I’ll take over your remainder.” [Num 6.8/2]

A continuation of Activity 1, for children in groups of 3. Its purpose is to teach them how to deal with remainders in each column.

Materials

For each group:

- HTOnes division board, laminated.*
- Non-permanent pen and wiper.*
- Examples card.**

* The same as for Activity 1.

** Here are 10 examples. Others may be taken from a suitable textbook. You may wish to grade the difficulty more slowly.

$$743 \div 3 \quad 851 \div 3 \quad 783 \div 4 \quad 680 \div 5 \quad 705 \div 6$$

$$241 \div 7 \quad 354 \div 8 \quad 559 \div 9 \quad 303 \div 4 \quad 600 \div 7$$

What they do 1. The first example is copied in the left hand space of the HTOnes division board by the child sitting in the middle, and read aloud, as in Activity 1. The board is then passed to the child on the left.

	H	T	Ones
$3 \overline{)743}$			

2. The left-hand child writes the hundreds figure in the hundreds column.

	H	T	Ones
$3 \overline{)743}$	7		

3. Next she writes, reading aloud as she does so: "How many 3's in 7 hundreds? Answer, 2 hundred, remainder 1 hundred."

	H	T	Ones
$3 \overline{)743}$	$3 \overline{)7} \begin{array}{l} 2 \text{ rem } 1 \end{array}$		

4. The question is now, what to do with the remainder?
5. The middle child writes the tens figure in the tens column.

	H	T	Ones
$3 \overline{)743}$	$3 \overline{)7} \begin{array}{l} 2 \text{ rem } 1 \end{array}$	4	

Num 6.8 Dividing larger numbers (cont.)

6. Next, she says, "I'll take over your remainder. 1 hundred is 10 tens, so that makes 14 tens." The remainder in the hundreds column is crossed out.

	H	T	Ones
$3 \overline{)743}$	$3 \overline{)7} \text{ rem } 1$	14	

7. He writes, and reads aloud: "How many 3's in 14 tens? Answer, 4 tens, remainder 2 tens."

	H	T	Ones
$3 \overline{)743}$	$3 \overline{)7} \text{ } 2$	$3 \overline{)14} \text{ } 4 \text{ r } 2$	

8. The right hand child deals similarly with the ones. He takes over the remainder of 2 tens, which are 20 ones, making 23 ones altogether.

	H	T	Ones
$3 \overline{)743}$	$3 \overline{)7} \text{ } 2$	$3 \overline{)14} \text{ } 4 \text{ r } 2$	$3 \overline{)23} \text{ } 7 \text{ r } 2$

9. The board is then passed back to the middle child, who completes the number sentence on the left.

	H	T	Ones
$3 \overline{)743} \text{ } 247 \text{ r } 2$	$3 \overline{)7} \text{ } 2 \text{ r } 1$	$3 \overline{)14} \text{ } 4 \text{ r } 2$	$3 \overline{)23} \text{ } 7 \text{ r } 2$

She reads it aloud. "How many threes in seven hundred and forty three? Answer, two hundred and forty seven, remainder 2.

10. The board is wiped clean, and steps 1 to 9 are repeated with the next example. The order in which the children sit should be varied.
11. When children have had plenty of experience of the above, transition to the condensed notation is straightforward. To begin with, the same calculation should be rewritten in the condensed notation to show the correspondence. In the above example, this would be:

$$\begin{array}{r} 247 \text{ r } 2 \\ 3 \overline{)7^1 4^2 3} \end{array}$$

This condensed notation is a good one, provided that its meaning is well understood. Now that calculators are widely available, I attach little importance to children's acquiring a high degree of skill in this kind of computation; but it is good to understand the principle.

Activity 3 Q and R ladders [Num 6.8/3]

This game is suitable for 4 or 6 children, playing in teams of 2. Its purpose is first, to expand the abilities developed in Activities 1 and 2 to larger numbers; and second, to develop fluency in these calculations.

- Materials*
- Game board.*
 - Markers: a pair of one colour for each team of 2.
 - Start cards 50 to 100.*
 - Die 5 to 10.**
 - Pencil and paper for each pair, or single player.

* See the photomasters in *Volume 2a*. The game board consists of two ladders, one for quotients and one for remainders.

** Make your own from 1 cm cubes.

- Rules of the game*
1. The start cards are shuffled and put face down.
 2. Each team puts one of its markers on the zero rung in each ladder. These represent climbers.
 3. The first pair begins as follows: one player turns over the top start card, and the other throws the die.
 4. They then co-operate in dividing the number on the start card, by the number on the die, using the method learned in Activity 2. This gives them a quotient and a remainder.
 5. These two numbers show how many rungs their two pieces may climb, each up its respective ladder.
 6. The other teams in turn repeat steps 3 to 5.
 7. The winning team is that which gets both of its two climbers to the top of the ladder, or beyond. The exact number is not required in this game.

Note This game usually takes 6, 7, or 8 rounds; occasionally, fewer. It has been designed so that the likelihood of being first at the top is approximately equal for climbers on the two ladders. There is also about a 1 in 5 chance of both getting there at the same time.

Activity 4 Cargo Airships [Num 6.8/4]

A game for up to 4 players. Its purpose is to give them plenty of practice in division of numbers up to 1000, by numbers from 3 to 9.

- Materials*
- Cards representing airports, one for each player.*
 - 7 cards representing airships.*
 - Cards representing loads.*†
 - Loading slips.*
 - Calculators.
 - Pencil and paper for each player.
 - A paper clip for each airship.

* Photomasters provided in *SAIL Volume 2a*.

† A suitable set of numbers is 325, 108, 473, 235, 90, 156, 209, 392, 267, 340, 440, 60, 83, 116, 180, in each case followed by ‘tonnes.’ (See photomasters.)

- Rules of the game*
1. This game is like One Tonne Van Drivers, but harder since it involves dividing.
 2. Each cargo airship can carry from 600 to 1000 tonnes, according to size. They also vary from 3 to 9 in their number of compartments, and the loads must be equally shared between the compartments to keep the airship level. This is the task of every loading officer.
 3. All the players except one act as loading officers at distant airports. The remaining player is the checking officer at the home airport. He has a calculator.
 4. Each loading officer has a card representing his airport, and an airship. The airships are assigned randomly.
 5. To begin the game, the load cards are shuffled and put face down in the middle of the table.
 6. The loads arrive by road at the distant airport. To represent this in the game, the first loading officer takes the top load card.
 7. This must be shared equally among whatever number of compartments the aircraft has. So the loading officer calculates the correct part-loads, equal for each compartment. These will be carefully weighed out on the scale before loading. E.g., if he has a 6 compartment airship, and a 473 tonne load arrives, he would instruct his loaders to make part-loads of 78 tonnes each, leaving a remainder of 5 tonnes. He writes this on a loading slip, as shown in step 10.
 8. Since the remainders are small compared with the part loads, the cargo officer on the airship uses his own discretion where to put these.
 9. Steps 6 and 7 are repeated in turn by the other loading officers, and this continues until one of them receives a load which would make his airship overloaded. He must then send that airship off to the home port, together with a loading slip which might look like this.

10.

	<i>Airship Sky King</i>		
	Max. load 900 tonnes, 6 compartments		
	<i>Load</i>	<i>6 part-loads, each</i>	<i>remainder</i>
	473	78	5
	108	18	0
	235	39	1
<i>Totals</i>	816	135	6

11. The other loading officers send home their airships when these can take no more.
12. As each airship arrives back at the home airport, the checking officer checks by calculator all the relevant calculations, including part-loads and remainders. There is good opportunity for discussion about how best to do this. The unused capacity is also calculated, in the above example 84 tonnes (900 - 816).
13. Miscalculations are penalized by adding 100 tonnes to the unused capacity for each error in a part load, and 5 tonnes for each error in a remainder.
14. The best loading officer is the one with the least unused capacity.

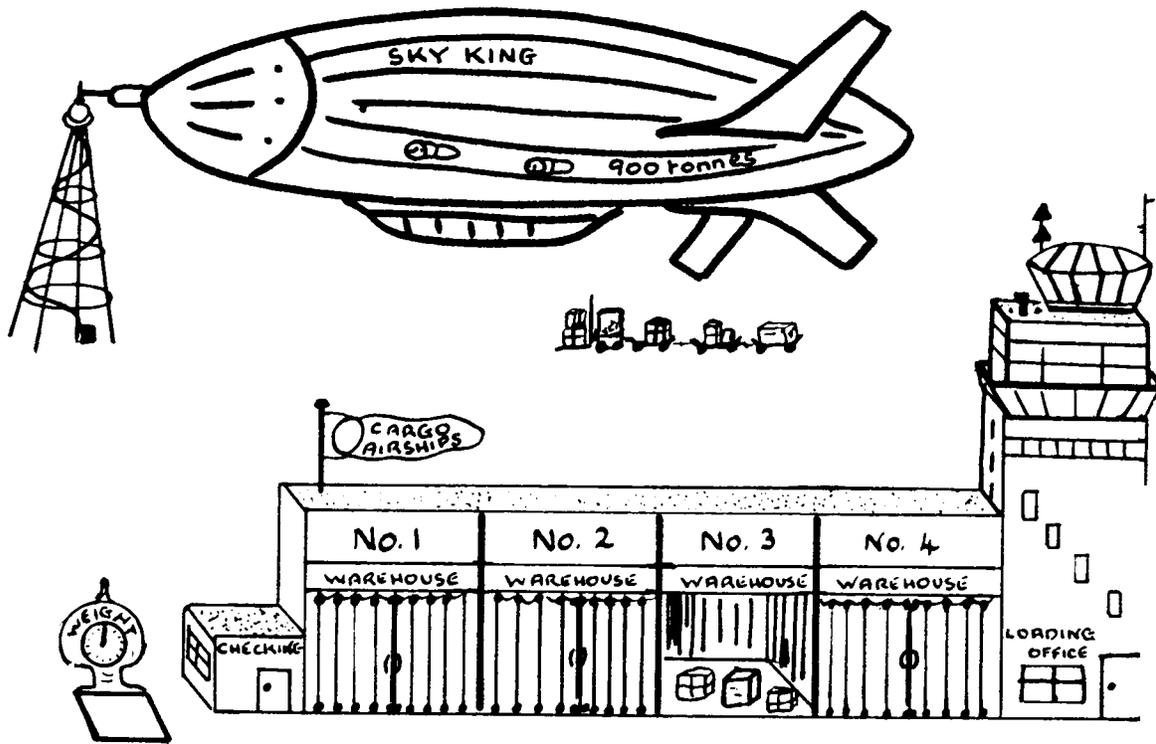


Figure 16 Cargo airships

Discussion of activities Activities 1 and 2 are for extending children's calculating skills to larger dividends, still with single figure divisors. These would conventionally be done by children working singly, using pencil and paper and very likely the conventional notation.

The thinking behind the present approach, in which three children work together using a shared calculation board, is as follows. By making it a social activity, it becomes more interesting and introduces discussion. Each child sees and checks the accuracy of the calculations of the others. In Activity 1, by having 3 children each say aloud, "I'm thinking in . . .," they emphasize the changes in thinking between columns. This is important preparation for passing down the remainders in Activity 2. Both the headed columns notation, and saying aloud what they are doing, are being used to retain awareness of what is happening at each stage, so that when this method is eventually replaced by the conventional (and highly condensed) notation, children will, we hope, have established their understanding of the thinking processes which it represents.

I have introduced the method with HTOnes from the start, rather than TOnes, because it is no harder and gives two examples of passing on the remainder each time rather than one. But if you prefer to begin with TOnes, by all means do so.

The method of calculating used in this topic is a hybrid. It is much easier to think of

$$\begin{array}{r} 3 \\ 2 \overline{)6} \end{array}$$

as “How many 2’s in 6? Answer 3” than as “Start with 6, share between 2, 3 in each share”; so it is presented the first way. But if we were working with physical materials, we would do it by sharing first the hundreds, then the tens, then the singles. (The reason for not using physical materials here is that very large quantities would be needed. With a divisor (say) 8, there could be a remainder of 7 hundred-cubes to be exchanged for 70 ten-rods.) And as already noted, “2 into 6 goes 3 times” does not describe what is being done – it is inaccurate and confusing.

My justification for this hybrid is that once we have established grouping and sharing as mathematically equivalent (topic 3), they become interchangeable for purposes of calculation. (We think in the same way when we know that 3 tens and 30 ones are equivalent, and therefore interchangeable for the purposes of calculation.) So when we are concerned with the mathematical operation of division, we are justified in using whichever of the two varieties best serves our thinking.

Activities 3 and 4 are games which give plenty of practice in these new methods. Activity 3 (Q and R ladders) involves only division of tens and ones; Activity 4, division of hundreds, tens and ones.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Cargo Airships [Num 6.8/4]

Num 6.9 DIVISION BY CALCULATOR

Concept The calculator as a tool by which we can divide, whatever the numbers, easily and quickly.

Abilities (i) To use a calculator correctly.
(ii) To interpret the results.
(iii) To deal appropriately with any decimal fractions which appear in the results.

Discussion of concepts The widespread availability of inexpensive calculators has a number of advantages for school mathematics. By reducing the amount of time which has to be spent on calculating, time is released which can be spent more usefully in other ways. They also allow the use of numbers which arise naturally: there is no longer any need to use artificially simplified numbers which ‘work out nicely.’

In general, I think that children should continue to learn the skills of addition, subtraction, multiplication, and division. This will be obvious from the amount of space given to these in the networks. But I cannot see any good reason why children should continue to learn long division. No new mathematical understanding results, and compared with the time and effort involved the benefits are negligible.

What I do see as important is that calculators should be used with understanding, and this is the aim of this topic.

Decimal fractions are dealt with in the Fractions network, which is where they belong. Rounding also comes there, with the help of the a number line. Both of these topics should be done before the present one.

Activity 1 **Number targets: division by calculator** [Num 6.9/1]

A game for up to 6 children, though a smaller number is better. Its purpose is to develop further the relationships between dividend, divisor, and quotient, with more difficult numbers. The use of a calculator frees the mind from the distraction of laborious calculations, and allows full attention to be given to these relationships. This is also a convenient time to introduce the terms ‘dividend’ and ‘divisor,’ if this has not been done already.

Materials

- Calculator.
- Three 0-9 dice.
- Paper and pencil.

Rules of the game

1. The three dice are thrown to give a 3-digit number. This is written on the paper as the dividend.
2. Two dice are then thrown to give a 2-digit number. This is written as the target.
E.g.

Dividend	Target
747	64

Num 6.9 Division by calculator (cont.)

3. The aim is to find a number which, when used as divisor, gives the target number as quotient when the calculator result is rounded to the nearest whole number. Each player in turn writes his attempt, and passes paper, pencil, and calculator to the next player. Example:

Dividend	Target
747	64

$$747 \div 12 = 62.25$$

(Good for a first trial.)

$$747 \div 13 = 57.461539$$

(This player changed the divisor in the wrong direction).

$$747 \div 11 = 67.909091$$

(This player learned from the other's mistake).

$$747 \div 11.5 = 64.956522$$

(Nearly there, but this rounds to 65).

$$747 \div 11.6 = 64.396552$$

(He wins this round.)

4. Another round may now be played, repeating steps 1 to 3. The winner of the last round starts, since the first attempt is the most difficult.

Discussion of activities Activity 1 uses a calculator to test predictions. It is quite a difficult activity, requiring children to take into account both the direction in which the divisor needs to be changed from the last trial, and also what is a likely amount. This seems to me a much better direction in which to develop their understanding than learning the laborious, time-consuming, and no longer necessary procedure of long division.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Number targets: division by calculator [Num 6.9/1]

[Num 7] FRACTIONS

Double operations

Numbers which represent these Fractions as quotients

Num 7.1 MAKING EQUAL PARTS

- Concepts*
- (i) The whole of an object.
 - (ii) Part of an object.
 - (iii) Equal parts and their names.

Abilities To make, name, and recognize wholes, halves, third-parts, fourth-parts, fifth-parts, etc., of a variety of objects.

Discussion of concepts Many children have difficulty with fractions, and I think there are several causes which continue to bring this about.

- (i) Fractions *are* difficult. Work with fractions is begun too early, and taken too far for children of elementary school age.
- (ii) The same notation is used with three distinct meanings. For example, $\frac{2}{3}$ can mean a fraction, or a fractional number, or a quotient.
- (iii) The authors of many text books for children appear to confuse these three meanings, and they pass on their confusion to the children. It is rather like confusing the different physical embodiments of $7 - 3$: take-away, comparison, complement. But in this case the confusion is at a more abstract level. The distinction is not an easy one, and a full discussion of this I would consider as at least high school mathematics.

In the present network I have tried to present only the truth, but for the reasons above not the whole truth. We begin as usual by introducing the concept in several different physical embodiments. The terms ‘third-part,’ ‘fourth-part,’ etc., are used to distinguish fractions from the ordinal numbers third, fourth, etc.; and also as reminders that we are talking about parts *of something*. This is implicit in the word ‘part,’ but to start with we need to say so explicitly. Note that in this topic we are not yet talking about fractions, but about wholes and parts. This is just the first stage of the concept.

Activity 1 Making equal parts [Num 7.1/1]

An activity for up to 6 children. Its purpose is to make a start with the concept of a fraction, using two different physical embodiments.

- Materials*
- SAR boards 1, 2, and 3 for ‘Making equal parts.’ *
 - Plasticine.
 - Blunt knives.
 - Cutting boards.

* See Figure 17. A full-size version of board 1, together with boards 2 and 3, will be found in the photomasters. For board 2, it is helpful also to have a cardboard template the size and shape of a fruit bar, which children can cut around.

Num 7.1 Making equal parts (cont.)

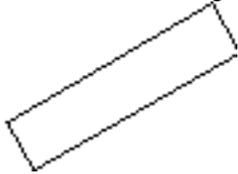
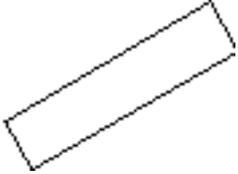
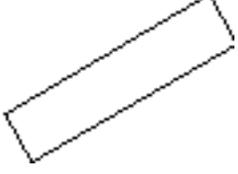
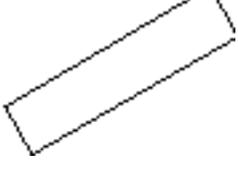
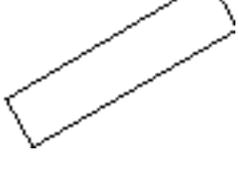
START	ACTION	RESULT	NAME
<p>A Plasticine sausage.</p> 	<p>Leave this one as it is.</p>	<p>(Put it here.)</p>	<p>The <u>whole</u> of a sausage.</p>
<p>A Plasticine sausage.</p> 	<p>Make 2 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>halves</u> of a sausage.</p>
<p>A Plasticine sausage.</p> 	<p>Make 3 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>third-parts</u> of a sausage.</p>
<p>A Plasticine sausage.</p> 	<p>Make 4 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>fourth-parts</u> of a sausage. Also called quarters.</p>
<p>A Plasticine sausage.</p> 	<p>Make 5 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>fifth-parts</u> of a sausage.</p>

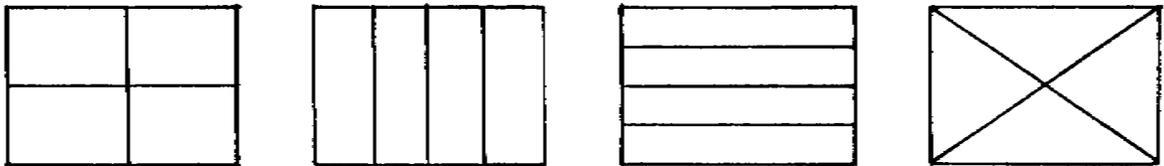
Figure 17 SAR board 1. Making equal parts

What they do (apportioned between the children)

1. Six equal-sized round Plasticine 'sausages' are made, by rolling 6 equal amounts of Plasticine and trimming the ends to fit the sausage shapes on the left of board 1. One sausage is put into each outline.
2. The children do the actions described on the board. The lines of division may be marked lightly before cutting. In this way, trials can be made and corrected by smoothing out the marks.

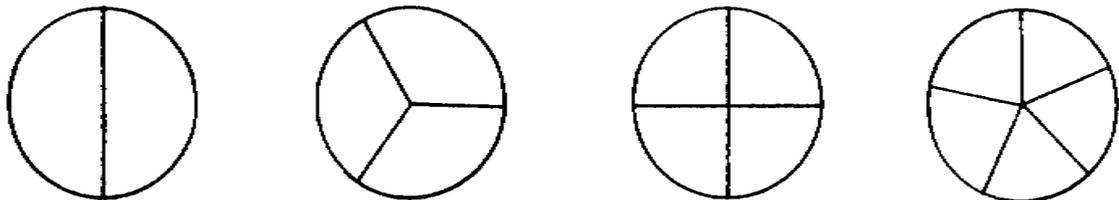


3. After cutting, the separated parts are put in the RESULT column next to their descriptions.
4. Steps 1 to 3 are repeated using SAR board 2 and fresh Plasticine. Board 1 should if possible remain on view. With board 2, a variety of division lines are easily found. E.g., fourth-parts:



At this stage accept any correct results. These possibilities will be explored further in the next activity.

5. Steps 1 to 3 are now repeated using SAR board 3. If possible, fresh Plasticine should be used, the other two boards remaining on view. The lines of division should be radial, as shown below.



Activity 2 Same kind, different shapes [Num 7.1/2]

An activity for up to 6 children. Its purpose is to develop the idea that parts of the same kind may not look alike. In Activity 1, this arose from the use of different objects. Here we see that this can be so even with the same object.

- Materials*
- SAR boards: halves, fourth-parts, third-parts, see Figure 18.*
 - Plasticine.
 - Blunt knives.
 - Cutting boards.

* Figure 18 illustrates the first of the SAR boards for this activity; the others are in the photomasters.

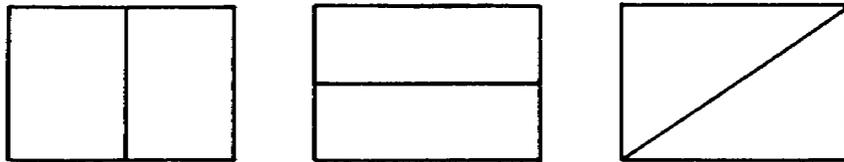
Num 7.1 Making equal parts (cont.)

START	ACTION	RESULT	NAME
A fruit bar. 	Make 2 equal parts.	(Put them here.)	These are <u>halves</u> of a fruit bar.
	Make 2 equal parts in a different way.	(Put them here.)	These are also <u>halves</u> of a fruit bar.
	Make 2 equal parts. in a different way.	(Put them here.)	These are also <u>halves</u> of a fruit bar.

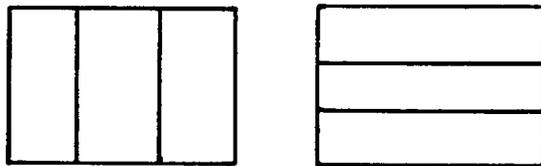
There are three simple ways. Later you may be able to find others.

Figure 18 SAR board 1. Same kind, different shapes. *Halves*

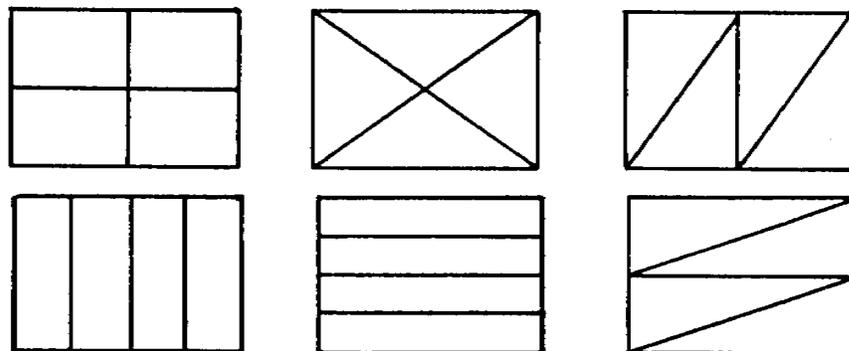
What they do 1. Begin with the halves board. This is used in the same way as the SAR board for Activity 1. The three straightforward ways are:



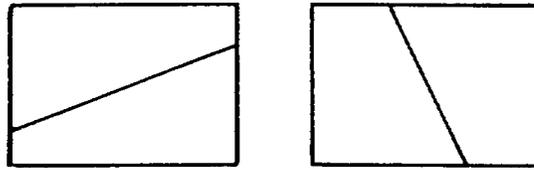
2. Next, they use the third-parts board. This offers only two straightforward ways.



3. Next, they use the fourth-parts board. There are six ways of doing this which are fairly easy to find.



4. Finally, they may like to return to the halves board, and try to find more ways. Here are 2 more, which can be varied indefinitely.



Activity 3 Parts and bits [Num 7.1/3]

A teacher-led discussion for up to 6 children. Its purpose is to emphasize that when we talk about third-parts, fourth-parts, etc., we mean equal parts. To make this clear, we use the word ‘bits’ when these are unequal.

- Materials*
- Plasticine.
 - Blunt knives.
 - Cutting boards.

- Suggested sequence for discussion*
1. Make 2 Plasticine ‘sausages’ of equal size. Cut one into 3 equal parts, the other into three bits. Ask whether both of these have been cut into third-parts.
 2. If necessary, explain the difference, as described above.
 3. Have the children make some more sausages, all of the same size, and cut some into halves, fourth-parts, fifth-parts, and some into two, four, five bits.
 4. These are put centrally together with the parts and bits made in step 1.
 5. The children then ask each other for parts or bits. E.g., “Sally, please give me a fourth-part of a sausage”; or “Mark, please give me a bit of a sausage.”
 6. The others say whether they agree.
 7. Steps 5 and 6 are repeated as necessary.

Activity 4 Sorting parts [Num 7.1/4]

An activity for up to 5 children. Its purpose is to consolidate the concepts formed in Activities 1 and 2, moving on to a pictorial representation.

- Materials*
- Parts pack of cards.*
 - 5 name cards.**
 - 5 set loops.

* Some of these are illustrated in Figure 19. The complete pack is in the photomasters.

** These are marked WHOLES, HALVES, THIRD-PARTS, FOURTH-PARTS, FIFTH-PARTS.

- What they do*
1. Begin by looking at some of the cards together. Explain that these represent the objects which they made from Plasticine, in the last activity – sausages, fruit bars, cookies; and some new ones. They also represent the parts into which the objects have been cut, e.g., third-parts, fourth-parts, halves, fifth-parts. Some have not been cut: these are wholes.

Num 7.1 Making equal parts (cont.)

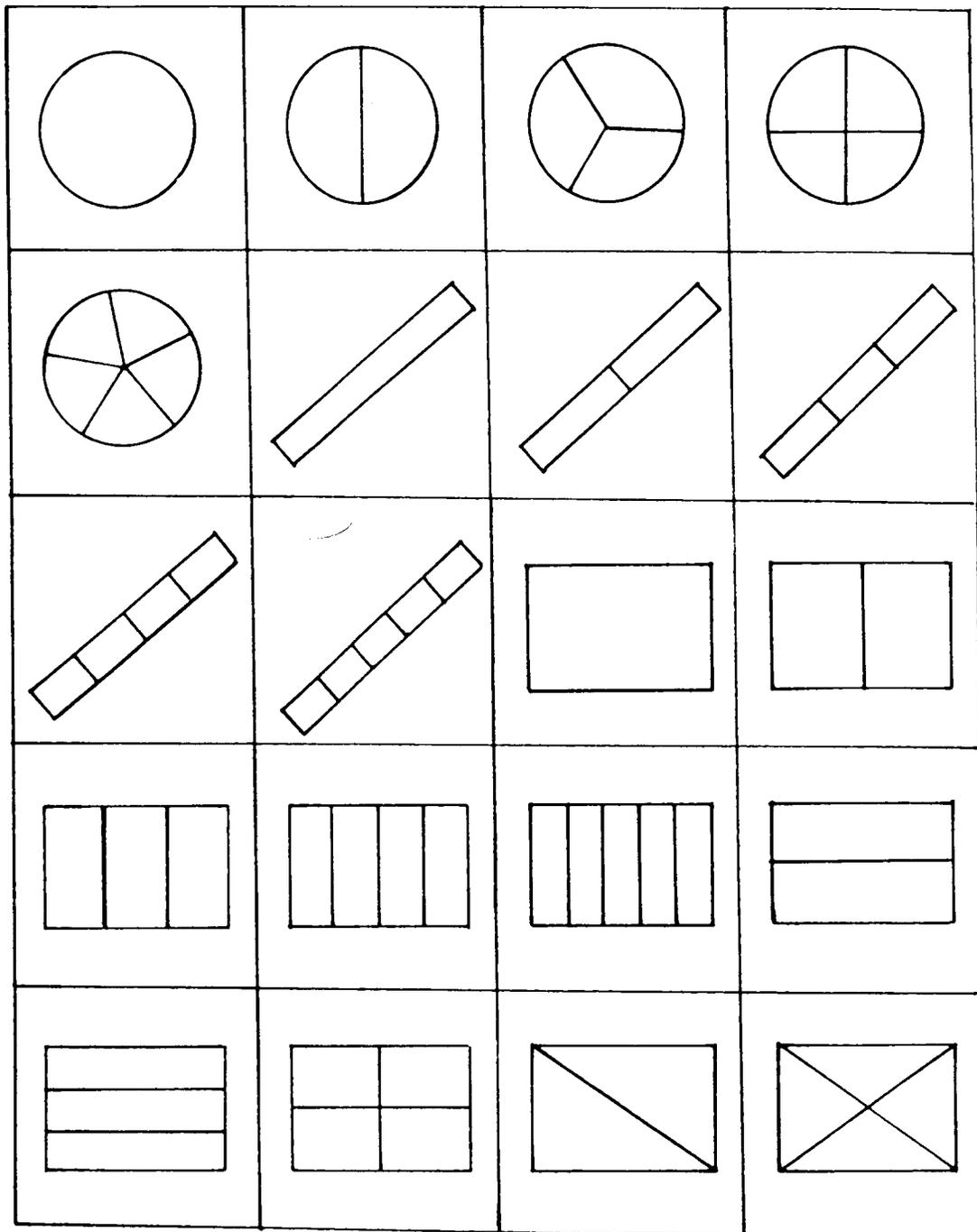


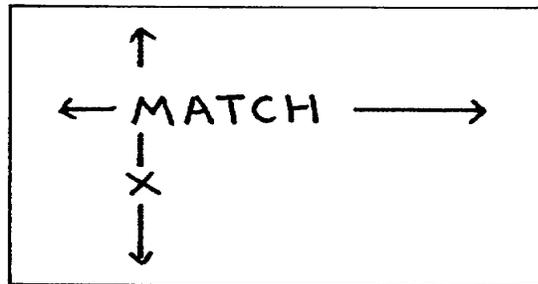
Figure 19 Cards for Sorting parts and Match and mix: parts

2. The parts pack is then shuffled and spread out face upwards on the table.
3. The name cards are put face down and each child takes one.
4. Each child puts in front of her a set loop with the name card she has taken inside, face up.
5. They all then collect cards of one kind, according to the name card they have taken.
6. They check each other's sets, and discuss, if necessary.
7. If there are fewer than 6 children, they may then work together to sort the remaining cards.
8. Steps 2 to 7 may be repeated, children collecting a different set from before.

Activity 5 Match and mix: parts [Num 7.1/5]

A game for 2 to 5 players. Its purpose is to consolidate further the concepts formed in Activity 1.

- Materials*
- Parts pack of cards.*
 - A card as illustrated below (provided in the photomasters).
- *The same as for Activity 4.



- Rules of the game*
1. The cards are spread out face downwards in the middle of the table.
 2. The MATCH and MIX card is put wherever convenient.
 3. Each player takes 5 cards (if 2 players only, they take 7 cards each). Alternatively the cards may be dealt in the usual way.
 4. They collect their cards in a pile face downwards.
 5. Players in turn look at the top card in their piles, and put cards down next to cards already there (after the first) according to the following three rules.
 - (i) Cards must match or be different, according as they are put next to each other in the 'match' or 'mix' directions.
 - (ii) Not more than 3 cards may be put together in either direction.
 - (iii) There may not be two 3's next to each other.
 ('Match' or 'mix' refers to the kind of part.)

Examples (using A, B, C . . . for different kinds of part).
A typical arrangement.

```

C
AAA
BBB
DDD
B
A
    
```

Num 7.1 Making equal parts (cont.)

None of these is allowed.

A	BBBB	CCC	AA	A
B		DDD	BB	B
C			CC	A
D				

5. A card which cannot be played is replaced at the bottom of the pile.
6. Scoring is as follows.
 - 1 point for completing a row or column of three.
 - 2 points for putting down a card which simultaneously matches one way and mixes the other way.
 - 2 points for being the first 'out.'
 - 1 point for being the second 'out.'
 - (So it is possible to score up to 6 points in a single turn.)
7. Play continues until all players have put down all their cards.
8. Another round is then played.

Discussion of activities

Activities 1, 2 and 3 make use of Mode 1 schema building (physical experience). This is every bit as important when introducing older children to fractions as physical sorting is when introducing young children to the concept of a set. In the present case, it is the physical action of cutting up into equal parts from which we want the children to abstract the mathematical operation of division. In this case, the operand is a whole object, so it is important to have something which can easily be cut up. Plasticine is ideal for this, since children can first make the right shapes and then cut them up.

Activities 4 and 5 move on to pictorial representation of these physical actions. This is where most text books begin. What is not understood is that the diagrams condense no less than six ideas, as will be seen in Num 7.3. These diagrams are likely to be helpful if and only if the children already have the right foundation schema to which these can be assimilated. Otherwise, here is the first place where children can get confused.

So in this topic, we begin with physical actions on objects: making equal parts. We then introduce diagrams representing these and no more. We are not yet into fractions; just objects, and parts of objects. The key ideas are that the parts must all be equal in size (or we call them 'bits'); and that the kind of parts we are talking about depends only on how many the object is cut up into, not on their shape, nor on what the object is. These ideas are encountered first with physical objects, then with diagrams.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 7.2 TAKE A NUMBER OF LIKE PARTS

Concept That of a number of like parts.

Abilities (i) To put together any required number of any required kind of part.
(ii) To recognize and name any such combination.

Discussion of concept This is the next step towards the concept of a fraction. It is much more straightforward than that of topic 1, which entailed (i) separating a single object into part-objects (ii) of a given number (iii) all of the same amount. Here we only have to put together a given number of these parts.

Activity 1 Feeding the animals [Num 7.2/1]

An activity for up to 6 children working in 2 teams. Its purpose is to introduce the concept described above.

Materials

- 3 sets of animals.*
- Menu for each set, on separate cards.*
- Cards for each menu showing standard sizes of eel, meat slab, biscuit. (One between 2 children.) **
- 5 food trays.
- Plasticine.
- Blunt knives.
- Cutting boards.
- 5 set loops.

* As listed below and provided in the photomasters. The animals may be models, or pictures on cards. Quite a lot of these are needed; sorry, but if they help children to like fractions it's worth it!

** A template for the meat slab is useful. See Num 7.1/1 under '*Materials*.'

SET 1

Animals 2 bears, 3 walruses, 5 seals, 6 penguins, 8 otters.

Menu

These all eat fish.

Today, the menu is eels (beheaded and tailed).

Each bear gets the whole of an eel.

Each walrus gets half of an eel.

Each seal gets a third-part of an eel.

Each penguin gets a fourth-part of an eel.

Each otter gets a fifth-part of an eel.

Num 7.2 Take a number of like parts (cont.)

SET 2

Animals 2 lions, 2 cheetahs, 4 wolves, 4 hyenas, 6 wild cats.

Menu

These all eat raw meat, supplied in large slabs.

Each lion gets the whole of a slab.

Each cheetah gets half of a slab.

Each wolf gets a third-part of a slab.

Each hyena gets a fourth-part of a slab.

Each wild cat gets a fifth-part of a slab.

SET 3

Animals 3 rats, 3 voles, 5 shrews, 5 house mice, 7 harvest mice.

Menu

These all eat biscuits.

Each rat gets the whole of a biscuit.

Each vole gets half of a biscuit.

Each shrew gets a third-part of a biscuit.

Each house mouse gets a fourth-part of a biscuit.

Each harvest mouse gets a fifth-part of a biscuit.

- What they do*
1. One team acts as animal keepers, the other works in the zoo kitchen. The latter need to be more numerous, since there is more work for them to do.
 2. A set of animals is chosen. Suppose that this is Set 1. The kitchen staff look at the menu and set to work, preparing eels, as in Num 7.1/1. The animal keepers put the animals in their enclosures (segregated, not assorted). They may choose how many of each.
 3. The animal keepers, one at a time, come to the kitchen and ask for food for each animal in turn. The kitchen staff cut the eels as required. E.g.,

Animal keeper

“Food for 3 walruses, please.”

“Food for 4 otters, please.”

“Food for 5 seals, please.”

“Food for 2 bears, please.”

“Food for 3 penguins, please.”

Zoo kitchen staff

“Here it is: 3 halves of an eel.”

“4 fifth-parts. Tell them not to leave any scraps.”

“Here it is: 5 third-parts of an eel.”

“Here you are : 2 whole eels.”

“3 fourth-parts. Lucky penguins.”

4. Each time, the animals' keeper checks that the amounts are correct, and then gives its ration to each animal. The keepers also check each other.
5. When feeding time is over, the food is returned to the kitchen for reprocessing. Steps 1 to 4 are then repeated with different animals, keepers, and kitchen staff.
6. Note that the eels, after their head and tails are removed, resemble the sausages of Num 7.1/1; and the slabs of meat are oblongs. Note also that the eels, slabs, and biscuits should be of standard sizes.

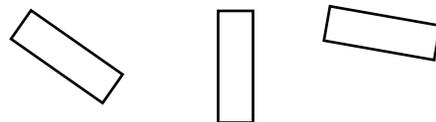
Activity 2 Trainee keepers, qualified keepers [Num 7.2/2]

An activity for up to 6 children. Its purpose is to consolidate their recognition of the combination: number of parts and kind of parts.

Materials • As for Activity 1, without the animals and set loops.

What they do *Stage (a) Trainee keepers*

1. The children are acting as trainee keepers. In this part of their course, they are learning to ensure that they can recognize the right food for all their animals.
2. Initially they work with one menu at a time. A standard card (eel, meat slab, or biscuit) is put at the top of the menu.
3. Each of them then makes up a food tray for a given number of animals of a particular kind.
4. In turn, one of the food trays is put in the middle and the others say what kind of animal it is for, and how many. E.g.,



This is for 3 penguins.

5. It is difficult to judge the difference in size between, e.g., fourth-parts and fifth-parts. So the unused parts should be left on the cutting board, with any remaining Plasticine cleared away. Thus in the example above, the remaining fourth-part should be left on the cutting board.
6. As they become more expert, they work with 2, then 3 menus.

Stage (b) Qualified keepers

1. When one of them thinks he is ready to take his test, the others prepare 5 trays including at least 1 from each menu.
2. If the one taking his test can identify all of these correctly, he qualifies. (A pass mark of 100% may seem severe, but every animal has to be correctly fed.)

Activity 3 Head keepers [Num 7.2/3]

An activity for up to six children. Its purpose is for children to write the two operations which together go to make a fraction in their own words, as preparation for the standard notation which they will learn in the next topic.

Materials For each child:

- The three menus already used, on one sheet of paper.
- Lists of animals to be fed (see below).
- Pencil and paper for each child.

- What they do*
1. A head keeper has to be able to write clear instructions for feeding any of the animals.
 2. Those who wish to be considered for this job are each given one list of animals to be fed. Specimen lists are given below.
 3. They write, in their own words, instructions for preparing a food tray for each cage of animals on their list.
 4. When they have finished, all but one of the children act as examiners. The remaining one reads out the questions, and his answers. The others decide whether these are both clear and accurate.
 5. If all the answers meet these requirements, the candidate is eligible for head keeper when there is a vacancy.

Write clear and accurate instructions telling someone how to prepare a food tray for each of the following cages of animals:

LIST A	LIST B	LIST C
3 house mice	2 seals	1 vole
2 rats	2 otters	3 harvest mice
2 cheetahs	3 house mice	4 wolves
6 hyenas	4 voles	2 hyenas
1 seal	7 wild cats	3 bears
4 penguins	3 lions	5 walruses

LIST D	LIST E	LIST F
3 wildcats	3 bears	3 cheetahs
4 lions	1 penguin	2 hyenas
1 shrew	2 wolves	5 walruses
5 voles	8 wild cats	2 otters
1 bear	3 voles	5 rats
2 penguins	5 shrews	4 shrews

Discussion of activities In this topic we bring in the second contributor to the concept of fraction, returning for this to the same physical embodiments as in Num 7.1/1. As in the previous activity, these physical embodiments are of two kinds. The Plasticine is of the first kind: a physical operand, in different shapes representing different imaginary operands. The actions of making a given number of equal parts, and taking a given number of these, are of the second kind. It is the second kind which leads to the mathematical concept of a fraction, independently of what the operand is.

However, a fraction has to have some operand. So in step 3 of Activity 1, I have kept in the words ‘... of an eel’ (etc.) to help us all to remember this. (You can skip the rest of this paragraph if you like.) This leads to a slight problem of syntax. Should ‘eel’ be singular or plural? After thought, I have settled for singular. ‘Third-part of an eel’ leads to ‘5 third-parts of an eel.’ It is true that they won’t all come from the same eel – but each part will come from an eel, not several.

In Activity 1, the fractional parts were the end-point of the activity, leading to the concept. In Activity 2, they are the starting point. Children have to use their newly formed concepts to recognize what number of what kind of parts they are looking at. Their decisions are confirmed or otherwise by the rest of the group (Mode 2 testing).

Activity 3 requires them to think about and describe the physical actions of making equal parts, and then taking a given number of these, instead of actually doing these actions. This takes them a step further towards fractions as mental operations.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Head keepers [Num 7.2/3]

Num 7.3 FRACTIONS AS A DOUBLE OPERATION; NOTATION

- Concepts*
- (i) A fraction as a mathematical operation which corresponds to two actions: making a given number of equal parts, and putting together a given number of these.
 - (ii) A notation which represents this double operation.

- Abilities*
- (i) To match this operation (as represented by its notation) with a physical embodiment.
 - (ii) To match this operation (as represented by its notation) with a diagram for the same fraction.

Discussion of concepts As soon as we are dealing with mathematical operations, which are purely mental, we require a notation by which to communicate, manipulate, and record these. So the introduction of a notation, and the transition to the concept of a fraction as a combined operation, come together.

The traditional notation is a good one,

$$\frac{1}{2}$$

how many parts
of a given kind (numerator)

what kind of
parts (denominator)

What kind of parts is determined by how many equal parts are made from a whole, so a number is sufficient to specify what kind of part. This is implied by the way we read it, “two thirds”; or as I think we should continue to say at this stage, “two third-parts.”

As has been said already, ‘part’ implies part *of* something. It implies an operand. There are now two further points to be made.

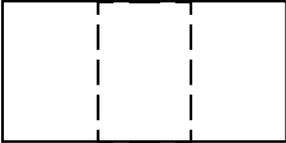
- (i) In the present treatment of fractions, the numerator is not the operand, and $\frac{2}{3}$ does not mean the same as $2 \div 3$. (The latter meaning is also correct, but to introduce it at this stage complicates the concept further, and I am trying to simplify.)
- (ii) The whole is not the operand either. The operand is what this whole is whole of.

The traditional way of representing fractions by diagrams is also good, but very condensed.

This diagram
for $\frac{2}{3}$



simultaneously represents no less than six ideas:

the object we start with (the operand);	
the number of cuts we make, and where we make them (cuts as actions);	
the cuts we have made (cuts as results), and the parts which result;	
and the number of parts which we are taking.	

So it is small wonder that children have problems when their learning of fractions *begins* with these diagrams.

In topic 1, these diagrams were introduced after children had met with these ideas separately, in physical form. Here we shall renew the connection, and link it with the notation.

Activity 1 Expanding the diagram [Num 7.3/1]

An activity for up to six children, working singly or in twos or threes. Its purpose is to relate the conventional fraction diagrams to the six concepts which they combine.

- Materials*
- SARAR board, see Figure 20.
 - Pack of fraction diagram cards, see Figure 21.*
 - Reminder card (as illustrated below).*
 - Plasticine.
 - Knives.

* Figure 21 shows two specimen cards. A full pack is provided in the *Volume 2a* photomasters. You can invent more, if further practice is needed.

REMINDER CARD FOR EXPANDING THE DIAGRAM	
The diagram shows:-	
the object we start with	
<i>first action</i>	the cuts we make
<i>first result</i>	the parts we make
<i>second action</i>	the parts we take
<i>combined result</i>	a fraction of an object

Num 7.3 Fractions as a double operation; notation (cont.)

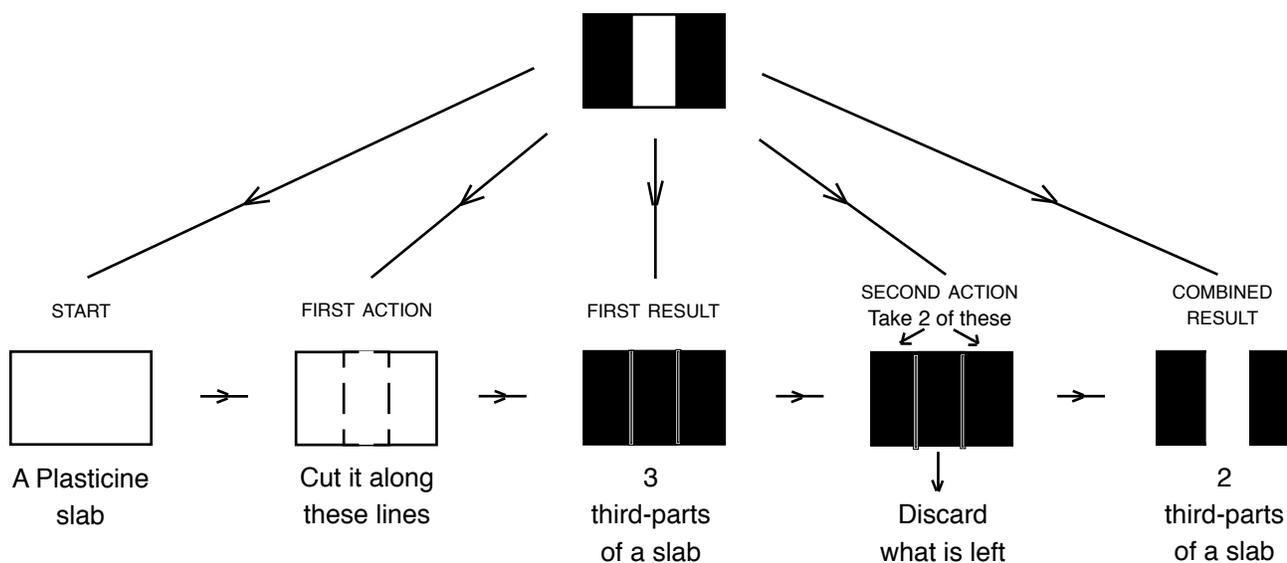


Figure 20 SARAR board for 'Expanding the diagram'

- What they do*
1. The SARAR board is put where all can see it. To begin with, all except the diagram at the top is covered
 2. Explain that this diagram shows the result of two actions combined. This combination is called a fraction.
 3. Uncover the rest of the board, and help them to work through the 5 steps shown. Help them to see how each step is embedded in the combined diagram. Leave this on view with the final result in Plasticine still in position.
 4. Choose a suitable fraction diagram card. Help them to work through the steps which it represents, using the reminder card as a guide. The Plasticine objects are put in the blank half of the card. Repeat this step if necessary.
 5. When they are ready to continue, each pair or trio takes a fraction card from the set. They may choose cards they think they can do. Note that the long thin oblong may be modelled as a sausage, and the torus as a ring doughnut. The children may use either description.
 6. For each card, they work through the steps which this represents, agreeing each step with each other.
 7. When finished, they agree what the result is called. (E.g., 2 fifth-parts of a ring doughnut.)
 8. Steps 5, 6, 7 are repeated until they are confident that they can interpret every part of the meaning of a fraction diagram.

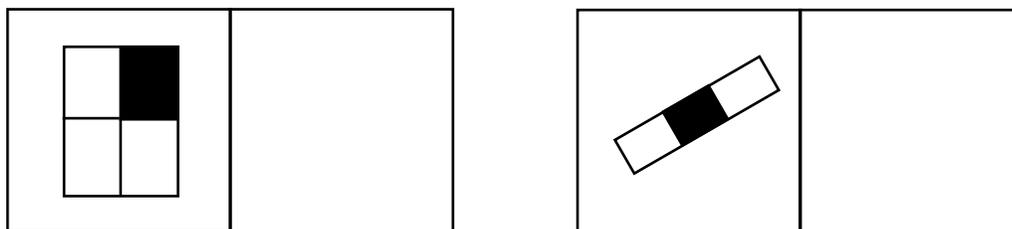


Figure 21 Specimen fraction diagram cards

Activity 2 “Please may I have?” (Diagrams and notation) [Num 7.3/2]

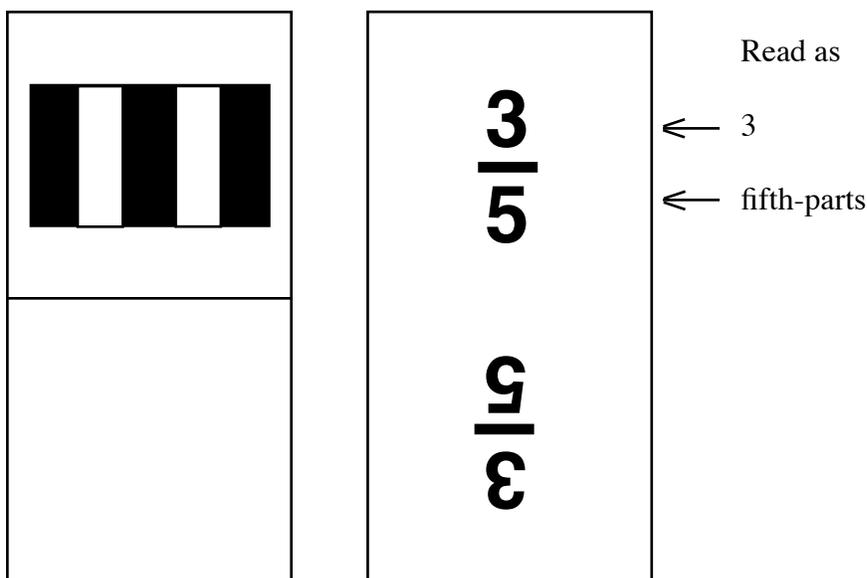
A game for 5 or 6 players. Its purpose is to link the diagrams with the conventional notation.

- Materials*
- Set of fraction diagram cards.*
 - Set of fraction notation cards.**

* As used for Activity 1.

** Illustrated in step 2 below and provided in the photomasters.

- Rules of the game*
1. All the cards are dealt to the players, each pack separately. If there are 5 players, each will thus get 3 cards of each sort. If there are 6 players, each will receive 2 cards of one sort and 3 of the other.
 2. The purpose is to get rid of one’s cards by making pairs. Each pair must consist of a notation card and a diagram card. E.g.,



3. After the deal, the players first put down any pairs which they already hold.
4. They then try to make more pairs by asking each other, taking turns. E.g., “Please, Susan, may I have a diagram for two third-parts?” Or, “. . . a notation for two third-parts?”
5. The winner is the player who has most pairs when all have played out their hands.

Discussion of activities Because so many children (and adults) have difficulty with fractions, I think that it is important to continue with physical embodiments of both the actions and the operands in order to give plenty of Mode 1 learning to establish a good foundation. This is particularly important when, as in this case, diagrams are used which themselves are highly condensed. So in Activity 1 we return to physical actions on physical operands.

The notation, which represents the same concept as a fraction diagram, must therefore inevitably be more condensed even than the diagram. So the children need – and I hope by now will have had – plenty of experience of the separate operations, before they learn the conventional notation.

In Activity 2, the words ‘of . . .’ have been dropped for two reasons. The same operator can act on a variety of operands, so we have now begun to treat fractions independently of any particular embodiment.

OBSERVE AND LISTEN

REFLECT

DISCUSS



“Please may I have?” (Diagrams and notation) [Num 7.3/2]

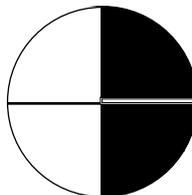
Num 7.4 SIMPLE EQUIVALENT FRACTIONS

Concept That different fractions of the same operand may give the same amount.

Ability To recognize when this is the case, i.e., to match equivalent fractions.

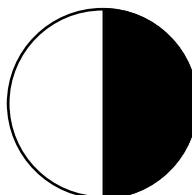
Discussion of concept

2 fourth-parts
of a biscuit



and

1 half
of a biscuit



are not identical, but we get the same amount to eat (assuming no crumbs). So we say that $\frac{2}{4}$ and $\frac{1}{2}$ are equivalent fractions, meaning that if applied to the same operand they result in the same amount: though these may not look exactly alike.

The idea of equivalence is an important one throughout mathematics. (For a fuller discussion, see chapter 10 of *The Psychology of Learning Mathematics*). It is important in this network because it makes the bridge between fractions as double operations, and fractions as representing numbers.

It is the idea which is important, and in this topic I have therefore kept to simple examples.

Activity 1 “Will this do instead?” [Num 7.4/1]

An activity for up to 6 children. Its purpose is to introduce the concept of equivalent fractions. It also introduces eighth-parts.

- Materials*
- Plasticine.
 - 8 card templates for fruit bars (See *SAIL Volume 2a*, Photomaster 156).
 - Knife.
 - Board to cut on.
 - A small rolling pin is also useful.

- What they do*
1. One of the children acts as the owner of a candy store. He sells very good fruit bars.
 2. Before the store opens, he makes a number of these, each the size of one of the cards. The cards are used as supports.
(Initially you may find it better to do this yourself. It helps to prevent sticking if the cutting board and rolling pin are wetted).

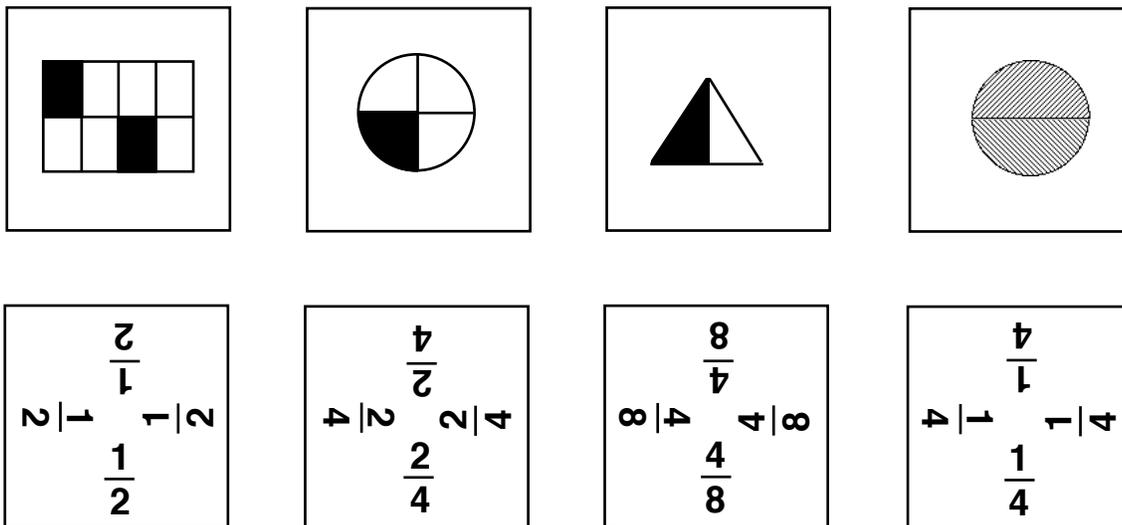
Num 7.4 Simple equivalent fractions (cont.)

3. Fruit bars are expensive, and pennies are scarce. So the store owner, a kind old man, cuts them up so that children can buy them in parts. Two he cuts into halves, two into fourth-parts, and two into eighth-parts. (This leaves two spare cards.)
4. He is also eccentric, and tries never to give exactly what is asked for.
5. When the store opens, a customer comes in and asks for (e.g.) half of a fruit bar. The store owner takes 2 fourth-parts, or 4 eighth-parts, and puts these on a spare card. He offers these to the customer, saying “Will these do instead?”
6. The customer says (in this case) “2 fourth parts – that will do nicely, thank you.” (Or “4 eighth-parts . . . ,” as the case may be.)
7. Another customer enters the store, and steps 5 and 6 are repeated.
8. The stock is replenished when necessary.
9. Sometimes the store owner makes a mistake. (If this does not arise naturally, he does so ‘accidentally on purpose.’) In this case the customer politely explains.
10. The game may be extended by the inclusion of other goodies, of various shapes.

Activity 2 Sorting equivalent fractions [Num 7.4/2]

An activity for up to 6 children. Its purpose is to expand the concept of equivalent fractions to diagrams and notation.

- Materials*
- Equivalent fractions pack 1 (see specimens below and the photomasters).
 - 3 set loops



- What they do*
1. The fraction pack is shuffled and spread out face upwards on the table.
 2. Children work together in ones or twos, according to how many there are. Each child (or pair) takes a set loop.
 3. To begin with, the notation card for a whole, 1 half, and 1 fourth-part are found. One is put in each set loop.
 4. They then collect all fractions which are equivalent to this, whether diagrams or notation.
 5. They check each others' sets, and discuss if necessary.
 6. As a continuation activity, each equivalence set may be sorted into subsets. E.g., within the set of fractions equivalent to one half, one subset would be all the diagrams representing the fraction 2 fourth-parts, together with its notation.

Activity 3 Match and mix: equivalent fractions [Num 7.4/3]

An activity for 2 to 5 players. Its purpose is to give further practice in recognizing equivalent fractions.

- Materials*
- Equivalent fractions pack 1.*
 - Match and mix card.**

* The same as for Activity 2.

** Like that for Num 7.1/5.

Rules of the game This is played in the same way as Match and mix: parts (Num 7.1/5). Here, ‘match,’ or ‘mix’ refers to equivalence of fractions, as represented by diagrams and notation.

Discussion of activities In Activity 1, the physical embodiments provide a good Mode 1 refresher of earlier work, and foundation for the concept of equivalent fractions. And the imagined situation embodies one of the meanings of equivalent (equi-valent: equal in value). Equivalent fractions of a fruit bar contain the same amount to eat, and one worth the same amount of money.

You will notice the omission of third-parts. While these fit well into the schema as so far developed, I see no object in children’s learning about sixth-parts just to provide something two of which are equivalent to third-parts. And looking forward to the equivalent decimal fractions, I have little enthusiasm for teaching children recurring decimals just in order to keep going with third-parts. Eighth-parts, on the other hand, form part of the sequence formed by successive halving and are still with us in everyday life. Both the rulers on my desk as I write still have inches along one edge, and these are subdivided into halves, quarters, eighth-parts, and sixteenth-parts.

In Activity 2, they extend the new concept to the more abstract representations of fractions provided by diagrams and notation. They have already learned the connections between these, in Num 7.3/2. Activity 3 consolidates these in a game which they have already learned to play, with simpler materials.

I would not encourage teaching the cancelling rule, which is a convenient technique but does not relate clearly to the concepts. Children will themselves notice the inverse, that equivalent fractions may be obtained by doubling numerators and denominator. This corresponds to cutting every part in two.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 7.5 DECIMAL FRACTIONS AND EQUIVALENTS

- Concepts*
- (i) Decimal fractions.
 - (ii) Equivalence between decimal fractions and fractions of other denominators.

- Abilities*
- (i) To find decimal fractions of a variety of operands.
 - (ii) To match decimal fractions with other equivalent fractions.

Discussion of concepts The common usage of the term ‘decimals’ confuses (i) a particular kind of fractions, and (ii) place-value notation. Unless we separate these two ideas, we would have to say that 0.5 is a decimal, while $5/10$ is not. So I hope that you will accept my proposal, that we use the term ‘decimal’ to mean what it says, i.e., related to 10 (as in ‘decimal coinage’); and ‘decimal fraction’ to mean a fraction whose denominator is a power of 10 (10, 100, 1000 . . .), whichever way it is written. If this is agreed, then the term ‘common fraction’ (or ‘vulgar fraction’) is no longer a useful one.

Decimal fractions have the great advantage that they can be represented in two notations: bar notation (the one used in this network up to now), and place-value notation. The latter offers considerable simplification when it comes to calculations – not least, that we can use a calculator!

In this topic, we shall confine ourselves to the introduction of decimal fractions, and their equivalences with fractions of other denominators. Their representation in place-value notation is a separate topic in itself.

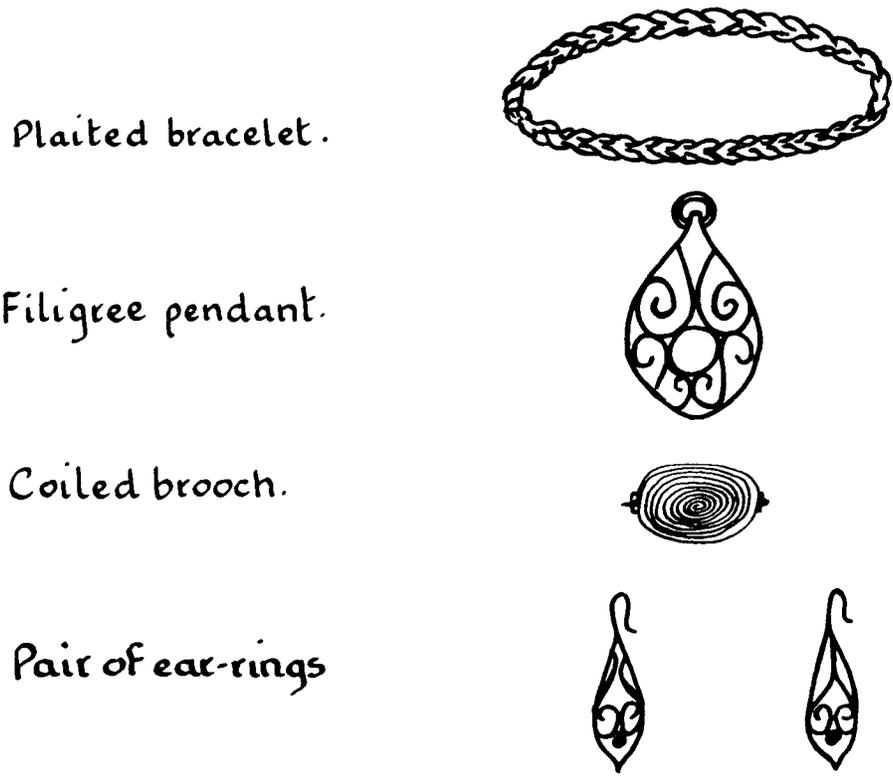
Activity 1 Making jewellery to order [Num 7.5/1]

An activity for up to 6 children. Its purpose is to introduce decimal fractions in verbal form.

- Materials*
- A metre measure, divided in decimetres and centimetres.
 - Jewellery catalogue, see Figure 22.*
 - Purchasing guide, see Figure 23.*

* Also in photomasters

- What they do*
1. (Introductory discussion.) They already know that 100 cm is the same length as 1 m; so they should find it easy to answer the question “What is a hundredth-part of a metre?” Next, they learn that the other length on the metre measure is called a decimetre, because there are ten of these in a metre. So a tenth-part of a metre is called . . .?
 2. Two children take the roles of jeweller and jewellers’ supply merchant. The others in turn act as customers.
 3. The jeweller works at home, making jewellery to order. She has a catalogue to show to the customers, and a purchasing guide for her own use.
 4. The first customer calls, looks at the catalogue, and chooses what she would like. E.g., a coiled brooch.



Plaited bracelet.

Filigree pendant.

Coiled brooch.

Pair of ear-rings

Figure 22 Jewellery catalogue

	1 metre of silver wire makes
Plaited bracelet	2
Filigree pendant	4
Coiled brooch	5
Pair of ear-rings	10

Figure 23 Purchasing guide

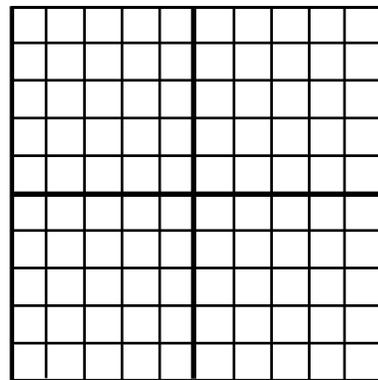
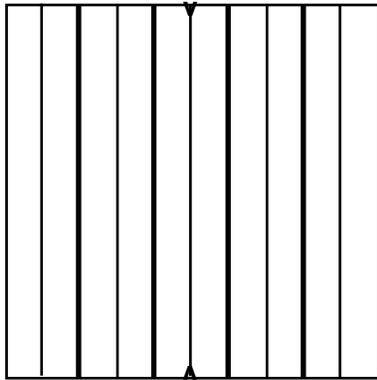
Num 7.5 Decimal fractions and equivalents (cont.)

