



Structured **A**ctivities for Intelligent **L**earning

An elementary school resource book

by

RICHARD SKEMP

Expanded North American Edition

Edited and adapted

by

MARILYN and BRUCE HARRISON

Volume 1

for the early years

STRUCTURED ACTIVITIES FOR INTELLIGENT LEARNING

The author Richard Skemp is internationally recognized as a pioneer in the psychology of learning mathematics based on understanding rather than memorizing rules. A former mathematics teacher, he has lectured in twenty countries, is a former Professor of Educational Theory and Director of the Mathematics Education Research Centre at Warwick University, and a Past President of the International Group for the Psychology of Mathematics Education.

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SAIL THROUGH MATHEMATICS:

**STRUCTURED
ACTIVITIES FOR
INTELLIGENT
LEARNING**

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Volume 1

EEC Ltd.
Calgary

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And finally, but by no means least, yourselves, for allowing our efforts to find their hoped-for destination in your own classrooms.

NOTES

SAIL through Mathematics, Volume 1 is intended for Pre-grade Level 1 through Grade Level 2, and *Volume 2* for Grade Levels 3 through 6. Because of the wide range of children's abilities, this division can only be very approximate. In particular, some of the later activities in *Volume 1* will be found useful for children in Grade Level 3, especially if they come from schools where this approach has not been used. Every administrative grade placement brings together children currently operating at many different developmental levels in mathematics. The *SAIL* program is uniquely designed to accommodate a wide range of abilities.

Since the English language lacks a pronoun which means either he or she, I have used these alternately by topics.



SAIL Activities in the Classroom

A TEACHER'S GUIDE

to Structured Activities for Intelligent Learning

Drawing on ten years of experience using Richard Skemp's structured learning activities with her own classes and with many other teachers and their pupils, Marilyn Harrison has written the following teacher's guide and the accompanying sequencing guides to share insights, ideas, and suggestions with fellow intelligent learning facilitators.

Setting Sail

You are about to embark on an adventure in learning mathematics with understanding. The vehicle for this adventure consists of a set of structured and sequenced mathematics learning activities that have been designed in accordance with contemporary learning theory. Cooperative, schematic learning and prediction from mathematical patterns provide the basis for a powerful and unique approach to problem solving. Clusters of activities enable students to construct networks of mathematical ideas intelligently by building and testing through *direct experience, communication with others, and creative extrapolation.*

The learning activities in the *SAIL* volumes provide opportunities for cooperative group work, discussion, conjecturing, analyzing, justifying, writing, exploring, and applying mathematics. Where appropriate, memorization is encouraged for fluency and efficiency, but never in place of developing reasoning and understanding. The activities are designed for students to learn effectively by doing, saying, and recording. They explore a variety of appropriate methods. Prediction activities occur frequently, providing an excellent context for practicing estimation and reasonable conjecture skills while learning to reason mathematically.

The use of concrete materials in the *SAIL* program is extensive and carefully designed. Exemplary techniques are modelled for drawing the most out of the concrete and the more abstract learning situations.

Suggestions for Smooth Sailing

Classroom Management

The activities are designed to engage students in pairs, in small groups of up to six students, or in whole class discussions. The optimal group size is indicated in the instructions for each activity. If students are not accustomed to working cooperatively in mathematics, their social learning skills need to be considered. They may have to learn such things as listening to each other, taking turns, discussing sensibly, and giving reasons rather than just arguing. The ways of learning mathematics which are embodied in *SAIL* both depend upon, and contribute to, social learning and clear speech. If these traits are already well established, you are off to a flying start.

Introducing the activities

It is the responsibility of a teacher to be familiar with each activity and its purpose before introducing it to students. To introduce an activity to the whole class, gather students in a circle and demonstrate the procedures by leading the activity with one or two students. Then have them do the activity in appropriate groups. Instruct them to check that all the materials are at hand before they begin and that all the materials go back in their proper places at the end. It is worth the effort to establish well-defined routines from the beginning and most students appreciate the importance of keeping everything in place for those who will next do this activity. As the activity period concludes, it is important that the teacher and the students together reflect on and discuss their discoveries.

An alternative to introducing a new activity to the whole class would be to initiate it with a small group of students while the others are engaged in activities which they have already learned. This provides an opportunity to give 'high quality input' to small groups. One of the advantages of organizing your classroom this way is that the other students are learning on their own, doing their own thinking. This kind of teaching includes ways of managing students' learning experiences which are less direct, more sophisticated, and more powerful than traditional approaches.

You will find that the activities fall into two main groups: those which introduce new concepts, and those which consolidate them and provide a variety of applications. Activities in the first group always need to be introduced by a teacher to ensure that the right concepts are learned. Once they have understood the concepts, students can go on to do the consolidating activities together with relatively little supervision. These activities could be introduced by another adult helper. In some cases students, especially older students, can teach it to others with the help of the printed rules.