5. The jeweller looks at her purchasing guide, and sees that a metre of silver wire will make five of these. She only wants enough for one.
6. So she goes to the supplier and asks for one fifth-part of a metre of silver wire.
7. The supplier looks at her metre measure, and says “I’ll measure you 2 decimetres; that is, 2 tenth-parts of a metre.”
8. The jeweller says “I agree,” or if she has any doubts, the matter is discussed with the help of the metre measure.
9. The jeweller returns home, another customer calls, and steps 4 to 8 are repeated.
10. When all the remaining children have acted as customers, two more children take the roles of jeweller and supplier, and steps 4 to 8 are repeated.
11. Note that since a filigree pendant takes a fourth-part of a metre, the supplier in this case says “I’ll measure you 25 centimetres; that is, 25 hundredth-parts of a metre.” For variety, centimetres may be used on other occasions. E.g., for a coiled brooch, the supplier could also say “I’ll measure you 20 centimetres; that is, 20 hundredth parts of a metre.”

Activity 2 Equivalent fraction diagrams (decimal) [Num 7.5/2]

A teacher-led discussion for up to 6 children. Its purposes are (i) to make explicit the concepts of a decimal fraction, introduced in Activity 1, (ii) to extend their use of fraction diagrams to those for decimal fractions, (iii) to use these diagrams for recognizing equivalences between decimal and non-decimal fractions, and also between decimal fractions with different denominators.

- Materials*
- For each child, a pair of equivalent fraction diagrams (decimal), as illustrated below and provided in the photomasters. These are best on separate cards.
 - Something to point with.



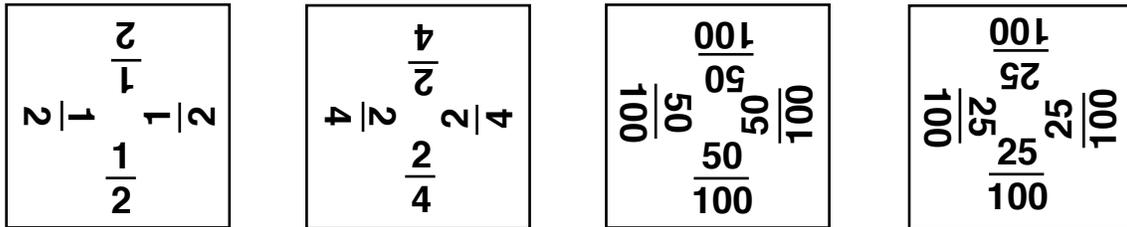
- Suggested sequence for the discussion*
1. Give each child a pair of equivalence diagrams. It doesn’t matter which way up they look at these. Begin by asking them to look at the one shown above on the left.
 2. What kind of parts are represented by the narrowest oblongs? (Tenth-parts.) By the oblongs between the thicker lines? (Fifth-parts.) And by the oblongs marked off by the arrowed line? (Halves.)
 3. What equivalences can we deduce? (1 fifth-part is equivalent to 2 tenth-parts. 2 fifth-parts are equivalent to 4 tenth-parts, etc. And 5 tenth-parts are equivalent to one half.)

4. Next, ask them to look at the other diagram. Discuss this in a similar way. (We can deduce that 1 fourth-part is equivalent to 25 hundredth-parts, etc. And that 1 half is equivalent to 50 hundredth-parts.)
5. Next, tell them to put the two diagrams side by side. What can they say by comparing these? (1 tenth-part is equivalent to 10 hundredth-parts. 1 fifth-part is equivalent to 20 hundredth-parts. Also, 3 tenth-parts are equivalent to 30 hundredth-parts, etc. And 2 fifth-parts are equivalent to 40 hundredth-parts, etc.)
6. Finally, tell them that there is a special name for the kind of fractions they have been talking about: decimal fractions. These include tenth-parts and hundredth-parts. Can they think of another decimal fraction? (What part of a metre is a millimetre? What part of a kilometre is a metre?)

Activity 3 Pair, and explain [Num 7.5/3]

A game for up to 6 children. Its purpose is to relate the equivalence they have learned in Activity 2 to the notation which is common to all kinds of fraction (common fraction notation).

- Materials*
- Fractions, pack 2 (see specimens below).*
 - For each child, a pair of equivalent fraction diagrams (decimal).**
- * A full set is provided in the photomasters
 ** The same as used in Activity 2.



- Rules of the game*
1. The fractions pack is shuffled and put face down on the table. The top card is turned face up and up and put separately.
 2. In turn, each player takes the top card from the pile and puts it face up on the table. The face-up cards are spread out so that all can be seen at the same time.
 3. The object is to form pairs of equivalent fractions. These will gradually show as more cards are turned face-up. Pairs may be made from identical fractions since these too are equivalent.
 4. The player who has just turned a card has first chance to make a pair. She takes these, puts them side by side in front of her, and explains the equivalence in terms of the equivalence diagrams. If the others agree, she keeps the pair. If not, there is discussion.
 5. If a pair is overlooked, the player whose next turn it is may claim it before turning over a card. She then turns over a card as in step 4. If another pair results, she may take this also.
 6. When there are no more cards to turn over the winner is the player who has most pairs.

Note The explanation in step 4 is a very important part of this activity. It must be in terms of the diagrams; not in terms of the cancelling rule, which is a convenient shortcut but makes no contribution to understanding.

Activity 4 Match and mix: equivalent decimal fractions [Num 7.5/4]

A game for 2 to 5 players. Its purpose is to give further practice in recognizing decimal equivalent fractions.

- Materials*
- Fractions pack 2.*
 - Match and mix card.**

* The same as used for Activity 2.*

** The same as used for Num 7.1/5 and Num 7.4/3.

Rules of the game This is played in the same way as ‘Match and mix: parts’ (Num 7.1/5), but ‘Fractions pack 2’ is used and ‘match’ or ‘mix’ refers to the equivalence of decimal fractions in bar notation.

Discussion of activities In this topic we are still using bar notation, for both decimal and non-decimal fractions. (See ‘Discussion of concepts.’)

In Activity 1, the metre, subdivided into tenth-parts, hundredth-parts, and thousandth-parts, provides a familiar and useful physical embodiment of the new concepts to be learned. I have introduced the decimetre here for two reasons. First, because we need it as a representation of the fraction a tenth-part; and second, because it is a useful unit which in my view deserves to be more widely used than it is.

In Activity 2, the new concept is made more explicit by discussion (Mode 2), and related to equivalence diagrams like those already used for non-decimal fractions. No new material is introduced in Activity 3. It consolidates the new concepts by Mode 2 testing. As we all know, there is nothing so good as trying to explain something to others, to make one get it clear in one’s own mind. Activities 3 and 4 work entirely with symbols, and so complete the process by which a concept is detached from any particular physical embodiment.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Match and mix: equivalent decimal fractions [Num 7.5/4]

Num 7.6 DECIMAL FRACTIONS IN PLACE-VALUE NOTATION

Concept Extension of place-value notation to represent decimal fractions.

Abilities (i) To represent, and recognize, decimal fractions in place-value notation.
(ii) To recognize the same fraction written in place-value and in common fraction notation.

Discussion of concept We are very dependent on notations of many kinds not only for communicating and recording our ideas, but as a help to the thinking process itself. Some notations do their job much better than others. Place-value notation is a very good one, so the combination of decimal fractions and place-value notation is a very useful one. With the advent of calculators and computers, I see no reason for children to learn to calculate in bar notation until such time as they are about to learn algebraic fractions.

Note that we are here talking about *equal* fractions.
These

$$\frac{4}{100} \quad \text{and} \quad .04$$

are two notations for the same fraction.

Note Children should already have completed topics NuSp 1.9 (interpolation between points on a number line) and NuSp 1.10 (extrapolation of place-value notation) before beginning the activities which follow.

Activity 1 Reading headed columns in two ways [Num 7.6/1]

An activity for up to 3 children. (They all need to see the activity board the same way up.) Its purpose is to refresh their knowledge of place-value notation for decimal fractions, and give additional practice in pairing the two notations for decimal fractions.

Materials

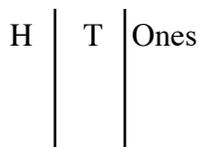
- Activity board.*
- Number cards.**

* See illustration in step 4 following. The full version is in the *Volume 2a* photomasters. Two should be provided, so that up to 6 children can play at a time.

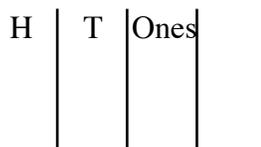
** 1 to 9, 3 cm square, 2 of each (see photomasters).

Num 7.6 Decimal fractions in place-value notation (cont.)

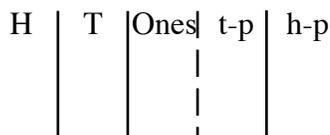
What they do 1. To introduce the activity, show the children the familiar headed columns.



Starting at the ones, each column we go to the left means that the numbers are 10 times bigger. So what would it mean if we had some more columns on the right of the ones column? Like this.

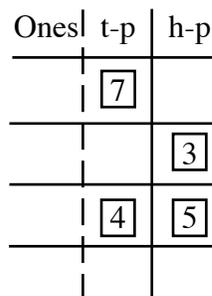


What are ten times smaller than ones? Tenth-parts. And what are ten times smaller than these? Hundredth-parts. To emphasize the boundary between wholes and parts, we will use a dotted line. So we now have headed columns like these, using t-p and h-p for tenth-parts and hundredth-parts. (Note the use of lower case letters for parts of a unit, as in the metric system of abbreviations.)



We can have as many columns as we like, extending to the left and right of the dotted line. For the present activity, three columns are enough. (See step 4 illustration.)

2. The activity board is then put on the table where all the children can see it right way up. The number cards are shuffled and put in a pile (or heap) face down.
3. In turn, the children take number cards from the pile and put them face up where indicated by squares on the activity board. They take one, two, or three cards as needed to fill a line.
4. They then read the resulting fraction in two ways, as in the examples below. (It does not matter which is said first, and it is a good idea to vary this.)



First child: “7 tenth-parts. Zero point 7 zero.”
 Second child: “3 hundredth-parts. Zero point zero 3.”
 Third child: “4 tenth-parts and 5 hundredth-parts. Zero point 4,5.”
 (“Four, five” not “forty-five.”)

(Next time around, they could speak these the other way about.)

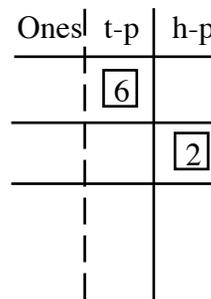
5. At this stage a zero is spoken for every empty column. This is never incorrect. Sometimes it is essential as a place-holder, sometimes it is not needed. This is a separate question which is dealt with in the next activity.
6. When all the lines are filled, the board is cleared and steps 3 and 4 repeated.

Activity 2 Same number, or different? [Num 7.6/2]

An activity for up to 3 children. Its purpose is for them to learn when a zero is essential as a place-holder, and when it is optional.

- Materials*
- Activity board.*
 - Number cards.*
 - Pencil and paper for each child.
- * The same as for Activity 1.

- What they do*
1. The activity board and number cards are put out as for Activity 1.
 2. The first child puts number cards on the board as in Activity 1.
 3. He then reads these in two ways, with and without zeros for the empty columns.
 4. The other two write exactly what is said, both ways. They then decide whether each of these is the same number as the one on the board, or different.
 5. Steps 2, 3, 4 are repeated, the next child putting down the cards and reading them.
 6. Examples:



(Top line)	Spoken:	“Zero point 6 zero.	Point 6.”
	Written:	0.60	.6
		same number	same number
(Second line)	Spoken:	“Zero point zero 2.	Point 2.”
	Written:	0.02	.2
		same number	different number

7. This continues until the board is filled.
8. Steps 2 to 7 may be repeated.
9. They may then try to put into words what determines the outcome – i.e., when zeros are necessary for place-value notation to represent correctly the fraction given by the headed columns, and when they are optional. This is not very easy, so I suggest it as suitable only for the brighter children.
10. A concise formulation is: “Leading and trailing zeros are optional. Sandwiched zeros are necessary as place-holders.” (‘Sandwiched’ means between two digits, or a digit and the decimal point.) This is a convenient rule, but children should be able to give reasons as well. Nor should they be given this to memorize. If they can produce a correct formulation of their own all the better.

Num 7.6 Decimal fractions in place-value notation (cont.)

- Notes*
- (i) In the context of measurement, .6 m and .60 m do have different meanings. The first means that we are measuring to the nearest decimetre; the second, to the nearest centimetre, the result being .60 as against .61 or .62.
 - (ii) Many people prefer to write and speak (e.g.) 0.4 rather than .4, on the grounds that this helps to prevent the decimal point from being overlooked. I see this as a purely personal preference. Mathematically, both stand for the same number, and this is what the present activity is about.

Activity 3 Claiming and naming [Num 7.6/3]

A game for 5 or 6 players. Its purpose is to give practice in recognizing the same decimal fraction written in two notations, bar and place-value.

Materials

- Fractions, packs PV and B.*

* See specimens shown in Figure 21. (A complete set of each is provided in the photomasters.). Pack PV is in place-value notation, pack B in bar notation. A limited range of digits is used to reduce the cues from these.

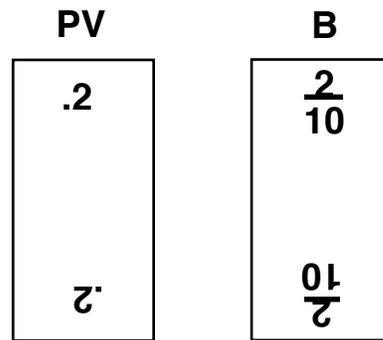


Figure 21 Fractions, packs PV and B (specimens)

- Rules of the game*
1. Pack PV is shuffled and dealt to the players. Pack B is shuffled and put face down on the table.
 2. Players look at their hands.
 3. Each player in turn picks up the top card from the pack on the table and puts it face up.
 4. Whichever player has in his hand the card for the same fraction says “I claim that”, and takes it.
 5. He puts the two cards together in front of him, and names both aloud. E.g., “4 hundredth-parts. Point zero 4.” The others check.
 6. The game continues as in steps 3, 4, 5. The first to put down all his cards is the winner, but the others play out their hands.
 7. Next time the game is played, the packs are used the other way about. That is, B is dealt to the players and PV is put on the table.

Discussion of activities We are now working at a purely abstract stage, without the support of physical materials or diagrams. So the earlier work, devoted to establishing the concepts firmly with the help of the above, is of great importance in preparation for this. If children have difficulty at this or later stages, then they need more time working at the more concrete levels.

This topic is concerned with notations rather than concepts (which emphasizes further the need for the concepts to have already been well established). Headed column notation continues to provide a way of showing what the numbers stand for which is clearer and more explicit than place-value notation, but which can be translated easily to and from place value notation. Bar notation provides a link with the earlier work on fractions of all denominators.

And the expansion of place-value notation to represent fractions implies also the expansion of the concept of number to include fractions. This implication will be made explicit in the next topic.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Claiming and naming [Num 7.6/3]

Num 7.7 FRACTIONS AS NUMBERS. ADDITION OF DECIMAL FRACTIONS IN PLACE-VALUE NOTATION

Concept Fractions as a new kind of number.

Ability To add decimal fractions, using place-value notation.

Discussion of concept

Fractions are often treated as numbers right from the start. This almost guarantees confusion for the children, since they are seldom told that we are *expanding* our concept of number to include a *new kind of number*. Place-value notation makes this transition smoother, but does not remove the need to explain what is going on, so that children have a chance to adjust their thinking.

Moreover, there are two distinct concepts involved, each with much interiority. The first is that of fractions – a new category of mathematical ideas. The second is that of other possible number systems. (Until now, children have only met one number system.) So one of the concerns of this topic is to introduce children to thinking of fractions as numbers. This we do in Activity 1. Here we show that fractions, in place-value notation, may be added in the same way as whole numbers; and that the results make sense. So in this respect they ‘behave’ like the numbers with which we are already familiar.

Whether we are justified in adding fractions in the same way as the familiar counting numbers is another question, and a more difficult one.

The problem is not unlike that of a naturalist who discovers a new species. First, he has to become aware of its existence. Then he has to decide how to categorize it, in the same way as one has to decide whether a bat is a bird or a rodent. And the most important characteristic for classifying may not be the most obvious, which in this case is flight.

To qualify for acceptance as a number system requires that these new (mental) objects ‘behave’ like the numbers we are already familiar with. The characteristics required by mathematicians are, again, not the most obvious. They have already been described in Num 5.10; namely, that addition is commutative and associative, likewise for multiplication, and multiplication is distributive over addition.

All this is, of course, far beyond our present needs, and I mention it for your information rather than for your detailed consideration. Also, so that you will understand why I do not treat it as a separate topic for the children. For the brighter children, however, I think it is worth touching lightly on this question, in the context of addition only. This is done in Activity 2.

Activity 1 Target, 1 [Num 7.7/1]

A game for up to 3 players (since they need to see the cards the same way up). Its purpose is to introduce the idea of adding decimal fractions, using place-value notation.

- Materials*
- Tenth-part cards.*
 - Hundredth-part cards.*
 - Pencil and paper for each player.
- * See illustration in step 2 below and the photomasters.

Introduction 1. “Now that we are writing decimal fractions like this they look rather like the ordinary numbers we’re used to. Suppose we decide to treat them like numbers, how would we add them?” E.g.,

$$.47$$

$$\begin{array}{r} .47 \\ + .32 \\ \hline \end{array}$$

2. Follow this with other examples, including those which involve ‘carrying’; first only to the right of the decimal point, e.g.,

$$\begin{array}{r} .26 \\ + .39 \\ \hline \end{array}$$

3. Then across the decimal point, e.g.,

$$\begin{array}{r} .74 \\ + .52 \\ \hline \end{array}$$

Easier version

- Rules of the game*
1. The two packs of cards are shuffled and put separately, face down.
 2. Each player in turn takes the top card from each pack. She may use one of these, or both combined, the smaller card on top; or she may decide to use neither if she wishes.



3. Each player writes this number, and adds it to the previous number or total to give a cumulative total.
4. The used cards are replaced at the bottom of the pile.
5. The player who reaches 1 unit exactly is the winner.
6. Overshooting is not allowed. If the numbers turned would take the total past 1 however they are used, then they are not used.

Harder version

This is played as above, except for step 4. In this version, all the cards which have been turned over remain face up, and are spread out so that all are visible. The player whose turn it is may use any one, or two, of these. So at each turn, there are two more cards to choose from.

Activity 2 “How do we know that our method is still correct?” [Num 7.7/2]

A teacher-led discussion, for the more able pupils only. Its purpose is first to put the above question, and then to answer it so far as addition is concerned.

- Materials*
- 2 decimal unit squares, see Figure 22 and the photomasters.
 - Base 10 material.
 - Pencil and paper for each child.

Suggested sequence for the discussion

1. Begin by putting the question, along the following lines. “When we started writing decimal fractions in place-value notation instead of bar notation, they looked much the same as ordinary whole numbers except for the decimal point. But of course they are not the same.”
2. Emphasize the difference by asking them to read each of the following in two ways.

37 “Thirty-seven.
Three tens, seven ones.”

(Now insert a decimal point.)

.37 “Point three seven.
Three tenth-parts, seven hundredth-parts.”

Note Never tolerate “point thirty seven”, which guarantees self-confusion.

3. “So when we’ve been adding decimal fractions like ordinary whole numbers, we have assumed that the way of adding which is correct for ordinary whole numbers is still correct for fractions. How do we know that our method is still correct, now that the figures mean something different?”

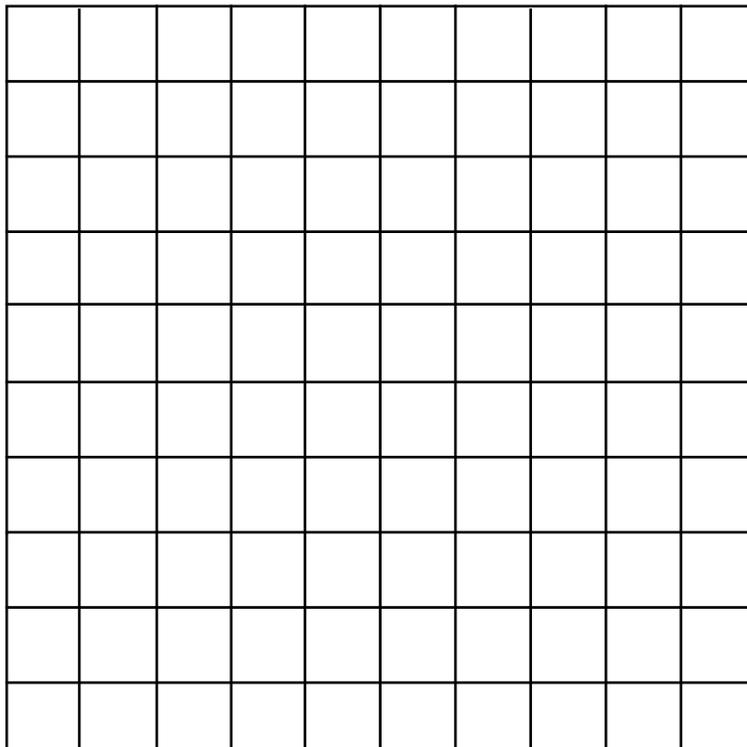


Figure 25 Decimal unit square

Num 7.7 Fractions as numbers. Addition of decimal fractions in place-value notation (cont.)

4. Show a decimal unit square. “If each of the little squares stands for a unit, what does the big square stand for?” (A hundred.) But if we change the meaning so that the *big* square stands for a unit, what does each little square now stand for?” (A hundredth-part.)
5. Put a rod from the base 10 materials in a column of the unit square. “And what does this now stand for?” (A tenth-part.)
6. So how can we represent (e.g.) .34 on this unit square? (3 rods, 4 cubes.)
7. The continuation to addition is now straightforward. The two numbers to be added are first represented separately on the two unit squares. The rods and cubes are then put together on one square, and the result interpreted. Begin with examples which do not require carrying; then, carrying from hundredth-parts to tenth-parts; then, from tenth-parts to ones (whole numbers). You can continue, according to your own judgement, by combining both kinds of carrying.
8. Again, use your own judgement about how many examples the children need to use in order to establish the deeper meaning of children’s use of the symbols for adding fractions.

Discussion of activities In this topic, we use our extrapolation of place-value notation to extrapolate the idea of number, from the familiar whole numbers to a new kind called fractional numbers. The word ‘fraction’ has now acquired a second meaning.

This meaning is assumed in topic 1, where the familiar methods for adding are applied to numbers of this new kind. This is concept building (Mode 3). Provided that this makes sense to them intuitively, I think that for many children it may be best to leave it at that.

However, there will be some who can see the point of the question put in topic 2. For these, the same base 10 materials which helped to provide good conceptual foundations for the early number work in place-value notation are useful in helping them to strengthen their conceptual foundations for the new number work. This topic makes a start with Mode 3 testing, that these new concepts are consistent with our mathematical knowledge.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 7.8 FRACTIONS AS QUOTIENTS

- Concepts*
- (i) A fraction as a result of sharing.
 - (ii) A fraction as a quotient.

- Abilities*
- (i) To share equally a number of objects greater than 1 when the result includes fractions of objects.
 - (ii) To predict the results of (i) as it would be shown in place-value notation by a calculator.
 - (iii) To use calculators to save time and labour, but thoughtfully.

Discussion of concepts One of the ways in which many textbooks create unnecessary difficulties for children is to confuse fractions as quotients with the other aspects of fractions which have already been discussed. This is like confusing the comparison and take-away aspects of subtraction; but worse, because fractions are harder, even when correctly explained.

If we read $\frac{2}{5}$ as 'two fifths,' short for 'two fifth-parts,' the corresponding physical actions are: start with an object, make five equal parts, take two of these parts, result two fifth-parts.

Now, however, we are going to think of $\frac{2}{5}$ in a different way, as the result of division. The corresponding physical actions are: take two objects, share equally among five, each share is how much?

At a physical level these are quite different, just as physically 3 sets of 5 objects and 5 sets of 3 objects are quite different, and mathematically $3(5)$ and $5(3)$ are different; although the results are the same.

So we should be more surprised than we are that the division $2 \div 5$ gives $\frac{2}{5}$ as result. And to teach that $\frac{2}{5}$ is just another way of writing $2 \div 5$ is to beg the whole question. $2 \div 5$ is a mathematical operation, a division. $\frac{2}{5}$ is the result of this operation, a quotient. The distinction becomes even sharper if we write this quotient as 0.4. To put this another way: would we say that 45 was just another way of writing 5×9 ?

Finally, I suggest that we should be more surprised than most people seem to be that a quotient is a number at all. It is, in fact, a new kind of number: a fractional number. A full discussion of this expansion of our schema for numbers, and what makes it legitimate and useful, is beyond the scope of what we are doing here; nor is it necessary. But I hope that the sequence in which the ideas have been offered, and the activities, will at least help to build a clearer understanding of the various aspects of fractions than children have usually been able to acquire.

Activity 1 Fractions for sharing [Num 7.8/1]

An activity for a small group of children. Its purpose is to introduce them to the connection between fractions and the sharing aspect of division.

- Materials*
- Plasticine.
 - Knives.
 - Cutting boards.
 - Fruit bar templates.†
 - SAR board: fractions for sharing, see Figure 23.*
 - Number cards 2 to 5.**
 - Non-permanent marker.
 - Wiper.

} Preferably
one per two
children.

* The full-sized one from the photomasters should be covered with film.

** To fit the spaces on the board, as provided in the photomasters.

† See *SAIL Volume 2a*, Photomaster 156.

- What they do*
1. About 5 Plasticine ‘fruit bars’ are made, using the cards as templates to give a uniform size.
 2. To begin with, put a ‘2’ in the left-hand space and a ‘3’ in the right-hand space of the SAR board. Ask the children to do as it says. The first time they may need a little help.
 3. One way to do this is to put two fruit bars, one on top of the other, and cut these into 3 equal parts. When taken apart, each share is 2 third-parts of a fruit bar. This may introduce practical difficulties, e.g., the layers of Plasticine may stick together if too warm; but conceptually it is probably the clearest.
 4. So they should write a number sentence something like this:

“2 fruit bars, shared between 3, result $\frac{2}{3}$ of a fruit bar each.”

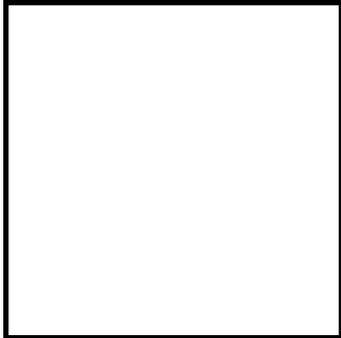
5. Let them experiment with other numbers in the START and ACTION spaces. To begin with, the smaller number should be in the START space.
6. When they fully understand the foregoing in physical terms, continue to step 7. This corresponds to the second instruction on the SAR board.
7. Remind them that sharing is one kind of division. So they can shorten their number sentences to (e.g.)

$$2 \div 5 = \frac{2}{5}$$

8. This is read as “2 divided by 5 equals (or ‘is equal to’) 2 fifth-parts.” Emphasize that we are not saying that $\frac{2}{5}$ is another way of writing $2 \div 5$. The left-hand side says what is *done*. It stands for making equal shares. The right-hand side says *how much* each person gets. It is called a quotient. (The word means ‘how much,’ or ‘how many.’)

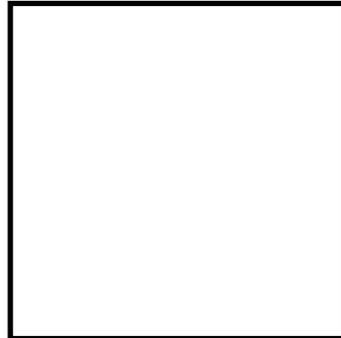
START

Take this number
of fruit bars.

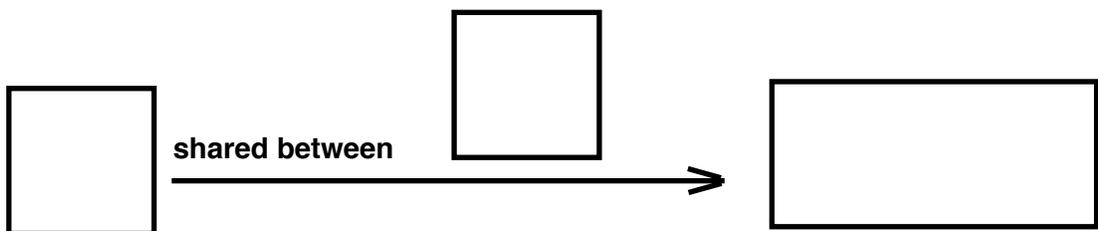


ACTION

Share them equally
between this number
of children.



1. Complete the number sentence below, recording what was done and the result.



2. Write a shorter number sentence with the same meaning.

Figure 26 Fractions for sharing

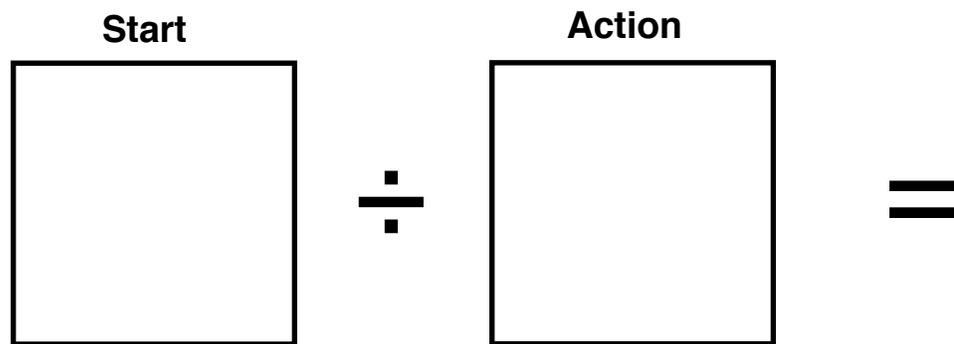
Activity 2 Predict, then press [Num 7.8/2]

An activity for children working in twos or threes. Its purpose is to consolidate the relation between fractions in bar notation and in place-value notation; and to bring the second of these into the present context of fractions as quotients.

- Materials*
- Activity board, see Figure 27.*
 - Start cards (numbered from 2 to 10).*†
 - Action cards (numbered 2, 4, 5, 8,10).*†
 - Calculator.
 - Pencil and paper.

* Provided in the photomasters

† To fit the squares on the activity board



Copy the above.

Write the quotient in fraction bar notation.

Then write it as it would be shown by a calculator.

Test your prediction.

Figure 27 Predict, then press

- What they do*
1. A card is put into each of the Start and Action squares. These may be randomly chosen (top of a shuffled pile); or children may be allowed to choose if they wish to experiment systematically.
 2. Each child then writes the quotient first in bar notation, and then in decimal notation. E.g.,

$$7 \div 5 = 1 \text{ and } \frac{2}{5} \\ = 1.4$$

(Thinking: 7 shared between 5 gives 1 each, and 2 over. These 2 shared between 5 give each, as in Activity 1).

3. Their prediction is then tested by using the calculator to give the quotient.
4. The 8 card should be taken out of the Action pack until the children are proficient with the easier divisors.

Activity 3 “Are calculators clever?” [Num 7.8/3]

A teacher-led discussion for up to 6 children. Its purpose is to encourage them to be critical about the results obtained by calculator.

- Materials*
- Calculators.*
 - Example cards.**

* At least one for every 2 children.

** Six are given in Figure 28 and in the photomasters. These should be on separate cards.

1. 7 people are to be shared between 2 cars, for a journey. How many will there be in each car?

2. 3 children have 2 parents between them. How many parents do they have each?

3. I have 9 carrots, to give to 4 horses. What would be a fair share for each horse?

4. I have 7 potted plants, to give to 4 friends for their birthdays. How many should I give to each?

5. A householder gives 3 children who do some gardening \$4.20 to be shared equally among them. How much should they have each?

6. A package contains 8 fish fingers. How should these be shared equally among a family of 3?

Figure 28 ‘Are calculators clever’ example cards.

- What they do*
1. Begin with this example.
“7 people are to be shared between 2 cars, for a journey. How many will there be in each car?”
 2. Ask them to use their calculators to work out the answer, just to see what happens.
 $7 \div 2 = 3.5$
What do they think about this?
 3. Ask if they can think of a number story for which this number sentence would make sense. Here is one. “7 biscuits are to be shared between 2 climbers. How many should they have each? It makes even better sense if instead of 3 whole biscuits and 5 tenth-parts each, they get 3 and 1 half! Did any one remember that 5 tenth-parts is equivalent to 1 half?”
 4. Repeat steps 1 and 2 with other number stories. Of the 6 given here, 3 make sense and 3 do not. For those that do not step 3 should be repeated. For those that do, they should invent a number story which does not make sense. (Number 6 may need discussion, and possibly even the introduction of Plasticine fish fingers!)

Activity 4 Number targets by calculator [Num 7.8/4]

A game for children to play in twos or threes, sharing a calculator. Its purpose is to increase their knowledge of the relationships between quotients and decimal fractions.

Materials For each group of 2 or 3:

- A calculator.
- Set of calculator target cards.*
- Pencil and paper for each child.

* See examples in step 2 below. A full set is provided in the photomasters.

- Rules of the game*
- Stage (a) Fractions only*
1. The fraction cards are shuffled, and put face down. (The unit cards are not used in stage 1.)
 2. In turn, each player turns over the top card from the pile. E.g.,

.4

3. He is then allowed 3 'shots' with the calculator. Each shot consists of 4 entries: (number) (\div) (number) (=). 1 point is scored for each shot which hits the target. In this case, some successful shots would be

$$4 \div 10 \qquad 8 \div 20 \qquad 2 \div 5$$

4. 3 hits could also be scored by

$$4 \div 10 \qquad 40 \div 100 \qquad 400 \div 1000$$

For beginners there is nothing against this – they learn something from it. But more experienced players may agree to allow only one shot with a power of 10 as divisor.

5. It is a good idea for one player to make his shots verbally, and another to enter these on the calculator and show the result.
6. After an agreed number of turns, the winner is the player who has scored most hits.

Stage (b) Mixed numbers

1. Both packs of cards are now used. These are shuffled and put face down in two piles.
2. In turn, each player turns over the top card from each pile, and puts these side by side to give a mixed number. There are 72 possible combinations; e.g.,

2	.35
---	-----

- When the ones pile is finished, it is shuffled and used again.
- 3, 4, 5. These are the same as steps 3, 4, 5 in Stage (a).

Discussion of activities In Activity 1, we go back to Mode 1 to relate the new meaning of fractions, namely quotients, to a typical physical situation. This helps to show that although the two aspects of division give the same result, this is not something to be taken as obvious from the beginning. This new meaning is of course additional to the other earlier meanings, not a replacement.

Calculators are the best way for doing all but the simplest divisions. It is however important that they be used with understanding, of two kinds. One kind is relating the answers obtained by calculator to their existing number schema; the other kind is relating these answers critically to situations such as those described in the number stories. These are the purposes of activities 2 and 3 respectively.

Activity 4 is a game for exercising and consolidating their understanding.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Number targets by calculator [Num 7.8/4]

Num 7.9 ROUNDING DECIMAL FRACTIONS IN PLACE VALUE NOTATION

Concept Rounding decimal fractions in place value notation to the nearest tenth or hundredth.

Ability To write and say a fractional number as described.

Discussion of concept This is not a new concept, but an extension of the concept already learned in Num 2.14 to the present context.

The discussion of concepts at the beginning of Num 7.5 is also very relevant here. Following on from this, the word ‘decimal’ is an adjective not a noun. When I hear people talk about ‘decimals’ I want to ask “Decimal what?” The answer in the present context is decimal fractions, and while children are learning I think we should say so.

Activity 1 Rounding decimal fractions [Num 7.9/1]

An activity for a small group of children, up to five or six. Its purpose is to apply their existing concept and ability of rounding, acquired in the context of whole numbers, to fractional numbers in place-value notation. This should be an easy transition provided that they have done Num 2.14 first. If this was some time ago, it might be as well to review this before continuing with the present activity.

Materials

- A calculator, for the group.
- Pencil and paper for each child.

What they do

1. The player whose turn it is to start has the calculator. She chooses two two-digit prime numbers and divides one by the other. (This is to be sure of generating an infinite decimal.) She reads out her result, and everyone (including herself) writes it down. They then continue in much the same way as in Num 2.14/3.
2. Example. The starting player chooses 31 divided by 59, result 0.5254237. . . . They all write this down: **0.5254237**
3. She then says “Rounded to the nearest tenth-part, this is zero point five. Rounded to the nearest hundredth-part, this is zero point five three. Rounded to the nearest thousandth-part, this is zero point five two five.” For each correct answer, she scores three points. If a player makes a mistake, she loses the rest of her turn and the next player starts again as in steps 1, 2, 3.
4. The others listen and check.
5. This continues until all have had their turns. A further round may be played if desired.

Notes

- (i) To say ‘Oh’ for zero is a mistake, and not less so because it is very common. ‘Oh’ is the name of a letter, not a number.
- (ii) It is also correct to say (e.g.) “Rounded to two decimal places.” The wording in the example is used because it also gives practice in remembering what the various places denote.
- (iii) This activity may of course be extended as far as you think fit. However, thousandth-parts are as far as it is usually necessary to go.

Activity 2 Fractional number targets [Num 7.9/2]

A game preferably for two or three, though larger numbers are possible, and it may also be played solo. It is a direct transfer of Num 6.9 into the present context, and gives plenty of practice in rounding. This also provides an interesting application, since without rounding it might sometimes be very difficult to achieve the target.

Materials For each group:

- A calculator.
- Three 0 - 9 dice (may be shared with another group). Alternatively, a 0 - 9 pack of number cards may be shuffled and used to give the starting numbers.

For each player:

- Pencil and paper.

Rules of the game

1. The three dice are thrown to give a 3-digit number. All the players write this at the top of their paper.
2. Two dice are then thrown to give a 2-digit number. This is preceded by '0.' and written as the target. E.g.,

Dividend	Target
549	0.86

3. The aim is to find a number which, when used as divisor, gives the target number as quotient when the calculator result is rounded to the nearest hundredth. Each player in turn writes her attempt, and passes paper, pencil, and calculator to the next player. Example:

Dividend	Target
549	0.86

First player:

$$549 \div 800 = 0.68625 \text{ which rounds to } 0.69$$

(Quite good for a first trial.)

Next player:

$$549 \div 900 = 0.61 \rightarrow 0.61$$

(This player changed the divisor in the wrong direction. A bigger divisor doesn't give a bigger result!)

Next player:

$$549 \div 700 = 0.7842857 \rightarrow 0.78$$

(This player learned from the other's mistake).

Next player:

$$549 \div 620 = 0.8854839 \rightarrow 0.89$$

Next player:

$$549 \div 630 = 0.8714286 \rightarrow 0.87$$

Next player:

$$549 \div 635 = 0.8645669 \rightarrow 0.86 \dots \text{ the target number.}$$

4. Another round may then be played.

Discussion of activities

Some calculators allow us to choose how many decimal places they show, but the less expensive ones give us all they can. This is not a disadvantage in a school situation, since it gives good practice in rounding. Activity 1 makes this the focus of the activity rather than incidental to whatever else the calculator is being used for, so that children become fluent in this skill, and have it available for use whenever it is required as part of some other task. Activity 2 is a situation of this second kind.

[Space 1] **SHAPE**

Shapes in the environment and in mathematics

Space 1.8 PARALLEL LINES, PERPENDICULAR LINES

- Concepts*
- (i) 'Is parallel to'
 - (ii) 'Is perpendicular to'
- as relationships between two lines.

- Abilities*
- (i) To recognize examples of parallel or perpendicular lines.
 - (ii) To construct examples of parallel or perpendicular lines.

Discussion of concepts Just as we have relationships between two numbers (e.g., 'is greater than,' 'is equal to'), so also we have relationships between two lines such as those in the present topic. If line a is parallel to line b , then line b is parallel to line a , so we may also say that these lines are parallel (meaning, parallel to each other). This is true also of the relationship 'is perpendicular to.' This reversibility does not hold for all relationships. E.g., it is true for 'is equal to,' but not for 'is greater than.' Here, however, our main concern is with the particular relationships named in this topic, not with the ways in which relationships themselves may be classified.

Activity 1 "My rods are parallel/perpendicular." [Space 1.8/1]

An activity for up to 6 children. Its purpose is to help children learn these two relationships.

- Materials*
- A pack of 20* cards. On 10 of these is written 'parallel' with an example, and on the other 10 is written 'perpendicular' with an example.**
 - For each child, 2 rods of different lengths. A square section is useful to prevent rolling.

* Any even number of cards will do, provided that there are enough to give a good variety of examples. In the examples, it is important that the pairs of lines should be of different lengths, and oblique relative to the edges of the paper. (See illustrations in 'Discussion of activities.')

Parallel and perpendicular are relationships between lines, independently of how these lines are positioned on the paper.

** Provided in the photomasters

What they do Stage (a), with cards

1. The cards are shuffled, and put face down.
2. Each child in turn takes a card and puts it in front of him face up.
3. The children then put their rods on top of the lines in the illustration.
4. In turn, they show their cards to the others and say, "My rods are parallel," or "My rods are perpendicular," as the case may be.
5. Each child then takes another card, which he puts face up on top of the card he has already. Steps 3 and 4 are then repeated.

Stage (b), without cards

1. Each child in turn puts his rods either parallel or perpendicular, and says "My rods are parallel/perpendicular" (as the case may be).
2. The others say "Agree" or "Disagree."
3. A child may deliberately give a false description if he chooses. The others should then all disagree.

Activity 2 “All put your rods parallel/perpendicular to the big rod.” [Space 1.8/2]

An activity for up to 6 children. Its purpose is to consolidate the concepts parallel and perpendicular.

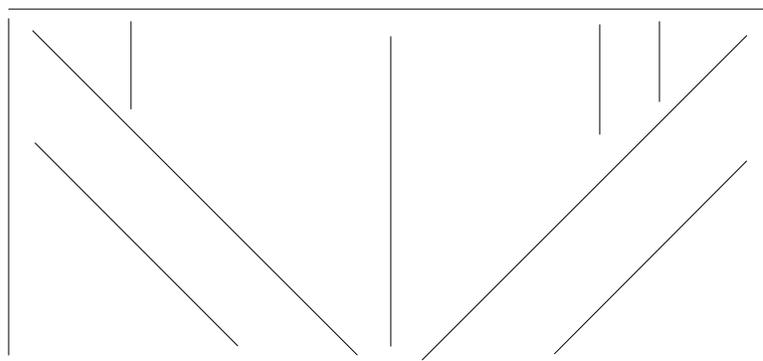
- Materials*
- Ruler or big rod.
 - Small rod for each child (It is good if the small rods are of assorted length.)

- What they do*
1. The teacher puts down the big rod and says “All put your rods parallel to the big rod.”
 2. The children do so.
 3. Steps 1 and 2 are repeated several times. Encourage variety in the placing of the children’s rods.
 4. Then, after step 2, the teacher removes the big rod and asks the children what they notice about their own rods. It should be brought out in discussion that the children’s rods are all parallel to each other.
 5. Steps 1, 2, 3, and 4 are repeated with the instruction “. . . perpendicular to the big rod.”

Activity 3 Colouring pictures [Space 1.8/3]

An activity for 2 to 6 children.

- Materials*
- A picture for each child, made up of lines which are all in parallel or perpendicular pairs. The pictures should be drawn with faint lines. (See examples in the illustration below and in the photomasters.)
 - A pack of parallel/perpendicular cards, as used in Activity 1.



- What they do*
1. The pack is shuffled and put face down.
 2. The first child turns over the top card.
 3. He is then allowed to colour two lines in his picture, which must be either parallel or perpendicular according to the card.
 4. The other children in turn do steps 2 and 3.
 5. Putting pencils on top of the lines helps to show up which lines are parallel or perpendicular.
 6. Continue until all the pictures have been coloured.

Discussion of activities Activity 1 is for building the concepts parallel and perpendicular from a variety of examples. Again I emphasize the importance of choosing examples which do not link these concepts either with length of line, or position on paper. If this mistake is not avoided, children will be able to recognize examples like these:



but not like these:



Activity 1 is also for linking the concepts with the appropriate vocabulary. 'Perpendicular' is quite a hard word, so you may decide to give children practice in saying it. Initially they are not expected to read these words from the cards. They learn them orally, linked with the visual examples of parallel and perpendicular lines. In this way they will gradually learn to recognize the written words.

Activity 2 uses the newly formed concepts to generate examples, with peer-group checking. Activity 3 is similar, but develops a situation in which more than two rods are involved. Of these, a given pair may be either parallel or perpendicular.

Activity 4 is the payoff. We now have a picture with many lines. Correct recognition of relationships between these lines allows children to colour their lines. This is a development from an earlier activity in *SAIL Volume 1*, Space 1.3/1, making use of more advanced mathematical ideas. Why more advanced? Because being straight or curved is a property of a single line; being parallel or perpendicular is a property of a pair of lines – a relational concept.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.9 CIRCLES

- Concepts*
- (i) Circles as shapes.
 - (ii) Diameter, radius, circumference of a circle.

- Abilities*
- (i) To recognize circles.
 - (ii) To name the above parts of a circle.

Discussion of concepts In this topic we focus on the shape aspect of circles. The various interesting relations between circles and other shapes, such as triangles and polygons, will be dealt with in a later topic, Space 1.17, 'Inter-relations of plane shapes.' So this is a simple, introductory treatment only, except for Activity 4.

Activity 1 Circles in the environment [Space 1.9/1]

A teacher-led discussion for up to 6 children. Its purpose is to teach the concept of a circle as the shape common to a variety of objects in the environment.

- Materials*
- A cutout circle for each child. The circles should be of varying colours and sizes, the smallest being at least 10 cm in diameter.

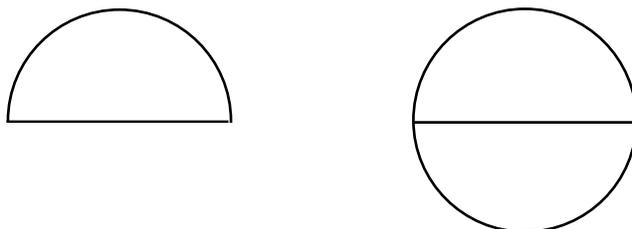
- Suggested sequence for the discussion*
1. Give each child a circle.
 2. Ask if they know the name of this shape. Probably they will: if not, tell them.
 3. Ask them to try and find objects in the environment which have this shape. Some (e.g., a traditional-style clock) will be seen flat-on, others, though in fact circular (e.g., top of a wastepaper container) will be seen foreshortened. These may need to be moved before it is clear to some children that the shape is like that of their cutout model circle.

Activity 2 Parts of a circle [Space 1.9/2]

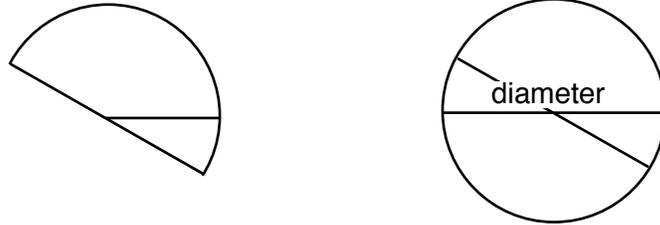
A teacher-led discussion for up to 6 children. This is a continuation of Activity 1. Its purpose is to add further detail (interiority) to the concept of a circle.

- Materials*
- The same as for Activity 1, with an additional circle for yourself.

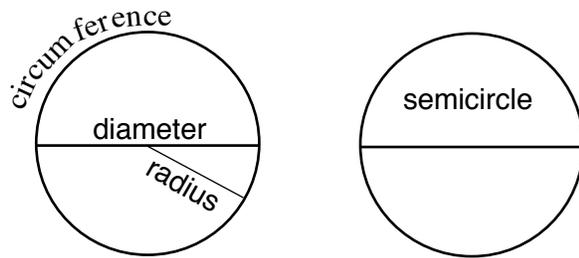
- Suggested sequence for the discussion*
1. Fold your circle in half, and open it out again.



2. Tell the children to do likewise.
3. Tell them the name for the new line (diameter). Label your diameter, and let them copy yours. You may like to have a line ruled along the diameter to make it show up more.
4. Fold along another diameter, preferably not perpendicular to the first, and open out again. Put a dot where the creases cross.



5. Ask the children to suggest names for this point. 'Middle' is a good name. The mathematical term is 'centre.'
6. Give them the names for the other parts, and let them label these as before. Only the radius (i.e., half diameter) should be pencilled in.



Activity 3 Circles and their parts in the environment [Space 1.9/3]

A teacher-led discussion for up to 6 children. This is a continuation of Activity 1. Its purpose is to consolidate the newly developed aspects of the concept of a circle.

Materials • The labelled circles from Activity 2.

- Suggested sequence for the discussion*
1. Ask the children to find circular objects in the environment where they can also find some of the other parts which they now have labelled in their own circles.
 2. A clock of the traditional kind provides a good example. The whole face is a circle; the rim round the edge is its circumference; the spindle of the hands is the centre; and the long hand is most of a radius.
 3. A more subtle example is provided by the rim of the kind of wastepaper container often found in school classrooms. This forms the circumference of a circle, but the rest of the circle (its interior) is missing. The bottom of the container has both.
 4. Radii and diameters are harder to find. They may be added to available objects (e.g., the wastepaper container again) by stretching strings across. One will provide a diameter. A second string crossing the first will give four radii.

Activity 4 Patterns with circles [Space 1.9/4]

An exploratory activity for up to 6 children. Its purposes are to develop skill in drawing, and to provide a ground from which further properties of a circle may be found.

Materials For each child:

- Safety compasses.
- Paper.
- Pencil.

- What they do*
1. Introduce the children to the use of the safety compasses, and allow them time to practice drawing circles of various sizes. In some of these they should draw and label the centre, a radius, and a diameter, to review and consolidate Activity 2.
 2. Each then experiments with making patterns from circles. Some interesting ones can be produced: a few examples are shown in Figure 29. It may be good for them near the beginning to copy these, since there are various things to be found out this way. Straight lines may be added if desired.
 3. One at a time, the diagrams are looked at together and discussed. The purpose is to find properties of the circle which are brought into view by the design. For example, in design A each small circle has half the diameter of the big circle. In B, the large circle is divided into two equal parts. (Why?). This pattern may be extended, as in C. In D, the lines joining the centres from an equilateral triangle, and pass through the points where the circles touch. With one more equal touching circle, we get a rhombus. (Why?).

Discussion of activities We begin with cutout circles, rather than objects in the environment, in order to provide physical examples with the minimum of other qualities which have to be ignored. The technical term is 'low-noise examples.' When the concept of a circle has been formed, it is consolidated by looking for and recognizing examples where there are more distractors – where more abstracting has to be done. This process is repeated in Activities 2 and 3 for those parts of a circle which we want the children to learn at this stage. These 3 activities should follow each other fairly closely.

Activity 4 is more advanced, and requires the physical skills involved in drawing; so it is good for developing these. Children enjoy making patterns with circles, and there are many important properties which can be found by studying these patterns. At this stage our task is to help children find them and put them into words: geometric proof is something for much later.

OBSERVE AND LISTEN

REFLECT

DISCUSS

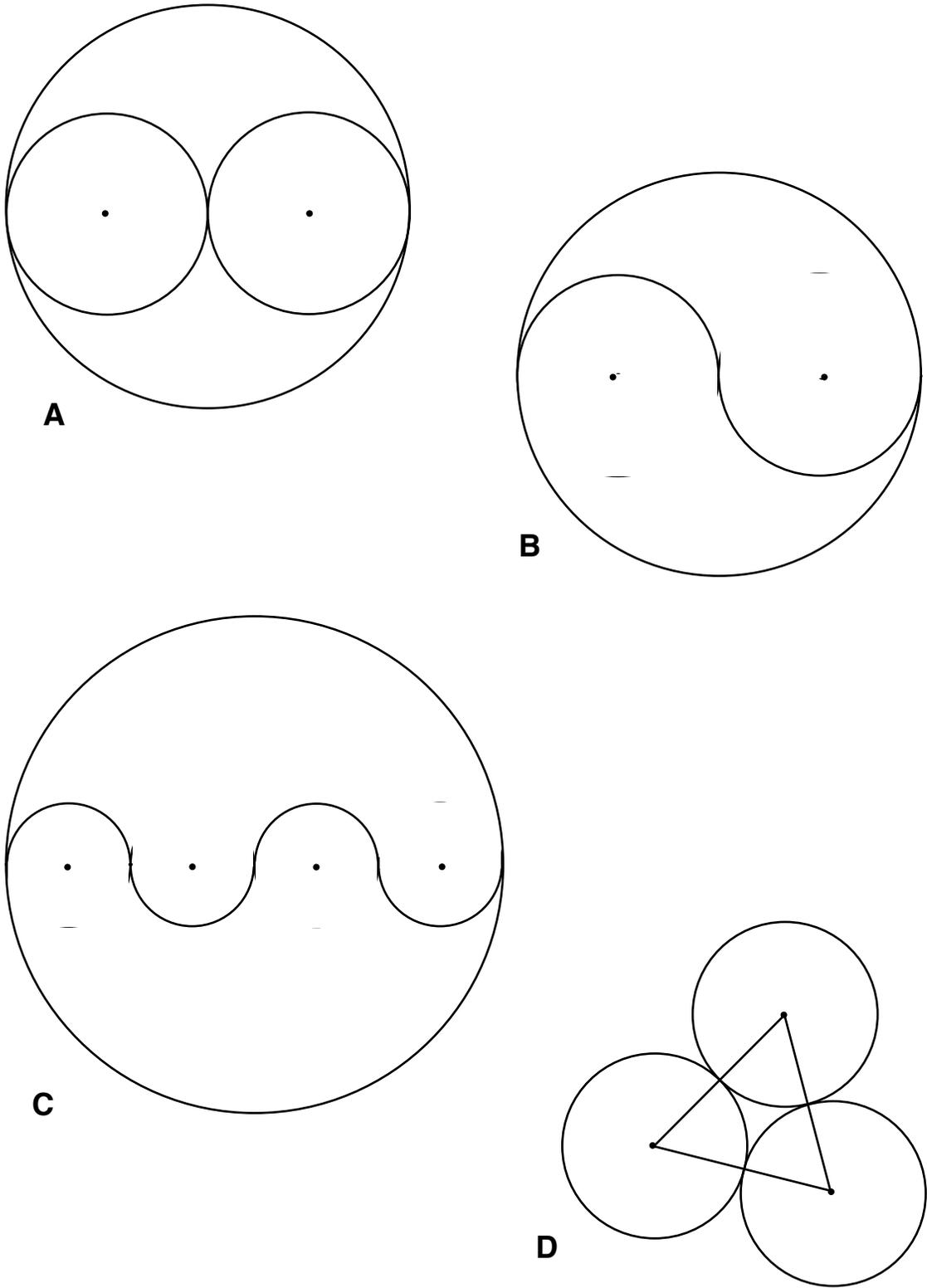


Figure 29 Patterns with circles

Space 1.10 COMPARISON OF ANGLES

Concept The shape and size aspects of angle.

Ability To decide which of two angles is the larger/smaller.

Discussion of concept When two straight lines meet, they form a shape which we call an angle. The everyday and mathematical meanings are in this case much the same. An angle can also represent a difference between two directions. Here we concentrate on angles as shapes, which may be considered either by themselves, as in this topic, or as contributing to descriptions of other shapes, as in Space 1.14.

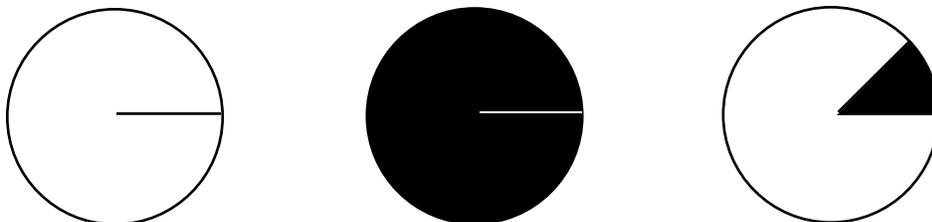
Objects may be ordered in various ways; e.g., persons by age, names alphabetically, lines by their lengths, sets by their number. Angles may be ordered by their size, which is independent of the length of their arms.

Activity 1 “All make an angle like mine.” [Space 1.10/1]

An activity for 2 to 6 children. Its purpose is to introduce the concept of an angle, in its shape aspect.

- Materials*
- An angle disc for every child, preferably each of a different colour.
 - A hinged angle made from two milk straws and a pipe cleaner.

Note An angle disc is made from two circular discs of thin cardboard, one white and one coloured. Each is cut along one radius, and the two are then interlaced so that one can be rotated relative to the other. In this way a coloured angle is formed which can be adjusted to any desired size.



What they do Stage (a) using angle discs only.

1. On receiving their angle discs, children are allowed a little time to explore.
2. Explain that the coloured shape is called an angle.
3. Child A sets his disc to any angle he likes, puts it in the middle of the table, and says “All make an angle like mine.”
4. The other children make angles of the same size, and put them near that of child A.
5. Child A then compares them with his own, and pointing to each in turn says “Agree” or “Disagree.” (The match need only be approximate, and visual comparison is all that is required at this stage.)
6. Another child becomes child A, and steps 3 to 5 are repeated.

Stage (b) using the hinged angle.

The steps are the same as in Activity 1, except that child A uses the hinged angle. This allows physical comparison if there is disagreement.

Activity 2 “Which angle is bigger?” [Space 1.10/2]

An activity for 2 to 6 children. Its purpose is to introduce comparison by size of angles.

- Materials*
- Two angle discs, of different colours. For *Stage (a)* these should be of the same size, for *Stage (b)* of different sizes.
 - Clear acetate sheet (as used with overhead projectors).
 - An overhead projector pen (or other non-permanent marker).
 - Damp rag.

What they do *Stage (a)* uses discs of the same size.

Stage (b) uses discs of different sizes.

Otherwise, the stages are alike.

1. Child A and child B each makes an angle. These are put in the middle of the table.
2. Child C then says (e.g.) “The red angle is bigger than the green angle.” (Or if he likes, “The green angle is smaller than the red angle.”)
3. In the event that the angles look alike, C would say, “These angles are about the same size.”
4. The other children say in turn “Agree,” or “Disagree.”
5. If there is disagreement, a check may be made by putting the acetate sheet on top of one angle and tracing this angle, using an overhead projector pen.
6. Other children take over the roles of A, B, C, and steps 1 to 4 are repeated.

Activity 3 Largest angle takes all [Space 1.10/3]

A game for 3 to 5 children, based on Activity 2. Its purpose is to consolidate the concept of comparison by size of angle.

- Materials*
- A pack of cards on which are drawn angles, 6 of each of the following sizes: 30, 60, 90, 120, and 150 degrees.* The arms of the angles should vary considerably in length among angles of the same size: see the ‘Discussion of activities’ following Activity 4.
 - Paper and pencil for scoring.
 - Acetate sheet, non-permanent marker, damp rag, as for Activity 2.

* Provided in the photomasters

- Rules of play*
1. The pack is shuffled. Five cards are then dealt to each player.
 2. Players put their cards in a pile face down.
 3. Starting with the player to the left of the dealer, each player turns over one card and puts it down face up in front of her.
 4. The player who puts down the largest angle takes all the others. She puts this set of cards in a pile in front of her. Comparison may be visual, or by the acetate sheet method as in Activity 2.
 5. If two angles are equal in size, each player takes one angle for her pile.

Space 1.10 Comparison of angles (cont.)

6. The winner of a round is the first to put down a card for the next round. The others follow in turn, clockwise.
7. When all the cards have been played, the players' scores are recorded according to the number of cards they have taken.
8. Another game may then be played, and the scores added to those of the previous game.
9. This game may also be played as 'Smallest angle takes all.'

Activity 4 Angles in the environment [Space 1.10/4]

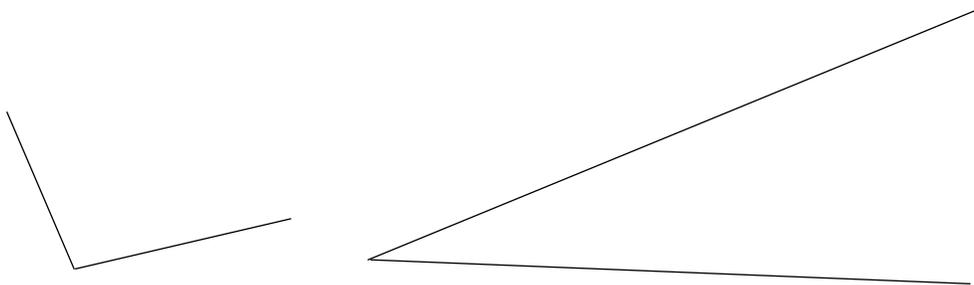
An activity for relating the concept of angle with examples in the environment.

Materials • An angle disc for each child, as in Activity 1.

- What they do*
1. Each child looks around the room and finds an angle. (Suitable sources are angles made by hanging strings, angles between books on shelves, corners of tables, hands of a clock. Many environmental angles are right angles. This does not matter: it prepares the way for the next topic.)
 2. Each child adjusts his angle to the same size as whatever he has chosen.
 3. In turn, each child holds up his angle and says (e.g.) "I've made my angle the same size as the angle of that string" (pointing).
 4. The others in turn say whether they agree.

Discussion of activities Activity 1 is for forming the concept of an angle, by making and copying a variety of angles of different sizes and colours. A different example of the same concept is introduced in stage 2, in the form of a hinged angle made from milk straws and a pipe cleaner.

Once the concept of an angle has been established children are in a position to compare angles of different sizes. Note that the size of an angle has nothing to do with the length of its arms. Of the two angles below, the left is the larger. This is the point of the second stage of Activity 2, and of Activity 3.



Activity 4 relates the concept of angle to environmental examples. Here there is more irrelevant detail to be ignored, so it comes last, after the concept has been established by simpler examples.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.11 CLASSIFICATION OF ANGLES

- Concepts*
- (i) Acute angle, obtuse angle, right angle.
 - (ii) (Later) Straight angle, reflex angle.

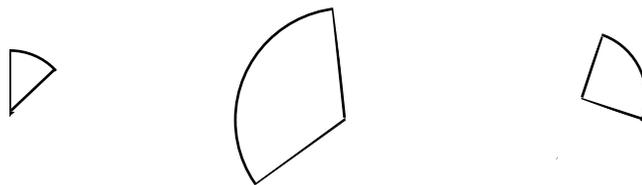
- Abilities*
- (i) To classify angles into these categories.
 - (ii) To produce examples of these categories.

Discussion of concepts These are simple concepts, once children can compare angles by size. An acute (sharp) angle is one which is less than a right angle, an obtuse (blunt) angle is one which is greater than a right angle. Among obtuse angles, we do not usually include angles greater than a straight angle, which is two right angles put together. These are called reflex angles.

Activity 1 Right angles, acute angles, obtuse angles [Space 1.11/1]

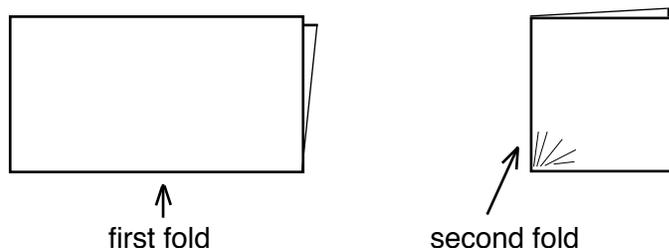
A sorting activity for 1 to 3 children. Its purpose is to help children form the concepts of acute angle, obtuse angle, right angle.

- Materials*
- A piece of paper for each child. (The size is not critical.)
 - Coloured crayon or felt tip pen.
 - An assortment of cutout angles, in the form of sectors of a circle. (See illustration below). There should be roughly equal numbers of acute, obtuse, and right angles. The circles should be of assorted sizes; assorted colours too, if you like. A suitable set is provided in the photomasters.
 - 3 set loops.
 - 3 small cards labelled 'Acute angles,' 'Obtuse angles,' 'Right angles,' respectively.



Sample cutout angles

- What they do*
1. Each child makes herself a right angle by folding her paper twice.



Space 1.11 Classification of angles (cont.)

2. The right angle (at the double fold) is then marked with coloured crayon or felt tip.
3. The cut out angles are mixed together in the middle of the table.
4. The 3 set loops are put out, with a card labelling each: respectively acute angles, obtuse angles, right angles.
5. The children work together to sort these into 3 sets: acute, obtuse, right angle. They do this by comparing each with their right angles.
6. A good way is for each child in turn to put an angle into a set loop, saying as she does so (e.g.) "Obtuse angle."

Activity 2 Angle dominoes [Space 1.11/2]

A game for 2 to 4 players. Its purpose is to consolidate the concepts formed in Activity 1.

- Materials*
- A set of 24 angle dominoes. These are of two kinds (see Figure 30). A full set is provided in the photomasters. Note that the right angles are not marked as such, so that the players have to decide for themselves, either by observation or by comparing with their own right angles.

- Rules of play*
1. The rules are the same as for ordinary dominoes. There are, however, various ways of playing. A way we find goes well is described in steps 2 to 5.
 2. 5 dominoes are dealt to each player. The rest are put face down in the middle.
 3. A player who cannot go takes a domino from the middle. This counts as her turn.
 4. When all are taken from the middle, a player who cannot go knocks once. If she cannot go a second time she knocks twice; if a third time, she knocks thrice and is out of the game.
 5. The winner is the first to use all her dominoes; or the one left with fewest, if none can go.
 6. In the present game, the match must be between an angle and a word, or between two angles of the same kind (acute, obtuse, right). Two words may not be matched, since this does not depend on understanding their meaning.

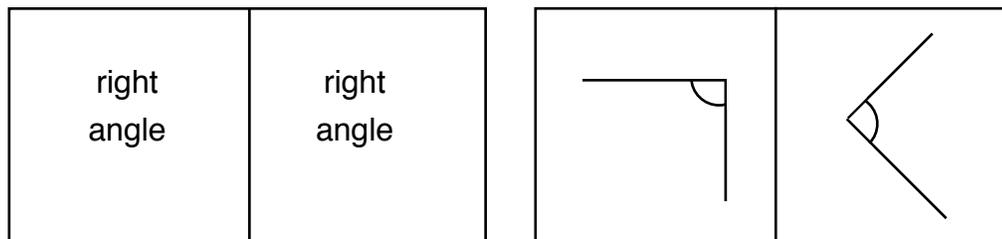


Figure 30 Angle dominoes

Activity 3 “Mine is the different kind.” [Space 1.11/3]

A game for exactly 3 players.

- Materials*
- 30 cards, on 10 of which are drawn acute angles, on 10 right angles, and on 10 obtuse angles. As provided in the photomasters, the arms of the angles on the cards should vary considerably in length.
 - They also keep their right angles from Activity 1, to check.

- Rules of play*
1. The cards are shuffled, and all are then dealt to the players.
 2. The players hold their packs face down.
 3. Starting with the player to the left of the dealer, each in turn puts down his top card face up in front of him.
 4. If all are alike or all are different, the players continue in turn to put down another card on top of their earlier one(s), until there are two alike and one different. Here ‘alike’ refers to the 3 categories acute, obtuse, right angle.
 5. The player whose pile shows the odd one out says “Mine is the different kind.” or “Mine is the odd one out.” and explains why. E.g., “Yours are acute, and mine is obtuse.” He then takes all 3 piles, which he puts face down at the bottom of his own pile.
 6. This player then puts down a card, and steps 3, 4, 5 are repeated.
 7. If a player has no more cards in his hand to put down, his pile stays as it is until step 5 applies.
 8. Play continues until one player has lost all his cards. The winner is then the one with most cards.
 9. If this takes too long, the winner is the player who finishes with most cards.

Activity 4 “Can’t cross, will fit, must cross.” [Space 1.11/4]

An activity for up to 4 children.

- Materials*
- An assortment of cutout angles.*
 - 12 instruction cards.** On 4 of these is written “Each of you take an acute angle”; on 4, “Each of you take a right angle”; and on four, “Each of you take an obtuse angle.”
 - One angle card as illustrated in step 2.**
- * The same as for Activity 1 ** Provided in the photomasters

What they do Stage (a)

1. The cutout angles are mixed together and spread out in the middle of the table.
2. The angle card is put on the table with the Side 1 showing.

Put two angles together with their points on the dot, on the same side of the line if you can.



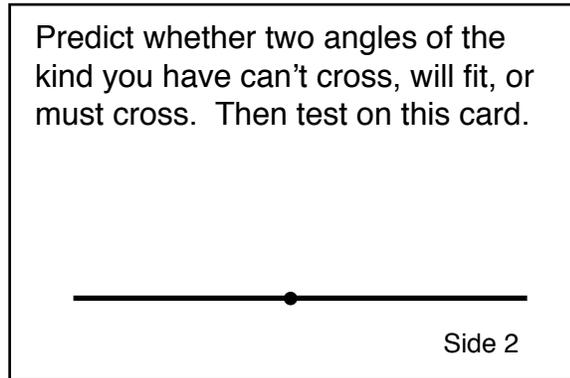
Side 1

Space 1.11 Classification of angles (cont.)

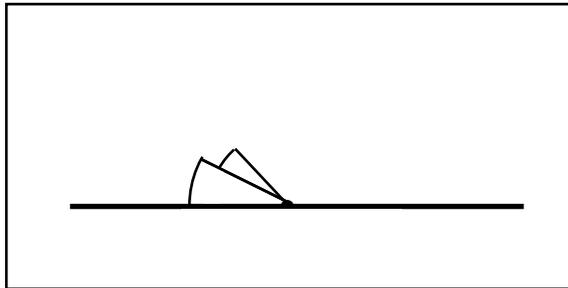
3. The instruction cards are shuffled and put face down.
4. The top instruction card is turned over, and the children do as it says.
5. They then follow the instruction on Side 1 of the angle card.
6. Steps 4 and 5 are repeated until all the instruction cards are used.

Stage (b)

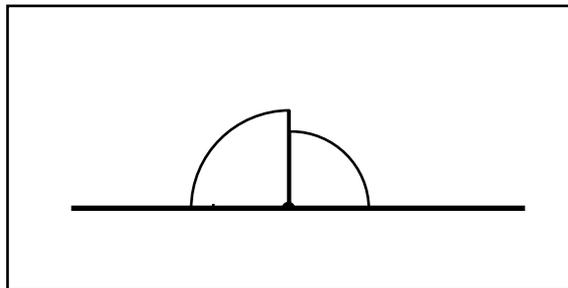
1. Steps 1 to 6 are the same as in Stage (a), except that Side 2 of the angle card is used. Examples of correct predictions are shown below.



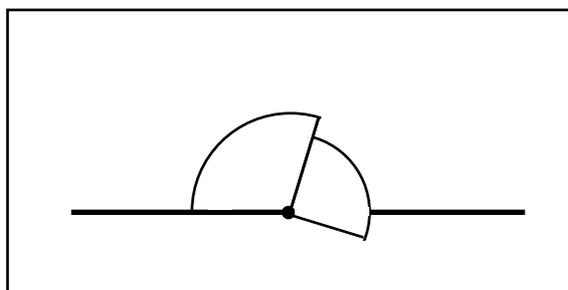
“Can’t cross”
(2 acute angles)



“Will fit.”
(2 right angles)



“Must cross.”
(2 obtuse angles)



7. The intention is that children discover the basis for prediction, as shown above in the three parentheses.

8. When they have done so, they may be invited to put this into words.
9. They may also be asked if they can think out why this is so. This, however, is difficult for children of this age.

Discussion of activities Activity 1 is for concept building from physical examples (Mode 1). Activity 2 is for consolidating the link between words and concepts, using a variety of examples. Testing in this case is by agreement and, if necessary, discussion (Mode 2). Activity 3 consolidates these concepts and their associated words in another game, involving finding two of a kind and one of a different kind. Activity 4 investigates some further consequences of the properties, and sets children on the path of making mathematical generalizations.

OBSERVE AND LISTEN

REFLECT

DISCUSS



“Can’t cross, will fit, must cross.” [Space 1.11/4]

Space 1.12 CLASSIFICATION OF POLYGONS

- Concepts*
- (i) That of a polygon.
 - (ii) Ways in which polygons may be classified:
 - (a) regular and non-regular
 - (b) number of sides.

Note Varieties of triangles and quadrilaterals are dealt with in later topics.

- Abilities*
- (i) To classify polygons in the ways described above.
 - (ii) To name them, and state their characteristic properties.

Discussion of concepts In this topic we make a start with thinking about the various kinds of shapes which can be made from straight lines. These are called polygons. We look for ways of classifying them, and start by using regular/irregular, and number of sides. In subsequent topics, further classifications within the set of triangles and the set of quadrilaterals will be introduced.

Activity 1 Classifying polygons [Space 1.12/1]

A teacher-led discussion for up to 6 children. Its purpose is to help children to differentiate among the various kinds of polygon.

- Materials*
- Polygons, pack 1.* This consists of 30 square cards, on which are 15 regular polygons and 15 non-regular polygons. The regular polygons consist of 3 equilateral triangles, 3 squares, 3 pentagons (5 sides), 3 hexagons (6 sides), 3 octagons. The 15 non-regular polygons are as above, except that no two sides or angles are of equal size. (So the triangle is scalene, and the squares are replaced by non-regular quadrilaterals.) The sizes of the polygons should be varied: but there should not be a clear large/small distinction.

* Provided in the photomasters

- Suggested sequence for the discussion*
1. Lay out an assortment of the polygon cards, about half the pack.
 2. Ask for suggestions how these might be sorted. The two ways we have in mind are (a) number of sides, and (b) regular and irregular. These we hope are the most noticeable.
 3. If other ways of sorting are proposed, give them a fair hearing and let children try to sort their way. Size is quite a reasonable one, but we have tried not to embody this in the material.
 4. Eventually the children will arrive at (a) and (b). Usually (a) is noticed first.
 5. Put down the rest of the pack, and let each child collect the set of all shapes which have a particular number of sides.
 6. When they have done this, let each describe in his own words the set he has collected. E.g., "I have collected the set of shapes with 5 sides."

7. Then tell them the mathematical names for these, and let them all describe their sets again.
8. Ask them to look at their sets, and see if they can sort them further. Usually they will now see the regular /irregular distinction.
9. When they have sorted into these subsets and described in their own terms, introduce them to the generally used words. E.g., “The set of regular hexagons, and the set of non-regular hexagons.”
10. Mix all the pieces, and repeat steps 5 to 9 with children collecting different sets.

Activity 2 Polygon dominoes [Space 1.12/2]

A game for 2 to 4 players. Its purpose is to consolidate the concepts formed in Activity 1, and link them with their names.

Materials • A set of dominoes, as provided in the photomasters.

What they do This game is played according to the usual rules of the game of dominoes, except that a match of word with word is not allowed. (See Space 1.11/2, ‘Angle dominoes,’ for these rules.) The following are some further suggestions which may be incorporated if and as desired.

1. It is good for each player to have a mixture of word dominoes and figure dominoes. To achieve this, first sort the cards into words and pictures. Then deal these two packs in succession.
2. If they want to score, the first to go ‘out’ scores 5, the next 4, etc. Players who do not go ‘out’ score zero.
3. When learning, it is good to help each other. Later, ‘No help’ may be agreed. Players can then often help themselves by looking at existing matches of words with polygons.

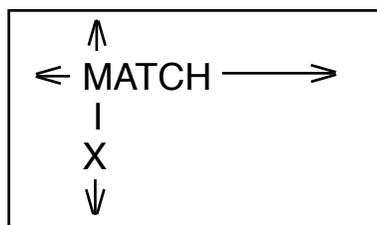
Activity 3 Match and mix: polygons [Space 1.12/3]

A game for 2 to 5 players. Its purpose is to consolidate further the concepts formed in Activity 1.

Materials • Polygons, pack 2.* This is a double pack of the regular polygons as in Activity 1, so there will be 30 cards, now including 6 of each kind of regular polygon. It might be helpful to have this pack of a different colour from pack 1.

- A card as illustrated below.

* Provided in the photomasters



Space 1.12 Classification of polygons (cont.)

1. The cards are spread out face downwards in the middle of the table.
2. The MATCH and MIX card is put wherever convenient.
3. Each player takes 5 cards (if 2 players only, they take 7 cards each). Alternatively the cards may be dealt in the usual way.
4. They collect their cards in a pile face downwards.
5. Players in turn look at the top card in their piles, and put cards down next to cards already there (after the first) according to the following three rules.
 - (i) Cards must match or be different, according as they are put next to each other in the 'match' or 'mix' directions.
 - (ii) Not more than 3 cards may be put together in either direction.
 - (iii) There may not be two 3's next to each other.
 ('Match' or 'mix' refers to the kind of polygon).

Examples (using A, B, C . . . for different kinds of polygon.)

A typical arrangement:

```

C
AAA
BBB
DDD
  B
    A
    
```

None of these is allowed:

```

      BBBB
A      CCC      AA      A
B      DDD      BB      B
C      CC       CC      A
D
    
```

A card which cannot be played is replaced at the bottom of the pile.

6. Scoring is as follows.
 - 1 point for completing a row or column of three.
 - 2 points for putting down a card which simultaneously matches one way and mixes the other way.
 - 2 points for being the first 'out.'
 - 1 point for being the second 'out.'
 (So it is possible to score up to 6 points in a single turn.)
7. Play continues until all players have put down all their cards.
8. Another round is then played.

Discussion of activities Sorting and classifying are among the most fundamental ways in which intelligence functions, and activities based on these have been in use from the very beginning. Activity 1 is a sorting activity, leading to two of the most noticeable ways in which polygons can be classified. Once these categories have been learned, they are consolidated by the games in Activities 2 and 3.

The cards used for sorting and classifying polygons have two advantages over the printed page. First, they show the figures in a variety of positions, since (being square) they may be any way up. This helps to establish that a figure is the same whatever its position, and accustoms children to recognizing figures in all positions. Second, by allowing physical sorting, it leads children to form the categories for themselves rather than the usual method of presenting them with the finished article.

Once again we use a combination of Modes 1 (physical experience) and 2 (discussion, communication of vocabulary by teacher) for schema building, followed by using the newly formed concepts in the co-operative situation provided by a game.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Match and mix: polygons [Space 1.12/3]

SPACE 1.13 POLYGONS: CONGRUENCE AND SIMILARITY

- Concepts*
- (i) Congruence
 - (ii) Similarity
 - (iii) Proportionality, in the context of similar polygons.

- Abilities*
- (i) To recognize when polygons are congruent or similar.
 - (ii) To identify and label corresponding vertices and sides.
 - (iii) To use the proportional property of similar polygons to calculate lengths of sides.

Discussion of concepts At an everyday level, we find it easy to perceive whether objects are exactly alike, or the same shape but different sizes. Congruence and similarity are the mathematical equivalents of these, and the concepts are easily formed from suitable examples. In the present topic, these examples are all polygons.

In mathematics we first go a little further, by noting which points and lines correspond to each other in the two figures. But the importance of similarity comes from its property that the ratios of the lengths of corresponding sides are all in the same proportion, and this applies also to the diagonals. It is this property of similar figures which is the foundation of all scale drawings, which includes maps (other than sketch maps). (It might be worth taking a few moments to consider how many areas of everyday and professional life are made easier by these.) So I thought we should give the children a preview.

I write 'preview,' because a full understanding of this property depends on an understanding of ratio and proportion; and to prove that it is always true (as against reasonable conjecture) depends on geometrical theorems which they will not learn until later. So I have tried to present this important property of similar figures in a way which, though necessarily incomplete, is accurate. This was in fact more difficult to write than if I had first included the topics just mentioned!

Activity 1 Congruent and similar polygons [Space 1.13/1]

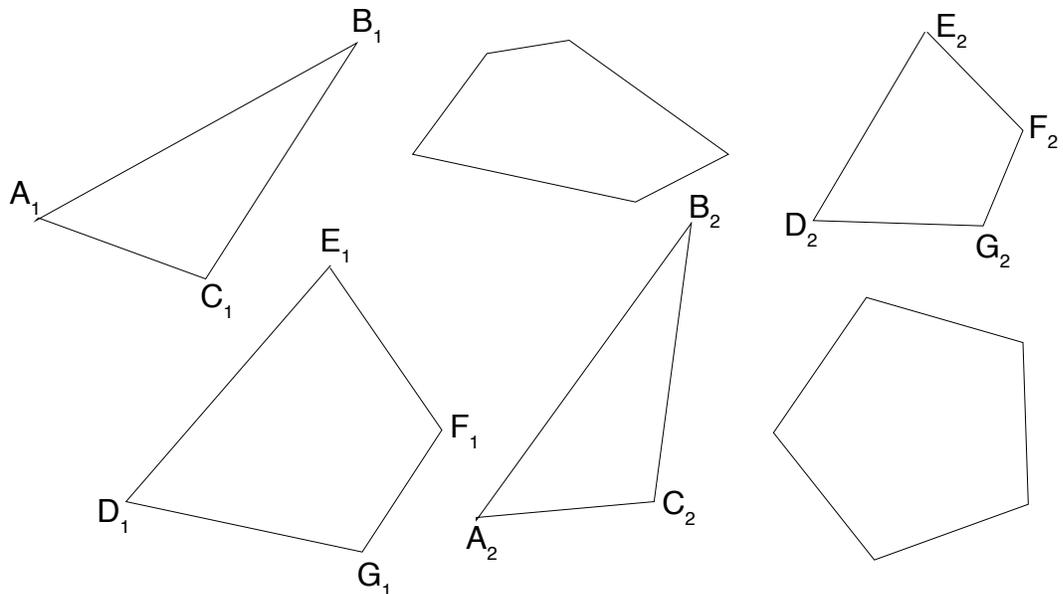
An activity for children to do individually, with guidance. Its purpose is to introduce the first two of the concepts listed for this topic, and help children to acquire the first two of the abilities.

Materials For each child

- A paper copy of Photomaster 193 from the *Volume 2a* photomasters, for use as a worksheet.*
- A transparent copy of the same. (Helpful but not essential.)
- Pencil and paper.

* Alternatively, a laminated copy may be used with water-soluble pens.

- What they do*
1. First, they spend a little time exploring the worksheet, to find some pairs of polygons which look alike.
 2. Are all the pairs alike in the same way? (No. Some are exactly alike except for position. Others are the same shape, but different sizes. Their angles are equal, but their sides are not. The transparent copy is useful for checking this.)
 3. Pairs of the first kind are called congruent. Pairs of the second kind are called similar.
 4. To show which figures belong to the same pair, they are lettered alike, with suffixes 1 and 2 for the first and second of the pair. In the example below, a pair of congruent triangles and a pair of similar quadrilaterals are shown in this way.



5. Notice that the letters are positioned so that they also show which vertices (corners) correspond, i.e., belong together. If this is done correctly, we can also use them to show which sides belong together. E.g., in the above pair of congruent triangles, the sides $A_1B_1 \leftrightarrow A_2B_2$ and $B_1C_1 \leftrightarrow B_2C_2$, etc.
6. Finally, they compare their results and discuss any points arising.

Activity 2 Sides of similar polygons [Space 1.13/2]

A teacher-led discussion in which children work individually on their own worksheets. Its purpose is to introduce the concept of proportionality.

Materials For each child:

- A copy of the worksheet (*SAIL Volume 2a*, Photomaster 194).*
- Ruler graduated in centimetres and millimetres.
- Pencil and eraser.

* Provided in the photomasters

- What they do*
1. They begin with the first pair of polygons, which are similar quadrilaterals, and letter them in the same way as in Activity 1.
 2. Here we are interested in the correspondences between the sides, and especially between their lengths. So they measure all the sides of both quadrilaterals, and write these in their worksheets.
 3. What do they notice? (Each side of the second quadrilateral is twice the length of the corresponding side of the first.)
They record this as follows.

$$\begin{array}{ccc}
 A_1B_1 & \xrightarrow{\quad \times 2 \quad} & A_2B_2 \\
 B_1C_1 & \xrightarrow{\quad \times 2 \quad} & B_2C_2 \\
 C_1D_1 & \xrightarrow{\quad \times 2 \quad} & C_2D_2 \\
 D_1A_1 & \xrightarrow{\quad \times 2 \quad} & D_2A_2
 \end{array}$$

4. They should now repeat steps 1, 2, 3 for pair 2, which is a pair of similar triangles. Does the same relationship apply? (Yes, except that now each side in the larger triangle is three times the size of that in the smaller.)
5. Now for the third pair of polygons, which are non-regular pentagons. They should measure and write in the lengths of all the sides of the smaller, and just one side of the larger — whichever they choose. Then, to think about whether they are ready to make an intelligent guess about the other lengths, but not to speak it until all have had time to think.
6. Any suggestions? We hope that at least one will say something like “In this pair, the multiplier is one and a half for all the sides.” (Meaning the multiplier from the smaller to the larger.) If not, they should measure another side of the larger polygon, and see if this gives them an idea. Likewise if necessary for a third side, by which time light should dawn! (If it does not, then they are probably not ready to continue with this topic and should return to it later.)

7. They should then write in one of their predicted lengths for one of the sides of the larger polygon, check by measurement, and, if confirmed, put a check mark against it; if wrong, a cross. Repeat for the other sides, learning, if necessary, as they go along.
8. Does this principle apply to all polygons? (We can't be certain, but it seems likely. Later on they will be able to prove that it does apply to all polygons.)
9. A general formulation would be appropriate at this stage, but is difficult without the language of proportionality. I would be content with something like the following: "For similar polygons, the multiplier is the same for all pairs of corresponding sides. So if we know it for one side, we know it for all of them."

- Notes*
- (i) (Step 5.) The mathematician's word for an intelligent guess is 'conjecture.' Some children may like to add this to their vocabulary.
 - (ii) The multiplier is not always a whole number. (E. g. the last pair.)
 - (iii) At some stage the question may arise whether the same principle applies to the areas of similar polygons. This is harder to answer. One way would be to apply the grid method of Meas 2.1/4 to find out. Alternatively, we could take two similar squares, one with the sides twice as long as the other. Here the answer becomes obvious — no, it isn't. In the case of the squares, the area of the second is four times that of the first, which opens up an interesting line of investigation for some of the children to follow, perhaps, on another occasion. If anyone suspects that the multiplier for the areas is the square of the multiplier for lengths, congratulations. They are quite right. If the question is not raised by any of the children, I suggest that at this stage it may be better not to introduce this additional aspect.

Activity 3 Calculating lengths from similarities [Space 1.13/3]

An activity for children to do individually. Its purpose is to develop the third of the abilities listed at the beginning of this topic, namely to be able to use the proportional property of similar polygons to calculate lengths of sides of polygons.

Materials For each child:

- A worksheet, see Figure 31.*
- Pencil and eraser.
- Ruler graduated in centimetres and millimetres.

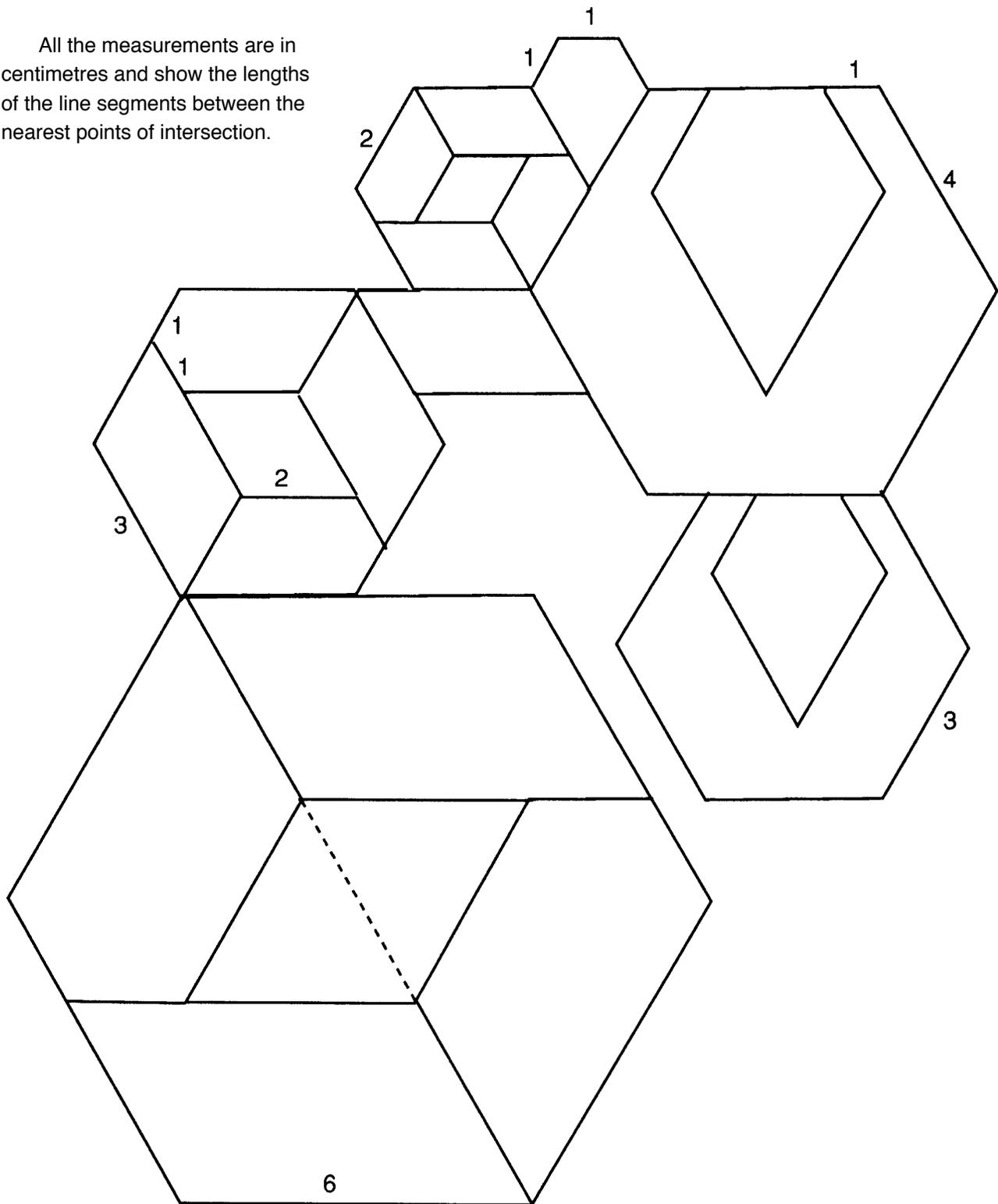
* A copy is provided in the photomasters.

- What they do*
1. A full size illustration of the worksheet appears on the next page. This includes instructions for the children to follow. I hope that they will find the quest satisfying.

Discussion of activities Activity 1 introduces the concepts of congruence and similarity at a perceptual level, and also introduces a notation which calls attention to the related concept of corresponding vertices and sides. Activity 2 develops the concept of proportionality from examples in which this shows clearly from the measurements; and Activity 3 consolidates this concept by using proportionality repeatedly to calculate many lengths from a few given measurements.

Space 1.13 Polygons: congruence and similarity (cont.)

All the measurements are in centimetres and show the lengths of the line segments between the nearest points of intersection.



This figure is made up of regular hexagons, parallelograms (including some rhombi), and non-regular pentagons. Some of the measurements are also given. See if you can use your knowledge of congruence and similarity to CALCULATE all the other measurements and write them in. The dotted line is there as a hint.

If a figure looks as if it is symmetrical, you may assume that it is.

When you have finished, check your results by measuring and/or comparing with someone else.

Figure 31 Worksheet for 'Calculating lengths from similarities' [Space 1.13/3]

Space 1.14 TRIANGLES: CLASSIFICATION, CONGRUENCE, SIMILARITY

- Concepts*
- (i) Ways in which triangles may be further classified: equilateral, isosceles, scalene, right angled.
 - (ii) Congruence of triangles.
 - (iii) Similarity of triangles.

- Abilities*
- (i) To classify triangles in the ways described above.
 - (ii) To recognize whether two triangles are congruent or not.
 - (iii) To recognize whether two triangles are similar, and say in what ways they are the same and in what ways different.

Discussion of concepts

(i) An equilateral triangle has all its sides equal in length, and all its angles equal in size. An isosceles triangle has two (but not three) equal sides and two equal angles. A scalene triangle has no two sides equal, and no two angles equal. A right-angled triangle has one right angle. Further categories are possible, e.g., obtuse-angled; but the above are sufficient for the present. The ‘right angled’ and ‘isosceles’ categories overlap, and so do ‘right-angled’ and ‘scalene.’

(ii) Two triangles are congruent if one fits exactly on top of the other, turning it over if necessary.

(iii) Two triangles are similar if they are the same shape but not necessarily the same size. (So congruent triangles are also similar, but would usually be called ‘congruent.’)

Activity 1 Classifying triangles [Space 1.14/1]

A teacher led discussion for up to 6 children. Its purpose is to help children form group (i) of the concepts above.

- Materials*
- Triangle pack 1.* This is like the polygons pack used for Activity 1 in the previous topic, but consists only of triangles. These are of 5 varieties, 6 assorted examples of each. The varieties are:

Equilateral	Isosceles, right-angled
Isosceles, not right-angled	Scalene, not right-angled
Right-angled, scalene	
 - Set loops.
- * Provided in the photomasters

Suggested sequence for discussion

This follows the same lines as Space 1.12/1, ‘Classification of polygons.’ There is a difference, in that the categories right angled and isosceles overlap. After children have discovered this for themselves, it may be shown using set loops. Start with the loops not overlapping and let children discover the need to overlap them. The same applies to right angled and scalene.

At this stage, informal descriptions are satisfactory. Children may then be told the mathematical names in preparation for Activity 2.

Activity 2 Triangle dominoes [Space 1.14/2]

A game for 2 to 4 players. Its purpose is to help children attach the mathematical names to the concepts formed in Activity 1, while also consolidating the concepts.

Materials • A set of dominoes as provided in the photomasters.

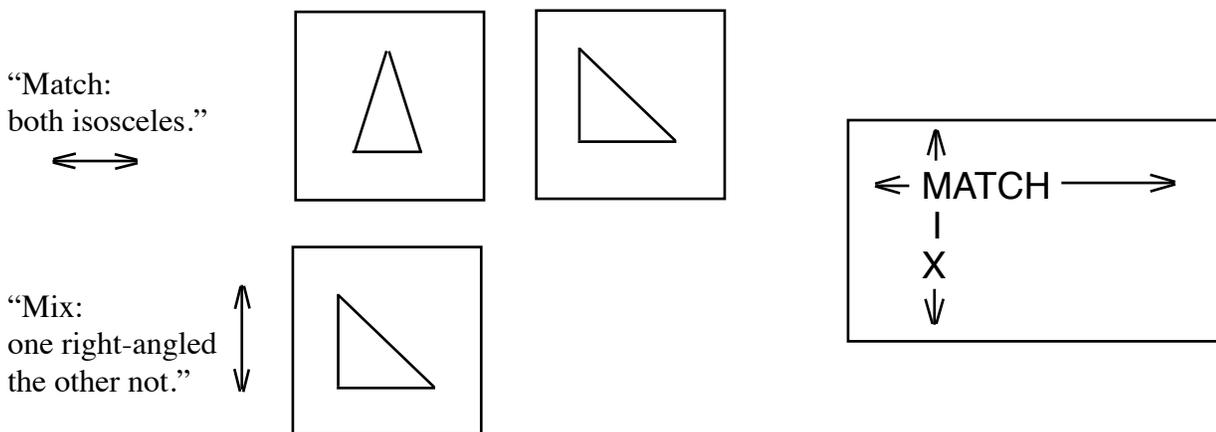
What they do This game is played as in Space 1.11/2 and Space 1.12/2. Reminder: each match must be between a word and a triangle.

Activity 3 Match and mix: triangles [Space 1.14/3]

A game for 2 to 5 players. Its purpose is to combine and practice further the concepts formed in Activities 1 and 2.

Materials • Triangles pack 1. This is the same as used in Activity 1.
• Match and Mix card as in Space 1.12/3.

Rules of the game The game is played in the same way as Space 1.12/3, ‘Match and mix: polygons.’ An interesting development is provided by the overlapping categories. These may be used for either matching or mixing, provided that the player states his reason. E.g.,



Activity 4 Congruent and similar triangles [Space 1.14/4]

A teacher-led discussion for up to 6 children. Its purpose is to introduce these concepts, with some of their further properties.

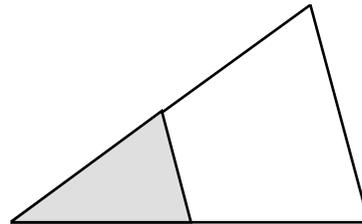
- Materials*
- Triangles pack 2.* This consists of 24 cutout triangles in 12 pairs. Of these, 6 are pairs of congruent triangles, and 6 are pairs of similar triangles. Most of these should be scalene, but 2 pairs in each set could be of one of the ‘special’ kinds (equilateral, isosceles, right angled). Sizes should be assorted. In 2 of the similar pairs, the larger should have sides twice as long as the smaller; in 2 similar pairs, three times as long; and in 2 similar pairs, one-and-a-half times as long.
 - * A suitable set of triangles is provided in the photomasters.

*Suggested
sequence for
discussion*

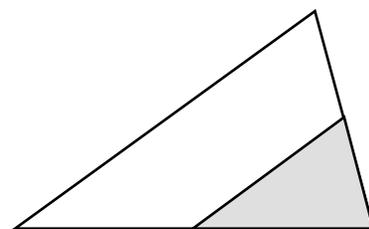
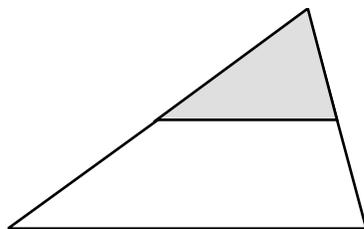
1. Put all the triangles in the middle of the table and ask children to find pairs of triangles which are alike. Depending on the number of children, each will get 2 or 3 pairs.
2. Ask them in turn to describe in their own words the ways in which the triangles in each pair are alike. E.g., (for congruent triangles) “They are exactly the same.” “They fit exactly.” (For similar triangles) “They are the same shape but different sizes.”
3. Tell them the mathematical names for these relationships, and let them in turn describe their pairs again. (E.g., “I have a pair of congruent triangles and a pair of similar triangles.”)
4. Ask them what else they can find out about their pairs of similar triangles.

Among the more important properties are

- (i) If you put the smaller on top of the larger like this, with two sides coinciding, the third sides are parallel.

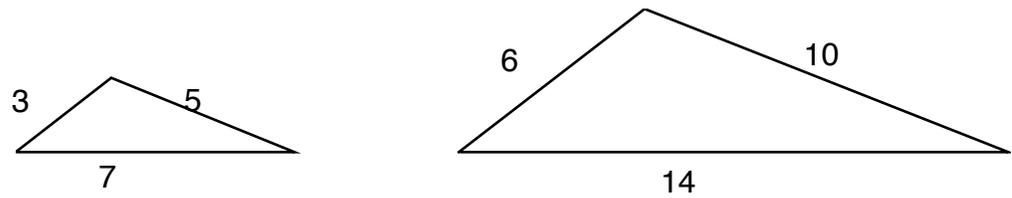


- (ii) This can be done in 3 ways. Here are the other two.



Space 1.14 Triangles: classification, congruence, similarity (cont.)

(iii) This is easier to follow in pictures than in words. E.g., (measurements in cm)



In mathematical language, the ratios of the lengths of corresponding sides are in the same proportion. This formulation lies several years into the future for the children. At the present stage, any kind of description which shows an intuitive grasp is acceptable.

Discussion of activities These activities parallel those in Space 1.12, so it may be worth rereading the discussion at the end of that topic. The advantage of the present approach over the printed page shows particularly clearly in Activity 4, step 4(ii), when the smaller triangle can be physically fitted onto the larger triangle in 3 different ways.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Congruent and similar triangles [Space 1.14/4]

Space 1.15 CLASSIFICATION OF QUADRILATERALS

Concepts Ways in which quadrilaterals may be classified:

- (i) kite
- (ii) trapezium
- (iii) parallelogram
- (iv) rhombus
- (v) rectangle
- (vi) square
- (vii) oblong

- Abilities*
- (i) To classify quadrilaterals in the ways described above.
 - (ii) To name them, and state their characteristic properties.
 - (iii) To relate these classifications.

Discussion of concepts In Space 1.12, we classified polygons by two criteria: number of sides, and regular/irregular. Then in Space 1.14, we took one of these categories, 3 sided polygons (triangles) and made further distinctions between different kinds of triangle. Now we do the same for 4 sided polygons (quadrilaterals).

These we sort by 2 criteria:

- (a) how many sides are equal, and which;
- (b) how many pairs of sides are parallel.

These are not independent. E.g., if a quadrilateral has two pairs of parallel sides, then these pairs of opposite sides must also be equal. This results in interesting relationships between categories: e.g., a square is also a rhombus, a rectangle, and a parallelogram.

Squares and oblongs, as we shall use the terms, are both kinds of rectangle. One sometimes hears people talk about squares and rectangles as though these were distinct categories, but this is rather like talking about girls and children. Girls and boys are both kinds of children. Likewise squares and oblongs both satisfy the criteria for rectangles, which is to have 4 right angles.

The categories are described and illustrated below.

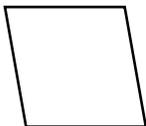
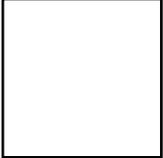
Kite 2 pairs of adjacent sides equal, but not 4 equal sides.



Trapezium 1 pair (only) of opposite sides parallel.



Space 1.15 Classification of quadrilaterals (cont.)

Parallelogram	2 pairs of sides parallel. (Also, 2 pairs of opposite sides equal)	
Rhombus	All 4 sides equal. (It is also a parallelogram.)	
Rectangle	4 right angles. (It is also a parallelogram.)	 or 
Square	4 right angles. 4 equal sides. (it is also a rectangle, a rhombus, and a parallelogram.)	
Oblong	A rectangle which is not a square.	

Activity 1 Classifying quadrilaterals [Space 1.15/1]

A teacher-led discussion for up to 6 children. Its purpose is to teach children the above categories and their names.

- Materials*
- Quadrilaterals pack.* This consists of 36 cards, with 6 examples each of categories (i) to (iv) as listed above, 6 squares, and 6 oblongs. (So there will in fact be 12 rectangles: 6 square rectangles and 6 oblong rectangles.) The examples should vary in size, shape (except squares), and orientation. E.g., the rectangles should not all be drawn with their sides parallel to the sides of the card.
 - Notebook or sheet of paper for each child.
 - Pencils.
- * Provided in the photomasters

- What they do*
1. Lay out an assortment of the quadrilaterals cards, about half the pack.
 2. Explain that the object is to sort these.
 3. Invite the children to look for shapes which belong together. Accept any reasonable suggestions, and let each child collect all the examples of that shape.
 4. Put down the rest of the pack, and let them continue until all are sorted.
 5. Let each child describe in his own words what he has collected, and then tell him the mathematical term for this.
 6. If any cards remain unsorted, sort these too. What happens now will depend on the categories which have been used.

7. If parallelograms and rhombi have been put together, ask if they can be sorted further.
8. If squares and oblongs have been separated, leave them so since this will be dealt with in Activity 2.
9. By now the categories are likely to be fairly near the ones illustrated. If not quite the same, explain that there are various reasonable ways of sorting. The way you are going to show them is a useful one which is widely used, and not very different from theirs.
10. The one to aim at in the present activity is, I suggest, the one shown above with oblongs and squares, parallelograms and rhombi, all separated.
11. Tell them the mathematical names for these, if they do not know already. At this stage, oblongs may be called rectangles (which they are), or oblongs.
12. Each child describes his set using its mathematical name. E.g., "I have collected a set of squares."

Activity 2 Relations between quadrilaterals [Space 1.15/2]

A continuation of Activity 1. Its purpose is to help children find the relationships between the various categories of rectangles.

Materials The same as for Activity 1.

- Suggested sequence for discussion*
1. Tell the children to keep 2 examples of each of their categories, and collect the rest.
 2. Ask for all the parallelograms.
 3. Depending on what you have been given, ask about other examples which you have not been given. E.g., (pointing to the oblongs) "Aren't those parallelograms too?"
 4. Discuss what properties a shape must have to be a parallelogram, and get them to formulate the outcome. E.g., "Rectangles are a kind of parallelogram."
 5. Repeat steps 2, 3, 4 until the children have come to realize that squares, oblongs, and rhombi are all 'kinds of parallelogram.'
 6. Repeat as in steps 2 to 5
 - (a) by asking for all the rectangles. You should get squares and oblongs.
 - (b) by asking for all the rhombi. You should get squares and other kinds of rhombi.
 In each case discuss the criteria, and get the children to formulate the outcome. E.g., "Squares and oblongs are both kinds of rectangle."

Activity 3 “And what else is this?” [Space 1.15/3]

A game for up to 6 children. Its purpose is to consolidate the shape concepts, and relations between them, learned in Activities 1 and 2.

Materials • Quadrilaterals pack. The smaller set with only 2 examples of each kind, used in Activity 2, is more convenient, but the full pack may be used.

- Rules of the game*
1. The cards are spread out face up on the table in such a way that it is easy to find an example of whichever kind is wanted.
 2. Then player 1 picks up a card and asks player 2 (on his left) “What is this?”
 3. Player 2 might reply “An oblong.”
 4. Player 1 asks player 3 (the next player after player 2) “And what else is this?”
 5. Player 3 might reply “A rectangle.”
 6. Player 1 repeats the question to players 4, 5 . . . in turn. Replies might be “A quadrilateral,” “A polygon with 4 sides” and eventually “That’s all.”
 7. Steps 2 to 6 are then repeated with the next player acting as player 1.

Activity 4 “I think you mean . . .” [Space 1.15/4]

A game for 4 or 6 children. Its purpose is further to exercise the shape concepts and relations learned in this topic, with emphasis on language and vocabulary.

Materials • Quadrilaterals pack, as used in Activity 1 but without kites or trapezia. This leaves 24 cards, all parallelograms of one kind or another.
• Card listing the pairs which may be formed: squares, oblongs, rhombi, and parallelograms which are none of the foregoing.*
* Provided in the photomasters

- Rules of play*
1. The cards are shuffled and dealt.
 2. Each player looks at his cards.
 3. The object is to collect pairs of the same kind: squares, oblongs, rhombi which are not squares, and parallelograms which are none of the foregoing.
 4. Any pairs which players already have in their hands are put down.
 5. They then try to acquire more pairs by asking in turn, starting with the player on the left of the dealer.
 6. They may ask whom they like, but must not ask for the card by name. So a player wanting, say, a rhombus might say “Please may I have a parallelogram with all its sides equal?” The one asked might reply “I think you mean a rhombus,” followed by “Here you are” or “Sorry.”
 7. A player asking by name loses his turn.
 8. When a player makes a pair, he puts it down.
 9. Scoring is as follows. 2 points for first ‘out,’ 1 point for second ‘out,’ plus 1 point for each pair.
 10. Another round may then be played.

Note Rule 6 requires that a parallelogram which is not a square, oblong, or rhombus, must not be called a parallelogram directly. It might be called “A quadrilateral with two pairs of parallel sides, not all equal.”

Discussion of activities Activity 1 is concerned with sorting and classifying – fundamental activities of intelligence which have been in frequent use from the beginning. Activities 2, 3, 4 are more sophisticated, however, since the present classes overlap. This gives the opportunity for reflection on the way these categories are related to each other. Activity 3 starts with an example, and asks in what ways it can be categorized. Activity 4 starts with a category, and ends with a particular member of that category (if the asker is lucky). It also emphasizes the language of description and of relationships.

OBSERVE AND LISTEN

REFLECT

DISCUSS



“I think you mean . . .” [Space 1.15/4]

Space 1.16 CLASSIFICATION OF GEOMETRIC SOLIDS

Concepts The defining characteristics of the better known geometric solids. I suggest the following as a basic list: cube, cuboid, prism (at least two kinds), pyramid, and cone. Your set may also include others, such as a tetrahedron.

Ability To state and discuss these accurately, first with and then without the help of physical models. (We are not, however, expecting definitions which are mathematically rigorous.)

Discussion of concepts They should be familiar with the concepts and names of the solids themselves, or at least most of them, having been introduced to them in Space 1.7 in *Volume 1*. We are now examining these in more detail, and moving towards a statement of defining characteristics for each.

Activity 1 What must it have to be . . . ? [Space 1.16/1]

A teacher-led discussion for a small group. Its purpose is to help them arrive for themselves at the defining characteristics of the solids listed above.

Materials • A set of geometric solids.

- Suggested outline for the discussion*
1. A cube is an easy example to start with. Put one out for them to look at, and ask what it is called. (A cube.)
 2. “What must it have, to be a cube and nothing else?” (A cube has six faces, all squares. It is probably enough to say that all of its faces are squares, since it is difficult to think of a solid of which the faces are all squares which is not a cube. But it might not be easy to prove that this is necessarily so.)
 3. Next, I suggest, a cuboid. “What must it have to be a cuboid and nothing else?” (A cuboid has six faces, all rectangles. Some, but not all of these rectangles may be squares. If all of the faces are squares, it is a cube.)
 4. Continue similarly with the other solids. For your convenience, I list their defining characteristics below. All of the descriptions are (I hope) sufficient, but not every detail is necessary for a complete definition.
 5. Prism. It has two parallel and congruent faces (the bases), which are both polygons. The lateral faces are usually rectangles (making a ‘right prism’), but your set may include a ‘skew prism’ which also has parallel congruent polygons for bases, but the lateral faces are parallelograms. Your set is likely to include a ‘triangular prism’ and a ‘pentagonal prism’ or a ‘hexagonal prism.’ A ‘square prism’ would be called a cuboid.
 6. Pyramid. Its base is a polygon, usually a square. Its sides are all triangles, sharing a single point as vertex at the top of the pyramid. A ‘triangular pyramid’ is also called a tetrahedron (see below).

7. Cone. Its base is usually a circle, and it has just one other surface, a curved surface with the base as one edge and a point as the other. There has to be a straight line from the summit to a point on the base. (For your own information, the concept has been generalized to include any closed curve as base. The cone is then 'swept out' by a moving straight line from the apex to a point which travels all around the base.)
8. Tetrahedron. This has four faces, all triangles.

Note All of the solids in the sets usually sold are likely to be of regular shapes. I do not think this matters, but I think we should tell the children that these are not the only kind. In topics 12 to 15 they have learned that polygons can be regular or non-regular, so we need just to mention that this is also the case with geometric solids, even though the set only includes regular solids.

<p>Discussion of activities This is a straightforward activity which requires children to reflect on their global concepts of the various solids, to analyze them, and to relate them to their existing knowledge of two-dimensional geometry.</p>

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.17 INTER-RELATIONS OF PLANE SHAPES

Concept A variety of part/whole relationships between smaller and larger geometric figures.

- Abilities*
- (i) Combining smaller figures to make larger figures.
 - (ii) Finding smaller figures within larger figures.

Discussion of concept The ability to see how parts fit together to make a whole, and to find shapes within other shapes, is an important one for later geometry. Here we are concerned with being able to think of, e.g., a parallelogram as made up of two congruent triangles. It is a good example of relational thinking in geometry, analogous to the relationships between numbers which are so important in numerical mathematics.

Activity 1 Triangles and polygons [Space 1.17/1]

An activity for up to 6 children. Its purpose is to teach relationships, including angular relationships, between triangles and regular polygons.

- Materials*
- Triangles/polygons set.* These are made by cutting out regular polygons, and then cutting these into isosceles triangles, as illustrated in Figure 32. All the angles should be marked, on both sides.
- * Provided in the photomasters

What they do Stage (a)

1. Each child takes a triangle of a different shape.
2. They collect all the triangles which are congruent with the one they have.
3. Each makes a regular polygon with their triangles. (A square may be made using 2 or 4 right angled isosceles triangles.)
4. They examine their polygons, and add together all the angles surrounding the centres.

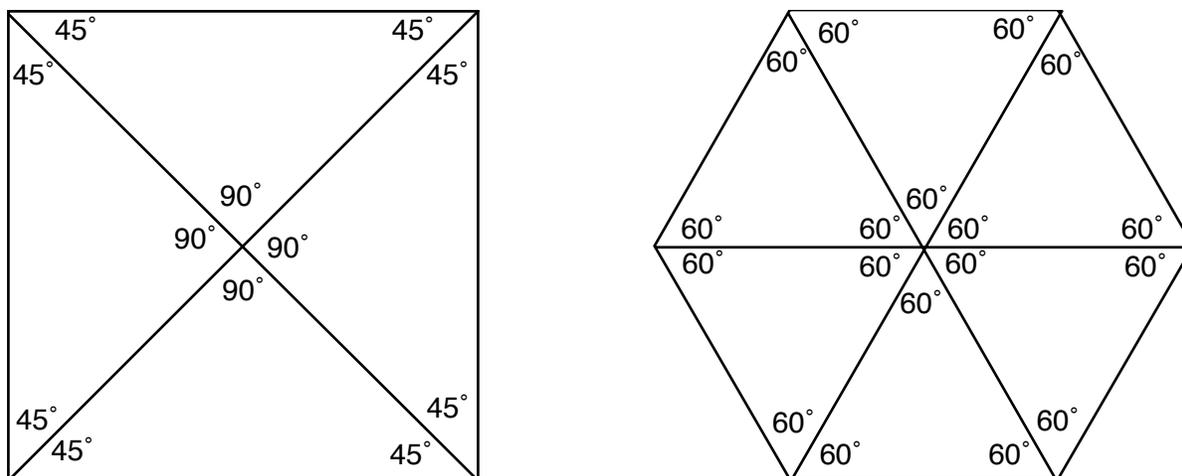


Figure 32 Triangles/polygons set

Stage (b)

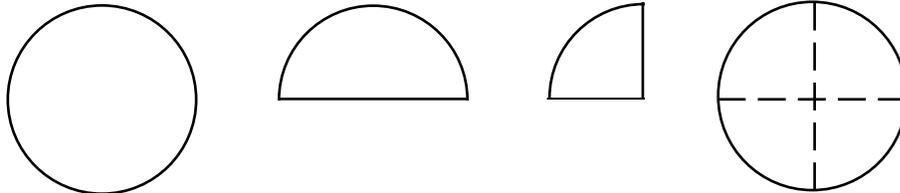
1. The first child shuts his eyes and takes a triangle.
2. He looks at it and predicts what kind of regular polygon can be made using it, and how many congruent triangles will be needed.
3. The others help him to collect all the triangles congruent with the one he has.
4. He tests his prediction.
5. If successful he explains how he knew.
6. Steps 1 to 5 are repeated by another child.

Activity 2 Circles and polygons [Space 1.17/2]

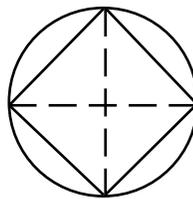
An activity for up to 6 children. Its purpose is to teach relationships between circles and polygons.

- Materials* For each child:
- Safety compasses.
 - A protractor.
 - Pencil and paper.
 - Scissors.

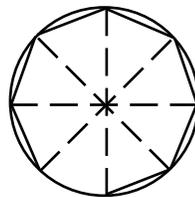
- What they do*
1. Each child draws a circle on paper and cuts it out.
 2. He folds it along a diameter, then again, and opens it flat.



3. He draws a square.



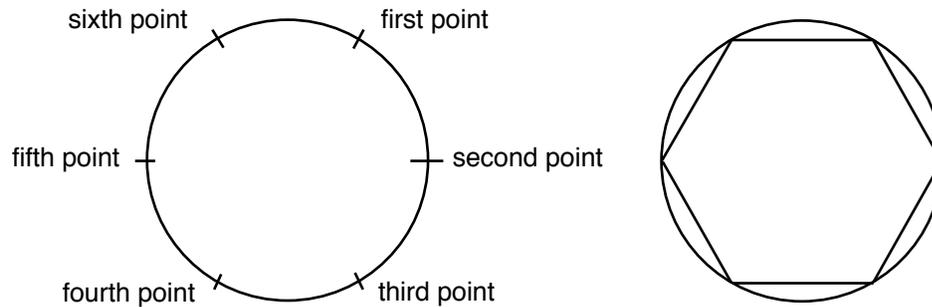
4. Each child cuts out another circle, and repeats step 2, but folds once more.
5. In this way he draws an octagon.



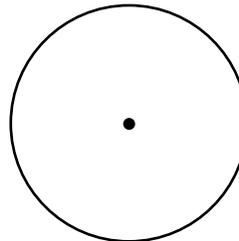
6. They mark the sizes of angles at the centre. These are arrived at by calculation, not measurement.
7. Next, a hexagon. The circle for this is not cut out.

Space 1.17 Inter-relations of plane shapes (cont.)

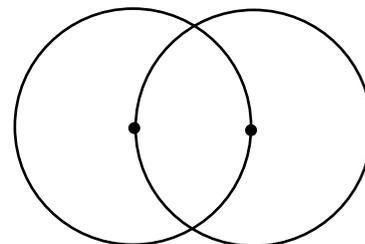
8. A first point is marked anywhere on the circumference. Using the same length of radius as that by which the circle was drawn, they use the compasses to mark second, third . . . points round the circumference, as shown below. The distance from the sixth to the first point should be found to be the same as all the others.



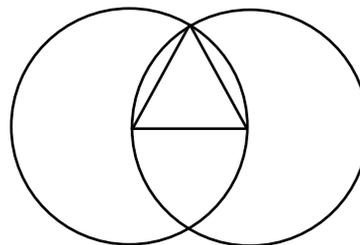
9. The points are joined to form a hexagon, as above.
 10. This is a very satisfying method. Ask “Why does it work?” Hint: join all the points to the centre. Second hint if necessary: look for equilateral triangles.
 11. Next, the equilateral triangle. This uses circles in a different way.
 12. Start with a circle.



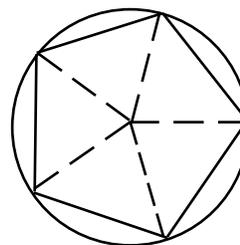
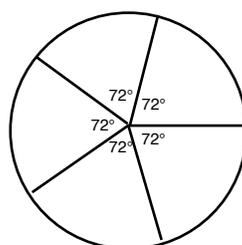
13. Take any point on the circumference as centre for another circle, using the same radius.



14. This gives two equilateral triangles, of which only one is shown here.



15. Ask: “Do we need to draw the whole circles? If not, how little can we do with?”
 16. A pentagon cannot be drawn by a method of the kind above. It is necessary to calculate the central angle, and use a protractor to draw angles of the required size, as below.



Activity 3 “I can see . . .” [Space 1.17/3]

A game for up to 6 players, but is probably best for 2 or 3. Its purpose is to develop the ability to see figures within figures.

- Materials*
- A complex figure, such as Figure 33. This should be covered with transparent film.*
 - Non-permanent markers, a different colour for each player.
 - Scoring rules on a piece of thin card.
 - A bowl of counters.

* Provided in the photomasters. Alternatively, the figure could be duplicated and expendable. In this case any kind of coloured marker will do.

- Rules of play
1. The figure is circulated from player to player.
 2. Each in turn ‘collects’ embedded figures, such as a pair of congruent triangles, a right angle, a pair of congruent angles or lines, a parallelogram.
 3. He names these, and claims them by marking them with his colour. This is done on or within the shape, as is best in the particular case.
 4. If the other players agree, he takes a counter from the pool.
 5. The same embedded figure may be claimed in various ways, but not twice in the same way.
 6. The winner is the one with most counters, when no one can think of anything new to claim and gain agreement thereto.
 7. One point is scored for each of the following (By agreement, this list may be extended.):

A pair of parallel lines.	An oblong.
A pair of congruent triangles.	A square.
A pair of congruent lines.	An isosceles triangle.
A pair of congruent angles.	An equilateral triangle.
A parallelogram.	

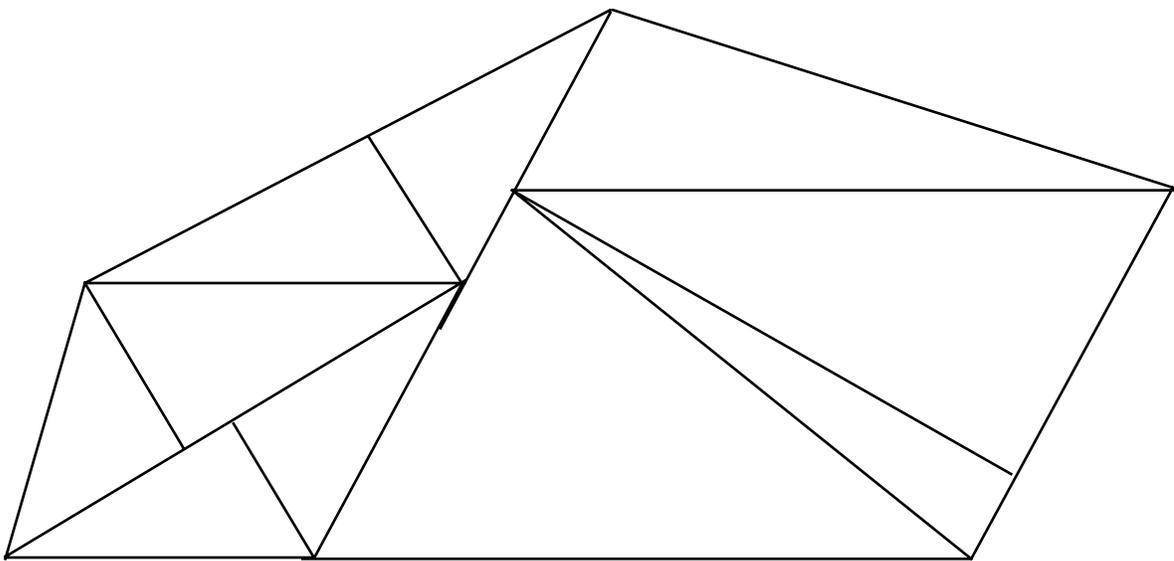


Figure 33 ‘I can see . . .’

Activity 4 Triangles and larger shapes [Space 1.17/4]

A game for up to 6 children, but is probably best for 2 initially. Its purpose is to develop awareness of the relationships between triangles and other shapes.

- Materials*
- A set of cutout triangles, as provided for in the photomasters. For the larger groups a double set will be needed. It is useful to make these sets of different colours.
 - Scoring rules on a piece of thin card.

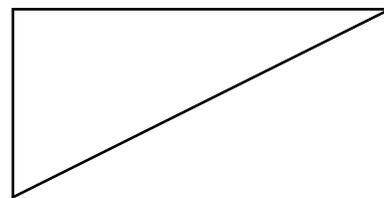
Preliminary Allow the children time to experiment with putting the cutout pieces together in various ways, and finding out what larger shapes they can make.

- Rules of play*
1. The object is to collect triangles and make larger shapes, e.g., parallelograms.
 2. The triangles are spread out in the middle of the table.
 3. Each player in turn takes a triangle, with which (from the second round on) he starts building larger figures.
 4. When there are no longer enough triangles for each player to take one more, one more round is played as in step 5.
 5. In this final round, each player may if he wishes exchange one of his triangles for one in the pool.
 6. Players then count up their scores, each in turn explaining to the others what points he claims and why.
 7. The scoring rules are as follows:

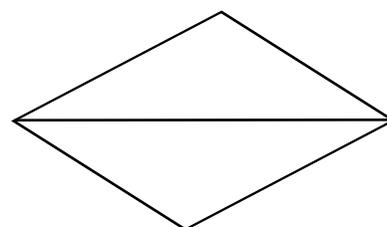
composite parallelogram	1
composite rhombus	1
composite rectangle	1
composite square	1
composite right angle	1
composite straight line	1
composite kite	1
composite oblong	1

The same figure may be scored in as many ways as possible. Example:

composite oblong	1
composite rectangle	1
composite parallelogram	1
2 composite right angles	2
total	5

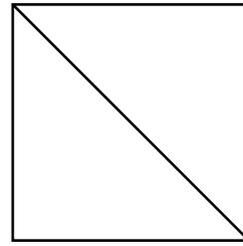


composite rhombus	1
composite parallelogram	1
total	2

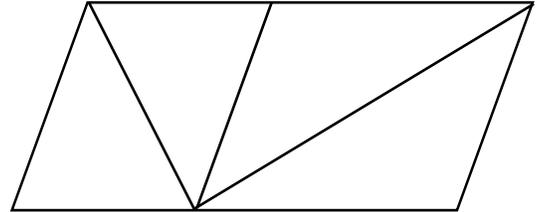


Space 1.17 Inter-relations of plane shapes (cont.)

composite rhombus	1
2 composite right angles	2
composite parallelogram	1
composite rectangle	1
composite square	1
total (if he says them all)	6



3 composite parallelograms	3
2 composite straight lines	2
total	5



Players may rearrange their pieces at any time up to their final declaration.

Discussion of activities

In Activity 1, physically putting shapes together is used as the means for learning part-whole relationships. Activity 2 takes a step towards pencil-and-paper geometry. It is still based on physical manipulation, but calculation and inference are also used. Activity 3 diversifies from regular polygons to all kinds of larger shapes. Prediction is also involved: triangles are chosen because, by combining them with other shapes in imagination, the player expects a desired result. He has to commit himself to taking the piece before this can be tested. So in these three activities, we have another clear example of Mode 1 schema building and testing.

The preliminary to Activity 4 consists of combining the cutout triangles physically, and observing the outcome. The activity itself involves combining the shapes first in imagination, and then testing the prediction physically. So here we have a very clear example of Mode 1 schema building followed by testing.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.18 TESSELLATIONS

Concept That of a tessellation.

Ability To discover whether a given shape will tessellate, or not.

Discussion of concept A *tessera* is one of the pieces of marble, glass, or tile from which a mosaic pavement is made, and tessellating is combining these so as to form a mosaic. The everyday meaning of tessellation (which is centuries old) and the mathematical meaning are the same; except that mathematicians, as one might expect, do it on paper. So the transition from cardboard or plastic *tessella* (little *tessera*) to drawing, which we follow in this topic, also follows its historical sequence.

A shape can be tessellated if a number of them, all the same size, can be fitted together (rotated if necessary) to cover a surface without gaps. Here we shall concern ourselves with tessellations using one shape only.

Activity 1 Tessellating regular polygons [Space 1.18/1]

A group investigation for up to 6 children, working singly or in pairs. Its purpose is to teach the concept of tessellation, and introduce some of the easier examples.

- Materials*
- About 20 of each of the following regular polygons,* cut out of cardboard or plastic, and stored in separate plastic bags. Some of those in each bag have their interior angles marked, on both sides. If the polygons in a bag can be of assorted colours, so much the better.
 - Equilateral triangles.
 - Squares.
 - Pentagons.
 - Hexagons.
 - Octagons.

* Provided in the photomasters

- What they do*
1. Using the equilateral triangles, demonstrate what is meant by tessellation.
 2. Each child or pair then takes a bag.
 3. Their first task is to discover, by experiment, whether the polygons they have can be tessellated, or not.
 4. Their second task is to find out why they can or cannot be tessellated. The marked angles provide a clue.
 5. In turn, the children tell the rest of the group what they have found out.

Activity 2 Tessellating other shapes [Space 1.18/2]

A group investigation for up to 6 children, working singly or in pairs. Its purpose is to expand children's concept of tessellation to include rectangles, parallelograms, all triangles, and a variety of other shapes. It also takes children from physical tessellation to doing it on paper.

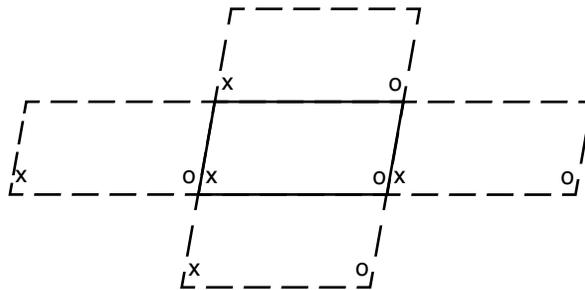
Materials For each child:

- Plain paper.
- Pencil and eraser.
- Ruler.
- Instruction card for the group, as below and in photomasters.

TESSELLATIONS

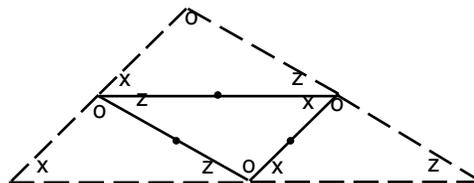
Which of these shapes can be tessellated? Which cannot?
 Rectangles, any shape.
 Parallelograms, any shape.
 Triangles, all varieties.
 What else can you find out?
 Share out the task among yourselves, and produce a combined report.