A set of 39 videoclips of the *SAIL* activities and of the theory which they embody has been produced. Each of the 3- to 11-minute activity videoclips models the introduction of a *SAIL* activity to a small group of children. Five of the videoclips demonstrate classroom management skills when the activities are used with a whole class. A 60-minute *Discussion Time with Richard Skemp* video provides a comprehensive overview of his theory of intelligent learning, illustrated with sample learning activities.

Charting the Course

Suggested sequencing

To help sequence the activities, especially for teachers new to the program, a suggested sequencing has been included on fold-out flaps on the covers of this book: Pre-grade 1 and Grade Level 1 at the front; Grade Level 2 on the back. The pre-grade 1 sequencing includes concepts appropriate for most 4- and 5-year-olds; Grade Level 1, for most 6-year-olds; and Grade Level 2, for most 7-year-olds. There are many possible routes through the Networks. The suggested sequencing, though not unique, is offered as an aid for those wishing to be assured that the curricular expectations of each level are covered.

Materials

Many of the materials are found in most schools. The following list will enable you to quickly check which materials may be needed.

Collectables

Enlist the help of children and parents to provide the following:

- thin cord or yarn for set-loops
- little objects for sorting (natural objects – we do not want children to think that mathematics only involves plastic cubes): buttons, sea shells, beans, pebbles, bottle tops, keys, pasta, bread tags, nuts and bolts, seeds, crayons, screws (store in small containers, boxes or see-through plastic bags)
- plastic animals and people (e.g., toy pigs are needed for the Space 1.4 activities; a model goat for Meas 1.3/2)
- sorting trays – trays with partitions; egg cartons; box lids
- catalogues and magazines for assembling pictures
- shaker for dice (though rolling the die without a shaker is fine)
- empty paper towel tubes
- containers: egg cups (or 35 mm film containers); identical drinking glasses; an assortment of glasses, jugs, jars, varying as widely as possible in height and width; two identical containers of capacity equal to or greater than any of the preceding
- a collection of objects which approximate in shape to spheres, cuboids, cylinders, and cones; and others which are none of these.

Purchased materials

- interlocking cubes – 1 cm (e.g., Centicubes)
- interlocking cubes – 2 cm (e.g., Unifix or Multilink cubes)
- attribute blocks
- spinner marked 'straight line'/'curved line' or die with 'straight line' written on 3 faces and 'curved line' on the other 3 faces
- opaque bag for hiding counters (e.g., a lunch bag or, better still, a cloth bag)
- '1-6' dice (Grade 1: two different colours; Grade 2: three different colours)
- '1-9' dice (Grade 2: '0-9' dice with '0' covered)
- dice bearing only 1's and 2's (e.g., cover the faces with sticky paper and write '1' on half of them and '2' on the other half)

Teacher's guide

- dice bearing only 'open' and 'closed'
- 1¢ coins (K); also 5¢, 10¢, 25¢, and \$1.00 coins (Grades 1 and 2)
- handkerchief or paper towel for hiding objects (Grade 1)
- milk straws and popsicle sticks (Grade 2 only)
- base ten blocks – ones, tens, hundreds (Grades 1 and 2)
- geometric models of spheres, cuboids, cubes, cylinders, cones, pyramids & prisms
- rubber stamps of animals or other pictures
- stamp pads
- plasticine
- non-permanent markers
- squared paper
- hundreds, tens, and ones charts
- rulers numbered in centimetres
- a metre rule
- paper clips – small, medium, large (1 box of each)
- 6 wooden rods 20 cm long; 6 wooden rods 24 cm long
- 3 model trucks (could be made from *Lego* in various widths and heights)
- resealable plastic bags (e.g., *Ziploc* freezer bags)
- balances
- bolts (see Meas 4.4/2)
- analogue clock face with hour hand
- analogue clock face with hour and minute hands
- Celsius thermometers
- calendars

Preparing the activity cards

Two routes are available. You can duplicate the cards from the photomasters in Volumes 1a or 2a or purchase sets of prepared cards. To keep the card preparation manageable, it is recommended that you work on one Network or one Level at a time.

Perhaps the best way to keep all the materials for each activity together is to store them in a labelled plastic bag or a cardboard box. Plastic bags can be stored upright in a suitably sized open cardboard box. Alternatively, the bags can be suspended from pegboards with hooks. Some teachers find it useful to have with each activity a copy of the relevant 'Concept,' 'Ability,' 'Materials,' and 'What they do' sections of the activity instructions. These may be mounted on coloured card with the relevant 'discussion boxes' included on the back. The instruction activity cards could be colour coded by Network.

i) Preparing the materials from the Photomasters:

The preparation of materials from the Photomasters is described in Volume 1a, page ix. Labelling the bags and underlining in red the materials (such as base 10 materials) which are not kept in these individual bags indicates what else has to be collected at the start of an activity.

ii) Sets of printed card are available either laminated or not.