- What they do*
1. They follow the instructions on the card.
 2. In fact, all these shapes can be tessellated.
 3. Rectangles are fairly obvious. Their right angles fit together to form straight lines.
 4. Parallelograms also tessellate quite easily. Opposite sides, being parallel, have the same direction; so they fit if the parallelogram is slid along any side without rotation.



From this we can see that the angles at each end of a side combine to give a straight line.

5. Any kind of triangle can be tessellated, since two congruent triangles make a parallelogram. To get the second in position, rotate the first half a turn about the mid point of any side.



From this we can see that the 3 angles of a triangle together make a straight line.

The dotted larger triangle is similar to each of the smaller ones, since the angles of the original triangle are each reproduced once.

Activity 3 Inventing tessellations [Space 1.18/3]

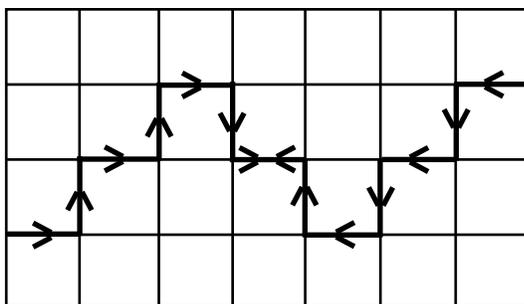
An investigation which may be followed singly or in pairs. Its purpose is to open up the concept of tessellation and teach children two ways of generating their own.

- Materials* For each child:
- Plain and squared paper.
 - Tracing paper.
 - Pencil and eraser.
 - Ruler.

- What they do*
1. Demonstrate one of the methods for generating tessellations described below.
 2. Then allow the children time for inventing their own.
 3. Let them show and discuss their results.
 4. Repeat steps 1 to 3 with the other method.
 5. These tessellations may be coloured.

Method 1 Dissection of a rectangle

If a rectangle is dissected into two halves of the same shape, these will tessellate. Squared paper is useful for this. A suitable dissection line may be obtained by working inwards from the two ends, as below.

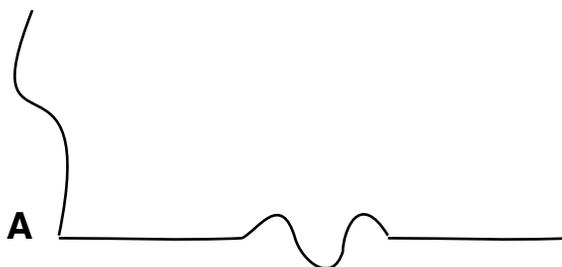


Method 2 Parasquigglegrams

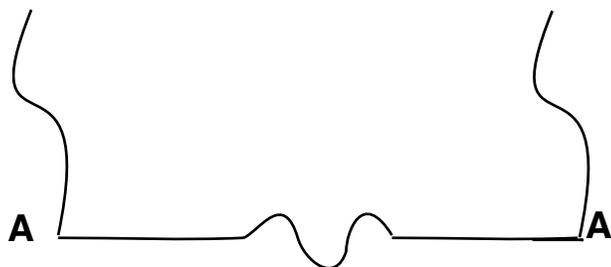
I discovered this method for myself, and have not yet seen it elsewhere, though it is too simple not to have been discovered by others. The name is my own. Tracing paper is useful for reproducing the squiggles.

This is a nice use for translations.

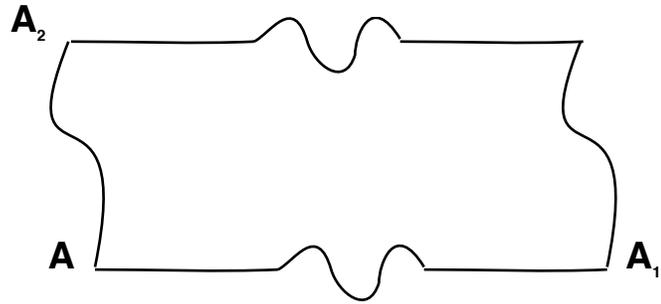
Start with 2 intersecting squiggles.



Move A without rotation to the other end of one squiggle, taking the other squiggle with it.



Now the other way.



Any shape made in this way can be tessellated.

Activity 4 Tessellating any quadrilateral [Space 1.18/4]

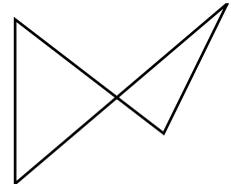
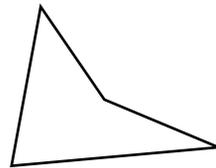
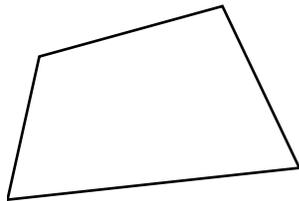
A more difficult investigation for children working singly or in pairs. Its purpose is to challenge the abilities of the brighter children.

Materials For each child:

- Pencil and plain paper.
- Tracing paper.
- Ruler.

- What they do*
1. Tell them that it is possible to tessellate a quadrilateral of any shape.
(Note: we mean quadrilaterals which are

convex
not concave
nor crossed.)



- Since the children have only encountered the first kind so far, we do not need to mention the other possibilities unless they do.)
2. They may like to try to find out how without help.
 3. The method is to rotate the quadrilateral about the midpoint of each side in turn. By drawing a diagonal in a dotted line, and including this in the rotated versions, two sets of tessellating parallelograms result. For clarity this has been omitted from Figure 34.
 4. From this we see that the interior angles of any quadrilateral add up to 360 degrees.
 5. Let the children draw their own tessellations of a quadrilateral of their own choice.

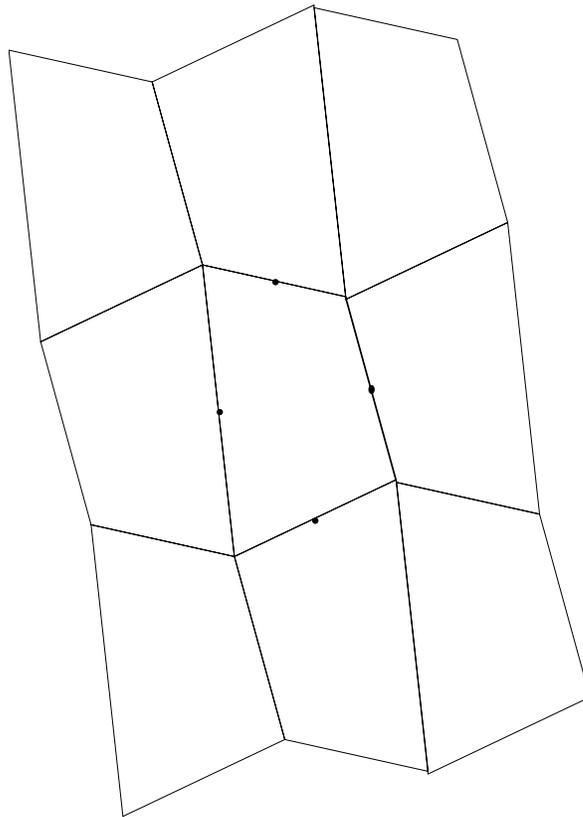


Figure 34 Tessellation of an irregular quadrilateral

Discussion of activities In a now-familiar sequence, we start with schema building by Mode 1, physical experience, and progress to Mode 3, creativity. Discussion plays an important part as always. This topic is open ended towards creativity of an artistic kind. If possible, do show the children some of the pictorial tessellations of the artist M. C. Escher. These are published both in book form, and as reproductions sold in art stores.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.19 DRAWING NETS OF GEOMETRIC SOLIDS

Concept That of a net of a given geometric solid.

Abilities (i) To draw the net of a given geometrical solid.
(ii) To recognize what solid a given net belongs to.

Discussion of concept This has been well prepared in the earlier topics, in *Volume 1*, “My pyramid has one square face . . .” [Space 1.7/5] and ‘Does its face fit?’ [Space 1.7/6]. It continues the development of the relationships between two- and three-dimensional objects, and Activity 2 calls visual imagination into play.

Activity 1 Drawing nets of geometric solids [Space 1.19/1]

An activity for children to do individually, after an introductory demonstration. Its purpose is to introduce children to above concept, and to develop the first ability.

Materials Shared among a group:

- A standard set of geometric solids, as already used in topic 16.

For each child:

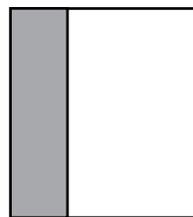
- Pencil, paper, eraser.

Suggested outline for the introductory demonstration

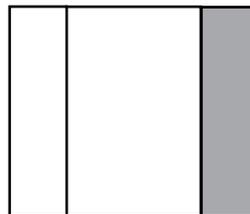
1. A cuboid is a good example to begin with. Put it on the paper with the largest face down, and draw around this. (In all these diagrams, the shaded area represents the face which is in contact with the paper.)



2. Roll it to the left, over the left-hand edge, and draw around it again.



3. Now roll it back to the centre position, and again over the right-hand edge to the position shown. Draw around this.

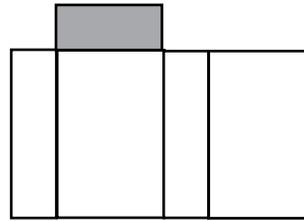


Space 1.19 Drawing nets of geometric solids (cont.)

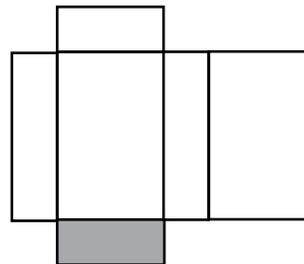
4 Likewise, again.



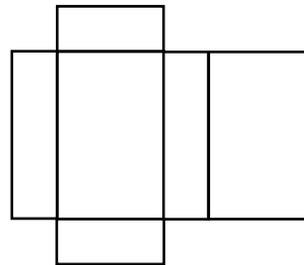
5. Back to position 1, roll this time over the upper edge, and draw around this face.



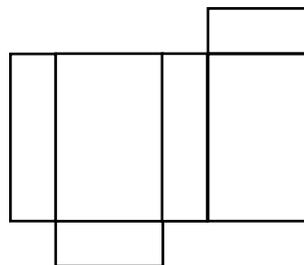
6. Repeat step 5, but this time roll over the lower edge.



7. Remove the cuboid, and we are left with its net:



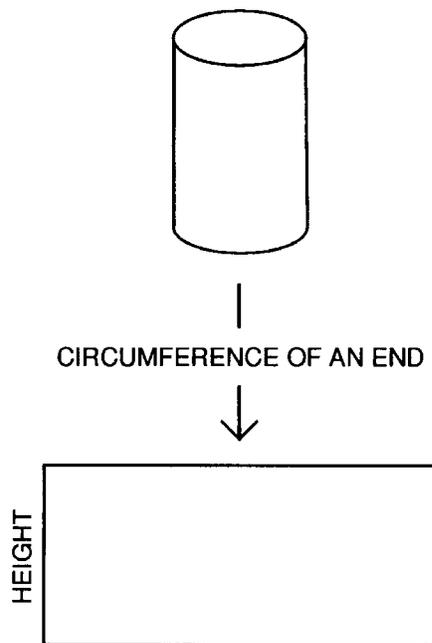
8. This can be done in several slightly different ways. Here is one of them.



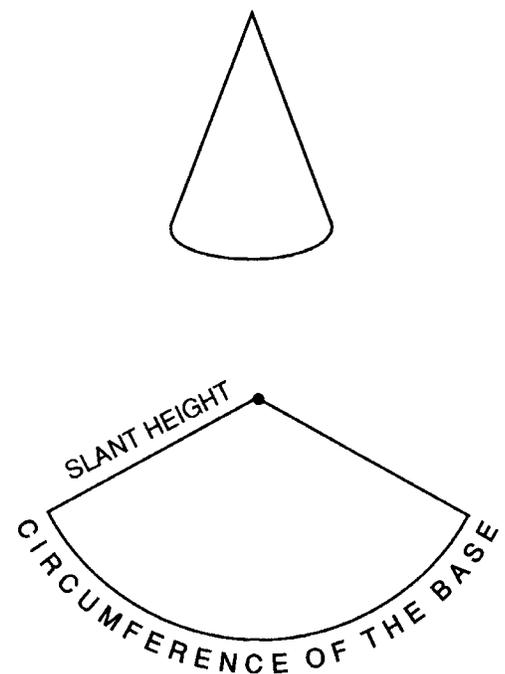
- What they do*
1. Each child chooses a solid, and draws its net in the way they have just seen.
 2. He then writes its name of the solid at the bottom of the paper.
 3. Steps 1 and 2 are repeated until nets have been drawn of all the solids.
 4. The solids are returned to a central pool, and the papers collected in readiness for Activity 2.

Some other nets:

Net of a cylinder



Net of a cone



A sphere does not have a net.

Activity 2 Recognizing solids from their nets [Space 1.19/2]

A sequel to Activity 2, also for a small group. Its purposes are to check the nets already drawn, and strengthen the connections between individual solids and their nets.

Materials • The solids and their nets, from Activity 1.

- What they do*
1. The titles at the bottom of the drawings are folded underneath so that they cannot be seen. The papers are then shuffled and put in a pile centrally.
 2. The solids are also put in a central pool. The player whose turn it is to start takes the top paper from the pile, and then takes the solid of which he thinks it is the net.
 3. The others say whether or not they agree.
 4. This continues until all the papers have been used.

Activity 3 Geometric nets in the supermarket and elsewhere [Space 1.19/3]

An activity for a small group. Its purpose is to relate the concept of a net to one of its applications.

- Materials*
- An assortment of empty cereal boxes, and the like. If possible, these should be in matching pairs.
1. The boxes are carefully pulled apart where they have been stuck, so that they can be flattened.
 2. Each is then put beside a similar box in its original form, for comparison. These can be seen to be similar to the nets of cuboids, with the addition of tabs by which to stick them together.
 3. Some of these pairs could be put on display, if desired .

Discussion of activities These are both straightforward activities, providing a simple introduction to the concept of the net of a solid.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Space 2] SYMMETRY and MOTION GEOMETRY

Space 2.1 REFLECTIONS OF TWO-DIMENSIONAL FIGURES

Concept The mirror image of a picture or other two-dimensional figure, also called its reflection.

- Abilities*
- (i) To recognize when one figure is the reflection of another.
 - (ii) To construct the geometrical reflection of a simple figure, for a given mirror line.

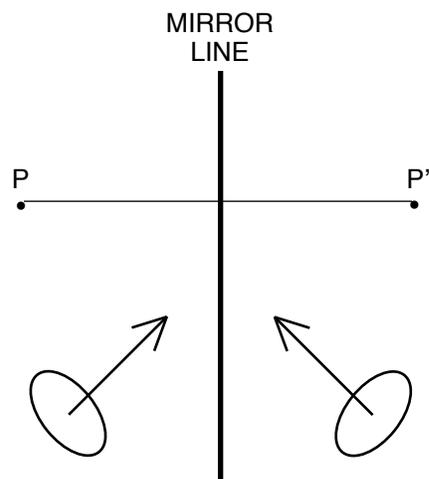
Discussion of concept 'Reflection' has the same mathematical meaning as that in everyday life. The only difference is that in mathematics we usually work in two dimensions, so while the objects are visible in full in the plane of the paper, the mirror usually only shows up as a line. However, there is nothing to prevent our using a real mirror, held along the mirror line perpendicular to the paper, and this is a good way to begin.

Mathematically, if P is any point, then its mirror image P' is on the other side of the mirror line and the same perpendicular distance away from it. This means that the mirror line is the perpendicular bisector of PP' .

The mirror image of any drawing or mathematical figure is made up of all the points arrived at in this way.

You can check that this definition does in fact give a mirror image which coincides with that obtained by holding a mirror edge on to the paper, along the mirror line. It also gives the same result as the optical one, based on how light is reflected from a mirror and how our eyes and brain interpret the result.

The terms 'mirror image' and 'reflection' (noun) are interchangeable. The meaning of 'flip' is slightly different, and is discussed in topic 10. Until then I suggest that it is better not used.



Activity 1 Animals through the looking glass [Space 2.1/1]

An activity for a small group of children. Its purpose is to introduce the concept of a mirror image in an easily recognizable embodiment.

- Materials*
- A pack of animal cards for the group.*
 - A small mirror for each child.†

* Provided in the photomasters. This consists of three pairs of pictures of each of five different animals, making thirty cards in all. In each pair, one picture is the mirror image of the other. If there are more than five children, a double pack may be used. In this case it is helpful to make them in different colours, so that they can easily be separated again for Activity 2.

† E.g., a pocket- or purse-sized mirror . . . or a MIRA.

- What they do*
1. The cards are shuffled and spread out face upwards on the table.
 2. Each child then chooses a different animal, and collects all the cards for that animal.
 3. They then take any one of their cards, and find another which looks like the reflection of the first one in their mirror. (The cards should be flat on the table, an edge of the mirror on the table, and the plane of the mirror at right angles to it. This is important for what will follow in later activities.)
 4. They continue until they have paired all their cards in this way.
 5. Further experiment should be encouraged, with sharing of what they have found out and verification by the others. E.g., it is possible to position the cards with one on each side of the mirror, in such a way that the second card is where the mirror image appears to be. When the mirror is removed, the second card takes the place of the image. What happens if the mirror is held facing the other way round? And what if one of the animals is not pointing directly towards the mirror?

Activity 2 Animals two by two [Space 2.1/2]

This game follows directly on from Activity 1, and its purpose is to consolidate the newly formed concept of a mirror image. The thirty card pack is suitable for a group of up to four children, above which it is best to have a second pack and another group.

- Materials*
- Animals pack, as for Activity 1.
 - A single mirror should now suffice for the whole group.

- Rules of the game*
1. The object of the game is to collect pairs in which each card is the mirror image of the other. This is done as follows.
 2. The cards are shuffled, and put in a pile face down on the table. The top card is turned face up and put separately.

3. In turn, each player turns over the top card of the face-down pile, and puts it face up separately from the other face-up card(s). If he sees a pair, he takes it and puts the cards side by side where the others can see. If the others agree, he keeps the pair. It is always good to check by using the mirror.
4. More pairs will gradually show as more cards are turned face-up.
5. The player who has just turned a card has first chance to make a pair. However, if a pair is overlooked, the player whose turn it is next may claim it before turning over another card. If his claim is correct (as agreed by the other players), he may also have his turn as in step 3. If however he is found to have claimed incorrectly, he loses his turn. (This is to discourage reckless claiming.)
6. If at any time there is no face-up card left, the player whose turn it is may turn over two cards.
7. When all the cards have been turned over, the winner is the player who has the most pairs.

Activity 3 Drawing mirror images [Space 2.1/3]

An activity for children to do individually, checking their results with others in their group. Its purpose is to introduce them to the method for constructing mirror images according to the mathematical definition described in the discussion of concepts, above.

- Materials*
- A worksheet for each child, see Figure 35.*
 - Several mirrors may be shared by a small group: it is not essential for them to have one each.
 - Coloured felt-tip pens (fine), which may be shared by a small group.
- * Photomaster provided in *SAIL Volume 2a*.

What they do 1. Full instructions are given on the worksheet, as illustrated in Figure 35.

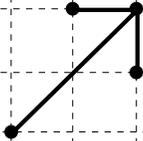
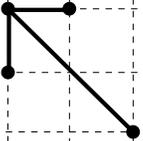
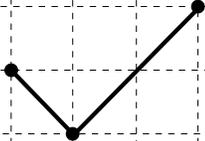
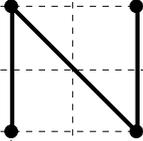
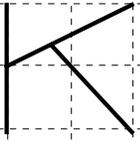
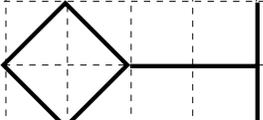
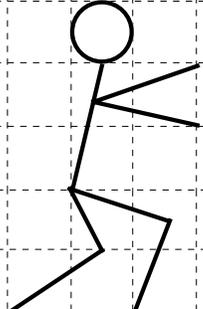
Discussion of activities Activity 1 introduces the mathematical concept of reflection at an intuitive level, closely related to their everyday experience. Strong cues are provided in the pictures, so the children provide for themselves a succession of examples of mirror images, which they check experimentally (Mode 1) and by discussion with peers (Mode 2). Activity 2 is a game for consolidating the concept, using the same materials. In Activity 3, the use of squared paper helps them to make the concept explicit, and to discover and use the mathematical method for constructing them.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 2.1 Reflections of two-dimensional figures (cont.)

OBJECT	MIRROR LINE	IMAGE
		
		
		
		
		
		
		

On the left is an arrow, and on the right is its mirror image. Check this by holding a mirror on the mirror line. Then use the dots to work out how the image was drawn.

Now see if you can draw the mirror image of the check mark. Start with the dots. Check two ways: with your mirror, and by comparing your drawing with someone else's.

Do the same with this one. (Be careful!)

From here on, provide your own dots. You may be able to manage without them.

Continue with the rest on this page, checking each of your drawings in the two ways described above.

When you have finished, you may like to colour over the lines with a felt tip pen (not too thick). Use the same colour for each object and its image, and a different colour for each pair.

Figure 35 'Drawing mirror images' worksheet

Space 2.2 REFLECTION AND LINE SYMMETRY

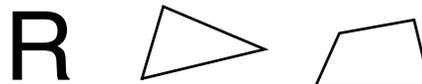
Concept Line symmetry.

Abilities (i) To recognize whether or not a figure has line symmetry.
(ii) To indicate all its axes of symmetry.

Discussion of concepts A figure is symmetrical if one half of it is the mirror image of the other half. In this case, the mirror line is called the axis of symmetry.

Some figures have more than one axis of symmetry. We say that it is symmetrical about each of these axes. So in this illustration, the letter A has one axis of symmetry, the letter X has two, and the pentagon has five axes of symmetry.

These figures are not symmetrical.



If a figure is symmetrical, the whole of it looks like its reflection wherever the mirror is placed. This is however not the defining property, but a consequence thereof. It does not show the position of the axis of symmetry, nor how many of these there are. For this reason I suggest that we do not mention it to the children unless they notice it themselves. If they do, we might tell them that this is another interesting property of symmetry — as indeed it is.

In this topic, ‘symmetry’ means line symmetry since this is the only kind which the children have met so far. In topic 9 they will learn about rotational symmetry, and from then on we shall need to say which kind we mean.

Activity 1 Introduction to symmetry [Space 2.2/1]

A teacher-led discussion for a small group, after which there are example sheets for them to do on their own. Its purpose is to introduce them to the concepts described above.

Materials For each child:

- A small mirror or a MIRA, as used in Space 2.1/1.
- An examples sheet.*
- Pencil and paper.

For the group:

- A set of six cards showing three symmetrical figures and three non-symmetrical figures.*

* Provided in the photomasters. This includes figures with one, two, and three axes of symmetry, and also some non-symmetrical figures.

Space 2.2 Reflection and line symmetry (cont.)

Suggested Stage (a)

outline for the discussion

1. Spread out the set of six cards face up, and tell them that there are two different kinds of figure here, three of each kind. Can they see which they are?
2. Accept and discuss any reasonable answers. We hope that at least one of them will perceive the symmetrical/non-symmetrical difference, in which case ask her to group together the ones which are alike. Then ask if they can describe the difference.
3. If no one gives the desired response, group the symmetrical figures together, and likewise the non-symmetrical figures, and ask if they can see the difference. If they cannot, they are probably not ready for this concept.
4. Use a mirror to show them that for three of the figures, one half is the mirror image of the other, both ways round. For this, the mirror has to be in the right position.
5. Let them experiment for themselves. If they do not find out for themselves, ask them whether there are any of the figures in which the mirror can be in more than one position to give this result. They should also try using the mirror with the non-symmetrical figures.
6. Tell them that the name for the property they have been investigating is *symmetry*, and that the mirror lines for these are called their *axes of symmetry*.

Stage (b)

1. They each receive an examples sheet, and follow the instructions thereon.

Activity 2 Collecting symmetries [Space 2.2/2]

A game for a small group of children. Its purpose is to consolidate the concept of symmetry, and to give practice in finding all the axes of symmetry.

- Materials*
- A pack of 48 symmetry cards.*
 - Mirrors.†
 - Pencil and paper for each child.

* Provided in the photomasters. The pack consists of 48 cards, of which 32 are symmetrical and 16 are non-symmetrical.

† As used in Activity 1.

Introduction I suggest that they are first given a preliminary look at the cards. This can be done by spreading out some of them, face up, and inviting them to take a look. Are there any cards with more than three axes of symmetry? (Yes: up to eight.) Are there any non-symmetric cards? (Yes: 16 out of the 48 are non-symmetrical.)

Rules of the game

1. The object is to collect figures totalling as many axes of symmetry as possible.
2. The pack is shuffled and turned face down. The top three cards are then turned and laid face up, separately so that all faces are showing.
3. The first player picks up whichever card she likes, from among the face-up cards. At this stage she has to choose by looking at it, without using the mirror. (This rule was introduced when we found that some players wanted to test all three cards carefully with their mirrors, before choosing. Meanwhile the other players got bored with waiting!)

4. Next, she turns over the next card from the face-down pile, and puts it face up to leave three cards for the next player.
5. She then writes down the number of axes of symmetry of the card she has just taken, using the mirror if desired. Meanwhile the next player is making her choice.
6. If at any time all three face-up cards have non-symmetrical figures, the player whose turn it is may declare “No symmetry.” She is then entitled to put aside all three of these, and lay out three new cards for herself to choose from. The game then continues as before.
7. Steps 3 and 5 are repeated until each player has five cards.
8. The players then total their scores, and check those of their neighbours.
9. If there is disagreement, all players check, with ultimate recourse to yourself as referee.
10. Another round may then be played.

Discussion of activities These activities relate the intuitive property of symmetry to the previously formed concept of reflection, and thereby bring it into a mathematical context. Activity 1 is for building up the concept from a variety of examples. Activity 2 consolidates it, with further emphasis on multiple axes of symmetry. It also diminishes dependence on the mirror, since the initial choice has to be made simply by examining the figure.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Collecting symmetries [Space 2.2/2]

Space 2.3 TWO KINDS OF MOVEMENT: TRANSLATION AND ROTATION

- Concepts*
- (i) Translation (in some contexts these may be called slides).
 - (ii) Rotations (turns).

- Abilities*
- (i) To distinguish between these two kinds of movement.
 - (ii) To use the right one for the required purpose.

Discussion of concepts Children will already have these concepts at an everyday level. The present topic introduces them in a mathematical context, and emphasizes the distinction between them. Also, in everyday life left and right turns are normally relative to oneself. Here the children are learning to apply them to outside objects, namely their markers.

Activity 1 Walking to school [Space 2.3/1]

A game for two to four children. Its purpose is to emphasize the distinction between the above kinds of movement.

- Materials*
- Game board, see Figure 36.*
 - Four differently coloured counters, with arrows.
 - Die, on which two sides are marked F (go forward), two sides R (turn right), and two sides L (turn left).

* Provided in the photomasters. The full-size game board has been made fairly large because often more than one will be waiting at the same junction.

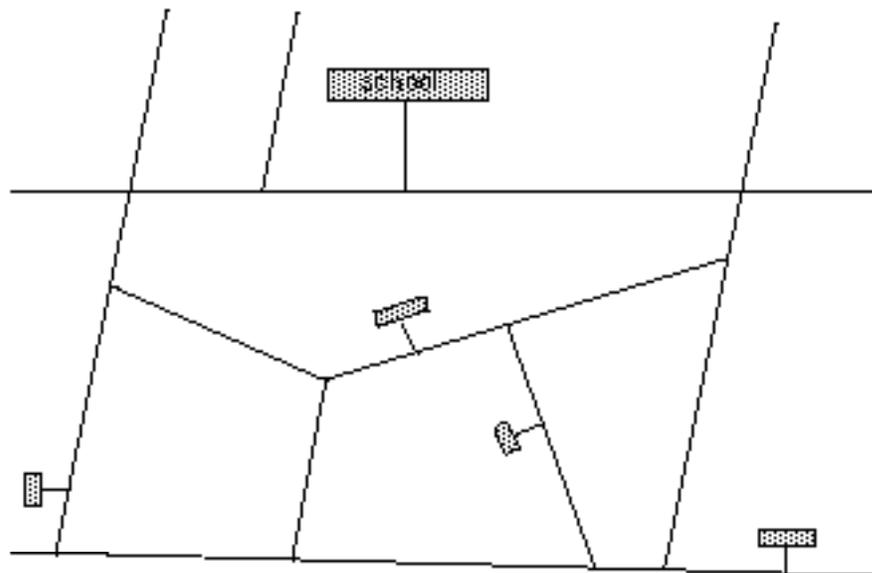


Figure 36 'Walking to school' map

- What they do*
1. The players agree on their choices of homes, which should be convenient to where they are sitting.
 2. Each also takes a marker. The arrows on these show the directions in which they are facing. They all start in their homes facing towards the road.
 3. Each in turn throws the die, and may then decide whether or not to make the movement indicated. F allows them to go forward to the next junction (not beyond), where they must wait until it is their turn to throw again. R and L respectively allow them to make a turn right or left, after which they must wait at the junction for their next throw. (So they all need to throw F to start with, which gets them from their homes on to the road.)
 4. They may go to school by any route they wish.
 5. The winner is the player who gets to school first. Clearly, the game should continue until all have arrived at school!

- Notes*
- (i) The board is designed so that there are the same number of junctions between each home and the school, by the shortest route.
 - (ii) The turns are not all right angles, since we do not want to limit their concept of turns to these.
 - (iii) The unstated assumption is that all turns are less than 180 degrees. However, if someone makes a three-quarter turn to the left to give the equivalent of a right turn, they are quite correct and may be congratulated on their ingenuity. The players should then agree whether this is allowed within the present game.

Discussion of activity This is a straightforward activity in which the children manipulate physical objects (markers with arrows) in ways which directly correspond to movements of their own bodies.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Walking to school [Space 2.3/1]

SPACE 2.4 LINES, RAYS, LINE SEGMENTS

- Concepts*
- The distinction between lines, rays, and line segments, as these terms are used by mathematicians.
 - Parallel lines.

Ability To use the above terms correctly.

Discussion of concepts In everyday life we use the word ‘line’ with a great variety of meanings, such as railway line, telephone line, airline, and also for the kind of line we draw on paper. This last is a representation of a mathematical line, which has no width and no thickness, only length. Mathematicians also have available three different names for this, according to whether it is of finite length, or extends indefinitely in both directions, or extends indefinitely in one direction only. Usually it is clear from the context which we mean, but sometimes we may need to be more exact, and the purpose of this topic is to make available an appropriate vocabulary.

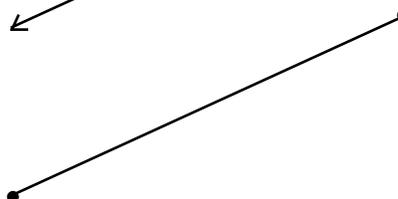
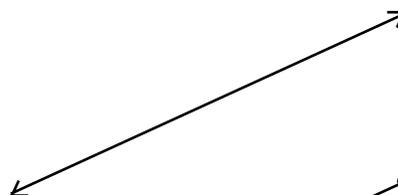
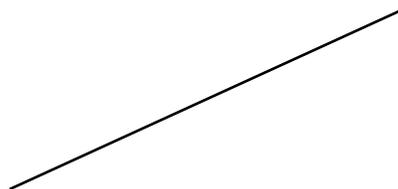
Activity 1 Talk like a Mathematician (lines, rays, segments) [Space 2.4/1]

A teacher-led discussion for a group of any size. Its purpose is to introduce children to the above concepts.

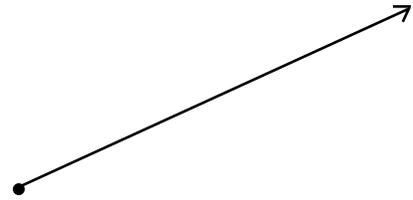
- Materials*
- Pencil and paper.
 - Ruler.

Suggested outline for the discussion

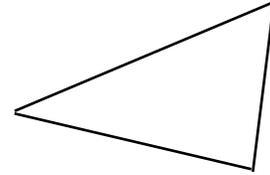
1. With the ruler, draw a short line and ask “What do we call this?” (“A line” is the likeliest answer. Or better, “A straight line.”) “For this activity we are talking only about straight lines, not curved ones.”
2. “That is what most people would call it: a straight line. However, Mathematicians use three different names for this, so that they can be quite clear exactly what they mean. We don’t always need to use these names, but it is useful to be able to talk like Mathematicians (with a capital M) sometimes, if we want.”
3. “By a line, Mathematicians mean something which goes on and on without end, in both directions. We can’t draw this, so we put arrows at each end instead.”
4. “If we mean something which has a beginning and an end, we put in end points like these. This is called a line segment, which means part of a line.”



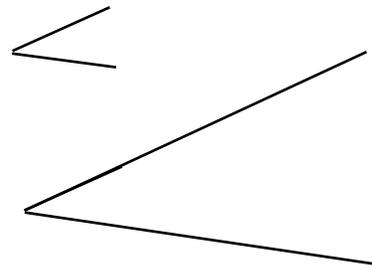
5. “And there is something in between, which has a starting point, but goes on and on in one direction. This is called a ray.



6. What would you call the three sides of this triangle? (Line segments. They end at each vertex (corner). In this case we don't need to put in end points — the vertices show where the sides end.) “Can you think of some other examples?” (The sides of a square, oblong, of any kind of polygon.)



7. What would you call the arms of this angle? (Rays.) “Why?” (Because the size of an angle is the same whatever the length of the arms.) So we'll put arrows on the ends of the arms. (These are not included in the present figure, which shows the first stage only.)



When they learn about compass bearings, they will recognize these as another example of rays. A compass bearing is a direction from a given point, the point of observation. This may relate to something a short distance away, or to a distant star.

8. Simple examples of lines, in the mathematical sense, are harder to think of. The number line is a good example, provided that we include negative numbers, but it is too advanced for children at this stage. Suggestions will be welcome. Here is one which I had from Laura Blick, aged 10: “The horizon, at sea.” Your own children might like to discuss this one.

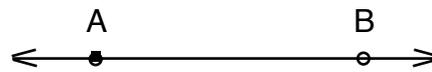
Activity 2 True or false? [Space 2.4/2]

This is a more difficult activity, for a group preferably of three or four players. It is not essential at this stage, but will exercise the wits of your more able children. You might also choose to return to it later with other children. Its more obvious purpose is to consolidate the distinction between lines, rays, and segments, and to link these with an accepted notation. Since at this stage in children's mathematics this is not a critical distinction, I would see as a more valuable purpose, though less obvious, that of developing children's ability to think and speak logically in a mathematical context.

- Materials*
- A pack of twelve cards, six bearing the word “True” and six bearing the word “False.” *
 - Pencil, paper, and ruler for each child.
- * Provided in the photomasters

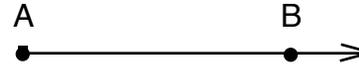
Space 2.4 Lines, rays, line segments (cont.)

Introductory explanation 1. The line AB means this:

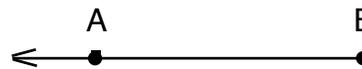


The line BA means the same.
Note that a line goes in both directions.

2. The ray AB means this:

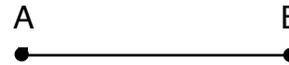


But the ray BA means this.



Note that a ray goes in one direction only.

3. The line segment AB means this:



The line segment BA means the same.

Note that a line segment does not normally have a direction. (If it does we call it something different.)

The activity 1. They begin by all making their own copies of this diagram. Point out that we may use any letters we like for labelling points.



(I stopped using ABCD because I found that when spoken, B and D are sometimes confused.)

2. "I'm going to make some statements, and I'd like you to tell me whether they are true or false. The line PQ and the line RS are both the same line. Is this true or false?" (True.)
3. "Here is another statement. The ray PQ and the ray RS are both the same ray. True or false?" (False.)
4. "Is the following statement true or false? The point R is on the line segment PQ." (False.)
5. "Is the following statement true or false? The ray PQ goes through the point R" (True.)
6. If necessary, give some more examples until they have grasped the idea. Here are three more true statements:
 - The point P is on the ray SR.
 - The line segment PS is part of the line QR.
 - The ray PR goes through the point S.
 And three false statements.
 - The point R is on the ray QP.
 - The line segment PS is part of the line segment QR.
 - The ray QR goes through the point P.
7. The cards are now shuffled, and each player takes a card. They look at their own cards, but do not let the others see.
8. Each player then writes down a statement which is either true or false according to what is on her card.

9. When all have done this, the player whose turn it is to start reads out her statement. The others in turn say whether they think this is true or false, with discussion if there is disagreement.
10. Steps 7, 8, and 9 may now be repeated.
11. Note that (e.g.) “R is on PQ” is not a proper statement, since we do not know whether it is true or false until we know whether PQ is a line, ray, or line segment.” “R is on the line segment PQ” would however be acceptable as a statement (it is false) since in this context R cannot be anything else but a point. “The point R” is, however, preferable (in the present activity).
12. They may like to know these quicker ways of writing down their statements.

The line AB

\overleftrightarrow{AB}

The ray AB

\overrightarrow{AB}

The line segment AB

\overline{AB}

Discussion of activities Activity 1 introduces the new concepts in a straightforward way, and relates them to geometrical knowledge which they already have. This is probably as much as is necessary for most children. I wrote Activity 2 somewhat as an experiment, and was pleasantly surprised to find how well it was received by a group of bright 5th grade children. They liked it as much as the easier activities they had done with me in a previous session, saying “It’s a challenge.”

OBSERVE AND LISTEN

REFLECT

DISCUSS



True or false? [Space 2.4/2]

SPACE 2.5 TRANSLATIONS OF TWO-DIMENSIONAL FIGURES (SLIDES WITHOUT ROTATION)

- Concepts*
- (i) Dimensions: one, two, and three dimensional objects.
 - (ii) Translations of two-dimensional figures (slides without rotation).
 - (iii) 'Motion lines' (my term) which give the location of each point of the image figure. (These are what in the future some of them will know as vectors.)

- Abilities*
- (i) To use motion lines to draw the result of a slide for simple figures.
 - (ii) To recognize this relationship between two figures.

Discussion of concepts

In topic 1, we learned about (mathematical) reflection. This is just one of several mathematical operations which can be done on a geometric figure. In the present topic we introduce another one, namely, translation, which is short for a slide without rotation; and this will be followed in topic 9 by rotation, which is a turning without a slide. These are geometrical counterparts of arithmetical operations such as addition and multiplication, which act on numbers and result in another number. Here the operations are on geometric figures, and the result is another geometric figure, or the same one in a different position. But the same sequence, 'Start, Action, Result,' can be seen in all of these.

It is perhaps a pity that the same word 'figure' is used for a geometric figure and for a symbol representing a number, i.e., a numeral. However, my dictionary gives no less than twelve distinct meanings for 'figure,' one of which is 'a diagram or pictorial representation.' This is the kind we are talking about in the present network.

I have also taken the opportunity to introduce, in a simple way, the concept of dimension.

Activity 1 Sliding home in Flatland [Space 2.5/1]

An activity for a small group of children. Its purpose is to help children to form the above concepts from a number of suitable examples. The idea is partly borrowed from a mathematical classic called *Flatland: a romance of many dimensions*, by a *Square* (author, E. A. Abbott, 1884).

- Materials*
- Game board, see Figure 37.*
 - Set of cutout polygons.*
 - Ruler.

* Provided in the photomasters

Note The polygons need to be cut out of thick cardboard, so that they do not get stuck under the ruler. The latter should be rigid so that it stays flat along its whole length.

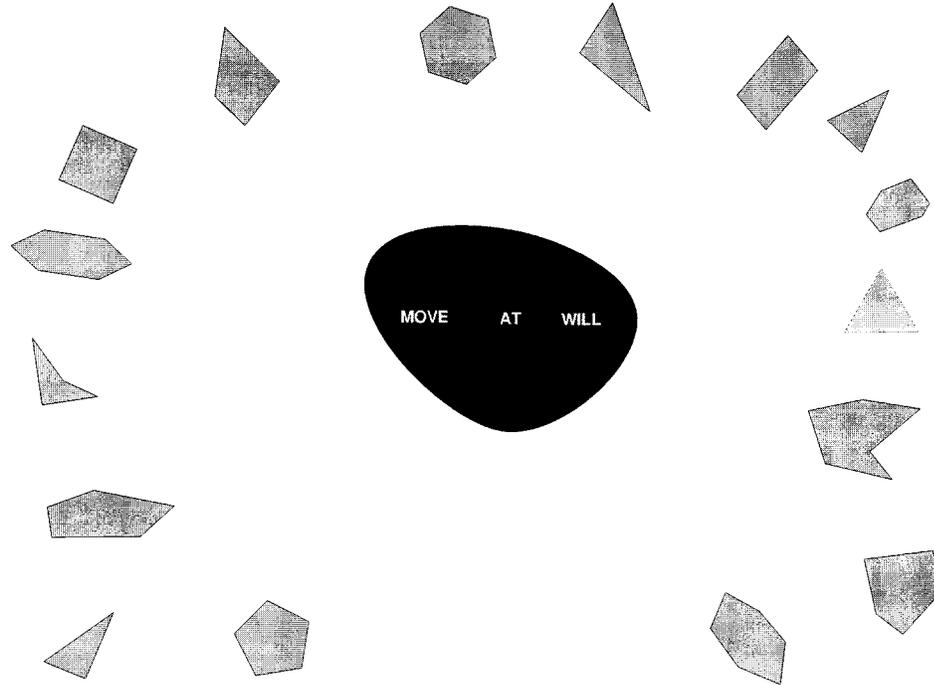
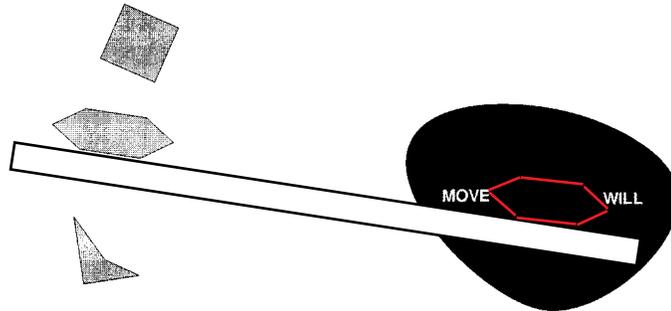


Figure 37 Game board for 'Sliding home in Flatland' [Space 2.5/1]

- Introduction*
1. Flatland is an imaginary country in which there are only two dimensions, like a sheet of paper with no thickness. Its inhabitants are polygons, which also have no thickness.
 2. The country is for the most part very cold, covered with ice, and therefore very slippery. There is a central area which is warm and not slippery, in which the polygons can meet and move at will. Their homes are also warm, but these are the exact size and shape of the polygons who live there. No movement is possible in these.
 3. The polygons like to meet each other in the move-at-will area. The problem is afterwards to get to their homes so that they fit exactly, since any parts which project will get frostbite. For this they have to position themselves so that once they push off from the move-at-will area, they can get home by a slide without rotation. They are not clever enough to do this for themselves.
 4. The country is governed by a ruler, one of whose most important functions is to keep his subjects safe by helping them to take up the right starting positions from which they can get home by a slide without rotation. Since it is not possible to see the homes from the central area, successful rulers are hard to find.

Space 2.5 Translations of two-dimensional figures (slides without rotation) (cont.)

- Rules of the game*
1. We, ourselves, are fortunate enough to have three dimensions — we have height, width, and thickness. So we can look at Flatland from above, which makes the task of positioning the polygons much easier. We also have the help of another kind of ruler, the sort used in mathematics.
 2. Players begin by taking polygons in turn until there are not enough left for another one each. (The ones we use do have thickness, or we would not be able to handle them.)
 3. The player whose turn it is has the ruler. He puts one of his polygons in the move-at-will area, and positions the ruler in the correct position for helping this particular polygon to take up the right starting position.



4. With help from the player, the polygon positions itself against the ruler, as in the diagram above, and then does a slide without rotation in the direction of its home. It is a relief to both polygon and ruler if he reaches the right home in the right position, but in any case the polygon has to stay there and the turn passes to the next player.
5. The game finishes when all the polygons are home. Another round may then be played.

Note When the children are familiar with the game, they will begin to notice that some of the polygons can fit into their homes in more than one position. In the above figure, the polygon against the ruler has two possible positions, and the square whose home is just above has four. This relates to the number of rotational symmetries, which they will study in more detail in topic 9.

Follow-up discussion If we are three-dimensional, and Flatland is two-dimensional, what would be one-dimensional? (A line. In mathematics a line has no width and no thickness, only length.) Can you think of something which has no dimension? (A point. This has only position, like a dot on paper. It has no length, no width, and no thickness. We are now talking about a mathematical point, which exists only in our imagination. A dot on paper does have some size or we would not be able to see it.) Some of the children might enjoy following their mathematical imagination a little further, as did the author of the original book.

Activity 2 Constructing the results of slides [Space 2.5/2]

An activity for children to do independently, using worksheets. Its purposes are (i) for them to examine in more detail what happens to figures which undergo translations (slides without rotations) and the relationship between each point of the original figure and its image; and (ii) for them to be able to construct geometrically the results of translations, for simple figures.

Materials For each child:

- Worksheet.*
- Pencil, ruler, eraser.

* Two are provided in the photomasters, one for Stage (a) and one for Stage (b).

- Suggested introduction*
1. Explain that so far they have made slides by actually sliding pieces of cardboard in various shapes. Now we move on to slides without rotation of figures drawn or printed on paper.
 2. Ask them to look at Worksheet 1, think about their answers to the question “What can you say . . .?”, but keep quiet until all have had time to think. You can then invite answers, in turn.
 3. The first point is that all the ‘lines’ are in fact line segments. They are of a definite length, with end points.
 4. Second, they are all the same length.
 5. Third, they are parallel.
 6. Why do they have arrows? Because the slide is from one location to the other. There is a starting position and a finishing position, and this is what the arrows are there to show. For this reason I call them ‘motion lines.’ (This fits in well with the term used in more advanced mathematics, which is ‘vector.’)

Stage (a) They should now be ready to complete the first worksheet, comparing results and discussing if necessary. Some children may invent other ways of drawing the image figure, which may be discussed also to see if they give a correct result. However, they should also learn the one indicated on the worksheet since this fits more closely with the concepts of the present topic.

- Suggested follow-up discussion*
1. Why is there only one line to show the slides for the check mark and the letter F? (This is all that is needed. The other motion lines are just the same except for their starting points.) What do we mean by “just the same”? (They are the same length and direction.)
 2. (Optional) They may like to know that directed line segments of this kind have a special name. These are called vectors, and are important in advanced mathematics.

Stage (b) This uses Worksheet 2, and should not be attempted until they have completed Activity 3. It is more difficult, and you may wish to postpone it until later.

Activity 3 Parallels by sliding [Space 2.5/3]

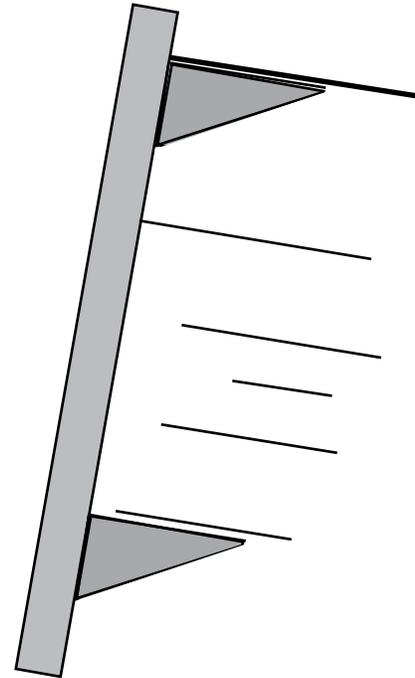
An activity for children to practise individually, after you have shown them how. Its purpose is to equip them with a new skill: drawing parallels to a given line in any desired position. This is a useful ability for many geometric purposes, and has an immediate use in Stage (b) of Activity 2.

Materials For each person

- Ruler.
- Set-square.
- Pencil and scrap paper.

Suggested introduction

1. Demonstrate the method for drawing one or more lines parallel to a given line, explaining as you go along.
2. First we position the set-square along the starting line (shown thicker, at the top of the diagram).
3. We hold the set-square firmly in position, and bring the ruler against it.
4. From now on, it is the ruler which has to be held firmly in this position. We slide the set-square along it (in the same way as we did with the polygons in 'Flatland') to any position we choose. Keep it here with a spare finger or thumb, and draw a line along it. Move it some more, and draw another line. The lines can be drawn of any length, and need not always start up against the ruler.



- What they do*
1. They should practise the above until they are proficient.
 2. Next, they should experiment starting with the hypotenuse against the starting line. Sometimes this may be more useful.
 3. Now they should set themselves the task of drawing a line parallel to the given line through a given point, experimenting with different positions of this point.
 4. They should practise the foregoing until they are proficient. They are then ready to return to Activity 2, Stage (b).

Suggested discussion of what they have done

1. Why does it work? (Because when any object slides without rotation, the direction of all its lines stays the same. Lines which have the same direction are parallel.)
2. Are we talking about lines, rays, or line segments? (All three. In everyday language, we don't have a word which means both 'he' and 'she.' It would be useful if we did. Likewise, when the mathematicians attached a special meanings to 'line,' 'ray,' and 'line segment,' they left us without a word which included all three. My personal view is that they should have thought about this!)

Discussion of activities The first activity, 'Sliding home in Flatland,' embodies the concepts in physical activities which are simple but capture the children's interest. The second activity is appreciably more difficult. It makes explicit much which was implicit in the activity with manipulatives, and also gives an opportunity to ask them to verbalize these. In Stage (a) the emphasis is on the method (drawing the motion lines, which are parallel line segments of equal length) rather than getting the parallels exact. Activity 3 introduces the method for drawing these accurately, which is itself another application of slides without rotation. They are then in a position to use it in Stage (b) of Activity 2. This is fairly demanding, but the way has been well prepared.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Parallels by sliding [Space 2.5/3]

SPACE 2.6 DIRECTIONS IN SPACE: NORTH, SOUTH, EAST, WEST, AND THE HALF POINTS

- Concepts*
- (i) North, south, east, west, as directions in space which are independent of where we are on the earth's surface.
 - (ii) The four half points, northeast, etc.

Abilities To use a compass and compass directions to identify and make movements in specified directions.

Discussion of concepts In the previous topic we introduced children to the distinction between two kinds of movement, translation and rotation. The idea of direction is an important preparation for developing both of these in further detail, as we shall see in later topics. Compass directions provide an excellent starting point, with important and clear real-life applications.

Activity 1 Compass directions [Space 2.6/1]

A teacher-led discussion for a group of any size. Its purpose is to consolidate, organize, and expand children's everyday knowledge of the compass, and compass directions.

Materials

- A magnetic compass.

- Suggested outline for the discussion*
1. What is this? What is it for?
 2. How does it help us to find our way? What else do we need? (A map. But we must also be able to match the directions on the map with directions on the ground, and a compass is sometimes the only way of doing this. We also need to know where we are on the map, and a compass can be used for this also if we know how.)
 3. What are navigators? Where do they work? (In ships and in aircraft. Their job is to tell the pilot in which directions to steer, so that they may all reach their destination safely.)
 4. In what direction is North? (. . . from where they are now)
- You might also wish to tell them about:*
5. Magnetic north and true north.
 6. The pole star and the north pole. (A star chart is useful here.)
 7. The apparent movement of stars. (Because of the rotation of the earth, which also brings day and night, the stars appear to rotate when seen from the earth. Just one star appears still, and this is the one which the earth's axis of rotation points towards. One end of this axis of rotation is the north pole, and the other is . . .? So the pole star shows us the true north.)

Activity 2 Directions for words [Space 2.6/2]

An activity for a small group of children, say up to six, or more if enough materials are available. Its purpose is to consolidate children's knowledge of the four main points of the compass, and the four half points.

Materials For each player:

- A game board, as illustrated in Figure 38.*
- A two-sided directions card showing on one side the eight main compass directions and on the other showing only N for north, also illustrated in Figure 38.*

For the group:

- Two sets of word cards, easier and harder.*

* Provided in the photocopiers

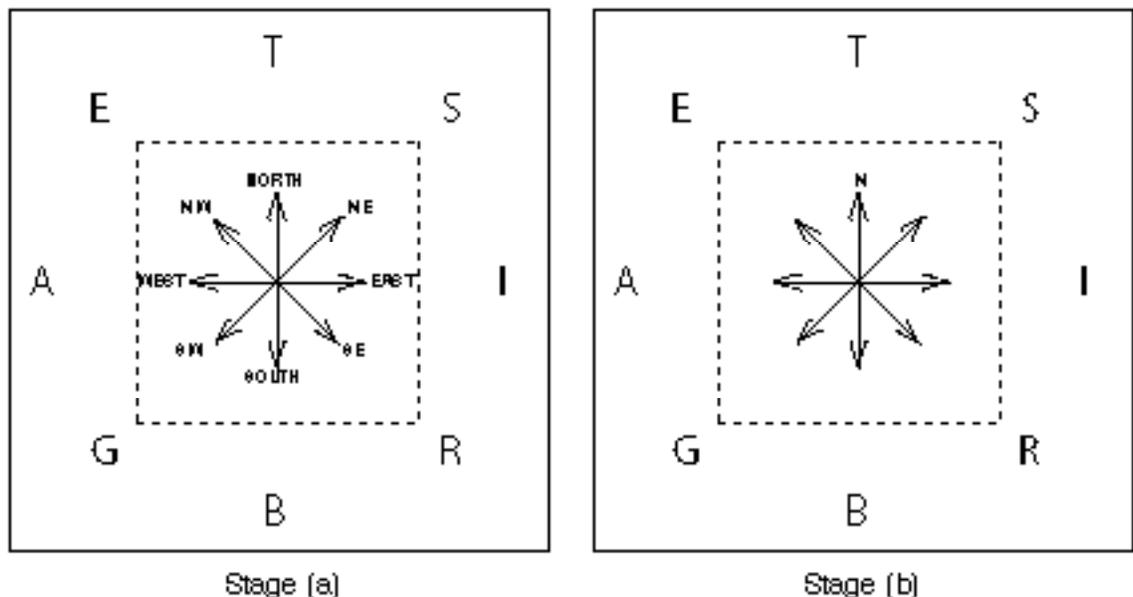


Figure 38 'Directions for words' game boards and two-sided directions card

What they do Stage (a)

1. Each player has a game board, and puts her directions card on the middle with the north direction pointing towards the top or the card. In this case, it will be pointing to the letter T. To begin with, they all have the side showing the names of all the directions face up.
2. One player, the caller, also has one of the sets of word cards. She picks one of these at random, which she looks at herself but does not let the others see. The caller should choose the word either by cutting the pack or picking one at random with the cards face down, not by looking through and choosing one. The latter makes it very easy to guess when another player is the caller.
3. She then calls out the directions for the letters in order. E.g., "First letter, south." The other players all write this. "Next letter, northwest." The other players write this, and so on, until the word is complete.
4. The caller then shows the word to the other players. If some do not have the same word, she repeats the directions given in step 3 while the others check what they have written to find whether it was they or the caller who made the mistake.
5. Another player then takes over the role of caller, and steps 1 to 4 are repeated.

Space 2.6 Directions in space: north, south, east, west, and the half points (cont.)

Stage (b)

This is played in the same way as Stage (a) except that they turn over their directions cards, so that the side showing is the one on which only north is marked. The caller (only) may if necessary take a look at the other side to ensure that her calls are correct, but without letting anyone else see.

Variation This goes best when using the harder pack. If a player thinks she knows the word, she may hold up a hand and call out the next direction (not the word itself). The caller will say whether or not this is correct. Only one guess per player is allowed until all have had a chance to guess.

- Notes*
- (i) The easier pack contains only words of four or five letters, including some plurals. The harder pack contains only words of five letters upwards.
 - (ii) You might wish to look through the word cards and remove any which you think the children might not know, e.g., 'arbitrate'. Alternatively, you might think it useful to leave these in, to enlarge their vocabularies.
 - (iii) I would have preferred not to include the letter E, since this is commonly used as an abbreviation for east. However, although I used a computer program for generating anagrams, modified to allow the same letter twice, I found that there were not enough words of more than three letters to be made from a set of eight without E. To reduce the possibility of confusion from this, the four main compass directions are therefore spelt in full on the directions cards.

Activity 3 Island cruising [Space 2.6/3]

A game for a small group of children, up to three or four. It provides a nautical embodiment of the concepts introduced in Activity 2, and further consolidates the ability listed above.

- Materials*
- Game board, see Figure 39.*
 - Pack of direction cards*
 - A bowl of counters.
 - For each player, a differently coloured marker in the shape of a boat.†

* Provided in the photomasters

† Ingenuity is required here. I have a set made of cutouts from coloured cardboard, with a blunt tack pushed through in the position of a mast (to make it easier to pick up); and another set made from half-shells of cashew nuts. Other suggestions will be welcome.

1. The ships all start in the harbour at the south end of the map.
2. The object of the game is to collect as many passengers as possible from the islands. This is done as follows.
3. The directions pack is shuffled and put face down on the table. The top three cards are now turned face up and put separately beside the pack.
4. The player whose turn it is may take just one of the face-up direction cards, and go to the first island which is in that direction from where she is. (In other words she may not continue past it.) At that island she may pick up just one passenger, and must then stay there until her next turn. Picking up a passenger is represented by taking a counter from the bowl.

Discussion of activities The real-world importance of compass directions is easily understood by children, and the detailed knowledge developed in this and the next topic can be directly applied to situations involving the use of maps to find one's way, and give directions to others.

Activity 1 is for underlining this relationship, after which Activity 2 develops the concepts in a small-scale situation. Activity 3 then enlarges the field of application to a board representation of a large-scale situation. When field testing this activity, I found that it did not matter that some of the children were sitting on either side of the board: they were still able to get the directions right. On reflection, I realized that this was a good learning situation. Although in a map, north is nearly always towards the top of the page, it is not always the direction in which we are looking at our surroundings when we want to find our way. So it is good to be able to face (e.g.) east, and know that west is behind us, north on our left, and south on our right.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Island cruising [Space 2.6/3]

SPACE 2.7 ANGLES AS AMOUNT OF TURN; COMPASS BEARINGS

- Concepts*
- (i) An angle as representing a turn.
 - (ii) An angle as representing the difference between two directions.
 - (iii) Clockwise and counterclockwise turns.

- Abilities*
- (i) To describe a given angle in terms of quarter, half, three-quarter turns and complete turns.
 - (ii) To do the same in terms of degrees for multiples of 30 degrees, and for half a right angle.

Discussion of concepts In the Space 1 network, angles were considered from the aspect of their shape. Here we think about them in two other ways.

First, as a difference between two directions. This is the same both ways around. E.g., the angle between north and northeast is 45° , and so is the angle between northeast and north, in the same way as the difference between 7 and 4 is the same as the difference between 4 and 7.

Second, as a turn from one direction to another. This is not the same both ways round. A turn from north to northeast is a turn of 45° clockwise, but a turn from northeast to north is a turn of 45° counterclockwise. We could also use positive and negative signs if we agreed which direction was positive, but unfortunately there are two conventions here. Compass bearings as used in navigation, etc., take clockwise as the positive directions; but mathematicians use counterclockwise as positive, since this fits in better with coordinate axes. This seems to me a good reason for staying with clockwise and counterclockwise.

A compass bearing is the clockwise angle of turn from north to the given direction, so zero, quarter, half, and three-quarter turns correspond to the major compass bearings north, east, south, west. For accurate direction-finding, smaller units are needed. The ancient Egyptians divided a complete turn into 360° , and this is the unit still in use. So the direction northeast corresponds to a bearing of 45° , and northwest to a bearing of 315° .

At this stage, we are concerned with building up the concept, and the use of a protractor for measuring angles to the nearest degree will not be introduced. Multiples of thirty degrees give twelve major directions, including the four main points of the compass, and I have chosen these as giving a good overview, and a framework within which more detail can easily be introduced as and when required.

Activity 1 Directions and angles [Space 2.7/1]

A teacher-led discussion for a small group. Its purpose is to introduce the idea of an angle as representing the difference between two directions, or a turn from one direction to another.

- Materials*
- Two pencils for each participant, including teacher. (Ballpoints will do equally well.)
 - Scrap paper for each child.

*Suggested
outline for
the discussion*

1. Ask them a question which is likely to be answered by pointing, such as “Show me where the fire extinguisher/clock/my desk is.” This is one of the most common ways in which we show a direction. On paper, we use an arrow. In our present discussion we are going to use pencils as pointers.
2. Put your own pencil on the table, and ask all the children to put their pencils so that they all point in the same direction as yours. Repeat a few times with the children taking turns to set the direction. In each case the pencils should all be parallel, with the sharp ends pointing in the same direction.
3. Now put two pencils with the unsharpened ends together and the pointed ends apart, to form an angle. “Here we are showing . . .” (two directions) “And the difference between these two directions is?” (an angle)
4. Ask the children to show other pairs of directions with roughly the same angle between them as yours. Repeat a few times with other children setting the angle. Emphasize that we are not trying to be exact: the idea is simply to show that different pairs of directions can make the same angle between them.
5. Now put the pencils to form a right angle. “Does anyone know what this angle is called?”
6. Likewise for a straight angle.
7. Now we want to introduce the idea of an angle as showing a turn. Put a pencil to show a starting direction, and another next to it. Rotate the second pencil through a full turn, and say “This has made a full turn, all the way around, until it finished in the same direction as it started.”
8. Next put two pencils at right angles to each other, and ask how much of a full turn this shows. (a quarter turn) So these are two names for the same angle.
9. Ask the same question for two pencils end to end in opposite directions. (a half turn, equal to a straight angle)
10. From the position in step 9, rotate one of the pencils clockwise until the two are again perpendicular. This angle needs discussion, since it can represent either a quarter or a three-quarter turn depending on the starting and finishing positions and the direction of rotation. (These are easier to demonstrate than to describe.)
11. Complete a full turn clockwise with the same pencil rotating, and elicit the idea that a full turn is the same as four right angles.
12. Next, we want them to connect the foregoing with the four main compass directions, which they already know. E.g., “What is the angle between east and south? Between south and north? Between east and north?” The last depends on whether the turn is clockwise (a three-quarter turn) or counterclockwise (a quarter turn). Explain that when using compass directions, turns are always clockwise unless we say otherwise. So the angle between west and south is a three-quarter turn, which is the same as three right angles.
13. Finally, we want to relate all these to the measurement of angles in degrees.
14. Explain that for describing angles of other sizes than the above, we use degrees. We also use degrees when greater accuracy is needed. A right angle is 90 degrees, a straight angle is a 180 degrees; so a three-quarter turn is? And a full turn, is?

Activity 2 Different name, same angle [Space 2.7/2]

A game for a small group. Its object is consolidate the ideas encountered in Activity 1, and especially to relate all the different ways of naming the same angle. It is played in much the same way as 'Collecting symmetries,' Space 2.2/2.

Materials • Game pack of cards.*

* Provided in the photomasters. This should be copied twice, to give a pack of 40 cards.

Introduction I suggest that they are first given a preliminary look at the cards, by spreading out some of them, face up, and inviting them to take a look. (This is a good time to explain the meaning of the sign for degrees.) Are there any cards which have different names for the same angle? Can they pair an angle with one or more of its names? Can they find two or more names for the same angle even if the angle itself is not among those showing? When they have familiarized themselves with the cards in this way, they are ready to play the game.

Rules of the game

1. The object is to collect pairs of cards which represent the same angle. These may either be a drawing of an angle with one of its names, or two names for the same angle. Two drawings of the same angle in different positions are not an allowable pair.
2. The pack is shuffled and turned face down. The first player turns over three cards, and if he sees a pair he takes it.
3. Thereafter, each player turns over the top card from the face-down pack, and puts it beside the other face-up cards. If he sees a pair, he takes it. Since this is unlikely to happen at every turn, the number of cards to choose from will gradually increase.
4. If the player whose turn it is sees an unclaimed pair, he may take it before taking his normal turn as in step 3.
5. Steps 3 and 4 are repeated until all the cards have been turned over, and every player has had a chance to take any unclaimed pairs. The winner is the player with most pairs.
6. Another round may then be played.

Activity 3 Words from compass bearings [Space 2.7/3]

An activity for a small group of children, similar to 'Directions for Words,' Space 2.6/2, but using compass bearings up to 360° instead of the compass directions north, south, etc. It thus uses a familiar setting to introduce the more difficult, but more accurate, method for giving compass directions. For this purpose we use just the multiples of 30° , which gives a choice from 12 letters. This establishes a full-circle framework within which further detail can be developed later, when children are learning to use protractors.

The directions are repeated here for convenience.

Space 2.7 Angles as amount of turn; compass bearings (cont.)

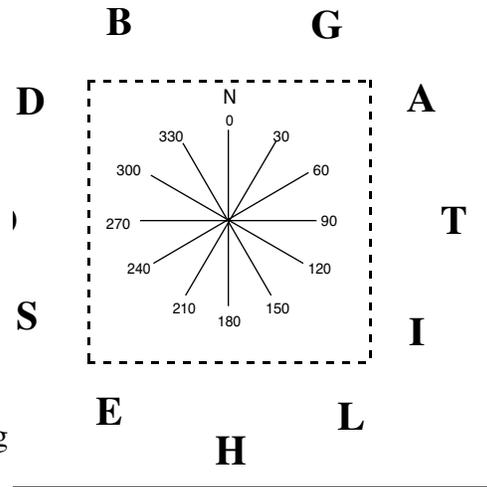
Materials For each player:

- A game board with letters placed on twelve compass bearings.*
- A two-sided directions card showing on one side the twelve compass bearings $0^\circ, 30^\circ, 60^\circ, 90^\circ, \dots, 300^\circ, 330^\circ$, and on the other side a single arrow for indicating north.*
- Paper and pencil.

For the group:

- Two sets of word cards, easier and harder.*

* Provided in the photomasters



Stage (a)

One side of two-sided directions card
(on game board)

What they do Stage (a)

1. Each player has an activity board, and puts his directions card on the middle with the north direction pointing away from him. To begin with, they all have the side showing all the directions face up, with the NORTH arrow pointing away from them. In this case, it will be pointing to the letter R.
2. One player also has one of the sets of word cards. He picks one of these at random, which he looks at himself but does not let the others see.
3. He then calls out the directions for the letters in order. E.g., “First letter, 330° .” The other players all write this. “Next letter, zero degrees.” The other players write this; and so on until the word is complete.
4. The caller then shows the word to the other players. If some do not have the same word, he repeats the directions given in step 3 while the others check what they have written to find whether it was they or the caller who made the mistake.
5. Another player then takes over the role of caller, and steps 2 to 4 are repeated.

Stage (b)

This is played in the same way as Stage (a) except that they turn over their cards, so that the side showing is the one on which only north is marked. The caller (only) may if necessary take a look at the other side to ensure that his calls are correct, but without letting anyone else see.

Variation This goes best when using the harder pack. If a player thinks he knows the word, he may hold up a hand and say the next direction (not the word itself). The caller will say whether or not this is correct. Only one guess per player is allowed until all have had a chance to guess.

- Notes*
- (i) The easier pack contains only words of four or five letters, including some plurals. The harder pack contains only words of five letters upwards.
 - (ii) You might wish to look through the word cards and remove any which you think the children might not know, e.g., ‘arbitrate.’ Alternatively, you might think it useful to leave these in, to enlarge their vocabularies.

Activity 4 Escape to freedom [Space 2.7/4]

An activity for a small group, working in pairs. Its purpose is to introduce the use of compass directions for finding the way. Stage (a) should not be hard for children who have done the earlier activities in this topic. Stage (b) is more difficult.

Materials For each child:

- An activity sheet representing a maze. (Figure 40 contains a reproduction of a maze designed by Jodie Palmer, one of the field-test schoolchildren.) *
- A non-permanent marker.

For Stage (b), also:

- Pencil and eraser.
- Ballpoint pen.
- Clear acetate file folder.

For the group:

- Damp sponge or equivalent.

* Photomasters are provided for three different mazes, and also for making one's own mazes. These need to be provided in pairs alike, and laminated so that lines drawn with the non-permanent markers can be erased afterwards.

What they do Stage (a)

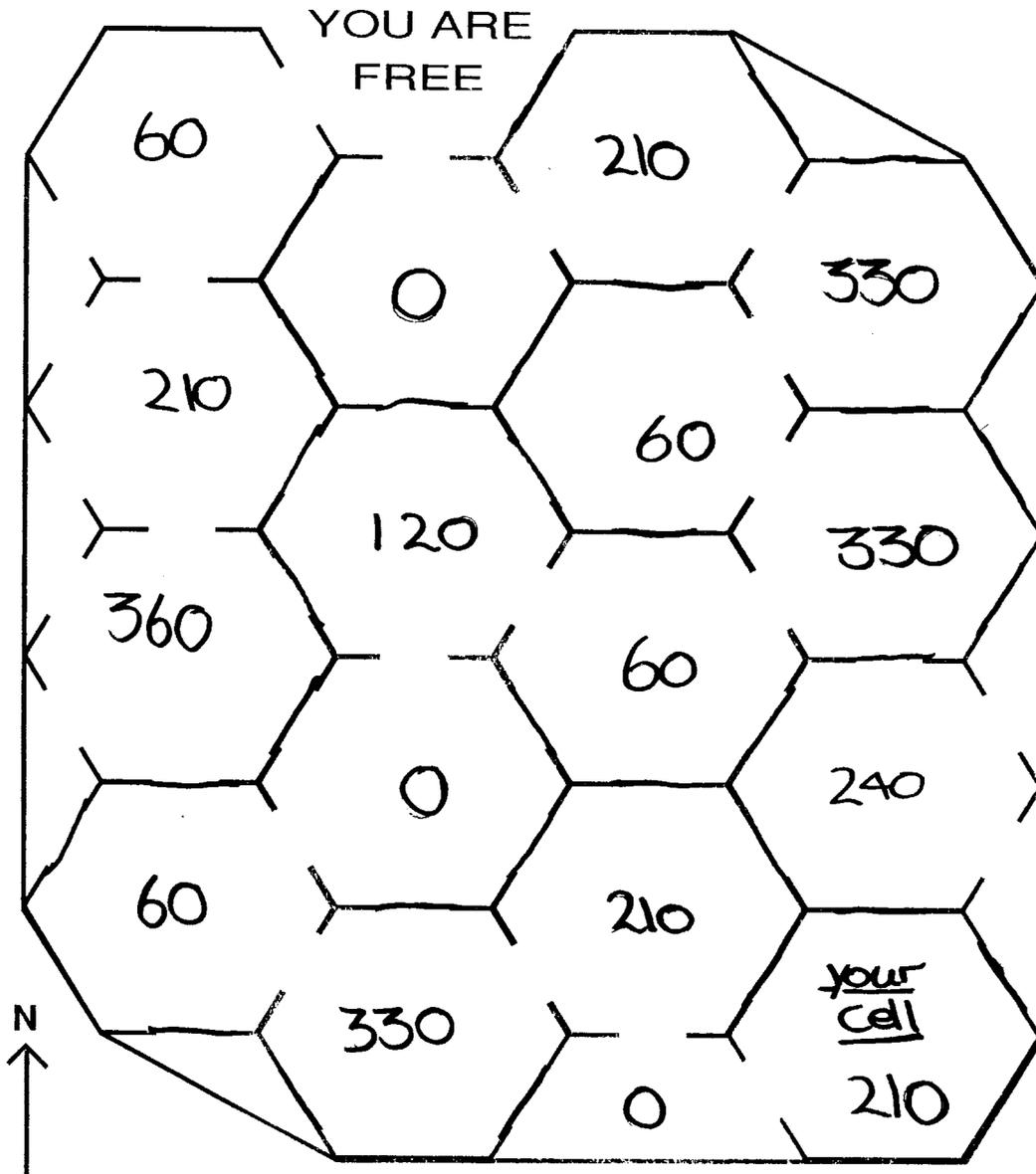
1. Each pair of children receives two activity sheets with the same maze on it.
2. They read the directions, which read as follows.

You have been imprisoned in a maze by a cruel tyrant. However, concealed in your clothing you have a compass. Moreover, a freedom fighter has scratched numbers on the floor of every cell, and these will help you to escape if you can interpret them. But if you get off the path, you may find wrong directions in some of the cells, so be careful. Good luck.

3. Explain that since they are looking at a picture of the whole maze, it is easy for them to see the way out. But we are asking them to imagine that they are always inside one of the cells, where they would only be able to look through the opening into the adjacent cell and no further, until they are out of the maze.
4. Explain that partners have the same mazes. So that they can compare the routes they have taken, they are to mark these as they go with their non-permanent markers.
5. When they have found the way out, each compares with his partner. Since there is only one correct way out for each maze, this provides a check that they have followed the directions correctly.
6. After erasing the routes, they may exchange mazes and repeat steps 1 to 5.

Stage (b)

1. They may now be ready to design their own mazes. The suggestions below offer one way for setting about this task. Alternatively, you may wish to leave the planning to them.
2. Each child receives a copy of the sheet MAKE YOUR OWN MAZE in a clear acetate file folder.



ESCAPE TO FREEDOM

You have been imprisoned in a maze by a cruel tyrant. However, concealed in your clothing you have a compass. Moreover, a freedom fighter has scratched numbers on the floor of every cell, and these will help you to escape if you can interpret their meaning. But if you get off the path, you may find wrong directions in some of the cells, so be careful. Good luck!

MAKE YOUR OWN MAZE This one was designed by *Jodie Palmer.*

Figure 40 A pupil-designed maze

3. He uses this to mark his starting point and exit route on the acetate, using the non-permanent marker.
4. He closes all openings which would allow short-cuts. Incorrect paths should not be prevented.
6. Next, he writes in the numbers for compass bearings which show the correct route.
7. This would be a good stage to exchange mazes with a partner for checking so far. (Partners will not in this case be working on the same maze.)
8. These numbers, if correct, are transferred in pencil to the paper copy, and other numbers are entered in the cells which do not form part of the route.
9. The acetate covers are now wiped clear, and the mazes tried out by someone else in the group. (If spare covers are available, the maze may be transferred to a clean cover and the original temporarily kept for reference until the maze is finalized.)
10. If satisfactory, the pencilled numbers are rewritten in ballpoint pen, and the maze may then be added to the school's collection.

Discussion of activities

The first three activities build on topic 6 to introduce angles as representing turns and directions, and degrees whereby these may be specified with closer accuracy. Activity 3 in particular is designed to give children fluency in relating angles to directions over the full range of 360° rather than the more frequently taught range 0° - 180° . The latter is usually sufficient for triangles, other polygons, and the shape aspect of angles in general, since reflex angles are not frequently encountered. However, all angles in the full circle are equally important in the context of directions.

Activity 4 relates compass bearings to movements in an imaginary situation. In the next topic, we shall apply these concepts to a real-life context.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Escape to freedom [Space 2.7/4]

Space 2.8 DIRECTIONS AND LOCATIONS

- Concepts*
- (i) A direction as pointing towards a location.
 - (ii) Back bearings, as the opposite direction to a given direction.
 - (iii) The intersection of two directions as determining a location.

- Abilities*
- (i) To use a map and compass to go to a chosen location.
 - (ii) To find one's way back to where one started.
 - (iii) To use map and compass to find out where one is.

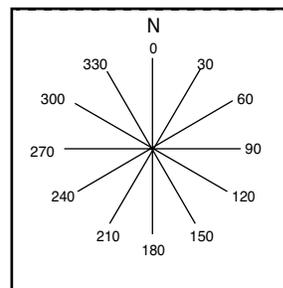
Discussion of concepts In Space 1, angles were seen as an aspect of shape. In the present network, they have been considered as a difference between two directions, and as the amount of turn from one direction to another. By choosing a fixed direction as a starting direction, usually north, other directions can be specified as amount of turn from this starting direction. In the present topic, directions are found to be an important way of determining a location, thereby introducing and developing further relationships among angles, lines, and points.

Activity 1 Introduction to back bearings [Space 2.8/1]

A teacher-led activity for a small group of children. The purpose of this and the next two activities is to teach them how to use a compass for finding their way back to where they came from. This knowledge might save their lives, one day.

- Materials* For each child:
- Pencil, paper, and ruler.
 - Compass bearings card *
- * Provided in the photomasters

- Suggested introduction* *Stage (a)*
1. Ask the children to look at their cards, which show the same set of compass bearings as they have already used in Space 2.7/3. "We are going to talk about directions and their opposites."
 2. "Suppose that you are pointing in a compass direction of 60° , and you make a half turn so as to point in exactly the opposite direction, what would the new direction be?" (240°)
"In other words, the opposite direction of 60° is 240° . You can check this by laying your ruler along the 60° line. This shows the two opposite directions."



Starting direction	Opposite direction
0°	180°
30°	210°
60°	
...	
...	
180°	

3. Let them repeat this with other angles both less than and greater than 180° .
4. Ask them to make a table of directions and their opposites, starting as shown below the compass bearings card. To start with, they are to stop at 180° .
5. Ask them to look at their tables and try to find a pattern. Hint: how much greater is each of the opposite directions than its starting direction? Is this amount always the same? If so, why? (Always 180° greater. To point in the opposite direction requires a half turn, which is 180° .)
6. Now ask them to complete their table, up to 330° , as before with the help of their compass cards. "Is it still the same pattern?" (No. For these, we have to subtract 180° .) Why is this? (Two ways of explaining. If we add 180° as before, this takes us past 360° , so we have to convert by subtracting 360 . Adding 180 and then subtracting 360 is equivalent to subtracting 180 . A simpler explanation is to turn anti-clockwise.)
7. Explain that these opposite directions are called back bearings. E.g., the back bearing for 60° is 240° . Ask for other examples using this name for them.
8. "Why do we need to be able to get back bearings by calculation?" (Calculation is more reliable for intermediate directions like 72° and 249° , for example. Also, we may not have a compass card with us. Sometimes we can read it off the compass itself, but this is not always easy. Finally, it is always good to have a way of checking our results.)

Activity 2 Get back safely [Space 2.8/2]

An activity for a small group of children, working in pairs. Stage (a) introduces the use of back bearings in an imaginary situation. Stage (b) then applies the knowledge in an outdoor situation which is a little closer to the real-life application.

Materials For the group:

- A compass.*

For each child:

- A copy of the map, as illustrated in Figure 41.†
- Transparent compass rose.†‡
- Pencil and paper.
- Eraser.
- Ruler.

* This is not needed until Stage (b). It would be good if a prismatic compass could be made available.

† Provided in the photomasters

‡ This needs to be copied onto transparent film, such as that used for overhead transparencies, and then cut out carefully. Alternatively, a 360° protractor may be used.

Note At this stage, I thought it better not to introduce the difference between true north and magnetic north, which is an additional complication. If in Stage (b) bearings using magnetic north are used on both outward and return journeys, the difference of 180° still applies and the back bearing thus calculated will take them back to where they started. This will be dealt with in the next activity, "Where are we?"

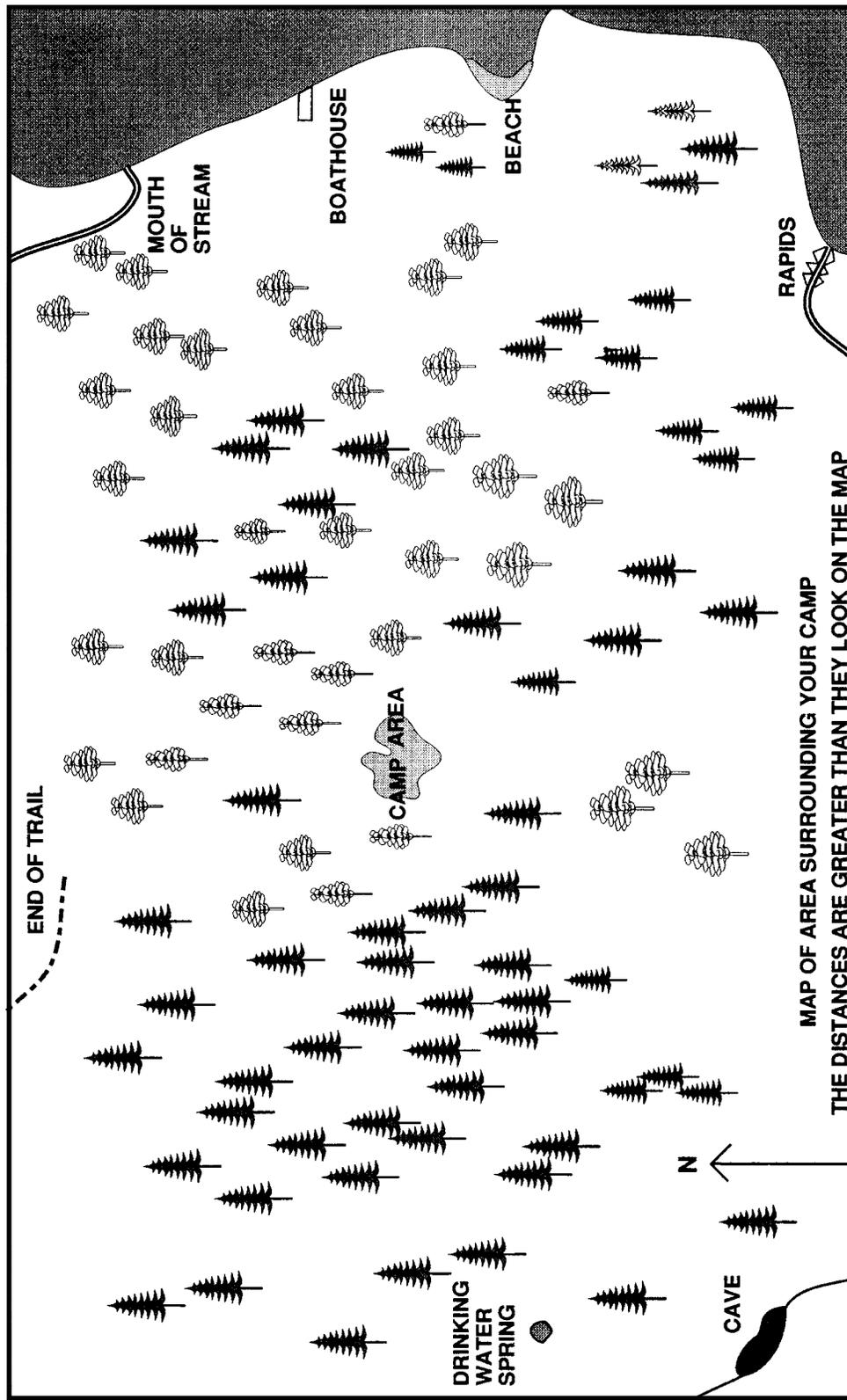


Figure 41 'Get back safely' map

What they do Stage (a)

1. To start with, they all write three headings:

Destination	Outward bearing	Back bearing
-------------	-----------------	--------------

2. Each of them then chooses, in turn with her partner, two or more different destinations, and enters these on her table. These should include some locations on the eastern side of the map, and some on the west and northwest.
3. They then fill in the outward bearings (only) by laying a ruler between the camp area and the destination, and putting the compass rose on top with the centre in the middle of the camp area. It is possible to get the north direction accurate to within one or two degrees by making it perpendicular to the top edge of the map.
4. They then calculate the back bearing which will get them back safely from their destinations, and add these to their tables. These results are then checked using the compass rose in the way the compass bearings card was used in Activity 1.
5. They exchange papers with their partners, and check each other's outward and return bearings. At this stage, the ruler and compass rose method is used for all the bearings. Since all of the locations are larger than a point, some variation in the figure is allowable. The question is, will the outward bearing get them to the correct location, and will the back bearing get them safely back? Both have to be correct, with reasonable accuracy, of course.
6. Any disagreements are then discussed, after which the process may be repeated until all the locations have been used.
7. They should then be ready for Stage (b).

Stage (b)

As and when opportunity permits, it will be valuable for them to use the method learned in Stage (a) for small distances outdoors with a compass. This could be, e.g., in a playground, playing field, park or other open area. At this stage no map is necessary. They simply choose a visible destination, find its bearing with the compass, calculate the back bearing and write it down. They then walk to the chosen location, and use the compass and back bearing to point them back to where they started. They can then see whether they would indeed get back safely if (say) a heavy mist descended. At this stage, they are in no real danger of getting lost; but they are consolidating the knowledge which will get them home in future situations when their destinations are no longer visible.

Activity 3 Where are we? [Space 2.8/3]

An activity for a small group of children. This is a further application of back bearings. In Activity 2, they learned how to use back bearings to re-trace their steps from a known destination to where they came from. It is a method for not getting lost. In the present activity, they learn how to find where they are if they do get lost.

Space 2.8 Directions and locations (cont.)

Materials For each child:

- A copy of the locations map.*
- Worksheet.*
- A transparent compass rose.†
- Ruler.
- Pencil and paper.
- Eraser.

* Provided in the photomasters. See also Figure 42.

† As used in the previous activity.

Suggested introduction for the group as a whole

1. They begin by looking at their location maps, and noting that these are like those which they used in Activity 2, with just two differences. First, the trees are replaced by a number of circles, representing locations. Each location is labelled with a letter or two letters. Second, three more objects are shown which are visible from nearly anywhere. These are a radio transmitter mast, an observation hut at the end of a high ridge, and a tall flagstaff near the boathouse. Explain that in this activity, they will imagine that they are lost somewhere in the area shown by the map, and will learn how to use observations of these landmarks to find where they are.
2. Next they look at their worksheets, the top part of which is illustrated here.

Name	Landmark	Bearing	Back bearing	Location
Alice	Flag	56°		
	Radio mast	296°		
Brian	Obsvn hut	232°		
	Radio mast	330°		

3. Alice is lost. Wisely, she has brought with her a map and compass, and knows how to use back bearings to find where she is. This is what she does.
4. First, she looks around for some prominent landmarks which are also shown on her map. She sees the flag by the boathouse, and the radio mast. So she takes bearings on these, and writes them down. (Yes, she has pencil and paper too!) These are the ones shown on the worksheet.
5. Next she calculates the back bearings from these, and writes these down too. The other children also calculate the back bearings by the method they already know, and write them into their worksheets.

Name	Landmark	Bearing	Back bearing	Location
Alice	Flag	56°	236°	
	Radio mast	296°	116°	
Brian	Obsvn hut	232°		
	Radio mast	330°		

Space 2.8 Directions and locations (cont.)

6. Alice now knows that her direction from the flag is 236° . If she draws a line on the map from the flag in this direction, she is somewhere on it. Likewise, if she draws a line from the radio mast in the compass direction 116° , she is somewhere on this line also. So where the two lines intersect on the map is where she is. Alice is no longer lost.
7. The children should plot Alice's position on their own location maps and check that they get the same location. (It is location AP.)
8. They may then do likewise to find where Brian is, having first erased the previous pair of lines. (Brian is at location AL.)
9. They should now be ready to consolidate this, working together in pairs.

- What they do*
1. Each chooses one of the circles representing locations on the map, and writes its identifying letter(s) on the back of their papers without letting their partners see it (them). The papers are then turned over again.
 2. They each enter their own names in the next space in the left-hand column. They then use their transparent compass roses to find the bearings from their chosen locations of two landmarks, and enter these in their tables.
 3. When both have completed this, they copy each other's entries into their own tables.
 4. When both have copied each other's entries, they find each other's locations by the same method as before. If these are what their partner has written on the back of her paper, congratulations to both of them. If not, they check each other's work to find who has made a mistake, and where.

- Notes*
- (i) Greater accuracy may be obtained by taking bearings of three landmarks. The three lines of the back bearings will now probably enclose a small triangle, and the likeliest location is at the centre of this.
 - (ii) The flagstaff is shown on the map as though it was vertical, for greater realism: so pupils will need to agree what part of it should be used for bearings. In the illustration I have used the flag, since this is likely to be most visible from afar.

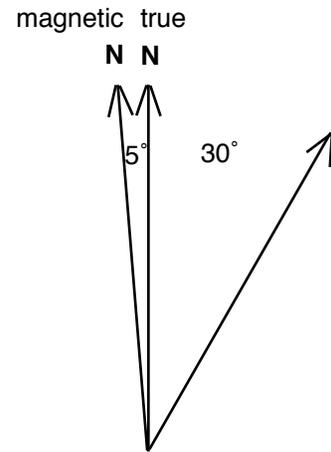
Activity 4 True north and magnetic north [Space 2.8/4]

This is some further information which should be known before using the method of Activity 3 in an outdoor situation. Its importance increases with the distance involved. For example, at a distance of 1 km from the observation point, the distance between true north and magnetic north is 87 metres, for a magnetic deviation of 5° .

- Suggested outline for the discussion*
1. North on a magnetic compass, and north on a map, are nearly but not exactly the same.
 2. Maps use true north, which is the direction of the earth's north pole. The easiest way to find this direction is by the pole star, which shows true north.
 3. Many maps also state the magnetic deviation, i.e., the difference between true north and magnetic north. This varies for different parts of the world, and also changes by a small amount annually.

A typical value is 5° west, which means that magnetic north is 5 degrees west of true north. A true bearing of, say, 30° is therefore equivalent to a magnetic bearing of (in this case) 35° . So magnetic bearings are converted to true bearings by (in this location) subtracting 5° .

4. However, some compasses have built-in ways for converting automatically to bearings based on true north. These are especially useful under difficult or stressful conditions.
5. The difference between true north and magnetic north does not matter for Stage (b) of Activity 2, for several reasons. First, the distances are small. Second, they are not using maps. Third, provided that bearings using magnetic north are used for both outward and return journeys, the difference is still 180° and the back bearing thus calculated will take them back to where they started.



Discussion of activities

In some parts of the world, it is arguable that the knowledge and abilities described in this topic are as important survival skills as learning to swim. The method for finding one's location described in Activity 3 has been a standard method for coastal navigation ever since the introduction of the magnetic compass. Using more sophisticated technology such as radio beacons and direction finding apparatus, the same principle is used worldwide by navigators of ships and aircraft.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Where are we? [Space 2.8/3]

Space 2.9 ROTATIONS OF TWO-DIMENSIONAL FIGURES

- Concepts*
- (i) Rotation of a two dimensional object in its own plane.
 - (ii) Centre and angle of rotation as being the two factors which determine a rotation.

- Abilities*
- (i) To apply a given rotation to a chosen two-dimensional object.
 - (ii) To choose a centre of rotation which will achieve a particular result.
 - (iii) To recognize when a figure has rotational symmetry, and to state the number and angles of rotation.

Discussion of concepts Attention was first focused on the distinction between linear and rotary motion in topic 3. Any linear motion (translations, slides) is linked with a direction, so this was the subject of topics 4 and 5. The related aspects of angle were also considered in these over seven activities. We now consider another feature of rotation. Again we start with an everyday concept, and bring it into a mathematical context.

The effect of a rotation when applied to a two-dimensional figure is determined by two factors: its centre and the angle of rotation, which may vary in amount and direction (clockwise or counterclockwise). Note that the centre of rotation may be within or outside the figure, or on its boundary.

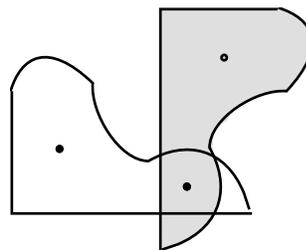
Rotational symmetry is another spatial example of a mathematical pattern. This further helps to integrate the numerical and spatial aspects of mathematics.

Activity 1 Centre and angle of rotation [Space 2.9/1]

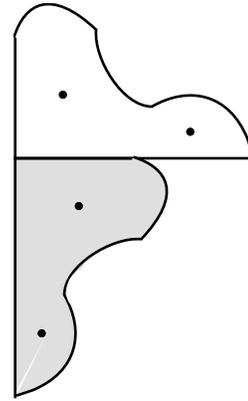
A teacher-led discussion for a small group. Its purpose is to focus attention on the two factors which determine a particular rotation, namely, ‘the centre of rotation’ and ‘the angle of rotation.’

- Materials*
- Baseboards and cutout shapes, one for each pair of children.*
 - Cocktail stick, toothpick, or the like.
- * Provided in the photomasters

- Suggested outline for the discussion*
1. Put out a baseboard with one of the cutout shapes in the matching outline.
 2. Show how, by pressing down through the hole in one of the dots, the figure can be rotated about this point as centre.
 3. Demonstrate rotation about the other dot as centre.
 4. The figure can, of course, be rotated about any point which we can keep still. The dots are just to show the centre of rotation if the stick is removed, and the outline shows the position of the figure before rotations.



5. We can also rotate a figure about a point outside it. In this case we have to use the figure on the baseboard itself.
6. Use two cutouts to show the effects of a quarter-turn (90°) rotation about two different vertices. What stays the same? (The new directions of all the lines.) And what is different? (The position of all the points.)
7. Is the above always the case, for the same amount of turn about different centres?
8. Let them have a board for each pair of children, with two cutouts; encourage them to experiment further. For example, they should try using one cutout to show the effect of a 90° rotation clockwise, leave this in position, and then, using the same centre of rotation with another cutout, compare the effect of a 270° rotation counterclockwise. Likewise with half turns. Can they find any other equivalent pairs?
9. Allow them to experiment, until they have understood the respective contributions of the centre and angle of rotation on the final position and are somewhat familiar with using different angles of rotation. They will then be ready for Activity 2.



Activity 2 Walking the planks [Space 2.9/2]

A game for two or three children: possibly four, but above three it is probably better to have a second set of materials. Its purpose is to consolidate the concepts formed in Activity 1 by using them in a new situation.

- Materials*
- Game board, see Figure 43.*
 - Pack of angle cards.*
 - Planks, one for each player.†

* Provided in the photomasters

† Made from popsicle sticks. They should have notches, or other distinguishing marks to tell them apart.

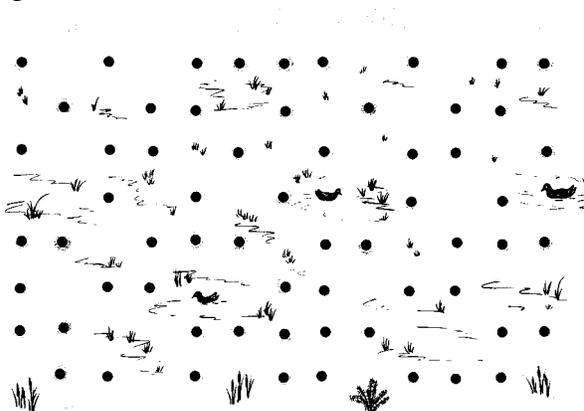
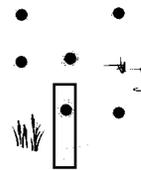


Figure 43 'Walking the planks' game board

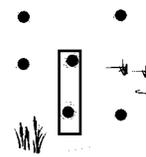
Space 2.9 Rotations of two-dimensional figures (cont.)

Rules of the game

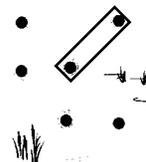
1. The game board represents a swampy area between two firm banks. There used to be several plank bridges across, but these have rotted away. Some of the supporting posts are still there, and these are shown by black circles. The lighter area around these represents the concrete in which the posts are embedded.
2. The aim is to cross the swamp with the help of single planks which each player carries with him as he goes. These can rest only on the posts themselves, but players can stand on the concrete surrounds, provided that they have good balance. They need to do this while moving their planks forward to the next position. It is not possible to have two planks resting on the same pole.
3. The game begins with each player in turn putting down his plank with one end on the bank and the other on any unoccupied post, pointing directly towards the far shore.
4. Each new position of a plank is achieved by a rotation. The centre of rotation may be chosen by the player, but the angle of rotation is determined as follows. (This is the same as in Island Cruising [Space 2.6/3], using different cards.)
5. The angles pack is shuffled and put face down on the table. The top three cards are now turned face up and put separately beside the pack.
6. The player whose turn it is may take just one of the face-up cards, and rotate his plank through that angle, clockwise or counterclockwise. If none of these is of use, the turn passes to the next player.
7. If a player takes a card, he replaces it by turning over the top card from the face-down pack so that there are always at least three face-up cards to choose from.
8. If a whole round goes without any player taking a card from the three face-up ones, these are put aside and three new ones are turned over. If the whole pack is used before all the players have crossed, all the cards are returned and the pack is shuffled ready for further use as above.
9. The winner is the first across, but play continues until all have reached the far shore.



Starting position



After 180° rotation in either direction



After 225° rotation clockwise or 135° rotation counterclockwise

Activity 3 Introduction to rotational symmetry [Space 2.9/3]

A teacher-led discussion for a small group. Its purpose is indicated by the title.

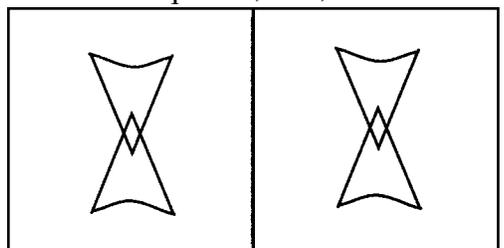
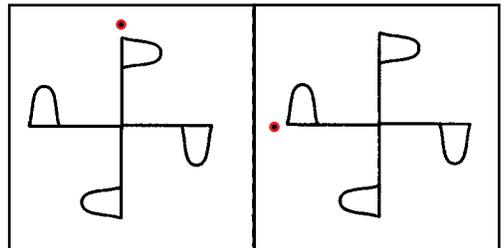
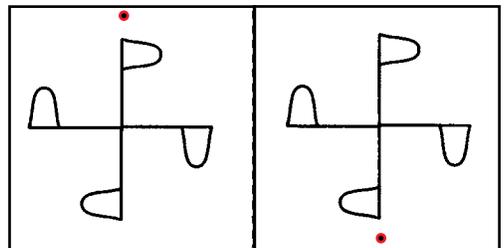
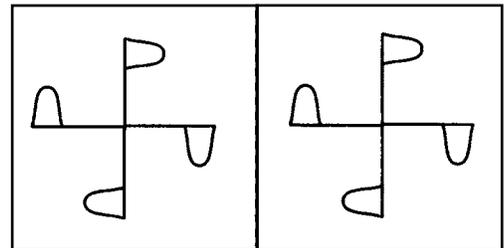
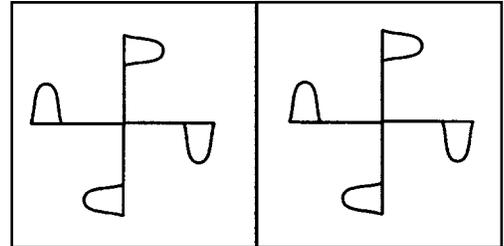
- Materials*
- Examples cards, sets A and B. *
 - A small piece of Plasticine, Blu-tak, or the like.

* Provided in the photomasters. Set A is used for Stage (a) of the present activity, and set B for Stage (b) of the present activity and also for Activities 4, 5, and 6.

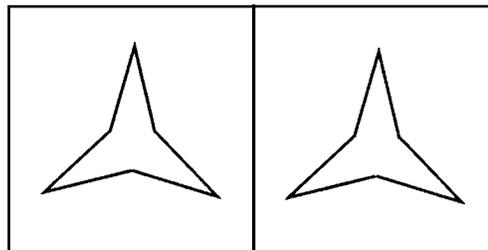
*Suggested
outline for
the discussion*

Stage (a)

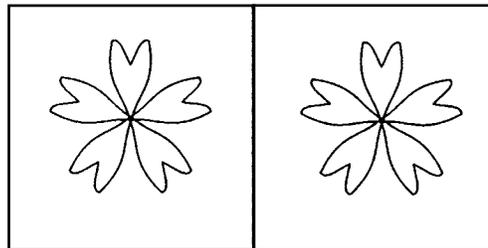
1. Put example cards 1 and 2 side by side. Explain that these are two exact copies of the same diagram, made by a computer.
2. Rotate the right-hand one through a quarter turn clockwise, and ask what they notice. (They still look the same.) Explain that if a figure has this property, we say that it has *rotational symmetry*.
3. Can they see any other angle for which this would be true? (Yes, a half turn.)
4. It is getting hard to see how much it has been turned, so let us start again and put little blobs of plasticine to show where it starts.
5. Now they both start in the same position, and as we rotate the right-hand figure we can see what turn has been made.
6. What rotational symmetries have we found so far? (Quarter, half, and three quarter turns.) And in degrees? (90° , 180° , 270°)
7. Have we got them all now? (And if they answer "Yes" . . .) Should we include a full turn? (It certainly looks exactly the same, but every figure has this symmetry, even a squiggle. So we call this one *trivial*. This figure has just three non-trivial symmetries, for rotations of 90° , 180° , and 270° , which is the same as quarter, half, and three-quarter turns.)
8. Now put out the next two cards, and ask similar questions for this figure. Test with the plasticine blob if desired. (It has just one non-trivial symmetry, for rotation of half a turn, or 180° .)



9. And this? (Two non-trivial symmetries, for 120° and 240° .)



10. The Plasticine will be useful for this one, which has four non-trivial symmetries, for rotations of 72° , 144° , 216° , and 288° . (They might be glad of pencil and paper here, or a calculator!)



Stage (b)

1. Replace the set A cards with set B. Spread out some of these face upwards, and invite the children to sort them into groups with the same number of rotational symmetries. (The pack does in fact have five cards for each of 0, 1, 2, 3, 4, and 5 rotational symmetries: total, 30 cards.)
2. Invite them to say how many rotational symmetries each group has, and also what the angles are.
3. They should then be ready for Activity 4.

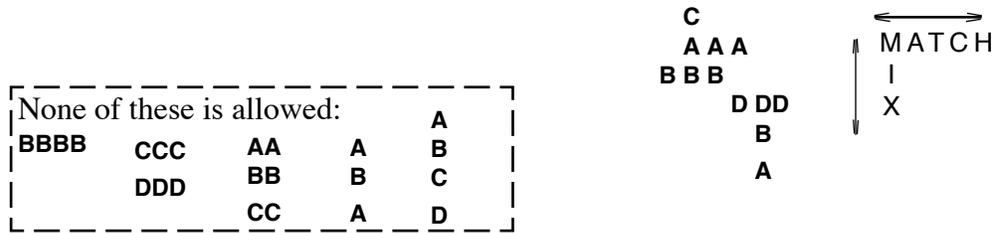
Activity 4 Match and mix: rotational symmetries [Space 2.9/4]

A game for 2 to 5 players, being another member of the ‘Match and Mix’ family. Its purpose is to consolidate the concepts formed in Activity 1, using a variety of examples. The rules are the same as for the other ‘Match and Mix’ games, but are repeated here for convenience.

- Materials*
- Cards set B, as used in Stage (b) of Activity 3.
 - Match and mix card. *
 - Scoring reminder card. *
- * Provided in the photomasters

- Rules of the game*
1. The cards are spread out face downwards in the middle of the table.
 2. The MATCH and MIX card is put wherever convenient.
 3. Each player takes 5 cards (if 2 players only, they take 7 cards each). Alternatively the cards may be dealt in the usual way.
 4. They collect their cards in a pile face downwards.
 5. Players in turn look at the top card in their piles, and put cards down next to cards already there (after the first) according to the following three rules.
 - (i) Cards must match or be different, according as they are put next to each other in the ‘match’ or ‘mix’ directions. Here, ‘match’ or ‘mix’ refers to the number of rotational symmetries, remembering that in some cases this number is zero.
 - (ii) Not more than 3 cards may be put together in either direction.
 - (iii) There may not be two threes next to each other. This is to keep an open pattern, with as many growing points as possible.

6. A card which cannot be played is replaced at the bottom of the pile.
A typical arrangement (using A, B, C . . . for different kinds of rotational symmetry):



7. Scoring is as follows:
1 point for completing a row or column of three, with an extra point for correctly naming all the angles of rotational symmetry.
2 points for putting down a card which simultaneously matches one way and mixes the other way.
2 points for being the first 'out'; 1 point for being the second 'out.'
(So it would be possible to score up to 7 points in a single turn.)
8. Play continues until all players have put down all their cards.
9. Another round may then be played.

Activity 5 Match and mix (line symmetry) [Space 2.9/5]

Since many of the cards used in Activity 4 also have line symmetries, we have here a good opportunity to review and consolidate this concept which was first introduced in topic 2. It also provides somewhat harder examples than were used then.

- Materials*
- The same as for Activity 4, with a minor difference in the scoring reminder card.*
 - Something with which to indicate the axes of symmetry, such as a cocktail stick, a straight piece of wire, a straightedge of clear plastic.
- * Provided in the photomasters

Rules of the game The same as for Activity 4, except that in this case the extra point is scored for correctly showing the axes of symmetry (mirror lines).

Discussion of activities

Having established the concept of angles as measuring amount of turn, we now apply this as a mathematical operation. Operations like addition and subtraction are applied to numbers (among other things). Here, the operands are two-dimensional figures, and the results are therefore more visible than numerical operations.

Following a familiar sequence, Activity 1 explores this new concept with physical embodiments, and Activity 2 then consolidates it in a board game. The latter underlines the distinct influences on the outcome of the centre and angle of rotation. Activity 3 combines their existing concept of symmetry with the just-acquired concept, and comes up with a new variety of symmetry. Activity 4 strengthens this concept by having them apply it to a substantial number of examples, some of them quite difficult. In Activity 5, they remind themselves of the other kind of symmetry which they know, and look for this in the same figures that they have just used for rotational symmetry.

As I write this, having just finished field testing the foregoing five activities with girls and boys of average ability, I continue to marvel at what the human child can apprehend, given suitable resources.

Space 2.10 RELATION BETWEEN REFLECTIONS, ROTATIONS, AND FLIPS

Discussion of concepts The terms 'reflection' and 'flip' are often used interchangeably. The purpose of the present topic is to examine this assumption more closely. The discussion which follows is for your own information, and there are no activities for the children. I leave it to your own judgement how much of the following you pass on to them.

A mathematical reflection (unlike that in everyday life) takes place entirely in two dimensions. It would be useful at this stage to review the discussion of the concept of reflection given at the beginning of topic 1 in this network. From this, and especially from the Space 2.1/3 worksheet, it can be seen that the original figure, the mirror line, the result of reflection, and all the construction lines are in the same plane, i.e., the plane of the paper.

Both in everyday life and mathematically, however, a flip takes place in three dimensions. If we start with a playing card face up and flip it, the result will be the same card face down. And although the start and the result are in the same plane, namely, the surface of the table, during the actual process of flipping the card has to come out of that plane. Twist it and turn it how we may, there is no way that a face-down card can move into a face-up position while remaining flat on the table. And in this case, the result is quite different from that produced by a reflection.

This is because the card is opaque. If it were transparent, then after the flip we would still see the same figure, but from the other side. So everything would be reversed, in the same way as in a reflection. In this case, although the process is different from that of a reflection, the result of a flip looks the same. And if the flip takes the form of a rotation about the mirror line, then the resulting location will be the same as the result of a reflection in the mirror line.

In everyday life there are thus two possibilities, since thin flat objects can be either opaque (like a playing card) or transparent (like an overhead projector transparency). Do these alternatives apply to a mathematical plane figure?

No. A mathematical plane is not just thin like a playing card; it has no thickness at all. So all points in a plane appear on both its sides. The *result* of flipping a mathematical plane figure is therefore like that of an everyday flip of a transparent object, as described above. The two *processes* however are importantly different.

It was with the foregoing in mind that I did not introduce the term 'flip' in topic 1 as being interchangeable with the term 'reflection.' The purpose of this discussion has been to help the reader to make an informed personal choice in this matter.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[NuSp 1] THE NUMBER TRACK and THE NUMBER LINE

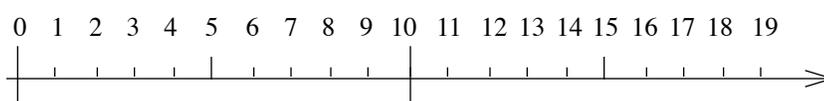
Powerful support to our thinking about numbers

NuSp 1.7 UNIT INTERVALS: THE NUMBER LINE

Concepts (i) Unit intervals on a line.
(ii) The number line.

Ability To use the number line in the same ways as the number track, in preparation for other uses of the number line.

Discussion of concepts



The differences between a number track and a number line are appreciable, and not immediately obvious.

The number track is physical, though we may represent it by a diagram. The number line is conceptual – it is a mental object, though we often use diagrams to help us think about it. The number track is finite, whereas the number line is infinite. However far we extend a physical track, it has to end somewhere. But in our thoughts, we can think of a number line as going on and on to infinity.

On the number line, numbers are represented by points, not spaces; and operations are represented by movements over intervals on the line, to the right for addition and to the left for subtraction. The concept of a unit interval thus replaces that of a unit object. Also, the number line starts at 0, not at 1. For the counting numbers, and all positive numbers, we use only the right-hand half of the number line, starting at zero and extending indefinitely to the right. For positive and negative numbers we still use 0 for the origin, but now the number line extends indefinitely to the right (positive numbers) and left (negative numbers).

Activity 1 Drawing the number line [NuSp 1.7/1]

This is a simple activity for introducing the concept.

Materials • Pencil and paper for each child.

- What they do*
1. Ask the children to draw a line, as long as will conveniently go on the paper.
 2. They mark off equal intervals on it.
 3. They number these 0, 1, 2 . . . as in the diagram above.
 4. At this stage, the two differences between the number track and the number line which need to be pointed out are: (i) With the number track, numbers are represented by spaces; with the number line, numbers are represented by points on the line. Though it is helpful to use different marks for tens, fives, and ones, it is the points on the line which represent the numbers. (ii) The number track starts at 1, the number line starts at 0.

Activity 2 Sequences on the number line [NuSp 1.7/2]

'Sequences on the number track,' NuSp 1.2/1 in *Volume 1*, may usefully be repeated here. Smaller markers like the one suggested in the following activity will be needed.

Activity 3 Where must the frog land? [NuSp 1.7/3]

A game for two players. Its purpose is to introduce the use of the number line for adding.

- Materials*
- As long a number line as you like (some number lines are provided in the photomasters).
 - A marker representing a frog for each player, occupying as small a base as possible.*
 - 1 die 1-6, or 1-10 for athletic frogs.

* A short length (about 2 cm) of coloured drinking straw, with a bit of Plasticine on the end, makes a good marker for this and many other activities.



- What they do*
1. The frogs start at zero.
 2. Player A throws the die and tells the frog what number it must hop to. This is done mentally, using aids such as finger counting if he likes. (To start with, children may use counting on along the number line, but should replace this by mentally adding as soon as they have learned the game.)
 3. Player B checks, and if he says "Agree" the frog is allowed to hop.
 4. If B does not agree, he says so, and they check.
 5. If A has made a mistake, his frog may not hop.
 6. Two frogs may be at the same number.
 7. They then exchange roles for B's throw of the die.
 8. The winner is the frog which first hops past the end of the line. (The exact number is not required).

Activity 4 Hopping backwards [NuSp 1.7/4]

The subtraction form of Activity 3, starting at the largest number on the number line and hopping backwards past zero.

Activity 5 Taking [NuSp 1.7/5]

Another capture game for two, but quite different from the 'Capture' in *Volume 1*, NuSp 1.4/4. Its purpose is to give further practice in relating numbers to positions and movements on the number line.

- Materials*
- 1 number line 0-20.*
 - 3 markers for each player.
 - 1 die 1-6.
- * Provided in the photomasters

What they do Form (a)

1. The markers begin at zero.
2. The die is thrown alternately, and according to the number thrown a player may jump his marker forward that interval on the line.
3. A piece which is jumped *over* is taken, and removed from the board for the rest of the game.
4. An occupied point may not be jumped *onto*.
5. A player does not have to move at all if he doesn't want to. (We introduced this rule when we found that starting throws of low numbers were likely to result in the piece being taken next throw, with no room for manoeuvre.)
6. The winner is the player who gets the largest number of pieces past 20. (It is not necessary to throw the exact number.)

Form (b)

This may also be played as a subtraction game, backwards from 20.

Activity 6 A race through a maze [NuSp 1.7/6]

This is a board game for up to 3 players. Its purpose is to bring out the correspondence between the relationships smaller/larger number and left/right hand on the number line.

- Materials*
- Board, see Figure 44.*
 - Number line 1-20 *
 - Number cards 1-20 *

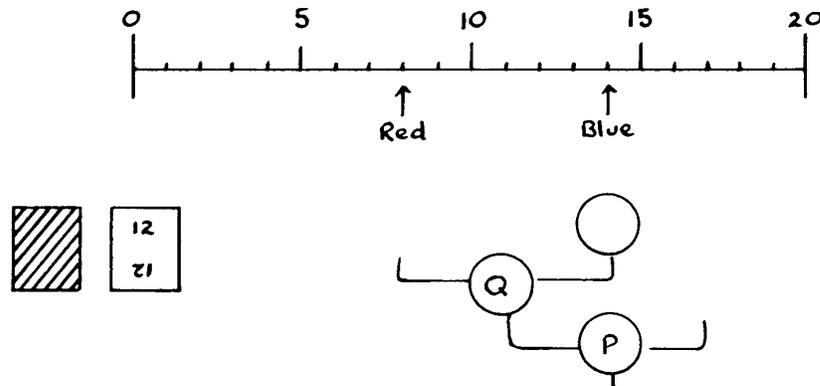
* Provided in the photomasters

For each player:

- 1 coloured pointer for the number line.
- 1 marker to take through the maze.

- What they do*
1. The pack of number cards is shuffled and put face down.
 2. In turn, each player turns over the top card and puts it face up, starting a new pile. He may then use this number to position his pointer on the number line, or decide not to (since an extreme left or right position is not helpful). In that case he repeats this step at his next move.
 3. When he does position his pointer, he also puts his marker at the start of the maze.
 4. After taking steps 2 and 3, players at subsequent turns move forward through the maze if they can. The number they turn over determines whether they can or not.

An example will make it clear.



NuSp 1.7 Unit intervals: the number line (cont.)

Since 12 is to the left of the blue pointer's position, player 'Blue' can move forward if he is at P on the maze, but not if he is at Q. However, the reverse is the case for player 'Red', since 12 is to the right of his pointer.

5. There is no limit to the number of players at a given position on the maze.
6. When all the cards are turned over the number pack is shuffled and used again.
7. The winner is the first player to reach the finish.

Note If the children have difficulty in remembering their left and right, you could help with labelled arrows.

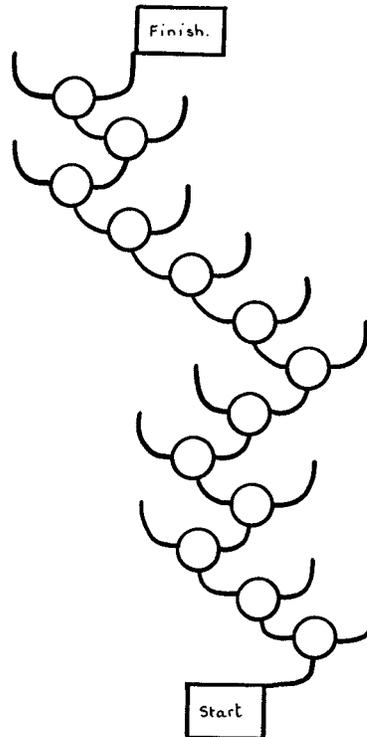


Figure 44 A race through a maze

Discussion of activities

The first four activities are concerned first with introducing the number line, and then with linking it to concepts which are already familiar. Activity 5, 'Taking,' is more difficult, since it involves mentally comparing a number of possible moves before deciding which one to make.

Activity 6 involves correspondence not between objects, but between two different relations. One is the relation of size, between two numbers, and the other is the relation of position, between two points on the number line. Here is another good example of the conceptual complexity of even elementary mathematics. Yet young children manage this without difficulty if we make it possible for them to use their intelligence to the full.

OBSERVE AND LISTEN

REFLECT

DISCUSS

NuSp 1.8 EXTRAPOLATION OF THE NUMBER LINE. EXTRAPOLATION OF THE COUNTING NUMBERS.

- Concepts*
- (i) The number line as extending further than we could conveniently draw the whole of.
 - (ii) The counting numbers as continuing further than we could conveniently count starting at zero.
 - (iii) The number line as having other points and numbers as well as those marked.

Abilities Given satisfactory ‘landmarks’ on the number line, to construct that part of the number line mentally; and thereby to state the numbers of other points on that part of the line.

Discussion of concepts When we have clearly seen a pattern, we are often able to extrapolate it, i.e., take the pattern further. Here we have two corresponding patterns: the spatial pattern of the number line, and the pattern of the counting numbers. Each of these has patterns within patterns: the 0-9 cycle is repeated anew, starting again at 10, 20, 30, The 10, 20, 30, . . . 90 cycle is repeated every hundred; and so on.

Since we can think of a line as continuing as far as we like, and as taking these patterns along with it, the number line is an excellent mental support for extending the more abstract concepts of the counting numbers.

Activity 1 What number is this? (Single starter) [NuSp 1.8/1]

An activity for children playing in pairs. Its purpose is to start them thinking in terms of a number line longer than one of which one could conveniently draw the whole.

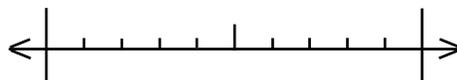
Materials For each pair:

- Two cards, each showing part of a number line.*
- Two pointers.**

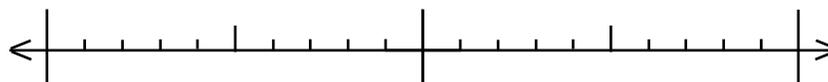
* See illustration below. Card (a) is used for Stage (a), Card (b) for Stage (b).

** E.g., Toothpicks or cocktail sticks. We have found that a paper clip, with one end straightened out, also makes a good pointer.

Card (a)



Card (b)



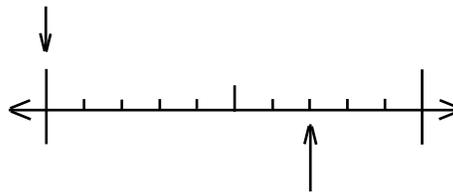
NuSp 1.8 Extrapolation of the number line. Extrapolation of the counting numbers. (cont.)

What they do Stage (a)

1. Explain that the card shows part of a number line. The arrows at each end mean that the line continues back to zero on the left, and as far as we like on the right. The cross-lines mark tens, the smallest marks show ones, and the fives are shown by slightly longer marks.
2. Child A begins by pointing to the left-hand mark, and says (e.g.)

“This number is 40.”

She points again, and says (e.g.)



“What number is this?”

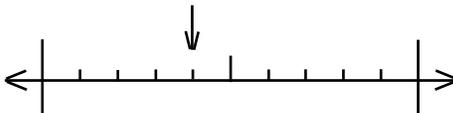
3. If child B says “47,” child A would reply “Agree” and they would change roles. If child B says something else, they would need to discuss.
4. Child A need not always name the left-hand point. E.g., she could begin by saying

“This number is 70”



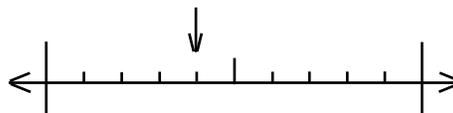
or

“This number is 34.”



5. But A must keep to the tens, fives, ones pattern of marking. E.g., she should not say

“This number is 22.”



If she did, B would say “Disagree” and they would need to discuss.

Stage (b)

This uses card (b), which covers 2 decades. Otherwise it is played as in Stage (a).

Activity 2 What number is this? (Double starter) [NuSp 1.8/2]

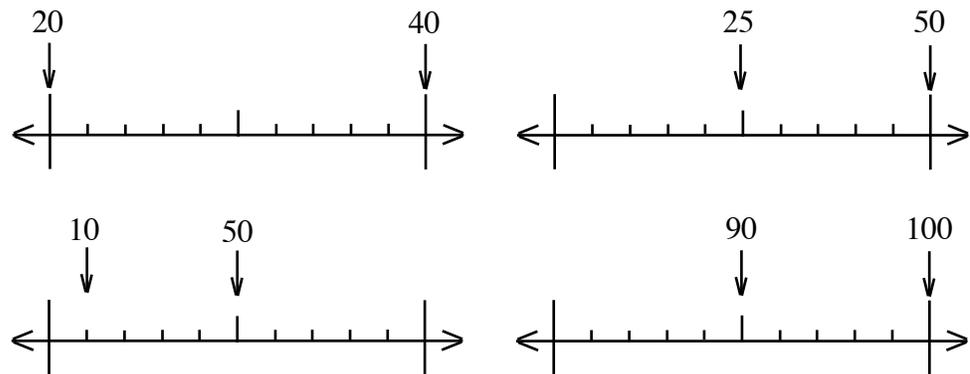
A continuation of Activity 1, to be played in pairs. This is appreciably harder.

Materials • The same as for Activity 1.

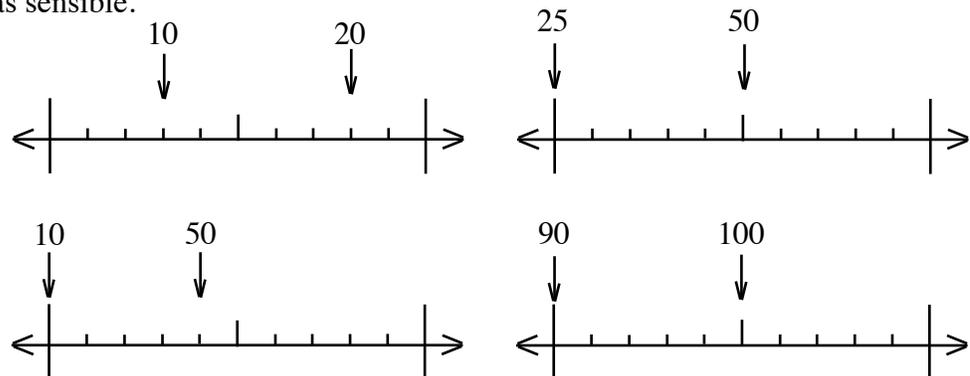
What they do Stage (a) using Card (a).

- Child A now begins by placing *two* pointers and saying what number they represent. The smaller marks may now represent either unit intervals as before, or intervals of 2, 5, or 10. The larger marks and numbers lines must form part of a ‘sensible’ system. This is best shown by examples.

Sensible.



- Child A then points to another mark on the line, and asks “What number is this?” If they agree, they interchange roles and steps 1 and 2 are repeated. If not, they discuss.
- If child B cannot think of a sensible system to explain where A has put the pointers, she may say “Challenge” and A must explain. What is a sensible system is to some extent open to discussion. Such a discussion would in itself be a valuable part of the activity. My own view is that the larger marks, and cross lines, should represent more important landmarks. Thus I would not see the following as sensible.



Stage (b)

This uses card (b); otherwise, as in Stage (a).

Stage (c)

This uses card (b). The only difference between this and Stage (a) is that the intervals used may be 1, 2, 5, 10, or multiples of these such as 20, 50, 100 etc.

Activity 3 Is there a limit? [NuSp 1.8/3]

A teacher-led discussion with a group of children. Underlying this is the difference between the finite nature of anything we can do physically, and the in-finite possibilities of thought.

Suggested sequence for discussion

1. Ask them how long a line they think they could draw. If they respond with answers like “100 miles,” emphasize that you mean *actually draw*. Lead them in this way to realize the practical limitations of a line drawn physically.
2. Next, ask them how big a number do they think they could actually count – meaning, say the number words? Again let them come to realize the practical limitations.
3. Combine these in the number line. Show them a number line (say, 0-20) drawn on paper and ask them how far do they think this number line could be continued?
4. Now ask them: suppose we don’t try to draw it, but just continue it in our imagination? What number would it take us to?
5. If they do not yet realize that there is no longer any limitation, then Activity 4 will help to lead them towards this realization. If they do, this activity will consolidate it.

Activity 4 Can you think of . . . ? [NuSp 1.8/4]

A game for two teams. I think it needs to be teacher directed. It might be played with the rest of the class as spectators.

Materials • Chalkboard.

What they do Stage (a)

1. Team A writes a very large number, as big as they like.
2. Team B tries to think of a bigger one.
3. The task becomes easy as soon as they realize that all they need to do is to add 1.
4. When they do, they are ready to go on to Stage b.

Stage (b) Can you think of a further point?

1. Team B have now to think of a very long number line, and name (give the number of) its last point.
2. When they have done so, team A tries to name a point beyond this.
3. Again all they need is to name the point one further on.

There is no largest number!
There is no furthest point!

Discussion of activities This topic is a good example of creative imagination. In Activities 1 and 2, our thinking is supported by visual representations; but the meaning of these becomes increasingly a matter of choice. This choice is, however, neither random nor arbitrary. It has to 'make sense' in terms of what we already know about numbers and the number line.

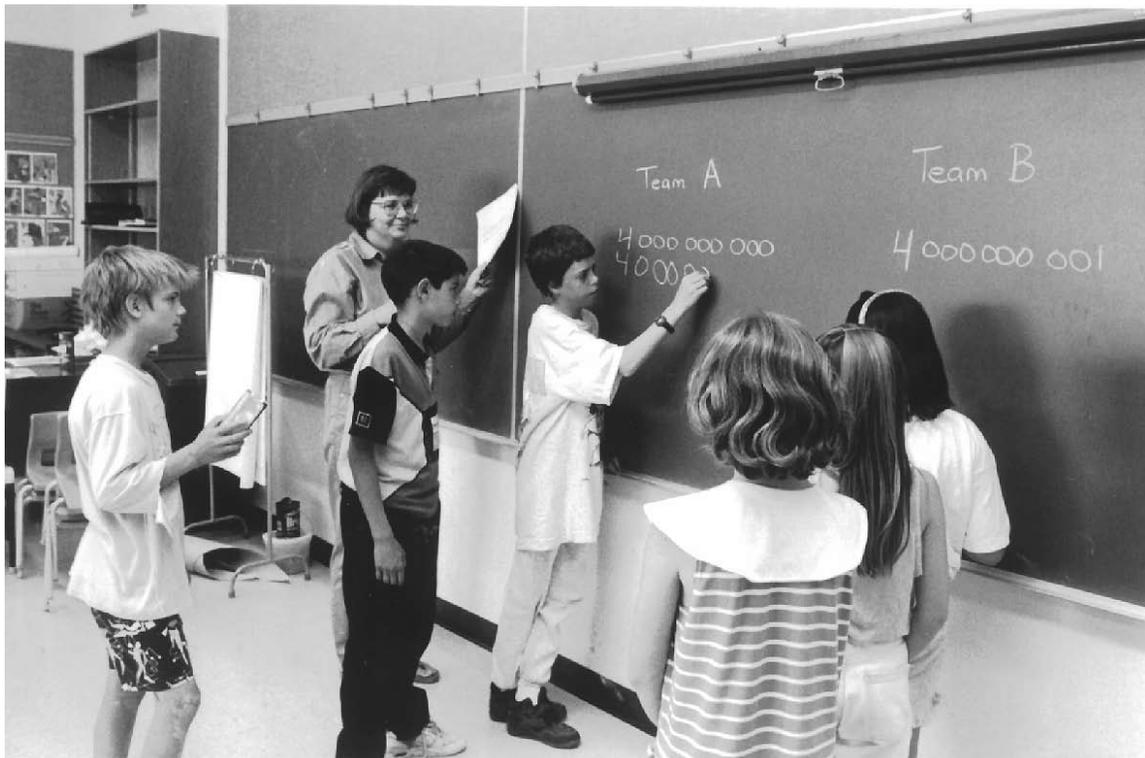
In both of these activities we thus have Mode 3 schema building, and schema testing by Mode 3 (consistency with existing knowledge) and Mode 2 (discussion). Activity 2 is suitable only for the more able children. It requires a well developed feeling for number patterns. With very able children, I have used it without giving them any restriction on what the marks may mean, beyond saying that they must form part of a sensible system.

In Activities 3 and 4, children are led to realize that the limitations of working with physical materials disappear when we move into the realm of thought. It then becomes all the more important to have an internally regulated orderliness, such as has been developed in Activities 1 and 2.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Can you think of . . . ? [NuSp 1.8/4]

NuSp 1.9 INTERPOLATION BETWEEN POINTS. FRACTIONAL NUMBERS (DECIMAL)

- Concepts* (i) Points marking tenth-parts of a unit interval.
(ii) New numbers, corresponding to these points, called decimal fractions.

Ability To identify tenth-parts of unit intervals on a number line and to associate them with appropriate decimal fractions.

Discussion of concepts There are still a lot of ‘unused’ points on the number line, that is, points which do not yet have a number. And we do not need to go in the direction of infinity to find them – they are in-between the points already in use, and correspond to fractional numbers. In this topic we shall be concerned mainly with decimal fractional numbers, usually and inaccurately called ‘decimals.’

The relation between a fraction and a fractional number is discussed in Num 7. As long as we know this difference, there is no harm in following the custom of using the word ‘fraction’ for both. The currently established usage of ‘decimal’ is however less acceptable. ‘Decimal’ simply means ‘related to ten’ (Latin, *decem*). So ‘decimal *notation*’ means base ten place-value notation as against (e.g.) base two, or binary, place-value notation. It is inaccurate to talk about ‘decimal *numbers*’ and ‘binary *numbers*,’ and very misleading. The *number* of objects in a set still remains the same if we *write* this number differently, just as it does if we use a French or German number-name.