Not laminated

Most of the cards are ready to be laminated but a few need to be taped together before laminating. These are clearly indicated. Laminate the cards on both sides to prevent curling.

Laminated card

Cut out individual cards, place elastic bands around decks of cards, and store them with the activity boards (if applicable) in plastic bags or boxes. Self-sticking printed labels are provided.

The prepared materials can be stored effectively in a central location or in individual classrooms. Even when school sets are available, teachers prefer to have copies of some of the consolidation activities in their classrooms. Extra elastic bands should be available in each classroom.

Parent involvement in the preparation of activities is discussed in detail on page 22.

It is worth remembering that, except for occasional replacement, the work of preparing the materials will not have to be repeated. The time spent is a capital investment, which will pay dividends in years to come. You are also contributing to children's long-term learning which makes all the hard work worth the effort.

Managing the Fleet

Data Management

Many opportunities arise in classroom settings which spontaneously lead to data management activities.

Ask children to contribute collectable materials. As the materials are collected record the type of material contributed by each child on a class graph. This will likely encourage them to bring more.

As they complete the activities, ask students to graph their results. Here are some examples.

Introduction to using interlocking cubes (Org 1.2/1)

When students have joined the cubes into rods, have them make a record of each colour by graphing the cubes that make the 'red set,' 'blue set,' etc. Discuss which set has 'more,' which set has 'less.'

Making picture sets (Org 1.2/2)

Graph the number of flowers in the 'set of flowers,' the number of milk bottles in the 'set of milk bottles', etc.

Lucky Dip (Org 1.3/1)

Make a record of the length of each rod at the end of the game. Discuss which is the longest rod, which is the shortest, etc.

Crossing (Num 3.2/4)

Make a record of the movement of each marker. e.g., first roll: '3,' colour in 3 spaces on the graphing sheet, next roll '6,' colour in 6 spaces for whichever marker is moved.

Evaluation

"In a book about teaching, the importance of assessment is that we must know how far children have reached in their understanding, to know what they are ready to learn next, or whether review and/or consolidation are needed before going on."¹

i) Teacher Observation

Observe students as they do the activities. Even when introducing the activities to a whole class, a teacher has the opportunity to observe individual students as the procedures are demonstrated. This is a valuable opportunity to focus on a few children followed by more as all of the children work in small groups. Notes can be assembled for individual students in the group by making brief comments on 'post notes' which can later be filed.

- ii) Evaluation checklists – keep a record of individual student progress using the Progress Record on the back cover fold-out flap.
- iii) Self/Peer Evaluation – discussions at the conclusion of each activity period will provide opportunities for individual and peer evaluation. Student journals can provide diagnostic assessment; e.g., “What I already know about subtraction” before introducing the topic, followed by “What I learned about subtraction” after completing the appropriate activities in the subtraction network.
- iv) Portfolio of students' work – as students record their work, keep their sheets in a folder or provide a booklet. Keep a record of the problems they write, the patterns they make, etc.
- v) Teacher-Child Interview/Conferences – when others are engaged in consolidation activities, teachers can have conferences with individual students.
- vi) Specific assessment – assessment booklets for each network of activities would provide teachers with opportunities for written feed-back.
- vii) Notebooks – as they do the activities, have the students record their written responses in a notebook. The amount of written work that they complete as they respond to the activities is often amazing!

Meeting individual needs

The concept maps on pages 27-61 provide a valuable vehicle for addressing individual needs by showing which concepts need to be understood before later ones can be acquired with understanding. The concept maps can be used diagnostically to provide activities at appropriate levels.

- i) Within each activity
Individual differences are accommodated by the very nature of the activities. For example, in **Crossing** (Num 3.2/4), students roll a die to move their counters up a 10-square number track. A student whose counter is on the ninth square, just needs to roll a '1' to finish. The other child is at square five. Each child rolls the die; the one at square five gets a '5' and wins the game. The first child says, “How did that happen? I was closer to the end.” The teacher discusses how probability works when you throw dice. One child may be working on addition: $5 + 5 = 10$ whereas the other child is thinking about probability.
- ii) Individual differences can also be accommodated by introducing a more advanced activity to a small group while others continue to work on an earlier activity. For example, in **Making equal parts** (Num 7.1 Activities 1, 2, 4 and 5) one group of students can be making physical representations in Activities 1 and 2 while others move on to pictorial representations in Activity 4 and others can consolidate the concepts in a challenging game, Activity 5.
- iii) Because the activities are open ended, students are able to work at their own pace and at a suitable level within each activity. One group of students may complete 20 addition questions using the 'Start,' 'Action,' and 'Result,' cards, while another group, using counters, may only complete 5. Each feels successful.