‘Decimal *fraction*’ does, however describe a different kind of number from the counting numbers. It is a fractional number whose denominator is a power of ten: ten, a hundred, a thousand, etc. So

$$\frac{1}{10}, \quad \frac{7}{10}, \quad \frac{8}{100}, \quad \frac{47}{100}, \quad \frac{305}{1000}$$

are all decimal fractions, written in ‘fraction bar’ notation. Decimal fractions have the advantage that we can also write them in the place-value notation already in use. So

$$.1 \quad .7 \quad .08 \quad .47 \quad .305$$

are decimal fractions written in (decimal) place-value notation. A ‘decimal’ may or may not be a decimal fraction, and a decimal fraction may or may not be written in place-value notation.

We do not need to explain all this to the children, but we must try to avoid muddling them, which means we must try to clear up a long-standing muddle in our own minds. It is no accident that ‘decimals’ have long been the bane of generations of school children.

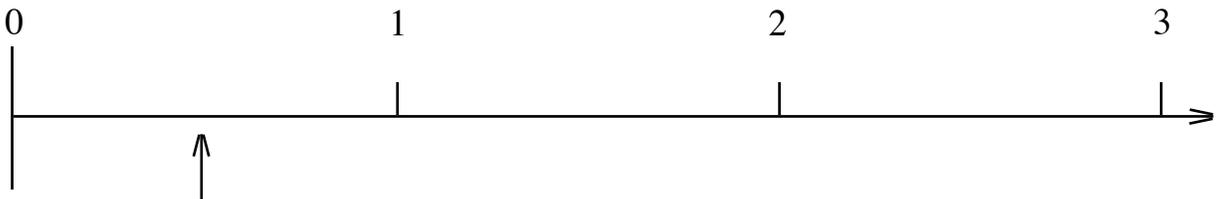
I suggest as a reasonable compromise that we accept ‘fraction’ with the dual meaning described above, but say and write ‘decimal fraction’ when we mean a fractional number of the kind described; and ‘place-value notation’ for (e.g.) 17, 0.17, 1.7, etc. This, though not identical with general usage, is not in conflict with it, and therefore should not muddle anyone.

Activity 1 What number is this? (Decimal fractions) [NuSp 1.9/1]

This is introduced as a teacher-led discussion with a group of children after which it continues as in NuSp 1.8/1.

Suggested sequence for the discussion

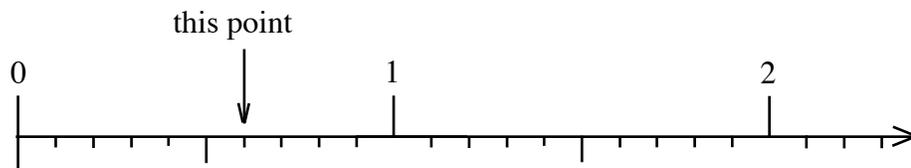
1. Draw, while you talk, an enlarged part of a number line, starting at zero, as shown below. Point halfway between 0 and 1, and ask



What number is this?

“A half” is a good answer at this stage; we have not yet arrived at decimal fractions. Mark this point in pencil, and do similarly for the points which correspond to a quarter and three quarters.

2. We now have the unit interval 0-1 divided into four equal parts. Explain that these are useful divisions, but they don't fit in with the tens pattern of the rest of the number line.
3. So we rub these out (the half one can stay if you like) and mark the interval 0-1 in ten equal parts. We call these ‘tenth-parts.’ (‘Tenth’ by itself has a different meaning, namely an ordinal number in the sequence first, second . . . ninth, tenth . . .) So the number for



we will call for the present ‘6 tenth-parts.’

And the number for



we will call ‘1 and 6 tenth-parts.’

Once they have grasped this, they are ready to repeat “What number is *this*?” as in NuSp 1.8/1, using the same diagrams, but with the smaller divisions representing tenth-parts.

Activity 2 Snail race [NuSp 1.9/2]

A number line game for up to about four children. Its purpose is to provide a simple introduction to fractional points on the number line.

- Materials*
- Snail race board, see Figure 45.*
 - Markers for each child to represent snails.**
 - 1 die 1-6.

** These need to take up as little space as possible, as in NuSp 1.7/2 and 3.

* Provided in the photomasters

- What they do*
1. Players in turn throw the die, but move forward at a snail's pace! Each number thrown represents only that number of tenth-parts.
 2. Passing is allowed.
 3. Two snails may not occupy the same point. Points are very small. They must not move if their roll would take them to an occupied point.
 4. Players should declare where they have landed, e.g., "1 and 3 tenth-parts."
 5. They must roll the exact number to finish.

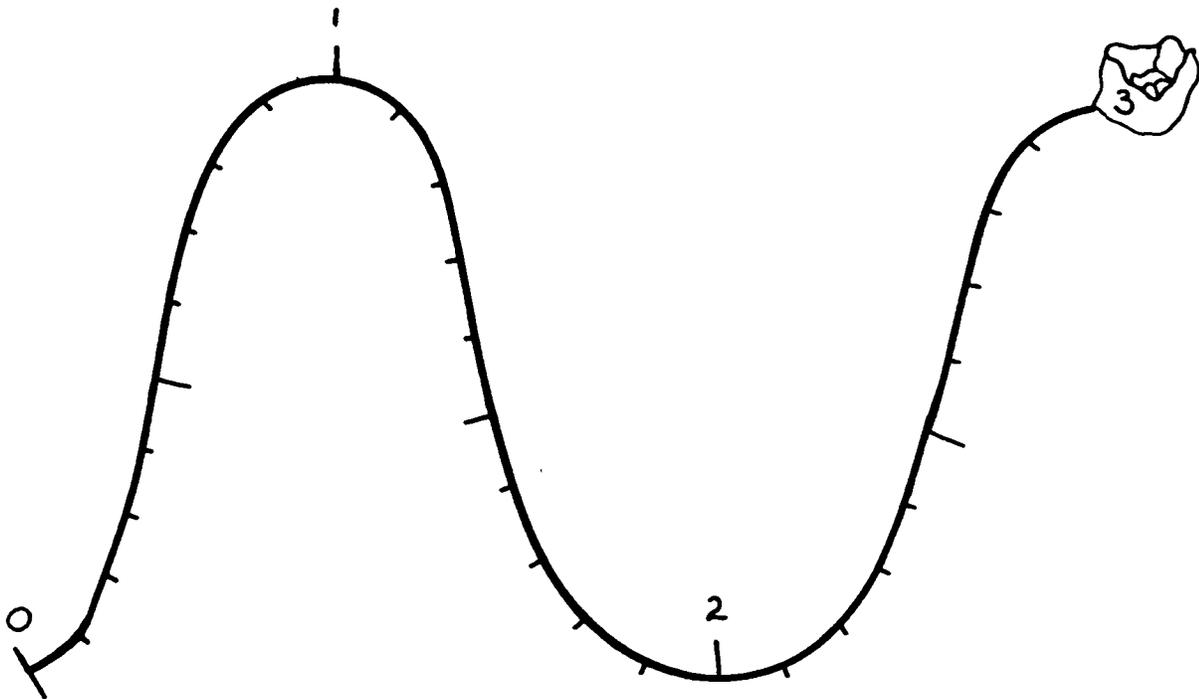


Figure 45 Snail race

Activity 3 Snails and frogs [NuSp 1.9/3]

This game is played like the one before, with the following modifications. Its purpose is to make a contrast between whole and fractional numbers.

- Materials*
- Snails and frogs board, see Figure 46.*
 - 1 die marked 1, 2, 3 only.**
 - Markers as before.

* Provided in the photomasters

** This can be made from a base-10 unit cube.

- What they do*
1. Normally a snail moves only that number of tenth-parts, as before.
 2. However, if a move takes him exactly to a whole number, he is magically transformed into a frog for his next move, and hops the number of whole unit intervals shown by the die.
 3. The spell is broken when he lands, however, and he must crawl snail-wise for his next moves.
 4. To increase his chance of landing on a whole number, a player may decide not to use his throw. A snail may go into his shell and wait for a beneficial throw. Some children may even calculate the odds that this is a good tactic!
 5. Passing is allowed, but two snails may not occupy the same position unless this is a whole number.
 6. Players should declare where they have landed, e.g., “4 and 3 tenth parts.” They must score the exact number to finish.

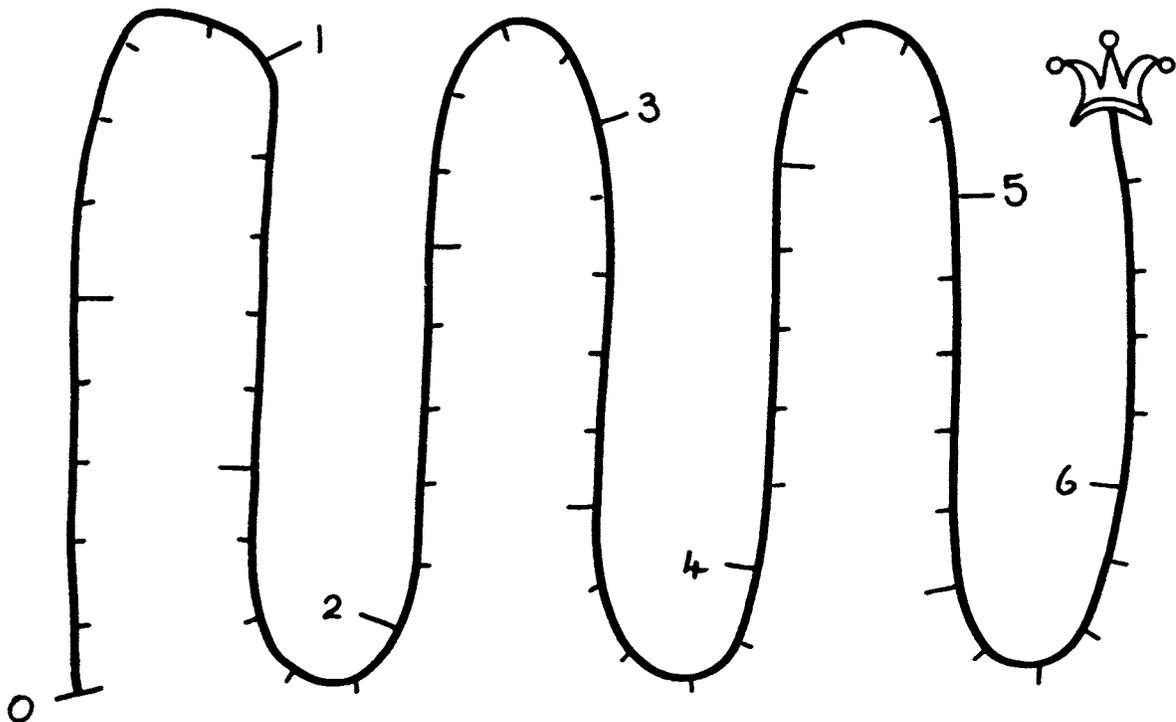


Figure 46 Snails and frogs

Discussion of activities Here again we have a clear example of extrapolation – continuing a known pattern into a new situation. This is of a more difficult kind, since a new variety of number is being mentally constructed. In doing this, the visual support of the number line is a useful aid. Mixed numbers have been used from the beginning here because this gives continuity with the established decimal pattern. (This does not apply in Num 7, where fractions other than decimal are encountered).

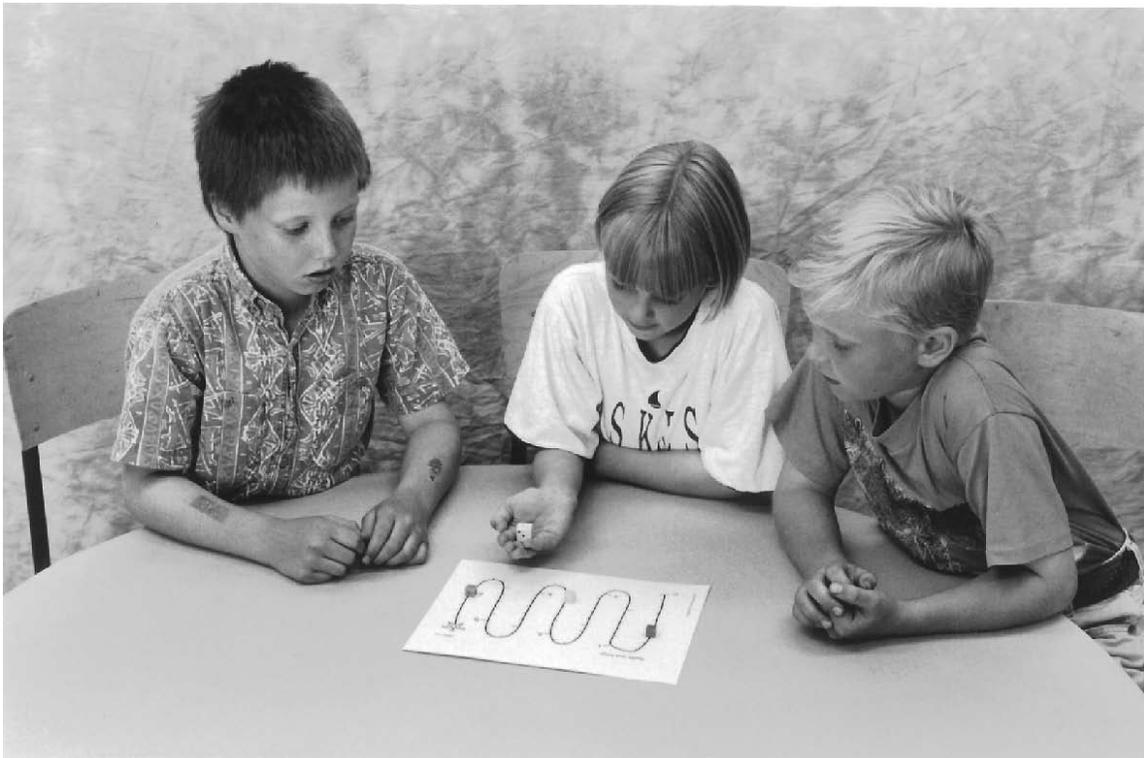
I hope that you will share my belief in the importance of correct vocabulary when developing this new area of mathematics. I have tried to keep it simple as well as accurate. “Tenth-part” may eventually be shortened to “tenth” (etc.) when the meaning is clear from the context – but not until this meaning is well established.

A muddled vocabulary is a hindrance to clear thinking. We correct bad English in our pupils, so we should not ourselves speak ‘bad mathematics.’

OBSERVE AND LISTEN

REFLECT

DISCUSS



Snails and frogs [NuSp 1.9/3]

NuSp 1.10 EXTRAPOLATION OF PLACE-VALUE NOTATION

Concept The use of place-value notation, including the decimal point, for writing mixed numbers as represented by points on the number line.

Abilities (i) To write in decimal notation the number for a given point on the number line.
(ii) To identify the point for a given mixed number.

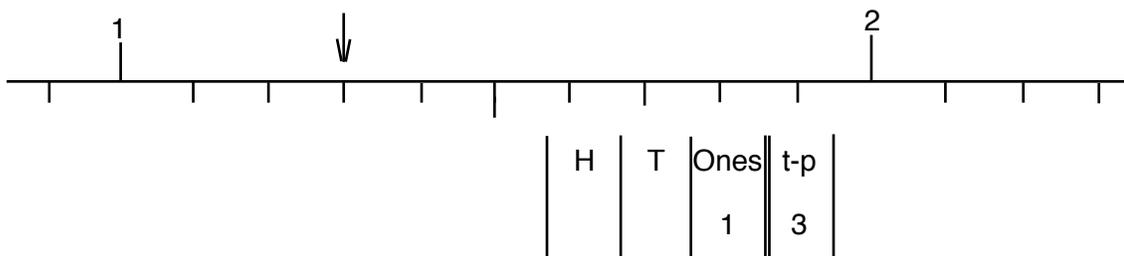
Discussion of concept In the preceding topic, children have made a start in forming the concept of a new kind of number, a decimal fraction, in the particular context of a number line. They also have spoken names for these new numbers. Now they learn how to write them.

Activity 1 “How can we write this number?” (Headed columns) [NuSp 1.10/1]

A teacher-led discussion with a group of children. Its purpose is to introduce the use of headed columns for writing fractional numbers.

- Materials*
- A number-line at least 30 cm long, using 1 decimetre (10 cm) as the unit interval.* Centimetres will now represent tenth-parts, and millimetres hundredth-parts. This is introduced in two stages.
 - Pencil and paper for everyone.
 - Pointers.
- * This may conveniently be made from graph paper.

- Suggested sequence for the discussion*
- Stage (a)*
1. Begin by saying, “We’ve now learned about new kinds of numbers, called decimal fractions, and we can say their number names. For example, the name of this number is . . . ?” (Here, use a few examples for review, as in NuSp 1.9/1. At this stage use only the centimetre divisions, representing tenth-parts.)
 2. “Now we need a way of writing them.” Begin by drawing headed columns, like those already in use for ones, tens, and hundreds. Discuss the pattern: ten times larger for each column to the left, ten times smaller for each column to the right. This suggests a new column for tenth-parts, on the right of the ones column. A double line separates whole numbers and fractions.
 3. Draw this, and let the children draw their own.
 4. What number is this?



NuSp 1.10 Extrapolation of place-value notation (cont.)

Ask how we should write the number on the previous page, spoken as “one and three tenth-parts” and achieve agreement that we should write 1 in the ones column, and 3 in the tenth-parts column. The children should say this aloud, and use the pointers to indicate what they mean.

5. The children may now take turns to point to positions on the line, keeping at this stage to the centimetre divisions. The others write the numbers for these points in their headed columns, and compare.
6. Repeat until they are confident.

Stage (b)

1. Point to one of the millimetre divisions and ask what they think these represent.
2. So now we need another column.

H	T	Ones	t-p	h-p
		1	3	

3. So what number is this?



“One and three tenth-parts, seven hundredth-parts.” It is written

H	T	Ones	t-p	h-p
		1	3	7

4. The children then take turns to point while the others say and write the numbers in their headed columns.

Activity 2 Introducing the decimal point [NuSp 1.10/2]

A teacher-led discussion, introducing the use of a decimal point instead of columns.

Materials • Writing materials for the teacher.

Suggested sequence for the discussion Remind the children that for whole numbers like this one:

H	T	Ones
4	8	5

we can save having to write the columns every time, and write just

4 8 5

as long as we remember that reading from right to left the digits mean ones, tens, hundreds.

This only works if we have nothing smaller than ones. As soon as we do, we can't simply leave out the columns, or these two numbers

H	T	Ones	t-p	h-p
		2	7	
		2	7	

would both be written:

2 7

We need to know where whole numbers finish and fractional numbers begin. This is emphasized in headed column notation by the double line. So if we want to drop the columns and use place-value notation, all we need to do is show where the double line would be. A decimal point does this.

We can now write the above as

2.7 meaning 2 ones, 7 tenth-parts, and
.27 meaning 2 tenth-parts, 7 hundredth-parts.

These are read aloud as “Two point seven” and “Point two seven.” It is a good idea to write the second of these as 0.27 to emphasize that there are zero whole numbers.

Activity 3 Pointing and writing [NuSp 1.10/3]

A game for a small group. Its purpose is to practise and test their use of place-value notation for decimal fractions, in conjunction with the number line.

- Materials*
- Number line marked in whole numbers, tenth-parts, and hundredth-parts.*
 - 2 packs of assorted number cards.*†
 - Pointer.
 - Pencil and paper for each child.

* Provided in the photomasters

† Mixed numbers in decimal notation. The pack for Stage (a) uses whole numbers and tenth-parts. The pack for Stage (b) uses whole numbers, tenth-parts and hundredth-parts.

- What they do*
- Stage (a) and Stage (b) are played in the same way, except that different packs of number cards are used (see above).
1. The cards are shuffled and put face down.
 2. Player A takes the top card, looks at it, but does not let the others see it yet.
 3. Player A now points to the appropriate point on the number line, and the others write the number in place-value notation.
 4. Finally the number card used by A is compared with what the others have written.
 5. If these agree, someone else acts as player A.
 6. If not they discuss before continuing.

Activity 4 Shrinking and growing [NuSp 1.10/4]

A group game. This is an Alice-in-Wonderland kind of activity, which will stretch the imaginations of the more able children.

- Materials*
- Number-line 0-10.*†
 - Card with decimal point, see Figure 47.*
 - Number cards 1-9.*
 - A paper clip for a pointer.

* Provided in the photomasters

† This should be numbered at every whole number, with divisions marked at every tenth-part and every hundredth-part. This may be made from graph paper with 10 mm and 1 mm squares. The unit interval will then be 1 decimetre, and the number line will be 1 metre long. A small part of this line is illustrated in Figure 47.

What they do Stage (a)

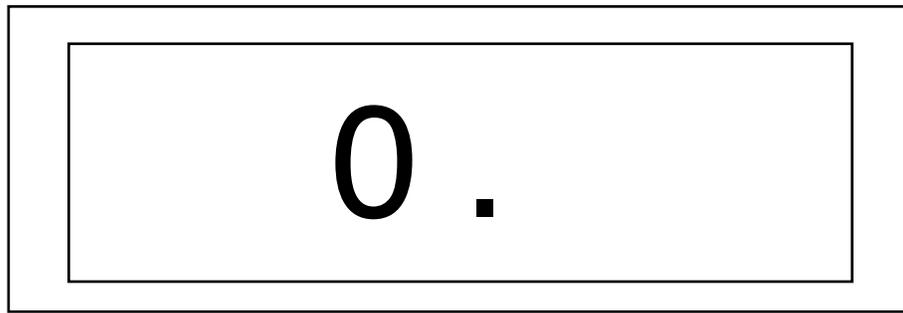
1. The number cards are shuffled and put face down.
2. Player A turns over the top card (say 4) and puts it to the left of the decimal point. (That is, on top of the zero, so that the zero is not showing but the decimal point is).
3. Player B points to the appropriate place on the number line and says “Four ones.”
4. The rest in turn say “Agree” or “Disagree.”
5. Player A moves the card to the right hand side of the decimal point. Player B should then say “Zero ones, four tenth parts” pointing as she speaks to the appropriate point on the number line. The others, as before, say “Agree” or “Disagree.”
6. Steps 2-5 are repeated using other number cards. Other players then take over the roles of players A and B, and the activity is repeated.

Stage (b)

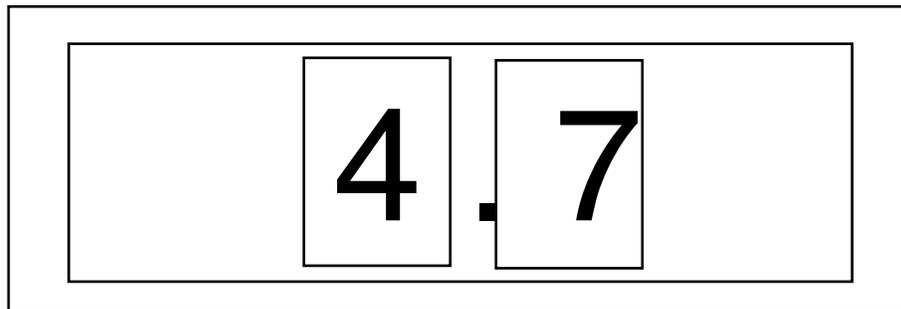
1. The number cards are shuffled and put face down as before.
2. Player A now turns over the top two cards and puts one each side of the decimal point (e.g., 4.7).
3. Player B points to the appropriate place on the number line and says (in this case) “Four ones, seven tenth-parts.”
4. The rest in turn say “Agree” or “Disagree.”
5. Player A moves both cards to the right hand side of the decimal point, giving (in this case) 0.47. Player B should then say “Zero ones, four tenth-parts, seven hundredth-parts” pointing as she speaks to the appropriate point on the number line. The others, as before, say “Agree” or “Disagree.”
6. Steps 2-5 are repeated using other number cards. Other players then take over the roles of players A and B, and the activity is repeated.

Stage (c)

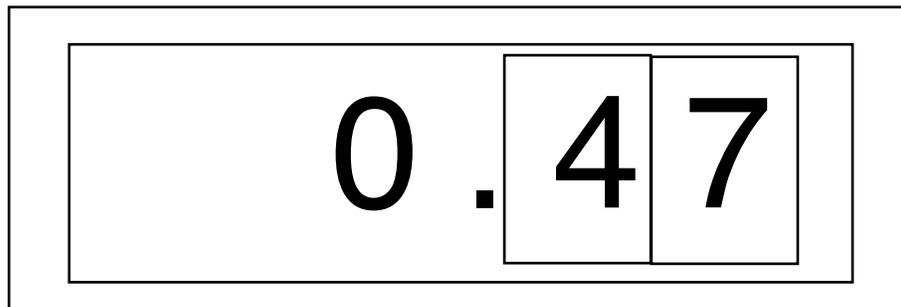
1. The number cards are shuffled and put face down as before.
2. Player A now turns over the top three cards and puts one to the left of the decimal point, the other two to the right (e.g., 4.72).
3. Player B points to the appropriate place on the number line and says (in this case) “Four ones, seven tenth-parts, two hundredth-parts.”
4. The rest in turn say “Agree” or “Disagree.”



In use



or



Number Line

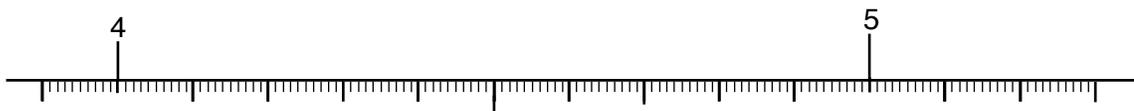


Figure 47 Shrinking and growing

NuSp 1.10 Extrapolation of place-value notation (cont.)

5. Player A moves all three cards to the right hand side of the decimal point (giving in this case 0.472). Player B now has a more difficult task. A suitable response would be to say “Zero ones, four tenth-parts, seven hundredth-parts, two thousandth-parts” pointing, as she speaks, to the appropriate point on the number line, and continuing “The thousandth-parts are too small to see.” When I was trying out this activity, Nicholas said “. . . and a very small move to the right.” The others, as before, say “Agree” or “Disagree.”
6. Steps 2-5 are repeated using other number cards. Other players then take over the roles of players A and B, and the activity is repeated.

Stage (d)

Activities 1, 2, 3 were about shrinking. Now we come to the growing. They begin with one number card, as in Stage (a).

1. The number cards are shuffled and put face down.
2. Player A turns over the top card (say 4) and puts it to the left of the decimal point. (That is, on top of the zero, so that the zero is not showing but the decimal point is).
3. Player B points to the appropriate place on the number line and says “Four ones.”
4. The rest in turn say “Agree” or “Disagree.”
5. Player A now moves the card one place to the left, giving (in the present example) 40 . This is off the number line, 4 metres from the zero point. So player B might then say something like “Forty. This will be off the line, somewhere in the middle of the next table.” The others, as before, say “Agree” or “Disagree.”
6. Steps 2-5 are repeated using other number cards. Other players then take over the roles of players A and B, and the activity is repeated.

Stages (e), (f)

Introduce these at your discretion.

The procedure is as in Stage (d), but using two and three cards respectively. The starting position is always with one card to the left of the decimal point (on top of the zero). Player A then moves them one place at a time to the left, with Player B responding as in step 5 of Stage (d). So in stage (e), she might have to deal with numbers such as 4.7, 47, and 470; and in stage (f), with the numbers such as 4.72, 47.2, 472, and 4720. In the last case, the corresponding point on the number line would be 4720 decimetres away, i.e., about half a kilometre away. The foregoing would be rather good answers. “Quite a long way down the street” would show sufficient appreciation of the kind of distance involved. This is less complicated to do than to describe, but it is still quite a sophisticated activity, suitable only for older children.

Discussion of activities Here again, children are extrapolating a known pattern: in this case a notation. Since a pattern is shown more strongly by a larger number of examples, Activity 1 shows hundreds and tens as well as ones, and also hundredth-parts although these are harder to point to on the number line. This is also an argument for using mixed numbers. Hundreds, tens, ones firmly establish the tens pattern which is now being extended downwards.

Activity 2 is for consolidating the three-way relation between headed column notation, place-value notation, and the decimal fraction concept as it is represented on the number line. The more the connections within a schema (conceptual structure), the better the understanding.

In Activity 3, we have gone over entirely to place-value notation. Since the notation itself is not new to the children, but only its present extension, we have not spent so long on the way as when it was first introduced. But it is easy enough for you to keep them using headed columns longer if in your judgement they still need the support of this much more explicit notation.

Activity 4 is partly for consolidation, but especially for giving children a feeling of the great differences in the sizes represented in this very concise notation. Using a decimetre as unit interval, a digit in the hundreds column represents a length possibly too big to have in the classroom, while the same digit in the hundredth-part column represents a length too small even to draw. This activity also makes frequent demands for adaptability in children's thinking, so it is a good mental exercise for their intelligence.

Is all this talk about "Four ones, seven tenth-parts" etc. really necessary? In my belief, emphatically yes. This verbalization is here being used, in conjunction with the number line, to help establish the concepts. The frequently-encountered problems with 'decimals' (decimal fractions in place-value notation) are conceptual problems, problems of understanding. The importance of these early stages in establishing the right concepts can hardly be over-stated.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Shrinking and growing [NuSp 1.10/4]



Different objects, same pattern [Patt 1.4/1]

[Patt 1] PATTERNS

Patt 1.4 TRANSLATING PATTERNS INTO OTHER EMBODIMENTS

- Concepts*
- (i) That the same pattern can be shown by different objects.
 - (ii) That a pattern is therefore independent of any particular embodiment.

Ability To show a given pattern with different objects.

Discussion of concept One of the reasons that mathematics is so useful and adaptable a mental tool is the great variety of particular situations to which the same mathematical knowledge can be applied. This means that we must help children's mathematical concepts to become context-free: that is, independent of any particular embodiment, and, in particular, independent of the manipulatives which are such a useful help in forming them initially. The present topic is a good example.

Activity 1 Different objects, same pattern [Patt 1.4/1]

A teacher-led discussion for a small group. Its purpose is to help children to realize that the same pattern can be made with a variety of different sets of objects.

- Materials*
- Containers with a variety of objects convenient for making patterns. For example, one container might have counters in three colours; another might have sunflower seeds, macaroni shells, and dried beans; another might have three different kinds of buttons. There needs to be enough of each variety so that children do not run out while making their patterns: I suggest not less than ten of each. This would allow, e.g., three repetitions of a pattern like
A B B B A B B B . . . with one to spare.
[See: 'Patterns with a variety of objects,' Patt 1.1/2, in *Volume 1*.]

- Suggested sequence for the discussion*
1. Make a simple pattern using just two kinds of objects from one of the containers. For example,
shell bean bean shell bean bean shell bean bean . . .
 2. Ask if any one thinks they can copy this pattern using objects from one of the other containers.
 3. If no one offers, it may be that they need more time with 'Patterns with a variety of objects' [Patt 1.1/2, *Volume 1*]. Alternatively, you could make a copy yourself, and ask whether they think that this is the same pattern, although the objects are different. If they appear doubtful, then they do need more time with Patt 1.1/2.
 4. Assuming that in step 2 someone does offer, let him try, and ask whether the others agree with what he has made. If they do not agree, discuss what changes are needed so that the second pattern is the same as the first, albeit with different objects.
 5. Working in pairs, the children may now choose other containers and make further embodiments of the given pattern. They check each other's results.
 6. The objects are replaced in the containers, and steps 1 and 5 are repeated.

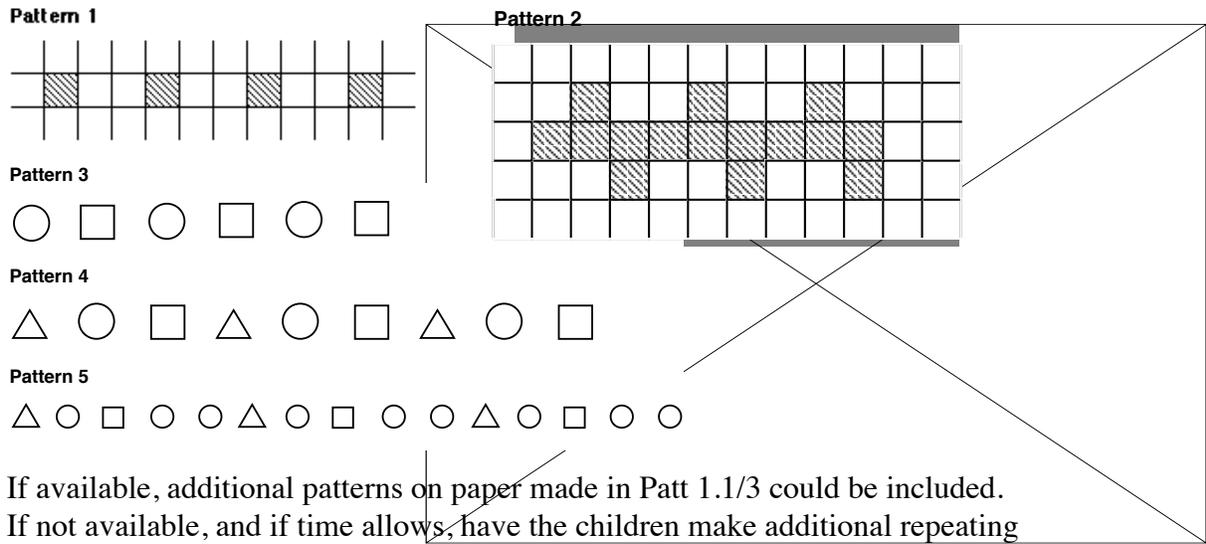
Patt 1.4 Translating patterns into other embodiments (cont.)

Activity 2 Patterns which match [Patt 1.4/2]

An activity for a small group of children. Its purpose is to consolidate the concept formed in Activity 1, by applying it to different materials.

Materials

- Example patterns from ‘Making patterns on paper,’ Patt 1.1/3 in *SAIL Volume 1*:



- If available, additional patterns on paper made in Patt 1.1/3 could be included. If not available, and if time allows, have the children make additional repeating patterns following the models in the example patterns (above) .

- What they do*
1. Begin by showing the example patterns. Ask if any of these show the same pattern with different objects.
 2. In the set of examples given above, Patterns 2 and 4 show the same pattern with different objects. In short, we say that they match. In this set there are no other matching patterns.
 3. The match will show even more clearly if we make a pattern matching Pattern 2 using the letters A B C If we now do the same with Pattern 4, we find that we get A B C A B C A B C for both.
 4. This is also a good way to make sure that we have not overlooked any other patterns which are alike.
 5. [Optional] Their own patterns (if available) are now spread out on the table. Each child takes a pattern, and collects all the patterns which match it. It is advisable for them to begin by checking that no two of them have matching patterns.
 6. Finally they check each other’s collections.

Activity 3 Patterns in sound [Patt 1.4/3]

A continuation of the two previous activities, for a small group. Its purpose is to show how the concept of pattern can be embodied in a completely different modality.

- Materials*
- Three or more objects which make different sounds when struck. Instruments from a percussion band offer one possibility.
 - One pattern from each of the sets of matching patterns from Activity 2.

- What they do*
1. The patterns are spread out where all can see them. It would be advisable to begin with, say, three or four, in which not more than three different objects are used.
 2. One child then chooses a pattern without telling the others which it is. He translates this into a sound pattern, in which each object in the visual pattern corresponds to a different sound.
 3. The other children try to identify which of the visual patterns is being played.
 4. The first child who thinks he knows says so (without indicating which pattern it is), and demonstrates by continuing the sound pattern.
 5. If the second child is right, that child continues until another child thinks he knows the pattern. If he is wrong, the first child says so and takes back the role of striker.
 6. Steps 3, 4, and 5 are repeated until all have identified the pattern.
 7. Steps 2 to 6 may then be repeated with a different child starting.

Activity 4 Similarities and differences between patterns [Patt 1.4/4]

A teacher-led discussion for a small group of children. Its purpose is to have children reflect on and organize what they now know about patterns, within the framework of similarities and differences.

- Materials*
- The example patterns from Activity 2.

- Suggested outline for the discussion*
1. Begin with the example patterns. Ask in what ways they can be alike, and in what ways different.
 2. We have already seen that they can be alike in that the pattern is the same although the objects themselves are different. So we can have two patterns which use the same objects but are different patterns, or which use different objects but are the same pattern.
 3. In the second case, we need a way of saying what the pattern is. I suggest that the translation into letters introduced in Patt 1.4/2 is convenient and universal.
 4. Another way in which patterns can be alike or different is the number of different objects used, and the number of objects in each repetition. These are independent. For example,
 A B A A B A A A B A B A A B A A A B A B A A B A A A B
 uses only two objects, but there are nine objects in each repetition (which we may call a cycle). The pattern
 A B C D A B C D A B C D A B C D A B C D A B C D A B C D
 uses four objects, and the number in a cycle is also four.
 5. Given a number of patterns on paper or in sound (though not limited to these) we can identify the following attributes:
 - The pattern itself.
 - The objects used to show the pattern.
 - The number of different objects used.
 - The number of objects in a cycle.
 6. These are also ways in which they can be alike or different, which prepares the way nicely for the next activity. I suggest that you make a start with this right away, after which they can do it on their own when it is convenient.

Activity 5 Alike because . . . and different because . . . [Patt 1.4/5]

An activity for a small group of children. Its purpose is to consolidate the concepts which were discussed in Activity 4.

- Materials*
- The Activity 2 example patterns, at least, supplemented by others made by the children, if possible.

- What they do*
1. The patterns are put in a pile face up. The top pattern is then put separately to start a second pile.
 2. The children then take turns describing the likenesses and differences between the two patterns showing. E.g., if these were the example Patterns 4 and 5, he might say:
“These are alike because they use the same objects: triangles, circles, and squares. They are different because one has a cycle of three and the other has a cycle of five, so they can’t be the same pattern.”
Or if they were Patterns 2 and 4, he might say:
“These patterns use different objects, but they both have three different objects and they both have a cycle of three. They are both the same pattern, which in letters would be A B C A B C A B C”
The other children say whether they agree or not, and may contribute further suggestions.
 3. When the first child has done this, the next child moves the top pattern from pile 1 across to the top of pile 2, so that there is a different pair showing (though one was in the previous pair). He then describes the likenesses and differences between these, as described in step 2.
 4. They continue as in step 3 until each child has had at least one turn.



Alike because . . . and different because . . . [Patt 1.4/5]

Discussion of activities In this topic children are expanding their concept of a pattern using all three modes of schema construction. They are learning from physical experience, using a variety of manipulatives; they are extrapolating pattern concepts to new embodiments; and they are checking for consistency with their existing schemas by frequent discussion.

[Meas 1] **LENGTH**

The measurement of distances

Meas 1.4 INTERNATIONAL UNITS: METRE, CENTIMETRE

Concepts (i) The need for international units.
(ii) Metre, centimetre.

Abilities (i) To measure in metres and centimetres.
(ii) To put this to practical use.

Discussion of concepts

We now introduce the metric system of units, based on the metre. The scientific name for these is S.I. units, where S.I. stands for 'Système Internationale.' S.I. units could also be thought of as referring to Standard International units, which would incorporate explicitly the idea of a standard unit.

The S.I. system has at least three important advantages. First, it is used internationally by scientists, and by an increasing number of countries for technology, commerce, and everyday life. Second, the system uses base 10 for relating units of the same kind. E.g., one kilometre is 1000 metres, one metre is 100 centimetres; whereas a mile was 1760 yards, a yard was 3 feet, a foot was 12 inches. So S.I. units are much more easily convertible, which is a great advantage for calculations using place-value notation. Third, S.I. units of different kinds often have simple relationships. E.g., one gram is the weight of one cubic centimetre of water (at 4 °C), one kilogram is the weight of one litre of water (1000 cubic centimetres). We shall gradually introduce children to all of these advantages.

Anyone who wants to use standard units needs to have something which is either a copy of the international standard, or a copy of a copy of a copy of . . . this. This is what we get when we buy a metre rule, or a smaller measure. So the use of standard units implies the use of standardized measuring instruments. The accuracy of these also depends on the transitive property of measurement.

Activity 1 The need for standard units [Meas 1.4/1]

A teacher-led discussion, for any number of children. Its purpose is to show the need for a standard international system of units.

Materials For each table:

- A ruler marked and numbered in centimetres.*
- A metre-stick.
- Source book for historical material.

* It does not matter if it is marked in centimetres and millimetres also, but any *numbering* must be in centimetres.

Meas 1.4 International units: metre, centimetre (cont.)

- Suggested outline for the discussion*
1. Ask: "Who can name one of the countries that we trade with? And another? And another? . . . (Countries of origin of our clothes, watches, playthings, calculators, . . .)"
 2. If we use things like paper clips to measure with, how do we know that they all use the same size of paper clip as we do? Why does it matter? (Recall Tricky Micky, Meas 1.1/2 in *SAIL Volume 1*.)
 3. What about units for longer distances, such as the length and width of a room, the distance from school to home? Paces would be useful here, but whose? Different people have paces of different lengths.
 4. And we need to know how many small units in a larger unit. The number of paper clips in a pace may not be a convenient number. What would be a convenient number?
 5. So we need at least two units, agreed internationally, one fairly small and one larger. And the large one should be equivalent to a convenient number of the smaller. Convenient for what? Especially, for conversions and calculations.
 6. Some will no doubt have heard of centimetres and metres. (Clothes these days are often marked with sizes in both centimetres and inches.) Let them examine a metre rule and a centimetre ruler. Explain that these units are now used in many countries, all over the world.
 7. There are 100 centimetres in a metre, just as there are 100 cents in a dollar. In both cases we use the larger unit (metres, dollars) for large amounts, and the smaller units (centimetres, cents) for smaller amounts. And we sometimes use them together, e.g., 3 metres, 45 centimetres; 3 dollars, 45 cents.
 8. Canada adopted the S.I. system relatively recently, but the United States has been unable to move as quickly in this direction. Consequently, many of the older units are still encountered (e.g., inches, feet, yards, and miles, as units of distance). Can anyone say how these relate to each other?
 9. Within living memory, even more units were in use which, from our present viewpoint, are quite hard to believe. The simplicity and convenience of the S.I. system is brought out more strongly by comparison with these. So I think that it is a worthwhile investigation for children to find out about these and write a short description, with comments.

Activity 2 Counting centimetres with a ruler [Meas 1.4/2]

An activity for up to 6 children, to be played in pairs. Its purpose is to establish the equivalence of counting a number of units, and reading a number on a ruler. This activity should also help children to place the starting point on the ruler correctly.

- Materials*
- Plenty of Centicubes.
 - A 30 cm ruler for each pair, marked in centimetres.

- What they do*
1. (Preliminary). All the children make pairs of rods of the same colour and length. No two pairs should be of the same colour. They should vary between 10 cm and 30 cm. These are mixed together and put in the middle for everyone to use.
 2. From now on, they work in pairs. Child A has the ruler.
 3. Child A names a colour, say blue.

4. Each then takes one rod of that colour. 'A' finds the number of cubes by counting, 'B' by using the ruler.
5. Whoever finishes first says the number. The other finishes her way, and says "Agree" or "Disagree."
6. If "Agree," they check by putting the rods alongside to see if they match.
7. If "Disagree," they change roles and measure/count again.
8. If counting cubes and measuring with the ruler do not agree, this might be because the starting point of the ruler has not been positioned correctly.
9. The rods are then replaced, and steps 2 to 7 are repeated.
10. This activity should be continued until children can always get a correct result more quickly with the ruler.

Activity 3 Mountain road [Meas 1.4/3]

A game for two, four, or six children working in pairs. Its purpose is to use measurements to make predictions of practical interest.

- Materials*
- 3 model trucks, on which loads of different sizes are secured. Some loads are high and narrow, some are wide and low. Furniture vans, trucks with cranes, and the like, could be used for the high narrow ones.
 - A table-sized enlargement of the road map on Photomaster 263 in *SAIL Volume 2a*, with models of bridges replacing the symbols. *
 - Centicubes.

For each child:

- Road map.†
- Ruler marked in centimetres.
- Non-permanent marker.

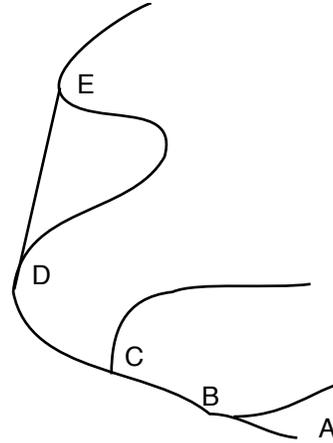
* The bridges are made from Centicubes to the heights or widths shown in the illustration. These measurements should be adjusted to the sizes of the model vehicles which you have. The high loads should always be able to cross the bridges of limited width (marked W), and the wide loads should be able to cross bridges with restricted heights (marked H).

† A copy from the photomaster, covered with transparent film to allow use and reuse with non-permanent markers.

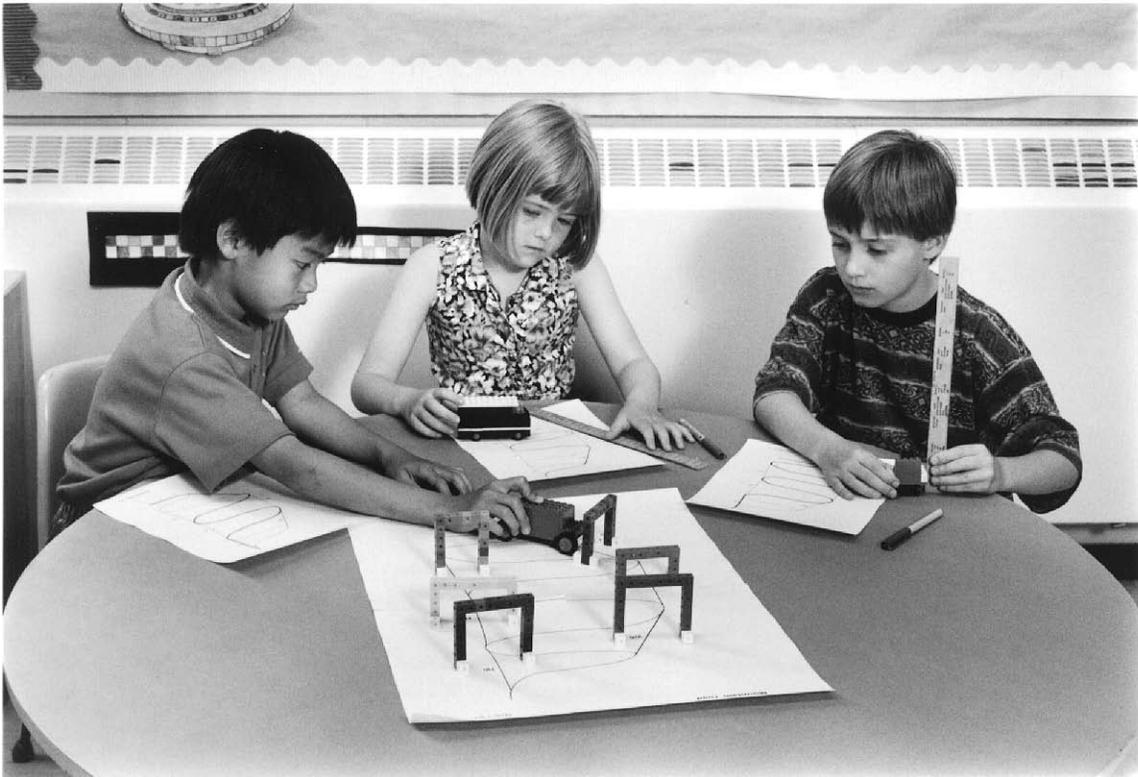
- Rules of the game*
1. The children are shown the table-sized model of a mountain road.
 2. Each is given a road map, and the correspondence between this and the road and bridges explained. The high vehicles can always use the bridges marked W (restricted width), but have to be careful with the bridges of restricted height (marked H). Likewise, the other way around, for trucks with wide loads.
 3. Another factor to be taken into account is that the trucks cannot get around hairpin bends.

Meas 1.4 International units: metre, centimetre (cont.)

So a truck coming from A could go in either direction at junctions C and D, but at B and E it could not take the hairpin turn to the right.



3. Back at their own table, each pair is given a vehicle. One acts as driver, the co-driver acts as navigator.
4. The drivers measure their vehicles, and together they plan their route. They mark this route on their road maps with a non-permanent marker.
5. Watched by the others, each driver in turn takes her vehicle along the chosen route. The co-driver navigates with the help of her road map.
6. Every pair which gets there without mishap, preferably by the shortest way possible for that vehicle, is a winner.
7. The vehicles may be re-allotted, and steps 4 to 6 repeated.



Mountain road [Meas 1.4/3]

Activity 4 Decorating the classroom [Meas 1.4/4]

An activity for as many children as you like. Its purpose is to introduce the use of the metre as a unit for larger measurement.

- Materials*
- Metre measures. *
 - Party paper streamers.
 - Plasticine.

* These do not need to be graduated in cm, so additional measures can be made from bamboo, dowel, and the like.

- What they do*
1. A suitable reason is found for decorating the classroom.
 2. In small groups, children are allotted distances in which the streamers are to be hung. E.g., a wall, the space between two windows.
 3. They use the metre rulers to measure these distances, and then to measure the right lengths of streamer.
 4. The streamers may be fixed in position with Plasticine, or otherwise.
 5. We are not here concerned with great accuracy, but with the use of a larger unit for larger distances. So measurements like “3 and a bit over” or “3 and a half” are suitable for this job.

Discussion of activities Activity 1 is a general discussion, calling attention to the need for international standard units, and some of the requirements to be satisfied. It relates the knowledge they have acquired while working on this network to their general knowledge.

Activity 2 introduces the use of a ruler, which like a number track greatly speeds up the counting of units. It calls attention to the correspondence between counting a number of units and reading this number directly from a ruler, using both Mode 1 (physical matching) and Mode 2 (“Agree”).

In Activity 3, I have tried to provide a game which would make use of measurement to arrive at a sound plan of action in the same kind of way as in the adult world. This was not easy, since full-sized objects are not easy or convenient for children to move around. Models are much more manageable, and the amount of mathematics used in a given time is greater. But please note that we are not here using scale models. The sizes are actual sizes, in centimetres. If children spontaneously suggest that 1 centimetre could represent (say) 1 metre, good for them. But conceptually, this is much more advanced, since in their general form scale models use ratio and proportion. At this stage, a centimetre is a centimetre, and the models are objects in their own right.

In Activity 4, measurement is put to another of its major uses in the adult world: making things fit. Here we are concerned with introducing the concept, rather than with accuracy. Party streamers provide an convenient and inexpensive material.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 1.5 COMBINING LENGTHS CORRESPONDS TO ADDING NUMBERS OF UNITS

Concept The correspondence stated in the heading.

Ability To make predictions about total length based on sums of individual lengths.

Discussion of concept Now that we can use numbers to quantify something continuous, as well as sets of separate objects, we can extend our use of mathematical operations into new realms of application. Indeed, each new kind of unit makes possible a new realm of applications.

The first of these applications is in the realm of distance. We find that numbers of units can be added in the same way as pure numbers, and the results put to similar uses. This also works for subtraction.

Note that we must be careful when it comes to multiplying. We can multiply a number of units by a pure number. If we put end to end 4 matchboxes each 8 cm long, their total length is 32 cm.

$$8 \text{ cm} + 8 \text{ cm} + 8 \text{ cm} + 8 \text{ cm} = 4(8 \text{ cm}) = 32 \text{ cm}$$

But $(4 \text{ cm}) * (8 \text{ cm})$ represents an area, as we shall find in Meas 2.

This concept could have been understood earlier in the learning sequence. But its application is more powerful when used with standard units, and this is why I have postponed it until here.

Activity 1 Model bridges [Meas 1.5/1]

An activity for up to six children, working singly, or in pairs. This is a continuation of 'Building a bridge,' (Meas 1.2/1 in *Volume 1*). Its purpose is to teach the concept and ability described above.

Materials

- River.*
- Banks.*
- Box girders.*

For each child:

- A ruler, graduated in centimetres, up to 30 cm.
- Boards for the roadway.†

* The river is made of paper or card, painted to look like water, about letter-size (21.5 by 28 cm). The banks can be books, boxes, or the like. The box girders are made from cardboard, square in cross-section. You need 2 each of the following lengths, cut as accurately as possible: 14 cm, 16 cm, 18 cm, 20 cm, 22 cm, 24 cm.

† All the same length, say 15 cm. You need six each of widths 4 cm and 5 cm. The widths should be as accurate as possible.

- What they do*
1. The task is now to put boards across the box girders, to complete the model bridge.
 2. The banks and river are put in the middle, with the plans nearby.
 3. Each child has one pair of box girders. He measures the planks, measures the length of his girders, and works out what combination of planks put side to side will fill the required distance exactly.

Meas 1.5 Combining lengths corresponds to adding numbers of units (cont.)

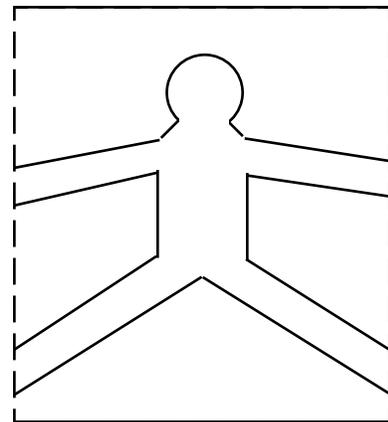
4. He writes this as an addition of numbers of units. This is a good time to introduce the abbreviation for centimetre(s), which up till now they have not needed to write. E.g.,
Girders: 18 cm Planks: 5 cm, 5 cm, 4 cm, 4 cm Total: 18 cm
5. Each child in turn then tests his prediction. The banks are adjusted to suit the length of his girders. He shows the others his calculation, takes the planks as listed, and puts them across the girders to make a roadway. They should fit exactly.
6. Steps 3, 4, and 5 may be repeated, the children using different girders.

Activity 2 How long will the frieze be? [Meas 1.5/2]

An activity for any number of children. It provides a different application of the same concept and ability as Activity 1.

- Materials*
- Newspaper.
 - Scissors.
 - Ruler marked in cm.
 - Metre rule marked in cm.

- What they do*
1. Each child cuts himself a strip of newspaper, and folds it accordion-wise into 3, 4 or 5 thicknesses. (More, if he likes.) This is trimmed at the free end(s) so that all the folded parts have the same width.
 2. He cuts this in any way which leaves part of the folded sides uncut. E.g.:
 3. He measures this, and predicts how long his frieze will be when unfolded.
 4. He tests his prediction.
 5. The total length of several friezes joined together may also be predicted.



Discussion of activities The new concept is a natural extension of their existing concept of addition, and in Activity 1 children are likely to arrive at it intuitively. By putting it to use in a physical embodiment, and testing it by making and testing predictions, this intuitive knowledge is made more explicit and conscious, and consolidated.

Activity 2 is similar. If children multiply rather than add the number of widths to be put together, this is an equally sound method. But in accordance with the discussion of concepts at the beginning of this topic, they should write it in a way which shows correctly what they are doing. I suggest $7 \text{ cm} * 3 = 21 \text{ cm}$, or $3 (7 \text{ cm}) = 21 \text{ cm}$.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 1.6 DIFFERENT SIZED UNITS FOR DIFFERENT JOBS: KILOMETRE, MILLIMETRE, DECIMETRE

- Concepts*
- (i) The millimetre as a unit for smaller and more exact measurements.
 - (ii) The kilometre as a unit for larger and less exact measurements.
 - (iii) The decimetre as another useful unit.

Ability To choose and use units suitable for the job to be done.

Discussion of concepts We need a smaller unit than the centimetre whenever our purpose requires greater accuracy than is given by a centimetre. It also avoids the need for fractional units, for distances less than one centimetre. Likewise, we need a larger unit than the metre for the kind of distances we go by car, train, or plane – distances between places, where even the starting and finishing locations are larger than a metre in extent.

For these requirements we use the millimetre and the kilometre, respectively: abbreviations mm and km. In addition to these, the decimetre provides a convenient unit between the centimetre and the metre.

In this topic the main emphasis is on using the right unit, giving the right degree of accuracy, for the job to be done. The relations between these units will be dealt with more thoroughly in the next topic.

It may be of interest that our word ‘mile’ comes from the corresponding Roman unit of distance, ‘mille passus,’ a thousand paces. In marching, they found it more convenient to count a pace every time the same foot touched the ground, so a Roman pace was what we should call two paces. Their average single pace was shorter than a yard, or we should have had a mile which was 2000 yards, rather than the seemingly ridiculous figure of 1760 yards.

Activity 1 “Please be more exact.” (Telephone shopping) [Meas 1.6/1]

An activity for children working in pairs. Its purpose is to introduce children to the use of the millimetre as a unit for measuring small distances, where greater accuracy is needed than is provided by the centimetre.

Materials For each pair:

- 6 or more nuts and bolts.*
- 2 rulers marked (and preferably numbered) in millimetres.

* These must all be of different sizes. The differences between their diameters (inner/outer respectively) should be not less than 3 mm.

- What they do*
1. One player acts as a customer, the other as a hardware clerk. The customer has all the bolts in a bag, the hardware clerk has all the nuts on a tray.
 2. The customer takes one bolt out of the bag, and ‘telephones’ the hardware clerk to ask her if she has a nut for it.
 3. The customer says “I need a nut to fit a bolt.”
 4. The hardware clerk replies “How thick is the bolt?”

Meas 1.6 Different sized units for different jobs: kilometre, millimetre, decimetre (cont.)

5. The customer first answers using centimetres. E.g., “A bit less than 2 cm across.”
6. The hardware clerk replies “Please be more exact. Tell me in millimetres.”
7. The customer then measures the bolt and answers, say, “18 mm.”
8. The hardware clerk chooses a nut, checks the inside measurement carefully, and ‘sends’ it to the customer, who checks to see that it fits the bolt.
9. If not, she sends it back. Steps 7 and 8 are repeated until a fit is achieved.
10. This continues until all the bolts are fitted with nuts. Though it is not essential to include steps 3 to 6 every time, I am in favour of doing so since this little interchange emphasizes one of the concepts which this activity is intended to teach: the right unit for the job.

Activity 2 Blind picture puzzle [Meas 1.6/2]

An activity for children working singly or in pairs. Its purpose is to consolidate the skill of measurement in millimetres, and the use of measurement to make things fit.

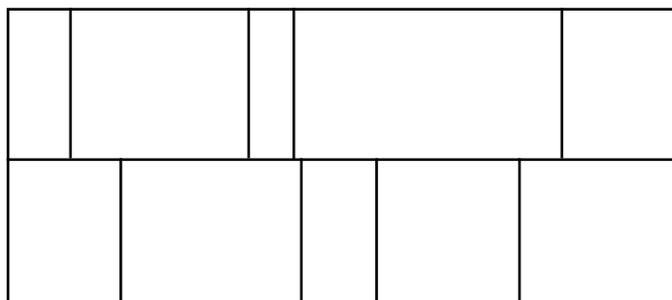
- Materials*
- A shallow box with a transparent plastic panel in the lid. This may also be used to contain the puzzle.
 - A number of cut-up pictures no larger than the transparent cutout.*
 - A fitting guide for each picture.†

* The picture should be cut up into oblong pieces as indicated in the first illustration shown below in ‘*Some hints on making . . .*’. These widths should differ by amounts of less than 10 mm. It is easiest to cut first, and measure the pieces afterwards. You will need to check that the totals of their widths come to that of the original picture, or at least that the upper and lower rows come to the same total.

† The guide should show only widths and relative locations. The arrows should not be the same lengths as the widths of the corresponding pieces. In other words it is a diagram showing sizes, not an accurate pattern. See the SPECIMEN FITTING GUIDE on the next page.

- What they do*
1. Each child (or pair) has a tray, a lid, and one of the pictures with its fitting guide.
 2. The pieces of the picture are spread out face down, and measured one at a time. Each width is then used, in conjunction with the fitting guide, to put the piece face down in its correct relative position on the transparent plastic lid. This is like doing a jigsaw puzzle ‘blind,’ i.e., without being able to see the picture.
 3. The box is put on top, wrong way up inside the lid, so that it will hold the picture in place when the whole is turned over. If all the measurements have been correct, the pieces will be correctly fitted together to make the picture.

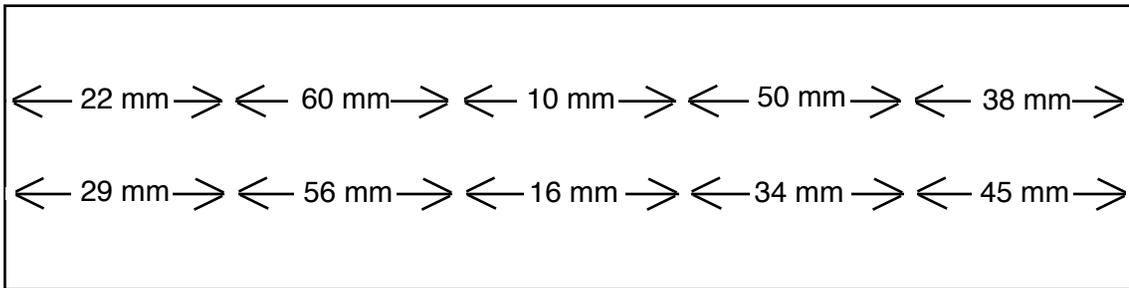
Some hints on making the puzzles The pictures need to be cut into oblongs, in two rows of equal height as shown below. It is only the widths which differ, and these are used to decide the locations of the pieces.



Meas 1.6 Different sized units for different jobs: kilometre, millimetre, decimetre (cont.)

The locations are shown by a fitting guide like the one shown here. All the widths must be different, by amounts not less than 4 mm.

SPECIMEN FITTING GUIDE



You may find it convenient to base your own cutting guide on the above. The total width here is 180 mm, so if your picture is (say) 200 mm wide, all you need to do is increase each width by 4 mm. Using the same set of widths, the positions in the upper row can be rearranged to give 120 permutations in all, and likewise for the lower row, making (if you are interested) a total of 14 400 possible sets of dimensions for a given overall width.

Interchanging widths in the upper and lower rows is likely to cause confusion, since the overall widths are unlikely to remain the same.

The first puzzle requires a certain amount of trouble to make, but after that it is easy to make more. This activity gives lots of practice in making careful measurements, and if these are correctly made and used the result is immediately apparent in the final picture.

Activity 3 “That is too exact.” (Power lines) [Meas 1.6/3]

An activity for two teams. It may also be done by children working in pairs. Its purpose is to expand the concept of ‘the right unit for the job’ in the other direction, that of a larger unit for larger distances. It introduces the kilometre, and also the idea of ‘rounding up.’

- Materials*
- For each team, a diagram of the power lines to be erected. *
 - Pencil and paper for each child.
- * Provided in the photomasters

- What they do*
1. One team forms a party of electrical engineers; the others are in charge of the warehouse and transportation.
 2. The engineers decide which line they are going to work on first, and ask the warehouse for a length of cable equal to the distance shown on the diagram. E.g., “Please issue 5 283 metres of cable.”
 3. The warehouse reply is “That is too exact. The cable is on reels of 1 kilometre, which is 1000 metres. We’ll give you 6 km of cable.”
 4. Here is the warehouse record:

<u>Requested</u>	<u>Supplied</u>
5 283 m	6 000 m
 5. When that cable has been laid, the engineers choose which line is to be built next, and steps 2, 3, 4, are repeated.

Activity 4 Using the short lengths (Power lines) [Meas 1.6/4]

A continuation of Activity 3, which deals with the obvious question of the lengths left over. It also very simply introduces calculation in these units.

Materials As for Activity 3.

What they do Steps 1, 2, 3 are as in Activity 3.

4. When the partly-used reel is returned to warehouse, a record is made as before not only of the requested and supplied lengths but also a calculation of how much is left on the reel:

<u>Requested</u>	<u>Supplied</u>	<u>Surplus</u>
5 283 m	6 000 m	717 m

5. Next time the engineers ask for cable, the warehouse personnel check whether they have a suitable short length. E.g., in response to a request for 3 578 m of cable, they might reply “We’ll give you 3 one-km reels and a short length of about 717 m.” (“About,” because there may be a small amount wasted.)
6. Steps 1 to 6 are then repeated.

Activity 5 “That is too exact.” (Car rental) [Meas 1.6/5]

An activity for two teams. It may also be done by children working in pairs. Its purpose is to use the ideas learned in Activities 3 and 4 in a different situation, and also to introduce the ideas of rounding down.

Materials • Car cards.*

* About 10 pictures of cars, stuck on cards. Underneath each is written a test figure for fuel consumption, e.g., “When tested, this car did exactly 52 837 metres on 1 litre of gasoline.”

- What they do*
1. One team are customers, the others are car rental agents.
 2. The first customer chooses a car, and asks what its fuel consumption is.
 3. One of the agents points out that each car has been individually tested, and points to what is written underneath.
 4. The customer replies “That is too exact. My map gives distances in kilometres.”
 5. So the agent says (in the example above) “It does 52 kilometres to a litre, and a bit over.” (Why in this case should he round down?)
 6. Steps 2, 3, 4 are repeated with a different customer and agent.
 7. The children may also be asked whether they think the cards are sensibly written, and suggest possible improvements.

Activity 6 Chairs in a row [Meas 1.6/6]

An activity for a small group of children working together, say two or three. Its purpose is to introduce the decimetre as a useful unit between the centimetre and the metre. It is introduced by a teacher-led discussion.

Suggested outline for discussion

1. This needs to be done at a time and place where it will be convenient to move chairs around. The task is to calculate how many chairs of the same size can be put in a row across the room or hall, with at least one metre of free space at each end for access.
2. Discuss what unit should be used. Centimetres would be possible, but are inconveniently small and unnecessarily exact. Metres are clearly too large, since this is not much different from the width of the chairs themselves. We need something in-between.
3. The decimetre, which is 10 centimetres in length, looks like a good unit for this job, so let us try it out. (How many decimetres are there in a metre?)

What they do

1. The width of a single chair is measured in decimetres, to the nearest unit above.
2. The maximum allowable length is measured, in decimetres.
3. The maximum number of chairs in a row is calculated.
4. This prediction is then tested physically.

You may care to follow this up by discussion of other jobs for which a decimetre would be convenient. For example, the dimensions of wall-to-wall carpeting would need to be measured in centimetres; but for a loose carpet or rug, decimetres would be sufficient.

Discussion of activities

Activities 1 and 2 both use measurement to make things fit together, which is one of its major uses. The more exact the fit, the more exact are the measurements required. Both of these activities use Mode 1 schema building (physical experience) and also Mode 1 testing (of predictions).

Nuts and bolts provide a good example of an exact fit. In some ways this activity is the harder of the two, since it also involves measuring diameters of two kinds, thickness of a bolt and distance across a hole. After some thought, I have put it first because the result of each pair of measurements is immediate and decisive. Activity 2 allows mistakes to persist unchecked until all the pieces are in position.

Activities 3, 4, and 5 are somewhat simple. It is difficult to introduce Mode 1 activities involving distances measured in kilometres, so here we rely on Mode 2 building and testing, discussion, and agreement. It should not be difficult to extend the discussion to distances between towns, airports, and other places, in which the locations themselves may be more than a kilometre across. What do we mean, for example, by “the distance from Calgary to Ottawa”? This could have somewhat different meanings according to whether we were talking about a car journey (door to door), or the distance between these cities according to a map.

Activity 6 uses the decimetre in a predictive situation.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 1.7 SIMPLE CONVERSIONS

Concept That the same distance may be expressed in different units or combinations of units.

Abilities (i) To make simple conversions between units and combinations of units.
(ii) To recognize the same measurement written in different ways.

Discussion of concept The central idea here is that we do not change the distance itself when we measure it in different units, or when we convert between these.

In this topic I have kept to simple conversions, using relationships which the children already know:

$$1 \text{ km} = 1000 \text{ m}$$

$$1 \text{ m} = 100 \text{ cm} \qquad 1 \text{ m} = 10 \text{ dm}$$

$$1 \text{ cm} = 10 \text{ mm} \qquad 1 \text{ dm} = 10 \text{ cm}$$

In the next topic we shall see how all of these are related: but for practical purposes, we do not need to use more than two kinds of unit together. (To give a measurement in metres and centimetres implies accuracy of the order of 1 percent; in kilometres and metres, 0.1 percent. So to give a measurement in 3 kinds of units implies a most unlikely accuracy of measurement.)

Activity 1 Equivalent measures, cm and mm [Meas 1.7/1]

An activity for children working in pairs. Its purpose is to teach children that the same distance can be measured using different units, and to develop recognition of pairs of equivalent measures. It also gives practice in accurate measurement.

Materials

- An assortment of objects to be measured. *
- Plasticine (or 'Blu-tak').

For each child:

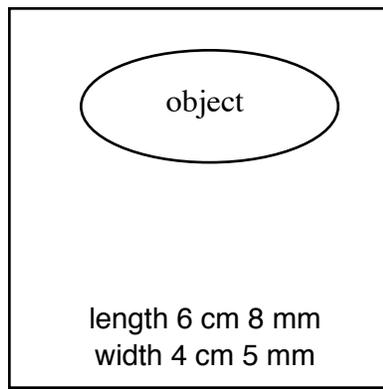
- Pencil, scrap paper.
- A scale marked in cm and mm but not numbered at all. †

* In some objects, e.g., milk straws, length is the salient quality. In others, e.g., oblongs, ovals, lolly sticks, both length and width should be measured.

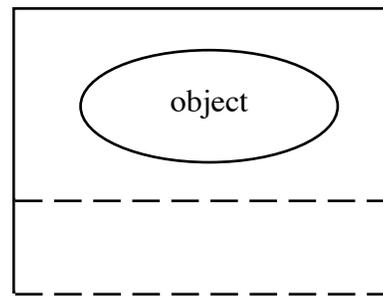
† These are easy to make from graph paper. 15 cm is long enough.

- What they do*
1. The objects are put in the middle for everyone to use.
 2. In each pair, one child measures in millimetres only, and the other in centimetres and millimetres. They begin by agreeing who will do which.
 3. Each child chooses an object, and measures its length, or length and width. He writes his measurement near the bottom of the paper, sticks the object at the top of the paper with Plasticine or Blu-tak, and folds a strip of paper under to hide his writing, like the example at the top of the following page.

Meas 1.7 Simple conversions (cont.)

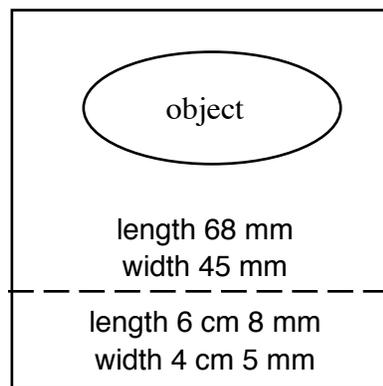


first this



then this

- Each then changes with his partner, who measures the object on his paper again, and writes the result in his own way below the object.
- They then unfold the bottom strip, and compare results.



- If the results do not match in this way, they show each other how they have measured and discuss.
- It may need practice before children can measure accurately to a millimetre, and we need to separate mismatches due to this difficulty from those due to mistakes about units. For example, if Paul records 67 mm and Mary records 6 cm 8 mm, you could ask Mary “What would yours need to be, to agree with Paul’s? Not to be all in millimetres, but to agree?” And a similar question could be put to Paul.
- Steps 3 to 6 are repeated, with other objects.

Activity 2 Buyer, beware [Meas 1.7/2]

A shopping activity for children working in pairs. Its purpose is to consolidate the concepts and abilities learned in Activity 1 in a social situation which does not depend on actual measurement.

Materials • Two notices, as illustrated in the photomaster provided in *SAIL Volume 2a*.

- What they do*
1. Children form pairs, in which one acts as parent and the other, a little child (about a year younger than the child taking the part of the parent).
 2. They are walking along the sea front or fair ground when the little child sees the notice MORE FOR YOUR MONEY, and says: “Mom/Dad, I’ve got some money left. Can I buy a 12¢/14¢/20¢ stick of rock candy? (whichever he or she likes).
 3. The parent sees the other notice nearby and says that the stall with the notice ALL PRICES PLAINLY MARKED gives better value. The child acting as a little girl/boy pretends not to understand why this is so, until the parent has explained so clearly that one could hardly fail to ‘catch on.’
 4. Two more children take the parts of parent and child, and steps 2 and 3 are repeated.
 5. Children may then make up some misleading notices of their own, with plainly marked prices for comparison.

This activity should now be repeated with another commodity, such as lengths of wood, the alternative prices now being in centimetres at the overpriced stand, and in decimetres at that of the better-value stand.

Activity 3 A computer-controlled train [Meas 1.7/3]

An activity for children working in pairs. Its purpose is to introduce conversion both ways between distances given in kilometres and metres, and in metres only.

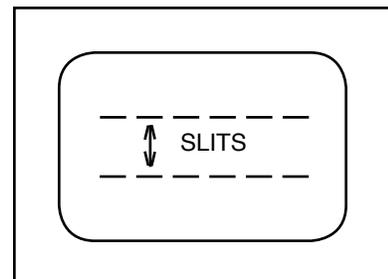
- Materials*
- Card representing computer screen, letter size.*
 - List of stops and distances.†
 - Card giving choice of prepared screen messages.†
 - Non-permanent marker.
 - Wiper.
 - Pencil and paper for each child.

* See illustration below.

† Provided in the photomasters.

The card with prepared screen messages is threaded through the slits, showing the chosen message on screen. The whole may be laminated or covered with a blank sheet of clear acetate.

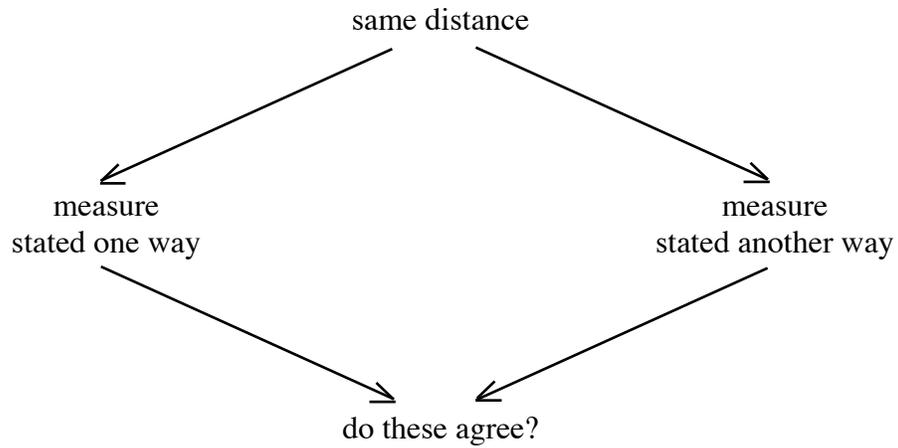
COMPUTER SCREEN



Meas 1.7 Simple conversions (cont.)

- What they do*
1. One child acts as a computer-controlled train, the other as the controlling operator.
 2. The train is controlled to stop at whatever distance is entered, to the nearest metre. (The nearest kilometre would hardly be accurate enough!)
 3. The activity begins with a screen display of the prepared message:
WHAT IS DISTANCE TO NEXT STOP?
 4. The controller writes the distance on the screen, using the non-permanent marker. The first time, he writes this as it appears on his list, which is the way humans understand more easily.
 5. In this case, the computer responds with the screen message
PROGRAM WORKS ONLY IN METRES
 6. So the controller has to convert into metres. This is quite easy, since each km is 1000 m. E.g., Distance is 7 km 437 m
 $7000 \text{ m} + 437 \text{ m}$
Distance is 7437 m
 7. The controller writes this on the screen.
 8. The computer changes the message to
TRAIN MOVING UNDER COMPUTER CONTROL
 9. At any time while this is on screen, the controller may change the message to
REPORT DISTANCE AFTER LAST STOP
 10. The computer clears the screen of the distance(s) previously written, and writes any number less than the distance it is supposed to go before stopping. This gives distances in metres, which the controller has to change into kilometres and metres.
 11. If this distance appears unlikely, the controller puts on screen the message
EMERGENCY- ERROR SUSPECTED
 12. The computer checks, and puts a corrected distance on screen.
 13. When the trains has reached the programmed stop, it first clears the screen of all numbers and then puts on screen
ARRIVED
followed by
WHAT IS DISTANCE TO NEXT STOP?
 14. Steps 3 to 13 are then repeated. (Steps 4 and 5 are optional, but the controller still has to convert each distance before entering it.)
 15. At step 7, the controller may write a number which does not match the computer's internal check with expected distances. In that case the emergency message (step 11) comes on screen. This time it is the controller who has to check.

Discussion of activities Activity 1 relates every pair of equivalent measures to the same object, which emphasizes the central concept of conversion. This is Mode 1 concept building, with Mode 2 testing.



Activity 2 underlines the point that if we look at the number alone, without thinking about the units, we are liable to be misled. This activity is more abstract, as introduced. But I think that the parents who are most successful in explaining will be those who show their children what these lengths actually look like. (They might also point out that the stall-keeper is unlikely to have measured the rock to an accuracy of a millimetre.)

In Activity 3, the conversions are again straightforward, provided that care is taken over zeros.

OBSERVE AND LISTEN

REFLECT

DISCUSS



A computer-controlled train [Meas 1.7/3]

Meas 1.8 THE SYSTEM OVERALL

Concept An overview of how kilometres, metres, centimetres, and millimetres all relate to each other.

Abilities (i) To convert any distance to metres.
(ii) To recognize equivalent measures.

Discussion of concept The use of base 10 throughout the standard international system fits very nicely with place value notation, including decimal fractions. So 5 cm 3 mm can be written 5.3 cm, since a millimetre is a tenth-part of a centimetre.

From a practical point of view I see little or no point in doing this, since a major advantage of the system is that we can avoid fractions by using smaller units. Why write a distance as 0.47 m when we can write it more simply as 47 cm?

However, it seems a natural conclusion for this network to let children see an overview of how all the units they have learned relate to each other. So the activities of the present topic have been designed with the latter as its main purpose. The only use of decimal fractions will be to exemplify relations of other units to the metre.

Activity 1 Relating different units [Meas 1.8/1]

An investigation for children working singly or in small groups. Its purpose is to help children to an overview of the relationships between all the units they have learned so far.

Materials

- Headings card.*
- Pencil and paper for each child

* Illustrated below, in step 2.

What they do

1. Ask them first to write down everything they know already about kilometres, metres, centimetres, and millimetres. E.g., 1 km = 1000 m. They should check by comparison with each other.
2. Then ask them each to copy this heading from the card.

Th	H	T	Ones	t-p	h-p	th-p

3. Ask what they think is the main unit in the metric system, on which all the other units are based? (Answer, the metre. The word 'metric' tells us.)
4. So they all write the abbreviation m in the ones column.
5. Ask them if they can put km, dm, cm, mm in their correct columns, using what they wrote down earlier.

6. When finished, they again check by comparison. Their results should look like this:

Th	H	T	Ones	t-p	h-p	th-p
km			m	dm	cm	mm

7. These headed columns should be kept for the next activity.
8. Explain that ‘kilo’ means a thousand times, ‘deci’ means a tenth-part, ‘centi’ means a hundredth-part, and ‘milli’ means a thousandth-part. Also, that although there are names for the other spaces, these are not often used.
9. On future occasions, t-p, h-p and th-p may be used as abbreviations for the fractional parts. Note the use of lower case letters for these, and capitals for the multiples.

Activity 2 “Please may I have?” (Metre and related units) [Meas 1.8/2]

A card game for 5 or 6 children. Its purpose is to exercise their thinking about the relations between the metre as main unit and kilometres, centimetres, and millimetres. It requires previous knowledge of decimal fractions.

- Materials*
- Pack of cards (metre and related units).*
 - The headed columns from the last activity.

* Provided in the photomasters. These are doubled-headed, 30 in number, with the following inscriptions:

<i>Pack 1</i>				<i>Pack 2</i>			
1 km	1000 m			5 km	5000 m		
1 m	10 dm	100 cm	1000 mm	5 m	50 dm	500 cm	5000 m
1 dm	10 cm	100 mm	0.1 m	5 dm	50 cm	500 mm	0.5 m
1 cm	10 mm	0.01 m	1 cm	5 cm	50 mm	0.05 m	5 cm
1 mm	0.001 m			5 mm	0.005 m		

To get a pack of 30, use either two sets of *Pack 1* with one pair removed (easier) or one *Pack 1* and one *Pack 2* (harder). Players should note that the only fractions which occur are fractions of a metre. The game may be played first with, and then without, the visual aid of a metre-stick.

1. All the cards are dealt. This will give 6 or 5 cards, each, for 5 or 6 players, respectively.
2. Players look at their cards. The object is to collect pairs of equivalent measures, e.g., 1 cm and 0.01 m. But *identical* measures, e.g., 1 cm and 1 cm, are not allowed.
3. At the start of play, players put down any pairs which they have.
4. They then ask in turn for a card which would enable them to make a pair.

Meas 1.8 The system overall (cont.)

5. E.g., if Sean holds a card showing 10 cm, he might ask “Please, Tina, may I have zero point one metres?”
6. Tina gives this card to Sean if she has it. Otherwise she says “Sorry.” In that case, the player who does hold 0.1 m might deduce that Sean holds the 10 cm card, thereby ensuring a pair when her turn comes up.
7. When a pair is collected, it is put on the table, face up, for the others to check. The winner is the first to put down all her cards. The others may play out their hands, however.

Discussion of activities The last two are not practical activities directly connected with measurement, but activities which involve first thinking about, and then using, the interrelations of the units which they have been using. They are activities in the use of reflective intelligence.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Relating different units [Meas 1.8/1]

[Meas 2] **AREA**

The measurement of surfaces

Meas 2.1 MEASURING AREA

- Concepts*
- (i) Tiling a surface.
 - (ii) Measuring area by counting tiles.
 - (iii) The square centimetre as a standard unit.
 - (iv) The centimetre grid as a measuring device.

- Abilities*
- (i) To tile a surface in a variety of ways.
 - (ii) To measure a surface by counting tiles.
 - (iii) To measure a surface by using a grid of centimetre squares.

Discussion of concepts The term ‘area’ is in general use with two meanings: a region (“There is an area of grass in front of the town hall.”), and the size of a region (“The area of our garden is approximately 600 square metres.”). Usually the context makes it clear which meaning we are using. If we want to distinguish explicitly between the two meanings, we can say that area is the measure of a surface, in the same way as length is the measure of a distance.

Whatever we choose for our unit of area must be easily reproducible, and we must be able to cover the surface with a number of these, without gaps and without overlap. (If there are gaps, some of the surface is not included. If there is overlap, some is included twice.) Doing this is called tiling, or tessellating. And the simpler the relationship with other units, the better. (Simpler than 2240 square yards = 1 acre!) The square centimetre satisfies these requirements well for small areas.

Activity 1 “Will it, won’t it?” [Meas 2.1/1]

A teacher-led activity for a small group of children. Its purpose is to introduce children to the first requirement of a unit for area measurement, that it can be used for tiling a surface.

- Materials*
- Eight different sets of identical shapes, of which some will and some will not tile.*

* Suitable sets are provided in the photomasters.

- What they do*
1. Start with the set of circles. Use these to demonstrate how if we try to cover part of the table top with these, either they leave gaps, or they overlap.
 2. Put these away and give the children one set each. Begin with the oblongs. Ask the child who has these to put just one of them in the middle. The children then say in turn whether they think that these shapes will or won’t fit together exactly, without gaps and without overlap. When all the children have replied, the child with the oblongs tests their predictions using the rest of the set.

Meas 2.1 Measuring area (cont.)

3. Explain that when (as with the oblongs) we can cover a surface with these without gaps or overlap, this is called tiling.
4. In turn each child similarly puts out one shape from his set, and when all have said whether or not it can be used for tiling a surface, tests their predictions using the rest of the set. When it is clear that they have the idea, they may be left to continue on their own.
5. Note that the triangles must all be the same way up. The markings are there primarily for this purpose, but some children may make other interesting discoveries.

Activity 2 Measuring by counting tiles [Meas 2.1/2]

A teacher-led discussion for a small group. This is to introduce the idea of measuring the area of a surface by counting how many tiles are required to cover it. It follows immediately on from Activity 1, and need not take long.

- Materials*
- The tileable shapes from Activity 1 should be available for use if needed.
 - A sheet of blank paper for teacher.

- Suggested outline for the discussion*
1. Draw an irregular shape, smaller than 10 cm by 15 cm (anticipating Activity 4), and ask how we could use tiling to measure how big it is.
 2. The answer is, of course, to tile it and count how many tiles it takes.
 3. There are however two questions which arise. One is that different tiles will give different answers, so we shall have to choose one as a standard which every one uses. This is discussed in the next activity. The other is that the tiles will not fit tidily at the edges of the region. This will be dealt with in Activity 4.

Activity 3 Advantages and disadvantages [Meas 2.1/3]

A teacher-led discussion for a small group. Its purpose is to introduce the square centimetre as the standard unit for small areas.

- Materials*
- As for Activity 2, and also
 - Some 1 centimetre square tiles.

- Suggested outline for the discussion*
1. Each child uses one shape to give an example of tiling with that shape.
 2. They discuss their advantages and disadvantages. For example, the hexagons are not easy to make lots of. The squares and oblongs are easy to use, but are not yet of any standard size.
 3. The 1 centimetre squares fit in exactly with the unit already in use for length. They are however very small and fiddly to use. Apart from this disadvantage, it is a natural choice of unit.

Activity 4 Instant tiling [Meas 2.1/4]

A demonstration and teacher-led discussion for a small group. It introduces children to the use of a centimetre grid as an instrument for measuring area.

- Materials*
- The shape you drew in Activity 2, step 1 (or another if you prefer).
 - A centimetre grid on transparent material. *
 - Non-permanent marker.

* A photomaster is provided in *SAIL Volume 2a*. If you have access to a suitable copier, the grid can be copied on to a transparency.

- Suggested outline for the discussion*
1. Show the shape whose area is to be measured. As they already know, this is to be done by counting the number of unit tiles, each 1 cm square, which will cover it.
 2. Produce the centimetre grid, and by putting this on top of the shape show how it instantly covers it with unit tiles. All we now have to do is count them.
 3. There is still the problem of the part-squares around the boundary. These can be dealt with in two ways. One is to count those which are more than half as wholes, and not to count at all those less than half. The other, which is more accurate, is mentally to combine part-squares to make whole squares.
 4. Whichever method is used, one needs a washable marker to keep track of which squares and part-squares have been counted.

Discussion of activities In this topic we aim to establish the true concept of area, as the number of unit squares which can be combined to cover the given surface. Area is not length multiplied by width, as so many children incorrectly learn. This is just a convenient shortcut for finding this number of unit squares, which applies only to rectangles.

The first three concepts listed are closely related, and so are the activities. The last activity provides a convenient technique for covering the given region with unit squares. All three make full use of Mode 1 schema building from activities with physical materials, and testing of predictions; and of Mode 2 schema construction, by communication from a teacher and discussion.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 2.2 IRREGULAR SHAPES WHICH DO NOT FIT THE GRID

Concept Consolidation and expansion of the concepts learned in topic 1 to new situations.

Ability To measure the area of a surface with an irregular outline, by counting unit squares.

Discussion of concepts No new concept is introduced in this topic. The aim is to consolidate the concept of area in its general form, before teaching short-cuts for particular cases.

Activity 1 Shapes and sizes [Meas 2.2/1]

An activity for a small group of children. Its purpose is to give them practice in the use of the centimetre grid for measuring area.

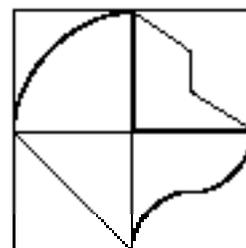
- Materials*
- Six cards with outlines of cookies of different shapes.*
 - Centimetre grid transparency.
 - Non-permanent marker and wiper for each child.
- * A suitable set is provided in the photomasters.

- What they do*
1. Each child receives one card at random.
 2. The first child puts her card where all can see it.
 3. The next child looks at her own card, and tries to decide whether it is smaller or larger in area than the one already there. If smaller, she puts it in the left; if larger, on the right.
 4. The other children in turn put down their cards at the end of the row, or between others, in what they think is the order of size.
 5. The order is recorded, and each child then takes one card and measures its area by using a centimetre grid using the method described in Meas 2.1/4.
 6. Finally the cards are arranged in the correct order of size. Any differences between this and the order recorded in step 5 may then be discussed.
 7. If desired, steps 1 to 6 may be repeated.

Activity 2 “Hard to know until we measure” [Meas 2.2/2]

An activity for two small teams of children. Its purpose is to give them further practice in the use of the centimetre grid for measuring area, and in estimating relative areas.

- Materials*
- Sixteen cards, each providing part of an outline for a surface, as illustrated.*
 - Non-permanent marker and wiper for each team.
 - Pencil and paper.
 - Centimetre grid transparency.
- * A suitable set is provided in the photomasters.



- What they do*
1. The cards are spread out where all can see them. The first team then chooses four, and puts them together to enclose an area, as shown in the illustration.
 2. The second team then uses four of the other cards to make another outline, to enclose an area as nearly as possible equal to the other.
 3. Are they of the same area, or is one larger and, if so, which? It is hard to tell until they measure.
 4. So each team measures the shape they made, and the results are compared.
 5. Steps 1 to 4 may then be repeated.

Activity 3 Gold rush [Meas 2.2/3]

(This embodiment was suggested by Helen Barrett, teacher at the International School, Hamburg.)

Part (a): A harder activity, for children to work at individually. It takes the form of an investigation, gives much practice in measuring area by counting squares, and thereby further consolidates the concept of area as already described. Plenty of time is likely to be needed.

Part (b): A teacher-led discussion, when all have completed Part (a).

Part (a)

Materials For each child:

- A loop of thin string,* pipe cleaner, or waxed string (e.g., Wikki Stix).
- A centimetre grid.
- Non-permanent marker and wiper.
- Pencil and paper.

* Cotton is better than plastic, which is too springy. String lies better if wetted.

- What they do*
1. They are asked to imagine that they are prospectors in the gold rush. Claims may be of any shape, but the maximum length of boundary allowed is fixed. This is represented by their loops of string. (You could also tell them that the word for distance all the way round is 'perimeter.')
 2. Their first task is therefore to find out what shape they should make their claims in order to enclose the largest area of ground. For this they use their loops of string and centimetre grids in the way they have already learned.
 3. They work individually, and keep their results secret from their rival prospectors. When they think that they have found the best shape, they tell their teacher without letting anyone else know.

Meas 2.2 Irregular shapes which do not fit the grid (cont.)

Part (b)

*Suggested
outline for
the discussion*

1. Point out that what they have found out will be much less useful if the best shape varies according to the length of the boundary. To find out if it does, the lengths of their loops are not all the same.
2. We assume that they have all arrived at the correct result, a circle. This answers the question for their loops of string. Is the answer still true for larger loops?
3. What if we used an overhead projector, to show their loops much bigger on a screen? Would the result still be the same?
4. And can they imagine a giant overhead projector, to show their loops on the ground? Would the result still be the same?
5. The expected answer in both cases is “Yes.” Though the foregoing is not a rigorous proof, it is good mathematical intuition, and can be made into a logical mathematical proof based on similar figures.

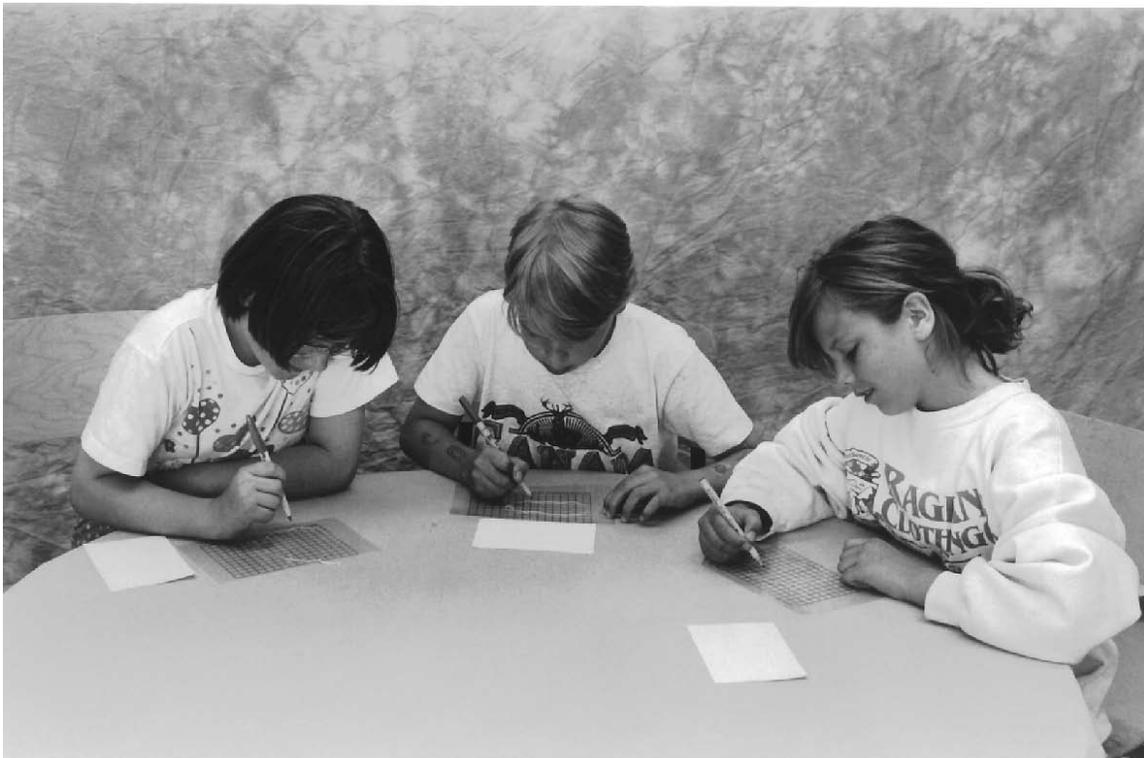
**Discussion
of activities**

The first two activities involve Mode 1 learning, in this case consolidation of the concept by practical activity, and Mode 2 testing, discussion within the peer group. The third activity brings in Mode 3 building (creativity), since they have to come up with new possibilities in a systematic way. Mode 2 building and testing then follows in the teacher-led discussion.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Gold rush [Meas 2.2/3]

Meas 2.3 RECTANGLES (WHOLE NUMBER DIMENSIONS); MEASUREMENT BY CALCULATION

Concept That in certain cases, the number of unit squares can be found by calculation.

Ability To calculate the area of a given rectangle.

Discussion of concepts The general concept is as stated above. The method of calculation for a rectangle is of course well known at an instrumental level, but after topics 1 and 2 we are now in a position to understand it relationally, and to distinguish between this particular method and the concept of area itself. 'Length times breadth' does however provide us with an excellent example of the lack of adaptability of instrumental understanding, since this pseudo-concept of area is of no use for any other shapes.

Activity 1 "I know a shortcut." [Meas 2.3/1]

A teacher-led activity for a small group of children. Its purpose is to introduce children to the above concept for the particular case of rectangles.

- Materials*
- About six different sets of identical cutout rectangles whose dimensions are a whole number of centimetres. In each set there needs to be one per child and one for the teacher.*
 - A centimetre grid for each person.*
- * Provided in the photomasters

- What they do*
1. Each person has one copy of the same shape. The aim is to be the first to say its area.
 2. Before starting, they are told that for rectangles there is a shortcut by which they can be fairly sure to come first if the others do not know it. If they think they know it, they shouldn't say what it is.
 3. When all have given their answers, they re-check if these do not agree. Then, if the first to answer was correct, he drops out of the race provided he did so by using a shortcut. (The first time this will, of course, be yourself.) He does not however drop out of checking.
 4. The last is given the choice of being told the shortcut, with explanation; or of continuing to try.
 5. Finally they check whether they were using the same shortcut. I only know of one, so this might be interesting!

Activity 2 Claim and explain [Meas 2.3/2]

An activity for a small group of children. Its purpose is to give practice in using the method with understanding.

- Materials*
- A set of cards with rectangles of various sizes all marked in centimetre squares. These are in pairs of equal area, e.g., 4 cm by 4 cm and 8 cm by 2 cm.*
- * A suitable set is provided in the photomasters.

- What they do*
1. The pack is shuffled and put face down on the table. The top card is turned over and put centrally.
 2. In turn, the players turn over the top card of the pile and put it centrally. If they see a pair of the same area, they claim it and explain how they know that the area is the same. (“Eight rows, two in each, sixteen square centimetres” is a better explanation than, say, “Eight twos are sixteen.”) If they cannot see a pair, they say “No claim.”
 3. The player whose turn it is may begin by claiming any pair which has been overlooked. He then still has his turn as in step 2.
 4. In the event that there is no face-up card when it comes to a player’s turn, he may turn over two cards.
 5. When all the cards have been turned over, the winner is the player with most pairs.

Activity 3 Carpeting with remnants [Meas 2.3/3]

An activity for a small group of children. Its purpose is to give further practice in finding the area of rectangles by calculation.

- Materials*
- For each player, a rectangle drawn on paper which is ruled in cm squares. The sides of these rectangles should fit the lines on the paper.*
 - Two lots of assorted rectangular cards, one larger and one smaller. These are not ruled in squares.**
 - Pencil.

* The sizes are not critical, but, as a guide, a suitable set to start with is included in the photomasters, measuring: 7 cm x 16 cm, 11 cm x 16 cm, 10 cm x 11 cm, 7 cm x 21 cm, 11 cm x 11 cm, 8 cm x 15 cm, and 11 cm x 14 cm. It would fit the narrative if the children were given the sizes and drew their own rectangles.

** Suitable examples are provided in the photomasters.

- What they do*
1. Each player’s rectangle represents at one-tenth size a floor to be carpeted as cheaply as possible by using remnants. These are represented by the smaller cards, which are roughly sorted into smaller and larger for shoppers’ convenience. These are likewise one-tenth size.
 2. Each player in turn chooses a remnant, and shades in a rectangle of the same area in his diagram. This need not be the same shape, but it must be a single rectangle. Once chosen, a remnant may not be returned.
 3. The last rectangle need not provide exactly the right area, but players must record the amount left over since the aim is to minimize waste.

Variation They still choose in turn, but do not shade their rectangles until finished. In this version, they may then return one remnant in exchange for one other, once only.

Discussion of activities In all these activities, the relation between the area as found by calculation and the area as a number of unit squares is preserved. Activity 3 introduces the idea of a scale drawing at an intuitive level, but this is not made explicit. Children who know about them will notice that the markings on the materials represent decimetres (which in my view are very useful units).

OBSERVE AND LISTEN

REFLECT

DISCUSS



Carpeting with remnants [Meas 2.3/3]

Meas 2.4 OTHER SHAPES MADE UP OF RECTANGLES

Concept No new concept, but expansion to more difficult examples.

Abilities To calculate areas which can be seen as made up of rectangles, either by adding or subtracting.

Discussion of concepts Having started with the simplest case of a surface whose area can be found by calculation from linear measurements, namely a rectangle, we start on the road of expanding the range of examples to which this can be applied.

Activity 1 'Home improvement' in a doll's house [Meas 2.4/1]

An activity for children to work on individually. Its purpose is to help children to enlarge their range of application of the method learned in the previous topic. It is introduced by a teacher-led discussion.

- Materials*
- Dimensioned floor plans of several rooms in a doll's house (not drawn to scale). It is useful to have two or three copies of the same plan. These may conveniently be laminated on card.*
 - Pencil and paper for each child.
- * Photomasters are provided for a kitchen, a bathroom, and a utility room.

- What they do*
1. Begin by showing them one of the plans, explaining that it represents the floor of a room in a doll's house. This needs to be covered with tiles 1 cm square. We want to know how many are needed, so that the right number can be bought.
 2. We cannot use a centimetre grid, since the diagrams are not full size, nor are they to scale as they were in Meas 2.3/3, Carpeting with remnants.
 3. There are two general approaches. One is to split up the area to be measured into rectangles, calculate the areas of each of these separately, and add. The other is to start with (in this case) the area of the whole floor, and then subtract all the areas not to be included. Usually one of these is neater than the other, the subtraction method being the less obvious.
 4. When they have the idea, give out plans for them to work on individually. Those with plans for the same room can then compare their results, and discuss each other's methods.

Activity 2 Claim and explain (harder) [Meas 2.4/2]

A harder version of Meas 2.3/2. Its purpose is to give further practice in the method introduced in Activity 1.

- Materials*
- A set of cards like those used in Meas 2.3/2, except that one card of each pair shows not a single rectangle but a shape made up of rectangles, e.g., a frame, border, L-shape. Some of these should be easily rearrangeable into a rectangle, thus preparing the way for the next topic.*
 - They will probably find pencil and paper useful sometimes.
- * A suitable set is provided in the photomasters.

What they do This game is played in exactly the same way as the earlier version (Meas 2.3/2), except for the different set of cards.

Note Some of these cards show a blank area surrounded by a frame, such as might represent a picture frame, a tiled pool deck, or a path around a rectangular flower bed. These are well-known examples for which the subtraction method is much quicker and neater. At school we were taught to find “the whole minus the hole”! I wonder if this is still in use, after more than 60 years?

Activity 3 Net of a box [Meas 2.4/3]

An investigation for children working in small groups. Its purpose is for children to apply the method learned in Activity 1 in a new situation, and to give them further practice.

- Materials*
- Two empty boxes (e.g., cereal boxes), with the tops stuck down again. One is left like this, the other is cut along some of its edges so that it can be spread out flat to give what is called a net. This can be done in several ways, and any one of these will do.
 - A dimensioned drawing of the net, large enough for all the children to see, with the dimensions rounded to the nearest centimetre.*
- * A ready-made, dimensioned net for a box is provided in the photomasters.

- Introduction*
1. Show them the box and its net, and tell them the term for the latter.
 2. Explain that the net is to be cut out of a single rectangle of card, the smallest possible.

The investigation

First part In the example that you have given them, what is the area of card used, and what is the area wasted?

Second part Can the waste be reduced by using a different net of the same box?

Third part Some children may like to investigate whether, if a lot of boxes are to be made, waste can be reduced by tessellating a number of these nets on a large sheet.

Note If anyone points out that a box cannot in fact be constructed from a net of this kind, they are of course quite right. We may then explain that we need to begin with a simplified example. The same principle will apply if we include all the tabs for sticking, etc. (except perhaps in the third stage of the investigation).

Activity 4 Tiling the floors in a home [Meas 2.4/4]

A more difficult version of Activity 1, for the more able children to work on individually or in pairs. It also provides an opportunity to introduce the square decimetre. Although this unit does not seem to be widely used, the (linear) decimetre is introduced in Meas 1 (topic 6), so it would seem sensible at least to mention the square decimetre. The activity is introduced by a teacher-led discussion.

- Materials*
- Dimensioned floor plans of three floors in a house (not drawn to scale). It is useful to have two or three copies of the same plan. These may conveniently be laminated on card.*
 - Pencil and paper for each child.
- * Provided in the photomasters

- What they do*
1. Begin by showing them the plans, explaining that each represents one floor of a house. The exposed parts need to be covered with tiles, and we want to know how many of these are needed.
 2. The tiles are available in different materials: carpet, vinyl, and non-slip ceramic. Suggestions may be invited as to which would be suitable for different rooms.
 3. The tiles are all 10 cm square, and (very conveniently) the dimensions of the surfaces to be covered are all multiples of 10 cm.
 4. The floor plans are now given out for children to work on individually or in pairs. Those with plans for the same room then compare their results, and discuss each other's methods.
 5. If this has not already arisen spontaneously, ask what we would call the area of one tile. So the floor surfaces to be covered measure . . .? What is this in square metres?

Discussion of activities These activities evoke children's creativity in extending the range of examples for which a desired area can be calculated by using the simple method for calculating the area of a rectangle. For Activity 2, mental agility is also needed. Since these are consolidation activities, I suggest that children are given plenty of practice in them, so that they may become fluent in adapting the method to many particular instances.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 2.5 OTHER SHAPES CONVERTIBLE TO RECTANGLES

- Concepts* (i) That other shapes can be converted to rectangles of the same area.
(ii) That in this way, their areas can also be found by calculation.

Ability To calculate the areas of given parallelograms, triangles, and circles.

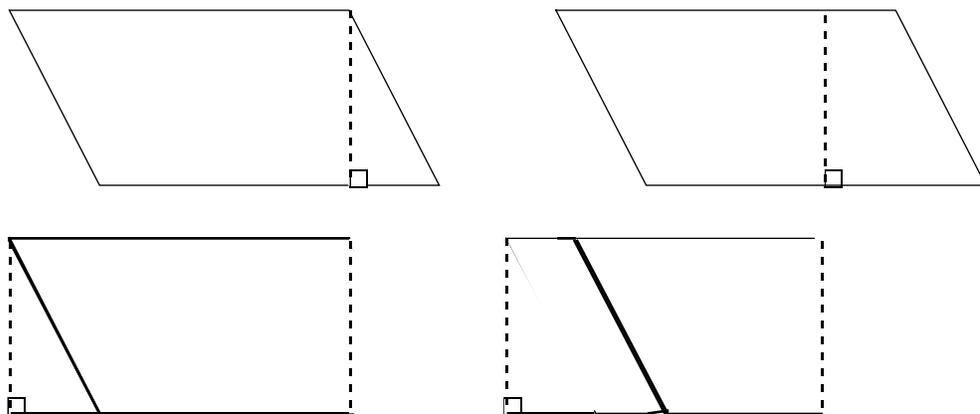
Discussion of concepts Here we continue along the road of calculating areas of surfaces from linear measurements. The beginning made in topic 3 is here expanded further, first to two more shapes whose sides are straight lines, and then to circles. To convert a circle into a rectangle of the same area seems to me an attractive example of mathematical ingenuity, and it also introduces the concept of ‘getting closer and closer to . . .’ which is mathematically quite advanced.

Activity 1 Area of a parallelogram [Meas 2.5/1]

An activity for children working in small groups. Its purpose is to help them find and use a method for calculating the area of a parallelogram.

- Materials*
- For each child, a sheet of coloured paper on which several parallelograms are drawn. In each parallelogram, the lengths of one pair of opposite sides and the perpendicular distance between them are whole numbers of centimetres.*
 - For each child, a straight-edge, a sheet of plain paper, and a pencil.
 - For the group, several pairs of scissors.
 - For the group, several gluesticks or equivalent.
- * A suitable set is provided in the photomasters.

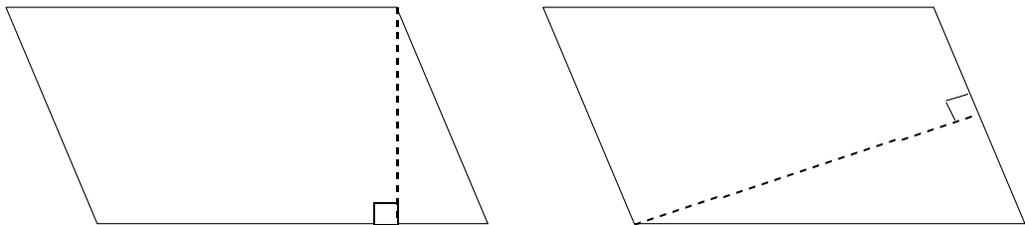
- What they do*
1. Each child cuts out the top parallelogram on their sheet (A).
 2. Tell them that by making just one cut and rearranging the pieces, their parallelogram can be converted into a rectangle of the same area.
 3. They all have a try, and then share their ideas and results. The successful conversions are pasted at the top of their sheets of paper. When they discover the importance of an accurate right-angle, they may find the corner of a sheet of paper helpful.
 4. By now we hope that at least one of the group has found the well-known method illustrated below, and thereby the area of the parallelogram.



Meas 2.5 Other shapes convertible to rectangles (cont.)

5. Ask them to find the area of the next parallelogram (B on the Meas 2.5/1 photomaster) by the same method, and compare results.
6. There are four more parallelograms whose areas are to be found. Tell them that there is a quicker way which doesn't need cutting. Ask them to think about it, and then to share their ideas. When a good method has been agreed on, they use it to find the areas of parallelograms C and D, and compare results.
7. There are two stages of this shortcut. The first is to replace cutting and pasting by drawing. The second is to by-pass this and simply calculate the area of the equivalent rectangle, without actually drawing it. If someone goes straight to the second stage, well done, but he should be asked to explain how he got there. This end result is easier to reach if the cut has been positioned as in the left-hand pair of diagrams.
8. Finally they use their new method to calculate the areas of the two remaining parallelograms, E and F.

- Notes*
1. For many parallelograms there are two directions in which the cut can be made, as shown here. Both are valid. In the examples provided in the photomasters, only one of these gives whole-number measurements.



2. We need a summary of the method which has been arrived at. The conventional one is 'base multiplied by height,' which is convenient provided we remember that the base can be any side, and the height has to be perpendicular to it and not straight up and down the page. As an alternative I suggest 'The length of a side multiplied by the perpendicular distance from its opposite side.' More accurate, but cumbersome. The first might be accepted provided that, if challenged, the user can expand it into something like the second.

Activity 2 Area of a triangle [Meas 2.5/2]

An activity for children working in small groups. Its purpose is to help them find and use a method for calculating the area of a triangle.

Materials For the group:

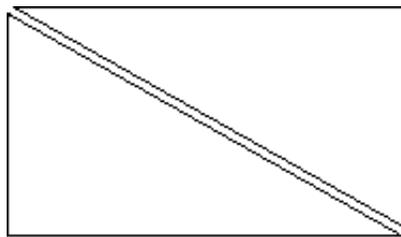
- One or more pages of identical right-angled triangles, suitable in size for cutting out if desired. There should be enough of these for each child to have two if they want.*
- One or more pages of scalene triangles.*
- Several pairs of scissors.
- Several gluesticks or equivalent.

For each child:

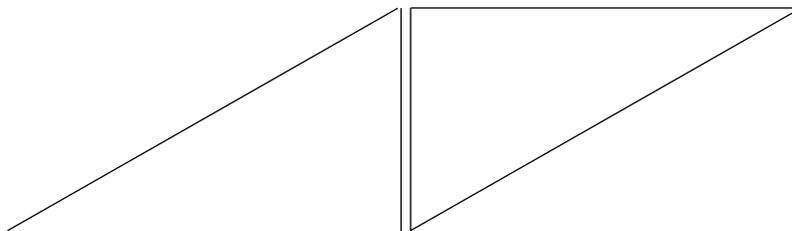
- A sheet of plain paper, and a pencil.
- * A suitable set is provided in the photomasters.

What they do Stage (a) *Right angled triangles*

1. Someone cuts up enough of the sheet(s) of right angled triangles for each child to receive an oblong of paper with a triangle on it. There should be some spare ones also. These are placed centrally.
2. Each child takes one of these, cuts it out, and tries to find its area by whatever means they like. Tell them that they may take another triangle if they wish and that the easiest way does not involve cutting up the triangle. The other materials are there for use if required.
3. The essence of the method is of course to put together two identical triangles to make a rectangle. This can be done either physically, by using a second triangle; or by drawing.



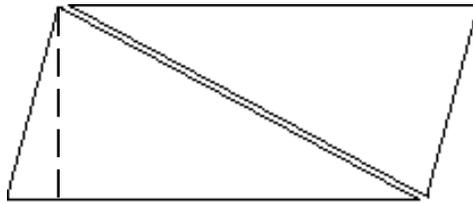
4. If the triangles are put with a pair of the smaller sides together, the result will be a parallelogram or an isosceles triangle, depending on the relative orientation of the right triangles. This is a little harder, and anticipates Stage (b).



Meas 2.5 Other shapes convertible to rectangles (cont.)

Stage (b) Scalene triangles

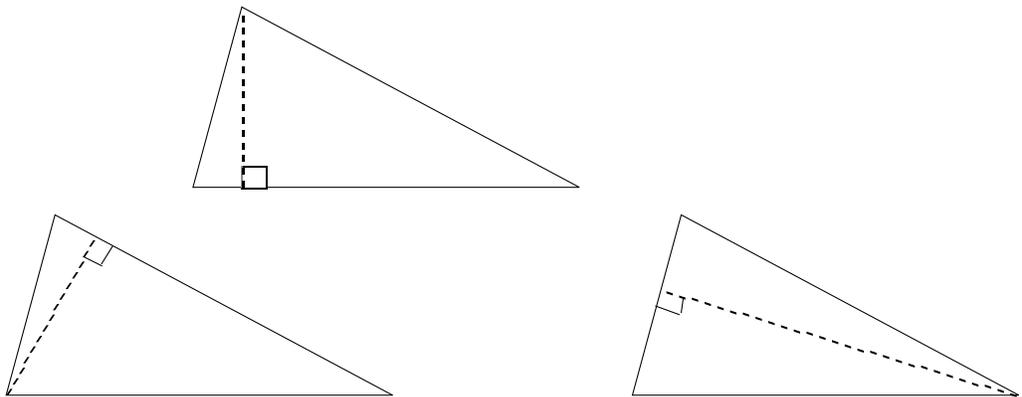
This repeats steps 1 to 4 of Stage (a), this time with scalene triangles. In this case the result will *always* be a parallelogram, and by now they are well equipped to deal with these.



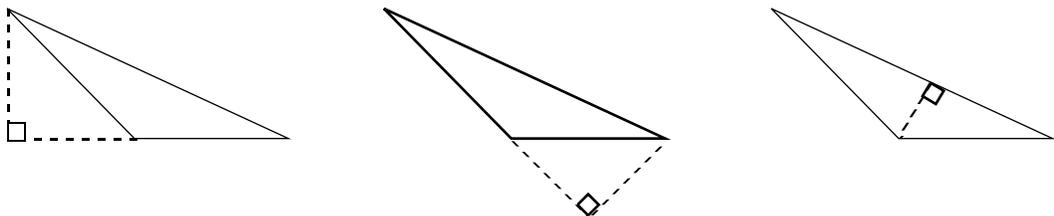
- The area of the original triangle is half that of the parallelogram. In the diagram above, this is half one of the length of one of the parallel sides multiplied by the distance between them, shown by the dotted line.

Steps 6 to 9 should not be taken until the final part of the method is well established.

- In the original triangle, this dotted line is called an altitude of this triangle.
- Two other altitudes can be drawn, each perpendicular to one of the sides. These are perpendicular to opposite sides of the two other parallelograms which can be made in the way described in above. The children should verify this.



- So in terms of the original triangle, the area can be calculated by multiplying the length of any side by the altitude perpendicular to this side, and halving the result.
- I suggest that the use of an obtuse-angled triangle is best dealt with as a teacher-led discussion. Here, two of the altitudes require dotted extensions of their opposite sides.



Activity 3 Area of a circle [Meas 2.5/3]

An activity for children to work on initially by themselves, followed by a teacher-led discussion. Its purpose is to introduce them to the method for calculating the area of a circle.

Materials For each child:

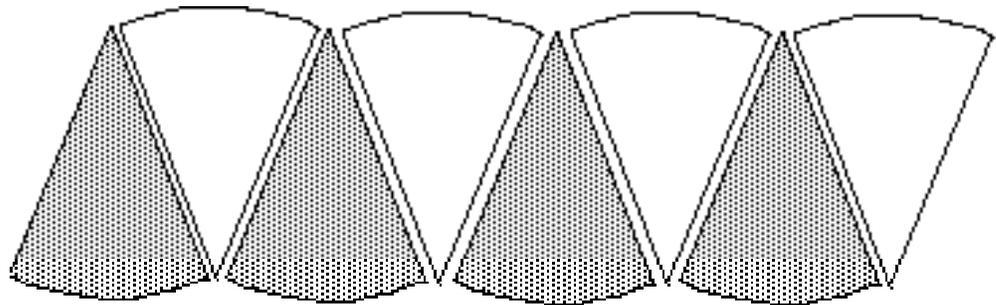
- A circle on paper, for cutting out.*
- A sheet of coloured paper.

For the group:

- Coloured felt-tips.
- Several glue sticks or equivalent.

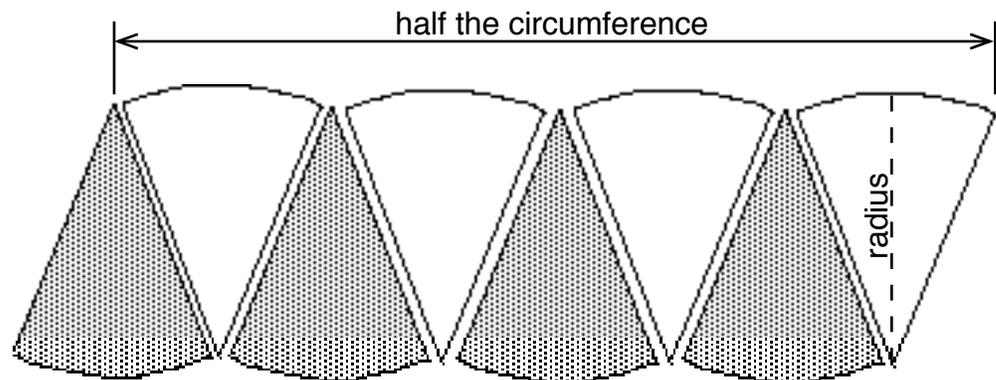
* A suitable one is provided in the photomasters.

- What they do*
1. They cut out their circles, fold them in half, open out again and colour one half (that is, one semicircle).
 2. They now fold again into halves, then quarters, then eighth-parts.
 3. They open out, and cut along the creases to get eight pie-shaped pieces. These are called *sectors* of the circle.



4. These sectors are then pasted on the sheet of coloured paper with the white and coloured sectors pointing in alternate directions, as shown.

- Suggested outline for the discussion*
1. What shape does this look like? (A parallelogram, with one pair of sides straight and the other pair wavy.)
 2. How long is each of the wavy sides? (Following the curve, it is half the circumference of the original circle.)



Meas 2.5 Other shapes convertible to rectangles (cont.)

3. What is the distance between them? (It depends where we measure. If we measure from the vertex to the arc of a sector, it is a radius of the original circle (and of each individual sector).)
4. So the area of this ‘nearly-a-parallelogram’ is approximately? (Half the circumference multiplied by the radius.)
5. And the area of the original circle? (The same.)
6. How could we make it closer to a parallelogram? (By folding into sixteenth-parts and using these. Some children may like to try this.)
7. Would this change its area? (No.)
8. Could we make it closer still? (Yes, by using smaller and smaller parts. In this case we could not do it by folding, but to begin with we could draw them.)
9. Could we go on making it closer and closer to a parallelogram, so close that the difference was too small to matter? (Yes; even when it became too hard to draw, we could continue in our imagination.)
10. And the area of this ‘can’t-tell-it-from-a-parallelogram’ is? (No change; it is still what we said before.)
11. So the area of the original circle is? (Half the circumference multiplied by the radius.)

Note Later, when they have learned about π , they will be able to use this to calculate the length of the circumference from the radius, and thereby to arrive at another form of the result given above.

Discussion of activities By the end of this topic, we have come a long way from counting unit squares with the help of a centimetre grid. Using our knowledge of how to calculate the area of a rectangle, we develop in succession methods for calculating areas of parallelograms, triangles, and finally circles, the last of which doesn’t look anything like a rectangle. These are activities in pure mathematics, and another nice example of mathematical creativity — using our existing knowledge to create new knowledge, while testing always for consistency with our existing knowledge.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 2.6 LARGER UNITS FOR LARGER AREAS (square metre, hectare)

- Concepts* (i) That larger units are need for measurement of larger areas.
(ii) Square metres, hectares.

Ability To calculate areas in these units.

Discussion of concepts These are straightforward expansions of the original concept of a square centimetre as a measure of area.

Activity 1 Calculating in square metres [Meas 2.6/1]

A teacher-led discussion, after which the children work by themselves, and compare results.

Materials For each child:

- Example sheets 1 & 2, with dimensioned drawings of areas to be calculated in square metres. These should be similar to those which they have already encountered in square centimetres. Sheet 1 has only whole number dimensions, but sheet 2 has some of the dimensions in metres and tenth-parts.*
- Pencil and paper.
- For sheet 2, a calculator.

* Suitable example sheets are provided in the photomasters. The second of these includes one new shape, a trapezium.

- Suggested outline for the discussion*
1. Suppose that we want to measure larger areas, such as that of a classroom, or a garden, or a swimming pool. Would square centimetres be a good unit for doing this? (No. The numbers would be too big, and we do not want to measure to an accuracy of 1 cm. In the case of a garden, we couldn't even if we wanted to.)
 2. So what could we use instead? (The answer we want to end up with is a square metre. If anyone suggests square decimetres, this too is a good answer. For measuring the dimensions of (say) a room, though, centimetres would usually be too accurate a measure; metres (in whole numbers) would not be accurate enough. However, while a square decimetre seems to me quite a sensible unit, it is not in general use. In the case of a room, the usual practice would be to measure in metres, or metres and tenth-parts (e.g., 7.3 metres by 5.7 metres), and in the second case to use a calculator for multiplying.)
 3. In square metres, what would be the area of a rectangle 3 metres by 5 metres? 6 metres by 4 metres? 30 metres by 90 metres?

- What they do*
1. Each child has a copy of example sheet 1. They all calculate the areas of each of these examples (on separate sheets of paper, not the example sheet).
 2. They compare results, and discuss any differences until they agree on the result.
 3. Steps 1 and 2 are repeated with example sheet 2.

Activity 2 Renting exhibition floor space [Meas 2.6/2]

An activity for children to work on individually or in pairs. Its purpose is to give practice in calculating area in square metres.

- Materials*
- Dimensioned plans of two or more exhibition halls, showing floor spaces for individual stands, and the daily rental charge for these per square metre. Enough copies are needed for each child to have one or the other of these plans.*
 - Pencil and paper for each child.
- * Suitable dimensioned plans are provided in the photomasters.

- What they do*
1. Each child has one of these plans, and calculates (a) the rental charge for each space at \$ ___ per square metre, and (b) the total daily income for the whole hall.
 2. When they have finished, children with copies of the same plan may compare results and discuss any disagreements. If these cannot be resolved, they seek help from their teacher.

Activity 3 Buying grass seed for the children's garden [Meas 2.6/3]

This is similar to Activity 2, with a different embodiment. It is harder, and may be better taken as a cooperative activity for two or three children.

- Materials* For each child:
- Children's garden plan, see Figure 48.*
 - Children's garden checklist.*
 - Pencil and paper.
- * Provided in the photomasters

- What they do*
1. Each child has a copy of the plan, which represents a garden of an English stately home. Originally this was owned by a rich family, but it is now open to the public, so children use the garden for playing in. The ornamental pool is now used for paddling, the yew trees for hide-and-seek, the summer house for quiet reading. Older children with their teachers sometimes measure the fountain.
 2. As a result it needs to be re-sown with extra hard wearing grass.
 3. Each child is asked to calculate the cost of doing this, given that the cost of grass seed is \$11.00 per 1 kilogram box, and each kilogram is enough for 20 square metres.
 4. When all have finished, they compare results and discuss any disagreements.
 5. (Optional) They are then told that the seed can also be bought in boxes of various sizes, costing \$6.00 for a 0.5 kg box, \$19.00 for a 2 kg box, and \$40.00 for a 5 kg bag. What would the cheapest way to buy it?

Note These prices were correct when going to press, but may need updating.

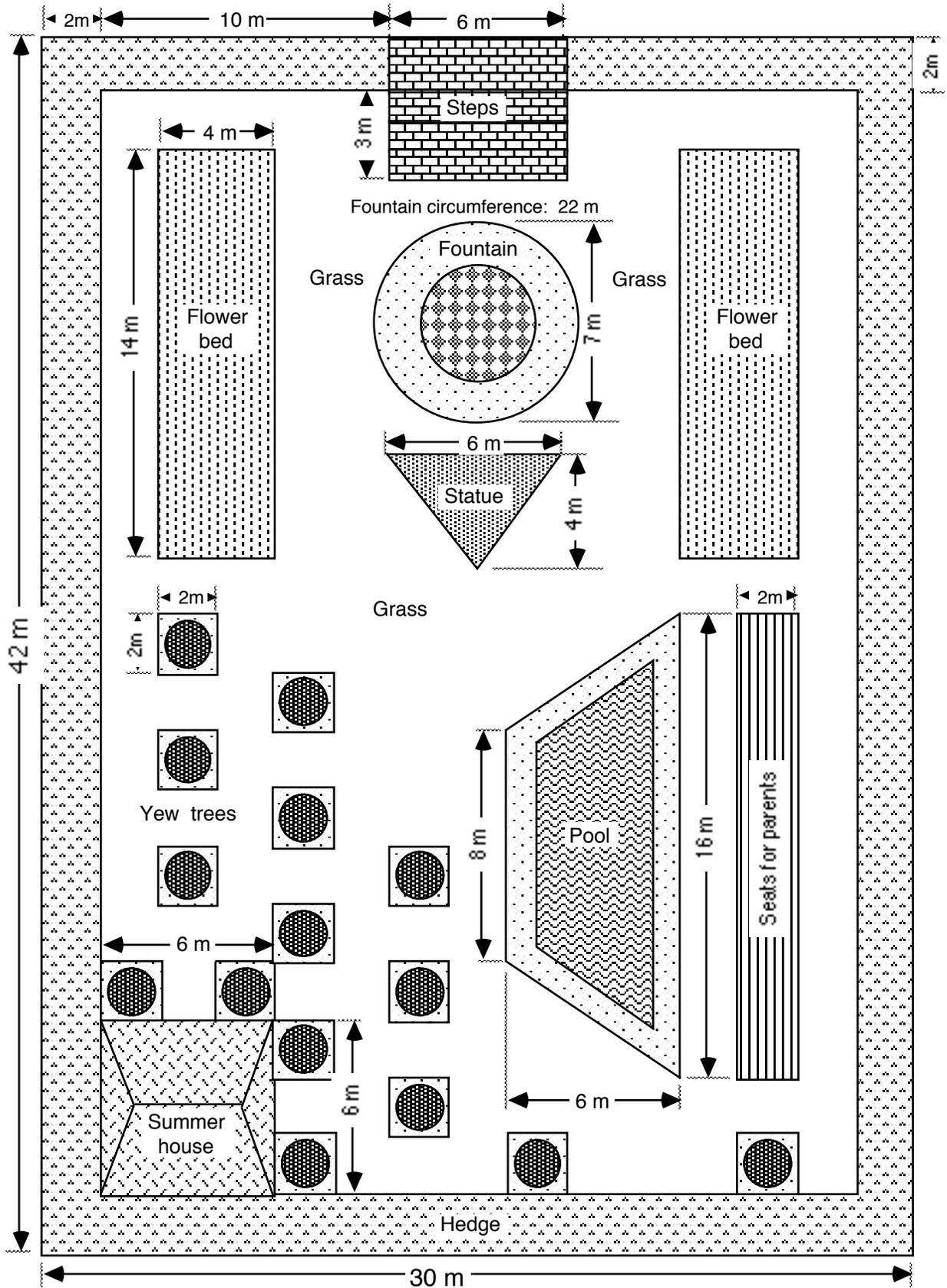


Figure 48 The children's garden

Activity 4 Calculating in hectares [Meas 2.6/4]

The purpose of this activity is to introduce children to the hectare as a unit for measuring large areas. It is introduced by a teacher-led discussion, after which the children work individually or in pairs. Finally they compare results and discuss any disagreements.

- Materials*
- For each child or pair, a copy of the example sheet provided in the photomasters.
 - Pencil and paper for each child.

- Suggested outline for the discussion*
1. Explain that for measuring land areas a unit often used is the hectare. This is the area of a 100 metre square.
 2. So has a 200 metre square an area of 2 hectares? (No. They should make a rough drawing, from which they can see that the area is 4 hectares.) And the area of a 300 metre square? Of a 400 metre square?
 3. Ask for some oblongs which would have an area of 1 hectare. (Examples: 200 m by 50 m, 20 m by 500 m, 40 m by 250 m, 80 m by 125 m, . . .)
 4. It may help to imagine the size of a hectare if we know that a lawn tennis court is 0.03 hectares, a soccer field can be as large as 1.08 hectares (max.) or as small as 0.41 hectares (min.), and a regulation Canadian football field, including the end zones, is about 0.87 hectares.
 5. How many square metres are there in a hectare? (10,000)
 6. What is the area of a rectangular plot of land 160 metres by 250 metres? (40,000 square metres, which is equal to 4 hectares.) And of one 200 metres by 120 metres? (24,000 square metres, which is equal to 2.4 hectares.)

What they do They should now be ready to calculate the areas of the plots of land shown in the example sheet. In shape these are rectangles, parallelograms, triangles, and circles. As in previous activities they work individually, compare results, and discuss any disagreements.

Activity 5 Buying smallholdings [Meas 2.6/5]

This activity is for further practice at calculating in hectares. Some of the shapes are less straightforward than those in Activity 4.

- Materials* For each child
- Example sheets 1 (easier) & 2 (harder).*
 - Pencil and paper
 - For sheet 2, a calculator.
- * Provided in the photomasters

What they do

1. Each sheet represents a number of adjacent plots of land which are to be sold as smallholdings. Prices are based on their areas, which therefore need to be calculated.
2. As before they work individually, compare results, and discuss any disagreements.

Discussion of activities In this topic, they encounter requirements for which their original unit of area, the square centimetre, is too small. The concept of measuring area should however have been well formed and consolidated in the five topics which come before, and thus be ready for expansion. This is Mode 3 concept building — expansion of existing concepts. Mode 3 testing (consistency with existing knowledge) is built in, since the new units are closely related to the old. Mode 2 building takes the form of communication from their teacher, and Mode 2 testing is provided by plenty of discussion, both teacher-led and within groups.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Buying smallholdings [Meas 2.6/5]

Meas 2.7 RELATIONS BETWEEN UNITS

Concept That of numerical relations between different units of area.

Abilities To convert in both directions between square centimetres, square metres, and hectares.

Discussion of concept One example of this concept has already been encountered in the previous topic, in the relationship between square metres and hectares. In this topic we make the concept explicit, expand it, and organize some of the particular cases in the form of a table.

Activity 1 “What could stand inside this?” [Meas 2.7/1]

A teacher-led discussion for a group of children, possibly the whole class. Its purpose is to consolidate their notions of the relative sizes of the units listed above and relate these to their everyday experience. In this activity we also include the ‘are’ (pronounced as in area), since it is from this that the hectare is derived. However, I do not think that the ‘are’ is widely used, so it is not included in any of the other activities.

- Suggested outline for the discussion*
1. Draw on the blackboard a metre square, and inside it stick a little piece of paper 1 cm square. Have ready a piece of cord 10 metres long.
 2. Start with the centimetre square. What could stand inside this? (A fly, a flea, a wood louse, . . .)
 3. Next, the metre square. What could stand inside this? (One of us, a bird, a squirrel, . . .)
 4. Stretch out the 10 metre cord, and ask them to imagine a square with this as side. The area of this is called an ‘are.’ What could stand inside this? (Several horses, a medium sized van, a small house, . . .)
 5. Ask them to imagine a square with side 100 metres. (A Canadian football field is 100 metres long.) What could stand inside this? (A large herd of cattle, a department store, a fleet of trucks, . . .)

Activity 2 Completing the table [Meas 2.7/2]

An activity for children to do individually. Its purpose is to summarize what they have learned about the relations between units of area.

Materials • Pencil, paper, and ruler for each child.

- What they do*
1. Each child copies and completes the table illustrated below.
 2. When completed, they compare results and discuss any disagreements. If these cannot be resolved, they consult their teacher.

TABLE

square centimetres		square metres		hectares
	=	1	=	
---	=		=	1
1	=			---

(The dashes indicate spaces which it would not make much sense to fill.)

This is what they should get.

TABLE

square centimetres		square metres		hectares
10 000	=	1	=	.0001
---	=	10 000	=	1
1	=	.0001		---

Activity 3 “It has to be this one.” [Meas 2.7/3]

An activity for children to work at individually. Its purpose is to give practice in estimating what would be a sensible answer, so that they can check that the result of using a calculator makes sense. Here it also gives practice in using what they have already learned about calculating area.

- Materials*
- One or more example sheets.*
 - Pencil and paper for each child.
- * Two suitable ones are provided in the photomasters.

- What they do*
1. The example sheet represents a page of diagrams as they come from the printer. The dimensions form part of the drawings, but the areas have been typeset separately and appear at the bottom.
 2. These areas were worked out by calculator, but for this exercise the children do not have calculators.
 3. But if the drawing is of a rectangle, (say) 71.4 metres by 185 metres, then out of the following, they should be able to choose the right one.
12 500 sq cm 1 260 sq m 1.3 hectares
 4. A good way to do this would be to round the 71.4 down to 70 metres, and the 185 up to 200 metres. These can be multiplied without a calculator: result 14 000 square metres, which is equal to 1.4 hectares.
 5. So for this rectangle it has to be the last figure, 1.3 hectares.
 6. When they have finished, they may compare results and discuss any disagreements. If these cannot be resolved, they seek help from their teacher.

Discussion of activities The first activity is for relating their newly-acquired mathematical concepts to their everyday knowledge, that is to say, relating their school learning to the world outside school. By assimilating their mathematical knowledge to a wider schema, they are broadening their understanding.

The second activity is for organizing and consolidating their knowledge of the numerical relationships between units, so that they have these available in the form of ready-to-hand methods.

In the last activity they are looking at a variety of possible answers made available to them and deciding which one makes sense in relation to what they already know. This is Mode 3 testing, and is followed by Mode 2 testing with others at their table.

OBSERVE AND LISTEN

REFLECT

DISCUSS



“It has to be this one.” [Meas 2.7/3]

Meas 2.8 MIXED UNITS

- Concepts* (i) That different units may sometimes be used for different dimensions of the same object.
(ii) That there is no recognized meaning for multiplying these as they stand.

Ability To calculate areas correctly in these conditions.

Discussion of concepts This is a straightforward development from their existing knowledge that different units are used for different jobs. Here, they are used for different parts of the same job, as described in Activity 1.

Activity 1 Mixed units [Meas 2.8/1]

An activity for children to children to do individually, introduced by a teacher-led discussion. Its purpose is to enable children to avoid being confused by mixed measures, and to calculate correctly under these conditions.

Materials For each child:

- Example sheets 1 & 2.*
- Pencil and paper.
- Calculator for the harder examples.

For yourself:

- An unopened package of paper towels.
- * Provided in the photomasters.

Suggested outline for the discussion *Note: Readers who have available a copy of Mathematics in the Primary School might find it interesting here to read the third paragraph on page 5, which begins "If the teacher asks . . ."*

1. Show them the roll of paper towels, and tell them the measurements given by the manufacturer. A typical example is 23 centimetres wide, 20 metres long.
2. Ask how they would calculate its area.
3. If someone wants to start by multiplying 23 by 20, the next question is:
4. What would the unit of measurement for the result be? The result can't be 460 square metres, which is much too big; for this, the roll would have to be 23 metres wide. It can't be 460 square centimetres, because we would get this if the roll were only 20 cm long. And an oblong 1 cm by 1 metre is not an accepted unit. (It could be used for a unit of area, but it never is because it would be a very inconvenient unit.)
5. We want our answer in either square centimetres or square metres; obviously, not in hectares.
6. For this, length and breadth would have to be either both in centimetres or both in metres, so one would have to be converted. Let's try both ways.
7. 23 cm by 2 000 cm. Result, 46 000 sq cm.
8. .23 m by 20 m. Result, 4.6 sq m.
9. Which gives a better idea of the total area? There is no right or wrong answer to this, though I, myself, prefer the second.

Meas 2.8 Mixed units (cont.)

- What they do*
1. They are now ready to work through the easier examples. When they have done these individually, then as usual they compare results and discuss any disagreements. If these cannot be resolved, they consult their teacher.
 2. Likewise with the harder examples.

Discussion of activity The teacher-led discussion is a good example of Mode 3 building — extrapolation of existing knowledge and methods to meet the requirements of a new situation. The teacher-led discussion leads the children in the right direction for doing this. Mode 3 testing, whether the incorrect results which children have been known to give make sense in terms of what we know already, and Mode 2 testing, whether an oblong 1 centimetre by 1 metre would ever be agreed on as a sensible unit of area, complete the process of concept formation. It is then consolidated by suitable examples.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Which of these can hold more? [Meas 3.4/1]

[Meas 3] VOLUME AND CAPACITY

Meas 3.5 MEASURING VOLUME AND CAPACITY USING NON-STANDARD UNITS

Concept The capacity of a container as distinct from the volume held at a particular time.

Ability To order containers according to capacity or volume held, using non-standard units.

Activity 1 Putting containers in order of capacity [Meas 3.5/1]

Discussion of concept	An activity for a small group of children working together. Its purpose is to extend their abilities to measure, using a non-standard unit of a different kind, and to introduce the term 'capacity.'
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- Materials*
- An assortment of glasses, jugs, jars, etc., no two alike and varying as widely as possible in height and width.
 - For each child, an identical container smaller than any of these, such as an egg cup.
 - Water.

- What they do*
1. Explain that their task is to arrange the containers in order of how much they can hold when full. This is called their capacity.
 2. To begin, each child takes one of the containers.
 3. After that, it is for them to work out the best method. That each has an egg cup gives a clue.
 4. One good method is, of course, to use the egg cup as a non-standard unit, recording the number of these that it takes to fill one's own container.
 5. If there are more containers, steps 2 to 4 are repeated. Those not thus engaged can use the results already obtained to begin putting in order those already measured.

Activity 2 "Hard to tell without measuring" [Meas 3.5/2]

A continuation of Activity 1 for two teams that underlines the need for measurement, and that gives children experience applying their concept of volume more generally.

- Materials*
- As for Activity 1, except that the containers are now lettered.

- What they do*
1. Using their egg cups, team A pours a different volume of water into each of the containers. The amount in each container is recorded. Team B does not watch while this is done.
 2. Team B now tries to arrange the containers in order of how much water is in them.
 3. While team B is doing this, team A writes the letters in order of how many egg cups full were poured into the containers.
 4. These two orders are now compared.
 5. Steps 1 to 4 may now be repeated with the teams changing roles.

Discussion of concepts	These two activities, like the earlier ones in <i>Volume 1</i> , extend the children's concepts and abilities by giving them problems to solve with physical materials. The first one concentrates on capacity, and the second on volume in general.
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Meas 3.6 STANDARD UNITS (LITRES, MILLILITRES, and KILOLITRES)

- Concepts*
- (i) The litre as a standard unit for measuring capacity.
 - (ii) The millilitre as a standard unit for relatively precise capacity measurement.
 - (iii) The kilolitre as a standard unit for measuring relatively large capacities.

Abilities To estimate, measure, and compare the capacities of containers using litres, millilitres, and kilolitres.

Discussion of concepts While non-standard units can be useful for comparing the capacities of two or more containers directly, standard units are essential when it comes to making purchases. For every day buying or selling of liquids, the litre (L) and the millilitre (mL) are usually the units of choice. The following activities focus on the sizes and usefulness of these units of measure.

Activity 1 Can you spot a litre? [Meas 3.6/1]

An activity for the whole class to draw attention to the relative size of a litre, to the many ways in which this unit is used, and to the variety of shapes that can be found that will hold a litre.

Materials For each group:

- A collection of containers brought in beforehand by the children.
- A variety of other containers, including some that hold exactly one litre, and, if available, a clear plastic cube open on one side and with internal measurements 10 cm by 10 cm by 10 cm (i.e., with a capacity of 1000 cubic centimetres or 1000 millilitres or 1 cubic decimetre or 1 litre . . . and occupying a volume of little more than 1000 cubic centimetres).

- What they do*
1. Well before the class activity, ask the children to bring empty containers from home which they think would hold about the same amount of liquid as, for example, a 1 L milk carton (available for the children to view). [They will likely bring in milk containers, apple juice boxes, ice cream containers, etc.]
 2. To begin with, each child takes one of the containers and determines whether it has the capacity of one litre, or more, or less.
 3. Then they demonstrate to the others in the group which of the containers have the same capacity (even though they are perhaps very different in appearance) . . . and which, in particular, can hold exactly one litre.
 4. When they are satisfied which containers have a one-litre capacity, the children make signs like the following: 'less than a litre (L),' '1 litre (L),' 'more than a litre (L).'
 5. They predict the capacities of each of the whole collection of containers, placing them with the appropriate sign.
 6. They then check their predictions by using the one-litre containers. Finally, they describe what they have done. Some may like to point out what they have noticed about the 10 cm by 10 cm by 10 cm container (if available).

Activity 2 “Special: Small Lemonade, 10¢” [Meas 3.6/2]

An activity for a group of six children. Its purpose is to call attention to the need for a smaller standard measure for volume or capacity than the litre.

Materials For each group:

- Paper cups of two sizes with shapes that make it difficult to judge by eye which has the greater capacity (e.g., cone-shaped versus nearly cylindrical).
- A jug of water.
- A plastic beaker graduated in mL.
- The litre containers from Activity 1.
- Clear containers on which the height of liquid within can be marked easily.

- What they do*
1. Two of the children take the part of lemonade stand operators. One runs the “Good-for-you” stand and the other the “Tasty Aide.” They each prepare a sign: “Special: Small Lemonade, 10¢.” Unknown to the operators (at least initially), the ‘small’ cups supplied to one stand have a larger capacity than those supplied to the other.
 2. The rest of the children act as customers. While the owners are preparing their signs, the customers (who have heard of the ‘specials’) write on a sheet of paper something like: “*Good-for-you* or *Tasty Aide* . . . where shall I buy some lemonade?”
 3. The customers will, of course, want to find out which stand offers the best buy. They can do this by pouring small cups of lemonade into a litre container but this would require buying *a lot* of lemonade and would likely take *a lot* of time. Can they think of another way to find out which is the better buy?
 4. When they think they have discovered an answer, they write a description of what they have found and compare with the other customers and the lemonade sellers.

Note Some may pour the cup of lemonade into a container, mark the height, pour the liquid out, and then pour the contents of the second cup into the same container, comparing the heights. Others may decide to use the graduated mL beaker. Advantages and characteristics of the beaker are discussed. Reading the scale is similar to using a ruler to measure length, but the children are likely to want to experiment to confirm that the markings on the beaker scale correspond to liquid volumes in mL and L.

Activity 3 Making and tasting (accuracy in the kitchen) [Meas 3.6/3]

A small group activity in which the result of any inaccuracies is likely to be very obvious! Some children may have had experience baking with grandparents or parents who successfully use a “pinch of this” and “about that much” of something else. Explain that when we are learning to bake, it is best to use exact amounts.

- Materials*
- Recipes with measurements in L and mL. There are many ‘no-bake’ children’s recipes available that could suit the purpose admirably.
 - The necessary ingredients for such treats as ‘Crunchy peanut butterballs,’ ‘cocoa rolled-oat cookies,’ ‘no-bake chocolate cookies,’ ‘banana bread,’ ‘yogurt dip,’ . . .

Suggestions The purpose of the activity is to provide experience with simple recipes while becoming more familiar with litres and millilitres. For younger children, one could use recipes which do not require very precise measurements, whereas older children could work with measurements such as 5 mL vanilla, 60 mL vegetable oil, etc.

Activity 4 Race to a litre [Meas 3.6/4]

An activity for 2 to 4 players to practise measuring mL in a game situation.

- Materials*
- A 1-L container for each player.
 - A plastic beaker graduated in mL.
 - A large container of water.
 - A die marked: 25 mL, 50 mL, 100 mL, 150 mL, 175 mL, and 250 mL.

- What they do*
1. Players roll the die in turn and measure the amount of water indicated by the die.
 2. Each observes carefully as the others measure the water into their own containers.
 3. The first player to reach 1 litre or more is the winner.

Activity 5 Completing the table of units of capacity or volume [Meas 3.6/5]

An activity for children to do individually. Its purpose is to summarize what they have learned about the relations between litres and millilitres and to introduce a third standard unit for measuring capacity or volume: the kilolitre.

- Materials*
- Pencil, paper, and ruler for each child.

- What they do*
1. Each child copies and completes the table illustrated below.
 2. When completed, they compare results and discuss any disagreements. If these cannot be resolved, they consult their teacher. It might be interesting to let them try to name the mystery unit, the ‘?,’ and to suggest where it might be used instead of litres (e.g., amount of water in a swimming pool or hot tub).

TABLE

millilitres		litres		?
mL		L		??
	=	1	=	.001
	=		=	1
1	=		=	

This is what they should get.

TABLE

millilitres		litres		kilolitres
mL		L		kL
1 000	=	1	=	.001
1 000 000	=	1 000	=	1
1	=	.001	=	.000 001

Discussion of activities These activities bring out the everyday, practical usefulness of measuring volume and capacity in standard units based on the litre. They provide opportunities for connecting home experiences with classroom activities.

[Meas 4] MASS AND WEIGHT

INTRODUCTORY DISCUSSION

Mass and weight are closely related, and often confused for reasons which will emerge in this discussion. Though we do not want to cause difficulties for children in the early stages by making too much of this distinction, it is easier to understand something which is accurate than something which is confused. So the object of the present discussion is to get the relations between mass, weight, and inertia as clear as possible in our own minds, and to outline the ideas which underlie the approach I have taken in this network.

At an everyday level, we can say that a mass is anything which has weight. So a stone, a book, a house, a person are masses, whereas a day of the week, a poem, a joke, are not. This is much the same at a scientific level, since weight is the gravitational attraction of the earth on all other masses. Every mass attracts every other mass with a gravitational force; but this is very small, unless at least one of the masses is very large, which is the case with the earth.

Weight is one of the characteristics of mass, the other being inertia. On the moon, the same mass would weigh less, and at some location in between, where the gravitational pull of the moon was equal and opposite to that of the earth, it would be weightless. But in all these places its inertia would be the same. When we push a car in neutral on smooth level ground, it is its inertia which requires a hard push to get it started, and an equally hard push backward to stop it. The forces required would be much less for a bicycle, and much more for a railway car, though the frictional resistance is small, even for the last of these. This would still be the case at each of the three locations described, though the weights would now be different.

Both weight and inertia are common everyday experiences. It is the weight which makes our arms tired when we carry a child; it is inertia which causes hurt when a child falls over. However, weight is much easier to measure (e.g., by bathroom scales). Given that the weights of two bodies are proportional to their masses, if both are the same distance from the centre of the earth, which for everyday situations they are, then the easiest way to measure a body's mass is to compare its weight with that of a standard mass. So for both these reasons, we shall approach the concept of mass through the experience of weight, and the measurement of mass by the measurement of its weight.

In this network, the hardest concepts come at the beginning and I would not expect children to grasp them in the form outlined above. My aim in the activities which follow is not to sweep anything important under the carpet, but to keep it visible though not necessarily fully discussed. The children will then, I hope, not have anything to unlearn later.

Meas 4.4 MEASURING MASS BY WEIGHING, NON-STANDARD UNITS

Concept Weighing as a way of measuring mass.

- Abilities*
- (i) To measure the mass of a given object in terms of non-standard units.
 - (ii) To compare the masses of two or more objects, using non-standard units.

Discussion of concepts Mass and weight are not the same, but, because they are very closely connected, when we compare weights of objects we are also comparing their masses.

The need for standard units has already been shown in the LENGTH network, Meas 1. In view of its importance, a similar sequence is followed here, in a more condensed form. The present topic is used to underline the utility of weighing, by introducing it in a problem-solving situation (Activity 1). Activity 2 again calls attention to the importance of standard units.

When measuring length, we combine units by putting them end to end. For area, we need units which tessellate, to cover a surface with no gaps and no overlap. Combining unit weights turns out to be straightforward, since if we put them in the same pan of a balance the gravitational forces on all of them are combined into a single downward force.

Activity 1 Problem: to put these objects in order of mass [Meas 4.4/1]

A group problem-solving activity. Its purpose is to lead children to the concept of weighing as a means of solving a problem which has been presented to them.

- Materials*
- An assortment of eight or more objects, most of them clearly distinguishable in weight, but with (say) three of the same weight. All should look different but with as little connection as possible between appearance and weight. So there should be small heavy objects and large heavy objects, small light objects and large light objects (i.e., 'light' and 'heavy' . . . relative to the other objects). All should be light enough for children to lift easily.
 - A suitable set of unit objects in a transparent bag (see *Notes* below). Collectively, these must be heavier than the heaviest objects in the assortment just described.
 - A balance.
 - Pencil and paper
 - Two envelopes containing hints (see steps 4 and 6 below and the photomasters).

- Notes*
- (i) It is not easy to think of unit objects for mass which can readily be found in schools and which fulfil the requirements of being all of the same mass, fairly heavy for their size (the plastic cubes so readily available are much too light) and about 50 to 100 grams in mass so that we can weigh up to 1 kilogram with up to 20 of them. After much thought, my suggestion is that you buy a bag of bolts (the metal kind which screw on to nuts) from a hardware store. It should be possible to find some of suitable size for our present purpose. If you think of something better, I hope that you will share it with me and other teachers.
 - (ii) You will need another bag of bolts (or whatever you choose) of slightly smaller mass for Activity 2, so I suggest that you read this activity also before shopping.

- What they do*
1. Give them all the materials listed, with the bag of bolts included among the others.
 2. The problem is to put all of these objects in order of mass.
 3. To do this by comparing them in pairs would be possible, but laborious. If they start out this way, it might be worth letting them find out how laborious this is, before asking if they can think of a better way.
 4. If they find themselves stuck, they may ask you for an envelope containing a hint.
 5. The first hint reads: “You may open the bag of bolts.” This suggests that they may have some special significance, and implies the possibility of using one or more of these separately.
 6. If after a time they are still stuck, they may ask for another hint. This reads: “Choose one of the objects, and see how many bolts it takes to balance this.”
 7. This, of course, as good as tells them the solution. After this, it is straightforward to order the numbers of bolts equal (or approximately equal) in mass to each object, and put these numbers in order.
 8. (Optional) Ask them to write a short account of the ways they tried, and how measurement was found to be the key to success.

Activity 2 Honest Hetty and Friendly Fred [Meas 4.4/2]

An activity for a small group of children. Its purpose is to call attention to the need for standard units, and thus prepare for the next topic.

- Materials*
- Twelve or more large potatoes (six or more for each stall).
 - Two balances.
 - The bag of bolts used in Activity 1, for Honest Hetty.
 - A similar bag, in which the bolts are like the others but lighter, for Friendly Fred.
 - Notices as illustrated below.*
- * Provided in the photomasters.

**COME TO HONEST HETTY
FOR YOUR
DELICIOUS BAKED POTATOES
CHOOSE YOUR OWN
AND PAY
ONLY ONE CENT
FOR EACH BOLT MASS**

**COME TO FRIENDLY FRED
FOR YOUR
FAVOURITE BAKED POTATOES
CHOOSE YOUR OWN
AND PAY
ONLY ONE CENT
FOR EACH BOLT MASS**

Meas 4.4 Measuring mass by weighing, non-standard units (cont.)

- What they do*
1. Honest Hetty and Friendly Fred are baked-potato sellers in the same mall. They lay out their stalls, with potatoes, balance, and the bolts which will be used for weighing, with their notices prominently displayed. The other children act as customers.
 2. Some customers are served by Honest Hetty and some by Friendly Fred.
 3. After several purchases from the two stalls, several of the more enterprising customers get together to find out which of the two stalls gives the most for the money, or whether they are both the same. They need to work out a method for doing this.
 4. Finally a report of the findings is given.

Discussion of activities Both of these activities involve plenty of social interaction and discussion. The practical usefulness of weighing is brought out in Activity 1, and the social importance of reliable measures is shown in Activity 2. Both of these, especially the latter, are small scale counterparts of applications of weighing which are important in everyday life.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Honest Hetty and Friendly Fred [Meas 4.4/2]

Meas 4.5 STANDARD UNITS (KILOGRAMS)

Concept The kilogram as a standard unit for measuring mass.

Abilities To estimate, measure, and compare the masses (weights) of objects using kilograms.

Discussion of concept As seen in the last topic, non-standard units may be useful for comparing the masses of two or more objects directly, but when it comes to communicating in the world of commerce, standard units are essential. For every day buying or selling by weight (mass), the kilogram (kg) is often the unit of choice. The following activities focus on what a kilogram is and its usefulness.

Activity 1 Making a set of kilogram masses [Meas 4.5/1]

A small group activity to familiarize children with the weight of a kilogram and to produce an economical set of kilogram masses for the following activity.

Materials For each group:

- A sturdy equal-arm, two-pan balance.
- A standard one kilogram mass (or 1 kg substitute).
- Seven empty, resealable containers which can hold up to 1 kg of sand (e.g., 500 mL milk cartons or small self-closing plastic freezer bags). [An alternative to the seven 500 mL milk cartons: one 500 mL, one 1 L and one 2 L carton]
- At least seven kilograms of sand, Plasticine, or other suitable material.
- A tablespoon or equivalent.
- Masking tape.
- Marking pen.

- What they do*
1. A single standard one-kilogram mass can be used to make a set of objects for weighing up to 7 kilograms, as follows.
 2. Each child carefully spoons sand into a resealable container on one pan of the balance until the standard kilogram mass on the other pan is balanced exactly. This is repeated until the group has seven newly-made 'kilograms.'
 3. Six of the newly-made kilograms are then taped together in pairs. (Alternatively, the newly-made 1-kg mass can be placed on the pan with the standard 1-kg mass so that a 2-kg mass can be made by placing sand in a suitable container.)
 4. Finally, two of the pairs of the 'new' 1-kg masses are taped together, resulting in a home-made set of 1, 2 and 4-kilogram masses. (Or the alternative described in step 3 can be taken a step further to make a 4-kg mass.)
 5. Question: What masses can be balanced using these 1, 2 and 4-kilogram masses?
 6. The children are then asked to find objects that have masses that they think can be weighed using their home-made weights and the balance. For each, they are asked to record an estimated weight (e.g., "more than 2 kg"), and then to use the balance and the 1, 2, and 4 kilogram masses to check their estimates.

Activity 2 Mailing parcels [Meas 4.5/2]

An activity for about four children to find out the cost to mail a parcel when its mass, rounded up to the nearest kilogram, is known.

- Materials*
- A sturdy equal-arm, two-pan balance.
 - At least one set of 1, 2, 4-kilogram masses (as made in Activity 1).
 - Suitable objects to wrap for ‘mailing’ (e.g., textbooks) to produce packages with masses in the range 1 to 7 (or more) kilograms.
 - Plain wrapping paper.
 - A ‘small packages’ postal rates chart.*
 - Marking pen.

* A sample is provided in the photomasters.

- What they do*
1. Ask each of the children to wrap a parcel (or parcels) to send to someone they know (arranging for a suitable range of masses, if possible).
 2. One child acts as postal clerk and each of the others presents a parcel for mailing. The clerk ‘weighs’ the parcel, using a marking pen to record its weight, to the next higher whole kilogram, on the parcel. Then, a chart like the following is used to determine the postage needed (typical rates within the same postal code area).

Mass (up to and including)	Postage
1 kg	\$2.80
2 kg	\$2.80
3 kg	\$3.16
4 kg	\$3.52
5 kg	\$3.88
6 kg	\$4.24
7 kg	\$4.60
8 kg	\$4.96
9 kg	\$5.32

3. Each child takes a turn as the postal clerk. Any differences in mass or postal rate for a given package should be discussed.

Discussion of activities Both of these activities provide Mode 1 experience with the kilogram as a unit of mass as well as Mode 2 discussions regarding accuracy and applications. The 1, 2, 4, . . . pattern and ideas of using materials of different density for ‘filler’ for the homemade kilogram objects may lead to some interesting Mode 3 extrapolations.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 4.6 THE SPRING BALANCE

Concept The spring balance as a convenient device for measuring mass.

Abilities To measure and compare the masses of objects in kilograms using a spring balance.

Discussion of concept The “spring balance” is not a balance at all and is often referred to, more appropriately, as a “spring scale.” It consists of a pan or hook that hangs from a spring or a pan or platform that rests on a spring. The object being weighed stretches or compresses the spring, moving a pointer relative to a scale to indicate the mass of the load. To begin with, the children should have the opportunity to verify that the scale readings agree with the values that they have given their homemade masses.

Activity 1 Checking a spring balance [Meas 4.6/1]

An activity for two children to check the reliability of scale balance readings.

- Materials*
- A spring balance suitable for measuring mass in the 1 to 10 kg range.
 - At least one set of 1, 2, 4-kilogram masses (as made in Meas 4.5/1).

- What they do*
1. The children cooperatively verify whether or not the spring scale readings agree with the masses of the homemade kilogram masses.
 2. If there are discrepancies, appropriate adjustments should be made before proceeding to the next activity (including ‘topping up’ any sand spilled).

Activity 2 Mailing parcels (spring balance) [Meas 4.6/2]

A repeat of Meas 4.5/2, finding the cost to mail a small parcel when its kilogram mass is found using a spring balance.

- Materials*
- A spring balance suitable for the 1 to 10 kilogram range.
 - The wrapped objects used in Meas 4.5/2.
 - The ‘small packages’ postal rates chart used in Meas 4.5/2.*
 - Marking pen.
- * A sample is provided in the photomasters.

- What they do*
1. As in Meas 4.5/2, one child acts as postal clerk and each of the others presents a parcel for mailing. The clerk weighs the parcel using the spring scale, noting the whole kilogram reading next above the position of the pointer. This is compared with the mass previously marked on the parcel. A chart like the one used in Meas 4.5/2 is used to confirm the postage needed.
 2. Each child takes a turn as the postal clerk. Any differences in mass or postal rate for a given package should be discussed.

Discussion of activities Though Activity 2 only requires whole-kg measures, the spring scale very likely provides for measurement of masses between whole kilograms, which can be used to set the stage for the introduction of ‘grams’ in the next topic.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 4.7 GRAMS, TONNES

- Concepts*
- (i) The gram as a standard unit for relatively precise mass measurement.
 - (ii) The tonne as a standard unit for measuring relatively large masses.

- Abilities*
- (i) To estimate, compare, measure, and order the masses of objects, to the nearest 10 grams.
 - (ii) To appreciate the size of a tonne mass and applications in which it is used.

Discussion of concepts

Scale balances suitable for use by elementary school children are not likely to be sensitive enough for reliable comparison of masses that differ by less than 10 grams. Some appreciation can be gained for the small size of a single gram (4 pennies have a mass of about 10 g) and for the usefulness of expressing masses in grams, even with reasonable expectations of precision only to nearest 10 g multiples.

A mass of 1000 grams is, of course, referred to as a kilogram, with which the children have had some experience by now. The following activities provide experiences with masses smaller than a kg and then lead to an extrapolation in the other direction. What is a 1000 kg mass called? What would be measured in tonnes?

Activity 1 “I estimate __ grams.” [Meas 4.7/1]

An activity for pairs in which 10 g and 100 g masses are made from Plasticine by balancing them against commercially-made masses. Then the masses of various objects are estimated and measured to the nearest 10 g.

- Materials*
- A sturdy equal-arm, two-pan balance.
 - A set of commercially-made gram masses.
 - At least 1 kg of Plasticine.
 - A collection of objects of various masses in the range 10 g to 1000 g.

- What they do*
1. A 10-gram mass is placed on one pan of the balance and a ball of Plasticine on the other. Plasticine is added to or removed from the ball until the two masses balance. The procedure is repeated until there are ten 10-g objects.
 2. Then, using a similar procedure, 100 g of standard mass(es) are placed on one pan and balanced by a single ball of Plasticine in the other pan. This is repeated until there are 8 more 100-g Plasticine balls.
 3. Question: Using all of the standard masses and all of the Plasticine masses, what is the largest mass that can be weighed?
 4. The masses of each of the objects in the collection are *estimated* and then *measured* (using the balance), *recorded*, and *compared* in a table like the one following. Comparisons are also made with measurements recorded by others.

Object	Estimated mass grams (g)	Measured mass grams (g)

5. Which of the objects is heaviest? . . . lightest? Can you graph your findings?

Activity 2 “How many grams to a litre? It all depends.” [Meas 4.7/2]

An activity for four providing further practice in estimating, measuring, and ordering mass in grams . . . as well as an experience with relative density.

- Materials*
- A sturdy equal-arm, two-pan balance.
 - A set of commercially-made gram masses.
 - The Plasticine masses made in Activity 1.
 - Two one-litre containers (exact capacity).
 - Appropriate quantities of materials of various densities, for example: styrofoam ‘peanuts’ (packaging material), puffed rice (or wheat), rice, marbles, beans, water.

- What they do*
1. Ask each pair to use a chart like that in Activity 1 to record their estimates and then their measures of the mass of *one litre* of three of the provided materials.
 2. They pool their findings, listing the materials in order from least to greatest mass.

Activity 3 A portion of candy [Meas 4.7/3]

An activity for 4 children. Its purpose is to consolidate their new ability to estimate and measure mass in grams.

- Materials*
- Several different kinds of ‘candy’ (whatever candy-sized objects are available).
 - Small bins or containers for the ‘match and mix’ candies
 - Small self-sealing plastic bags.
 - An equal-arm balance and a set of gram masses.
 - A pack of portion cards on which are written masses in multiples of 5 from 15 g to 60 g.*

* A suitable set is provided in the photomasters

- What they do*
1. Before beginning, a display of ‘match and mix’ candies is arranged in appropriate containers. As a special promotional event in the candy store, the players can win a portion of candy.
 2. Players take turns being the *checker*. The rest are *choosers*.
 3. Each *chooser* in turn takes the top card from the face-down pile of portion cards.
 4. Having taken a card, the *chooser* selects, by ‘matching and mixing,’ an amount of candy that she thinks will come close to, but not exceed, the mass on the portion card.
 5. The *checker* weighs the candy and informs the *chooser* of the measured mass. If the amount chosen is within 10 g of the measured mass, the chooser keeps the candies. If not they are returned.
 6. After an agreed number of turns, the winner is the player with the most ‘candy.’

Activity 4 The largest animal ever [Meas 4.7/4]

An activity for a small group of children working together. Its purpose is to introduce the tonne (t) as an appropriate unit when measuring and comparing masses of thousands of kilograms.

Materials • A good encyclopedia.

Suggested outline for discussion

1. Discuss with the children just how small a gram is and how useful a kilogram is for measuring the masses of many everyday things. How many grams in a kilogram?
2. Does anyone know the name of the standard mass that is 1000 kilograms? (If not, introduce the tonne, sometimes referred to as a 'metric ton.')
3. What things do you know about that have a mass of several thousand kilograms? (E.g., whales, elephants, dinosaurs, light rail transit cars, weights and capacities of vehicles, boxcars, ships, . . .) With masses that are so large, the tonne is a useful unit.
4. What is the largest animal that you know about? (One encyclopedia entry asserts that the whale is the largest animal that has ever lived, including prehistoric dinosaurs.)

What they do

1. In small groups, the children make lists of large animals, ordering them from those that they think have the greatest mass to those they think have the least.
2. They check their lists by reference to, say, an encyclopedia, where they will find masses of whales, dinosaurs, and elephants, at least, given in tonnes (also called 'metric tons,' especially in U.S. publications). Encourage them to make a chart with the masses in tonnes (t) and in kilograms (kg). [A blue whale, for example, can have a mass of up to 150 t or 150 000 kg.]
3. They discuss their predictions and findings and their understandings of the relative size of a tonne.
4. There might be interest in finding out the weight of a family vehicle and its load-carrying capacity, perhaps comparing minivans with regular cars.

Discussion of activities From the tiny gram to the massive tonne (and beyond, in each direction), the metric system provides logically interrelated units suited to a wide range of masses and levels of precision. As with all measurement tasks, the choice of an appropriate standard unit of measure and a suitable measuring device is central to the measuring process.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 4.8 RELATIONS BETWEEN STANDARD MEASURES

Concept That of numerical relations between different units of mass.

Ability To convert in both directions between grams, kilograms, and tonnes.

Discussion of concept Examples of this concept have already been encountered in the previous topics. In this topic we make the concept explicit and organize particular cases in the form of a table.

Activity 1 Completing the table of units of mass [Meas 4.8/1]

An activity for children to do individually. Its purpose is to summarize what they have learned about the relations between units of mass.

Materials • Pencil, paper, and ruler for each child.

- What they do*
1. Each child copies and completes the table illustrated below.
 2. When completed, they compare results and discuss any disagreements. If these cannot be resolved, they consult their teacher.

TABLE

grams g		kilograms kg		tonnes t
	=	1	=	
	=		=	1
1	=		=	

This is what they should get.

TABLE

grams g		kilograms kg		tonnes t
1 000	=	1	=	.001
1 000 000	=	1 000	=	1
1	=	.001	=	.000 001

Meas 4.8 Relations between standard measures (cont.)

Discussion of activity Completing the table of units of mass should not only help organize and consolidate the numerical relationships among them but also can provide a Mode 3 context for noticing and reflecting upon the visual non-symmetrical character of the symbols for, say, ‘one thousand’ and ‘one thousandth.’

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Meas 5] **TIME**

MEAS 5.6 LOCATIONS IN TIME: DATES

Concept 'Where we are' in time.

Abilities (i) To find and name given dates in a calendar.
(ii) To find how long a time it is from one date to another.

Discussion of concepts If we think of our passage in time as movement along a road, the date tells us where we are to the nearest day, and the time of day tells us within that day. The present topic deals with the first of these.

A calendar is a table of days in which each day is given a unique name by specifying the year, month, and day of the month. This name is called its date. In addition, it keeps track of the days of the week, within which much of our daily life is planned for work and leisure. And it enables us to work out how long in time it is between two given dates.

Activity 1 What the calendar tells us [Meas 5.6/1]

A teacher-led activity for a group of any size. Its purpose is to familiarize children with the processes of naming and locating dates in the calendar.

Materials

- A sufficiency of calendars.*
- If the print is small, they also need something to point with.

* If possible one per child, otherwise one between two. For a large group, it is useful to have also a wall calendar, large enough for all to see, and a pointer.

Suggested outline for the activity

1. Before the activity begins, have today's date written on the blackboard. (Many teachers make a daily practice of this.)
2. Ask someone to read this aloud, and ask what it tells us. (It tells us which day of the week we are in, which day of the month, which month of the year, and which year. That is to say, it tells us where we are in time and gives it a name. This kind of name is called a date.
3. Tomorrow we shall have moved on a day, so what will the date be then?
4. And what was the date yesterday?
5. Ask the children to find today's date in their calendars. If they need help, tell them to begin by finding the right month; Then, the number for the day of the month.
6. Point out how when we have found the right date in the month, the calendar also tells us which day of the week we are in. So we can check one against the other.
7. Ask them if they can find out from their calendars what the date will be one week from today. (If this takes them into another month, do step 8 before this one.)

Meas 5.6 Locations in time: dates (cont.)

8. Ask them to find what was the date a week *back* from today.
9. Repeat for two, three weeks *forward* and *back*. It becomes harder when the month changes.
10. What day of the week will it be one month forward from today? That is to say, on the day with the same number as today, next month. And one month back from today?
11. Likewise for other numbers of months forward and back.
12. Ask if they know the dates (month and day) of their birthdays. They can then use their calendars to find what day of the week it will fall on next.
13. Repeat for other important dates.

Activity 2 “How long is it . . . ?” (Same month) [Meas 5.6/2]

An activity for a small group of children, which they should be able to do on their own after the preparation they have had in Activity 1. Its purpose is to give further practice in finding their way around in a calendar. They may need help in getting started.

Materials

- As for Activity 1.

- Pencil and paper for each child.

- What they do*
1. One child chooses a date between the present day and the end of the month, and asks (e.g.) “How long is it until the 27th of this month?”
 2. They all count forward in weeks, and then the extra days. E.g., if ‘today’ was the 4th of the month, they would count, starting from 4 on the calendar, “1 week” (to the 11th), “2 weeks” (to the 18th), “3 weeks” (to the 25th), “and two days.”
 3. When they have all written down their answers, they compare results and discuss any disagreements.
 4. After all have had a turn, steps 1, 2, and 3 may be repeated for a date earlier in the month. In this case, the question asked would be “How long is it since . . . ?”

Activity 3 “How long is it . . . ?” (Different month) [Meas 5.6/3]

An extension of Activity 2. The steps are exactly the same, except that in step 1 the starting child chooses a date in the same year but a different month. E.g., she might ask how long it is until her next birthday, if it is not yet past; or since her last birthday, if it is.

Activity 4 “How long is it . . . ?” (Different year) [Meas 5.6/4]

A further extension of the previous two activities. An easy beginning would be to ask “How long is it since the day I was born?” They already know how many years this is, so it is only a matter of counting forward in months and days since their last birthdays. Other possibilities are historical events, preferably ones which have some interest for the children.

Computer programs are available which will print a calendar for any required year, past, present or future, and it adds to the interest if these can be made available for this activity. They can then find on which day of the week the event took place, as well as how long ago it was.

Future as well as past events should of course be included. It may be useful to have some suggestions ready in case they are short of ideas.

Discussion of activities The calendar is an excellent representation of the combination of three separate orderings by which we organize our experience of the passage of days, and give names to them. These have already been dealt with separately in previous topics on time in *SAIL Volume 1*, and the present topic brings them together. Giving them activities which involve repeated use of the calendar should help consolidate children’s knowledge of the relationships between days of the week, days of the month, and months of the year, and develop their ability to put this knowledge to use.

OBSERVE AND LISTEN

REFLECT

DISCUSS



“How long is it . . . ?” (Different year) [Meas 5.6/4]

Meas 5.7 LOCATIONS IN TIME: TIMES OF DAY

- Concepts* (i) Where we are during a day.
(ii) The clock as an instrument for measuring the time of day.

Abilities To tell the time of day by the clock, in both digital and analogue notations.

<p>Discussion of concepts The preceding topic was concerned with location in time to the nearest day. In the present topic, we continue the development of this concept to specify locations within the time stretch of a single day. The clock replaces the calendar as the means whereby we do this, and the symbolism by which we describe and record it.</p>

Activity 1 “How do we know when to . . . ?” [Meas 5.7/1]

A teacher-led discussion for a group of any size. Its purpose is to underline the importance of being able to tell the time, as a way of fitting one’s own activities in with those of others.

Materials None required.

- Suggested outline for the discussion*
1. Ask “How do we know when to . . . ?” This can be anything which must be done at a specific time, such as catch the bus for school, switch on the television for a favourite program, come home from visiting a friend. In some cases, e.g., end the lesson and go out for recess, the answer may well be that someone else says so: e.g., most children will be told by a parent when it is time to get up in the morning. In this case, we want to know how they know. If bells ring at programmed times in your school, you need to be prepared for the answer “Because the bell rings.”
 2. In nearly every case, the answer eventually comes back to because it is a particular time of day. It was in order not to imply this answer in the wording of the question that I did not name this activity “How do we know when it’s time to . . . ?”
 3. “So when do we know when it is . . . (whatever time is given)?”
 4. The answer will nearly always be “By looking at a clock or watch.”
 5. So we need to be able to read the time from a clock or watch, if we cannot already.

Activity 2 “Quelle heure est-il?” [Meas 5.7/2]

A teacher-led discussion for a group of any size. Its purpose is to introduce children to the analogue clock face.

Materials • An analogue clock face, with a movable hour hand only. This needs to be large enough for the whole group to see easily.

Suggested outline for the discussion

1. Explain that long ago, people only needed to tell the time to the nearest hour, so when they wanted to know the time they used to ask “What hour is it?” In French this is how they still ask what the time is.
2. So the early clock faces only showed hours, and this is how we are going to begin.
3. Earlier still, before there were any clocks, there was one time of day which they could tell by the sun (if it was shining). Can anyone say what this might be?
4. If anyone suggests sunrise or sunset, these are good ideas, but why won't they do? (These change from day to day.)
5. What doesn't change even though the days get shorter or longer is the middle of the day when the sun is halfway between rising and setting. This is when it is highest in its path across the sky. (Later, you might introduce the word ‘zenith.’)
6. So they divided the day into two equal parts, before midday and after midday. When writing, these are often written ‘a.m.’ and ‘p.m.’ for short. ‘A.m.’ stands for ‘*ante meridiem*,’ which is Latin for ‘before midday.’ ‘P.m.’ stands for ‘*post meridiem*,’ which means . . . ? Which is it now, a.m. or p.m.?
7. There are twelve hours in each half day. The a.m. half starts at . . . ? And ends at . . . ? And the p.m. part starts at . . . ? And ends at . . . ?
8. Now show the clock face, with the hand pointing to the actual time, to the nearest hour. Ask “Quelle heure est-il?”, or “What hour is it?”, whichever you prefer. Since we are imagining ourselves to be in the position of people long ago, they might like to answer as they did, and say (e.g.) “It is eleven of the clock.” Later this will be shortened to “o'clock.”
9. The next question is how we know whether the hour shown on the clock face is a.m. or p.m. This is usually clear from where we are and what we are doing. In the foregoing example, where would they be if it was eleven p.m.? We hope, in bed and asleep! If it is clear which we mean, we do not need to say a.m. or p.m. unless we want to.
10. They are then ready to practise reading the hour from the clock face, with the hour hand always pointing to an exact hour. It should not take long for them to become fluent at this, after which they will be ready for the next activity.

Activity 3 Hours, halves, and quarters [Meas 5.7/3]

A continuation of Activity 1, which uses the minute hand to show halves and quarters of an hour.

- Materials*
- The same clock face as used for Activity 2.
 - A second one, similar but with a minute hand as well as an hour hand.

- Suggested outline for the discussion*
1. Use the first clock face to explain that the hour hand moves slowly round the dial, taking an hour to get from one figure to the next.
 2. Position the hour hand halfway between two hours, and ask what they would call this time.
 3. Accept any sensible answers. Now show the second clock face, and explain that this helps us to tell the time more exactly. The longer hand makes a complete turn in every hour.
 4. Start the minute hand at 12, and ask: “Starting here, what part of a turn has it made when it gets to here?”, moving it to the figure 6. (Half a turn.) “So how long after the hour does it show?” (Half an hour.)
 5. Put the hour hand half way between any two figures, say, 9 and 10, with the minute hand at 6, and ask what time this is showing. “Half an hour after nine” is a good answer, which may be shortened to “Half past nine.”
 6. Move the hour hand to half way between two other figures, and ask “What time does the clock show now?” Repeat until they are fluent, which should not take long.
 7. Repeat steps 4, 5, and 6, as before, except that the minute hand now shows a quarter after the hour.
 8. Repeat steps 4, 5, and 6, as before, except that the minute hand now shows a quarter before the next hour. Though “A quarter before . . .” is the usual answer, I think that we should also accept “Three quarters after . . .”, first, because it is also correct, and, second, because it corresponds to the way it would be both spoken and written in hours and minutes, e.g., 9:45.
 9. Finally, repeat the above with the minute hand at 12. Explain that when the time is right on the hour, we read this as (e.g.) “Three o’clock.”
 10. On a subsequent occasion, the children may usefully practise reading the clock face as above, with the minute hand in one of the four positions above. They may take it in turns to set the hands while the others read the time.

Activity 4 Time, place, occupation [Num 5.7/4]

An activity for a small group of children. Its purposes are to give further practice in reading the time from a clock face, and to relate times of day to a wider context of where they go and what they do.

- Materials*
- A clock face, with movable hour and minute hands.
 - A two-sided card, with ‘a.m.’ on one side and ‘p.m.’ on the other.

- What they do*
1. One child has the clock face, and sets it to a time in which the minute hand is at one of the four quarters and the hour hand is in approximately the right position relative to the minute hand. The two sided card may be either side up, but for the first round the time should be earlier than the present time.
 2. He first asks the others “What time is this?”
 3. When they have answered, he then asks the other children in turn, “Where were you at *this* time (pointing to the clock) today? And what were you doing?”
 4. When all have answered, the turn passes to the next child who repeats steps 1, 2, and 3, as before, except that the time shown is later than the present time, and the questions in step 3 are changed to “Where will you probably be at *this* time (pointing to the clock) today? And what will you probably be doing?”
 5. The next child sets the time to any desired time of day, the questions in step 3 being changed to “Where were you at *this* time (pointing to the clock) yesterday? And what were you doing?”
 6. The next child does likewise, but asks “Where will you probably be at *this* time (pointing to the clock) tomorrow? And what will you probably be doing?”
 7. Four children will now have acted as questioner. If there are others who have not yet had a turn at this, they may make any sensible choice of day for their questions.

Discussion of activities

The object of all this is not only for them to be able to tell the time by the clock, but to understand this within meaningful contexts. Here I suggest two. First is the practical need of an agreed system for describing locations in time, for social cooperation. Many of the ways in which we coordinate our own actions with those of others would fall to pieces without clocks and watches. Because of this practical need, the origin and development of time measurement has a long history. I have tried to show just a glimpse of this.

Both analogue and digital clock faces are now widespread, and some teachers prefer to teach the latter first on the grounds that it is easier. This may be so, though it begins with hours and minutes rather than hours and quarters. Myself, I prefer to start with the analogue clock face, partly for the historical background which I have already mentioned, and partly because for some purposes I find it preferable. The analogue clock face shows intervals graphically, so that it is easy to read off how long it is (e.g.) from 9:45 to 10:20. At, say, 06:55 (digital) the position of the minute hand at 5 minutes before the hour shows directly that the time is coming up to seven o'clock. The digital reading requires this to be inferred from the difference between 55 and 60. Finally, the digital readings give a greater degree of accuracy than is needed for many purposes. When I look at my own watch, it tells me the time to the nearest minute and second, whether I want these or not. (Yes, it is digital, because I like the other facilities which come with this.) But in our living room we have an analogue clock to an old German design.

Children need to be at home with both symbolisms, the spatial and the numerical. Above, I have explained why I have begun with the former, but if you prefer the other way around, the foregoing activities can be taken in a different order.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 5.8 EQUIVALENT MEASURES OF TIME

Concept That the same duration of time can be expressed in different units.

Ability To recognize equivalent measures of time, i. e., the same duration expressed in different units.

Discussion of concept Equivalence between a single larger unit and a particular number of smaller units (e.g., 1 hour is equivalent to 60 seconds) has already been introduced, in ‘Time sheets,’ Meas 5.5/1 in *Volume 1*. Here we expand the use of the concept, to recognize equivalences such as $\frac{3}{4}$ hour and 45 minutes. I have kept to instances which are likely to be met with in everyday life.

Activity 1 Pairing equivalent times [Meas 5.8/1]

A game for a small group of children. Its purpose is to introduce and practise the more frequently used equivalences between days, hours, minutes, and seconds.

Materials • A pack of cards showing pairs of equivalent times (durations). *
* Photomasters are provided for a pack containing 15 pairs.

- What they do*
1. The cards are shuffled and put in a pile face down on the table. The top card is turned face up and put separately.
 2. The players then take turns to turn over the top card from the face-down pile, and put it face up with (not on top of) the others.
 3. If the player whose turn it is sees two cards which show the same duration of time in different units, she claims them.
 4. As the number of face-up cards increases, clearly the chance of there being two equivalent measure increases.
 5. If the player whose turn it is overlooks a pair, the next player may claim it before taking her turn in the normal way. And if this player overlooks it, the same applies to the next player and so on.
 6. The winner is the player with most pairs when all have played out their hands.

Discussion of activities This is a straightforward activity. However, it does encourage mental conversion of each new card as it is turned over, in order to look for an equivalent time among those already showing.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Meas 6] TEMPERATURE

MEAS 6.2 MEASURING TEMPERATURE BY USING A THERMOMETER

Concept The thermometer as an instrument for measuring temperature.

Abilities (i) To use a thermometer correctly for measuring temperature.
(ii) To sequence two or three objects in order of temperature.

Discussion of concept Children will already be familiar with the idea of measuring instruments, such as rulers for measuring length, kitchen scales, speedometers in cars. They will probably also have encountered a thermometer in some form or other — most children have had their temperatures taken by a parent or doctor. So here again, we are consolidating, organizing, and extending their everyday knowledge.

Activity 1 The need for a way of measuring temperature [Meas 6.2/1]

A teacher-led experiment followed by a discussion. Its purpose is to underline the need for a way of measuring temperature which is more accurate than estimation, and to think of other reasons for this. Below I suggest two forms of the experiment. You may like to try others.

Materials • Four containers for water, such as mugs, jam jars,

Suggested procedure for the experiment

1. The four containers are nearly filled with water at three different temperatures. One should contain cold water, chilled if necessary with a lump of ice which should be removed before the experiment. We may call this one **C**. Another, container **H**, should contain hot water, as hot as would be comfortable to wash one's hands in. The remaining two should contain warm water, both at exactly the same temperature. This should be just a little above room temperature. We will call these **W** (for warm). Note that these terms are for our own convenience: the containers should not be labelled with these letters.
2. First put out containers **C** and **H**.
3. A volunteer is now needed. Ask him to put two fingers of his left hand in **C** and two fingers of his right hand in **H**, and say which he thinks is hotter and which is colder. He should keep his fingers in the containers for not less than 20 seconds before the next step.
4. Now give him the two **W** containers and ask him to compare these. (In this case it is better to say "compare" rather than ask which is hotter/colder.)
5. Although they are in fact at the same temperature, one will *feel* warmer to the fingers coming from the cold water.
6. Now ask another volunteer to compare the two **W** containers. He is likely to say that they are both the same.

Meas 6.2 Measuring temperature by using a thermometer (cont.)

7. So here we have one reason why we need a way of measuring temperature more reliably than by feeling. Can anyone think of others? (E.g., too hot or too cold to be safe to touch; inaccessibility; greater accuracy required than would be possible by estimation.)

Variation In this simpler form of the experiment, only one **W** container is used. After step 3, the volunteer puts the fingers of both hands in **W**, which will feel warmer to one hand than to the other. Since this does not take long, more children can take part. May I suggest that in any case you try this for yourself beforehand? It is a little strange to have ones hands sending messages which are contrary to what one knows to be the case.

Activity 2 Using a thermometer [Meas 6.2/2]

An activity for a small group. After a teacher-led introduction, they should be able to continue on their own. Its purpose is to introduce children to the use of a thermometer.

- Materials*
- Two Celsius thermometers, scaled from 0° (or below) to 100° (or above).
 - At least two blocks of different heights. The more of these that can be made available, the better.
 - A number of containers for water, such as mugs, jam jars, . . .
 - Hot and cold tap-water.
 - Pencil and scrap paper for each child.

- Suggested introduction*
1. Fill two of the containers with water at clearly different temperatures.
 2. Have the children agree which is the hotter and which the colder, and let them show this as before by putting them on blocks of different heights.
 3. Now put a thermometer in each of the containers, and ask the children to watch carefully and report what happens.
 4. We hope that at least the nearer ones will observe that the thread of mercury or alcohol changes length, doing this more slowly until it is steady.
 5. Put the thermometers vertically side by side, in the same relative positions. We hope that they will now notice that the greater height in the thermometer corresponds to the higher temperature.
 6. Looking more closely, they can see that the thermometer is marked with a scale. These are degrees Celsius, which are the international units used to measure temperature.
 7. Pass them around, and let each say what the reading is. Unless the liquid was at room temperature, the reading will gradually change. What do they learn from this?
 8. They may now each take a container and fill it with water mixed from cold and hot taps. Working in pairs, one estimates which is hotter and which is colder, and shows this by putting them on blocks.

Meas 6.2 Measuring temperature by using a thermometer (cont.)

9. They then check with thermometers. Since there will certainly not be two of these per pair of children, they will have to make the comparison by reading the temperatures in degrees Celsius which they record on a small piece of scrap paper.
10. These are wetted so that they adhere temporarily to the containers.
11. (Optional further step.) Pairs may then cooperate to put all their containers in order of temperature.

Note The temporary nature of the labels' attachment corresponds well to the fact that the temperatures will not stay the same for long!

Discussion of activities Note that in Activity 2, the children begin by using their own senses to compare temperatures, so that the higher thermometer reading corresponds to what they already know to be the higher temperature. This relates the new experience to their existing schemas.

I have not included any explanation of how a thermometer works, mainly because these are resources for learning and applying mathematics rather than an introduction to science. It will, however, be as well to be prepared with an explanation if children ask. The most common kind of thermometer, and the one which you will need for use in the classroom, is the mercury-in-glass type, with alcohol thermometers available for sub-zero temperatures. Non-liquid thermometers, which use a bimetallic strip, are also fairly common, so you might want to check out this kind too.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Using a thermometer [Meas 6.2/2]

MEAS 6.3 TEMPERATURE IN RELATION TO EXPERIENCE

Concept The relationship between temperatures as shown by a thermometer and everyday experience.

Abilities (i) To say roughly what temperatures are to be expected in a variety of everyday examples.
(ii) Conversely, to say whether a given temperature is a likely one in any given case.

Discussion of concept The concept itself is the same as the children learned in Meas 6.2. Here they are expanding its field of application.

Activity 1 Everyday temperatures [Meas 6.3/1]

A project-like activity for a small group of children to work at cooperatively, as described below.

Materials

- Not less than two thermometers for the group; more if possible.
- Pencil and paper for each child.

What they do

1. Explain that now they know how to measure temperature accurately with a thermometer, you would like them to make a list of everyday temperatures which they could measure.
2. They should start by 'brainstorming' to produce a list of what they want to measure. Here are some suggestions to start with:
 - Inside temperatures, in the classroom and elsewhere in the school.
 - Outside temperatures, at different times of year.
 - Temperatures in the same room at floor level and as near the ceiling as practical.
 - Temperatures of liquids left in shade and sun.
 - Temperatures of ice water, and (under supervision) boiling water.
 - Body temperature using ordinary and (if available) a clinical thermometer.
 - Why is body temperature important?
3. They work in pairs to read and record the temperatures they have listed, having first apportioned this among themselves.
4. Finally they combine their information in a list for the classroom wall.

Activity 2 “What temperature would you expect?” [Meas 6.3/2]

An activity for any sized group of children, working in pairs. Its purpose is to give practice in using and extrapolating their knowledge of likely temperatures, based on the experience gained in Activity 1.

Materials For each pair:

- The query list, see below.*
- Pencil and paper.
- Their rough drafts of the lists made in Activity 1 would be useful, as an aid to memory.

* *Note* This is provided for your convenience, here and in the photomasters, but you may wish to extend this to include material of local and topical interest.

- What they do*
1. They discuss with their partners what would be reasonable answers to the queries on the list, and fill these in on their sheet. In most cases a range of temperatures, or “Around . . .”, would be sensible replies.
 2. All the pairs at the same table may then share their conclusions, and discuss any points of divergence. If these cannot be resolved, they will need to consult you.
 3. As in Activity 1, the final results might make a suitable display for the classroom wall.

Query List

What temperature would you expect to find in the following?

A classroom or living room.

A refrigerator.

A freezer.

An oven ready for cooking.

A shopping mall.

Water from a cold tap.

Water from a hot tap.

A swimming pool.

A hot bath or shower.

A mountain stream.

Meas 6.3 Temperature in relation to experience (cont.)

Activity 3 Temperature in our experience [Meas 6.3/3]

This is a long-term project, for the whole class, to be organized in whatever way you think best. It is based on the idea that temperature is an interesting source of data for environmental studies.

- Materials*
- Suitable display material for the classroom wall, to incorporate the data below as it is collected.
 - Celsius thermometers.
 - An outdoor thermometer, which can be read through the window, is an interesting source of data. A maximum and minimum thermometer would be even more interesting, and could be used to make a graph of year-long changes.

What they do (These are suggestions offered as a starting point.)

Temperatures are recorded and collated, in some cases graphically, for the following temperatures:

Outdoors (locally) in summer, winter, spring, fall, at the same times of day.

Highest temperature recorded, and lowest.

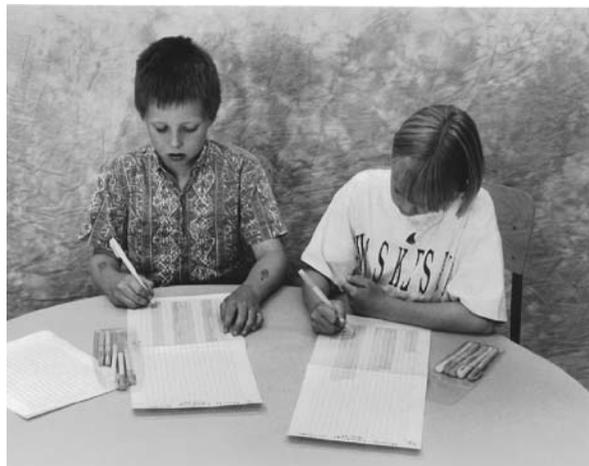
Hourly readings inside and outside for the whole of a school day.

Discussion of activities In Meas 6.1 we based the scientific concept of temperature on children's everyday experience of this. In Meas 6.2, we introduced the instrument (thermometer) and units (degrees Celsius) by which this is measured. In the present topic, children are developing connections in the reverse direction, from temperature as measured with a thermometer back to everyday experience. This continues the overall process by which we accept the importance and validity of the knowledge which children already have, lead them to consolidate and organize it, and then help them to expand it further.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Temperature in our experience [Meas 6.3/3]

GLOSSARY

These are words which may be unfamiliar, or which are used with specialized meanings. The definitions and short explanations given here are intended mainly as reminders for words already encountered in the text, where they are discussed more fully. This is not the best place to meet a word for the first time.

<i>abstract</i>	(verb) To perceive something in common among a diversity of experiences. (adj.) Resulting from this process, and thus more general, but also more remote from direct experience.
<i>add</i>	This can mean either a physical action, or a mathematical operation. Here we use it with only the second meaning, in order to keep these two ideas distinct.
<i>addend</i>	That which is to be added.
<i>base</i>	The number used for grouping objects, and then making groups of these groups, and so on. This is a way of organizing large collections of objects to make them easier to count, and is also important for place-value notation.
<i>binary</i>	Describes an operation with two operands.
<i>canonical form</i>	When there are several ways of writing the same mathematical idea, one of these is often accepted as the one which is most generally useful. This is called the canonical form. A well known example is a fraction in its lowest terms.
<i>characteristic property</i>	Property which is the basis for classification, and for membership of a given set.
<i>commutative</i>	This describes a physical action or a mathematical operation for which the result is still the same if we do it the other way about. E.g., addition is commutative, since $7 + 3$ gives the same result as $3 + 7$; but subtraction is not.
<i>concept</i>	An idea which represents what a variety of different experiences have in common. It is the result of abstracting.
<i>congruent</i>	Two figures are congruent if one of them, put on top of the other, would coincide with it exactly. The term still applies if one figure would first have to be turned over.
<i>contributor</i>	One of the experiences from which a concept is abstracted.
<i>counting numbers</i>	The number of objects in a set. The cardinal numbers, 1, 2, 3, . . . (continuing indefinitely). Zero is usually included among these, but not negative or fractional numbers.
<i>digit</i>	A single figure. E.g., 0, 1, 2, 3, . . . 9.
<i>equivalent</i>	Of the same value.
<i>extrapolate</i>	To expand a schema by perceiving a pattern and extending it to new applications.
<i>higher order concept</i>	A concept which is itself abstracted from other concepts. E.g., the concept of an even number is abstracted from numbers like 2, 4, 6, . . . so <i>even number</i> is a higher order concept than 2, or 4, or 6, . . .
<i>interiority</i>	The detail within a concept.
<i>interpolate</i>	To increase what is within a schema by perceiving a pattern and extending it inwards.

<i>low-noise example</i>	An example of a concept which has a minimum of irrelevant qualities.
<i>lower order concept</i>	The opposite of <i>higher order concept</i> , q.v.
<i>match</i>	To be alike in some way.
<i>mathematical operation</i>	See <i>operation</i> .
<i>Mode 1</i>	Schema building by physical experience, and testing by seeing whether predictions are confirmed.
<i>Mode 2</i>	Schema building by receiving communication, and testing by discussion.
<i>Mode 3</i>	Schema construction by mental creativity, and testing whether the new ideas thus obtained are consistent with what is already known.
<i>model</i>	A simplified representation of something. A model may be physical or mental, but here we are concerned mainly with mathematical models, which are an important kind of mental model.
<i>natural numbers</i>	The same as the counting numbers.
<i>notation</i>	A way of writing something.
<i>numeral</i>	A symbol for a number. Not to be confused with the number itself.
<i>oblong</i>	A plane figure having four sides and four right angles, with adjacent sides unequal. An oblong and a square are two kinds of rectangle, in the same way as boys and girls are two kinds of children.
<i>operand</i>	Whatever is acted on, physically or mentally.
<i>operation</i>	Used here to mean mental action, in contrast to a physical action.
<i>pair</i>	A set of two. Often used for a set made by taking one object from each of two existing sets: e.g., a knife and a fork.
<i>place-value notation</i>	A way of writing numbers in which the meaning of each digit depends both on the digit itself, and also on which place it is in, reading from right to left.
<i>predict</i>	To say what we think will happen, by inference from a suitable mental model. Not the same as guessing. Prediction is based on knowledge, guessing on ignorance.
<i>schema</i>	A conceptual structure. A connected group of ideas.
<i>set</i>	A collection of objects (these may be mental objects) which belong together in some way.
<i>subitize</i>	To perceive the number of objects in a set without counting.
<i>sum</i>	The result of an addition.
<i>symmetry</i>	A relation of a figure with itself, in which there is an exact correspondence of size and shape on opposite sides of a line or around a point. Both line symmetry and rotational symmetry are considered in this book.
<i>transitive</i>	A property of a relationship. E.g., if Alan is taller than Brenda, and Brenda is taller than Charles, then we know also that Alan is taller than Charles. So the relationship 'is taller than' is transitive.
<i>unary</i>	Describes an operation with a single operand.

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SEQUENCING GUIDES

Grade Levels 3 through 6

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A Sequencing Guide for Grade Level 4

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A Sequencing Guide for Grade Level 5

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