Across the curriculum

By their very nature, the ways of learning mathematics embodied in this program both depend upon and contribute to social learning and clear speech.

Many opportunities arise which spontaneously lead to activities across the curriculum. For example, in **Setting the Table** (Num 4.5/2) students may prepare place settings using patterning experiences, design attractive place mats, and learn nutrition facts.

In the **Num 1** and **Num 2** networks, a series of number rhymes relate familiar verses to finger counting.

Problem solving

Skemp's theory-based approach to problem solving has students learning one new concept at a time in the context of activities that are interesting and engaging but low in mathematically irrelevant material. This facilitates the process of abstraction as progress is made through the relevant concept maps and the child's knowledge structure is developed. The carefully sequenced activities lead to the development of appropriate mathematical models which can be used to generate solutions to problems, which in turn are tested in the original problem situation. This approach contrasts strikingly with approaches that begin with high-noise problem situations and lead to a disorderly development of concepts and processes.

Personalized number stories are included in each Network: addition, pages 170-174; subtraction, pages 212-215; multiplication, pages 254-256; and division, pages 260, 268.

Students learn how to solve problems. They are deliberately led from verbal problems to physical representations of the objects, numbers, and actions described in the number story (*modelling*), and from the latter to the mathematical statement, not directly from words to mathematical symbols. They learn how to produce numerical models in physical materials corresponding to given number stories, to manipulate these appropriately, and to interpret the result in the context of the number story. Later, they do this by using written symbols only. (For an example, see **Number stories: abstracting number sentences**, Num 3.4, Activities 1, 2, and 3.)

Students are encouraged to draw diagrams to show what they have done (concrete-pictorial-symbolic), e.g., **Different questions, same answer. Why?** (Num 6.3/1), to show the connection between grouping and sharing in division.

As they are engaged in the activities, students use problem solving skills. E.g., in **Crossing** (Num 3.2/4) they are applying problem solving skills as they predict the best move for each of three markers.

The students are helped to invent their own real-world problems, applying the concepts they have learned in practical situations. E.g., **Front window, rear window – make your own** (Num 4.9/4).

Meeting the NCTM Standards²

The close correlation between the NCTM Standards and the *SAIL* program written by Richard Skemp is undoubtedly the result of the convergence of two independent, thorough, and insightful explorations to the heart of what is needed for the intelligent learning of mathematics.

Accepting that the NCTM Standards have established a broad framework for guiding needed reform in school mathematics, an examination of some of the ways in which the Skemp materials fit that framework follows.

Mathematics as Problem Solving (Standard 1). The Skemp learning activities are, themselves, problem-solving tasks. The students are led to construct mathematical concepts and relationships from physical experiences designed to appeal to their imagination and to build on their real-world experiences. Students work cooperatively on well-designed, goal-directed tasks, making predictions, testing hypotheses and building relational understandings that facilitate routine and non-routine problem solving. (This is addressed at length above).

Mathematics as Communication (Standard 2). The Skemp learning activities are designed to foster communication about mathematical concepts between students and between students and adults. A typical cooperative-group activity, **Number Targets** (Num 2.8/1), engages the students in communicating about place value concepts with physical embodiments, spoken symbols, written symbols, oral and read symbols, all the while exploring the underlying mathematical meanings while using problem solving strategies to predict their best move.

Mathematics as Reasoning (Standard 3). Using patterns and relationships to make sense out of situations is an integral component of the Skemp learning activities. As mentioned previously, the activities frequently lead the students to explore, conjecture (make predictions), and test their conjectures. The program builds on relational understanding, and useful instrumental (habit) learning is promoted as appropriate.

Mathematical Connections (Standard 4). Skemp's detailed conceptual analysis of the elementary school mathematics curriculum has produced a set of well-defined concept maps or networks. The networks arrange the activities in optimal learning sequences and provide teachers with the framework to make relational connections within and across networks. Many of the activities require assembling previously learned concepts and processes to deal with the task at hand. An entire network is devoted to the **Number track and the number line**, which are of importance throughout mathematics, from kindergarten through university-level mathematics, and beyond. They provide a valuable support for our thinking about numbers in the form of a pictorial representation. Skemp's unerring notions about contexts that appeal to student imaginations have produced interesting life-like settings in which the students learn. They compare possible outcomes of the moves they might make in **One tonne van drivers** (Num 3.10/3), and they are introduced to a budgeting activity in **Catalogue shopping** (Num 3.10/4). In adult life, planning the use of money

and other resources – time, labour – is one of the major uses of arithmetic. Because of interesting real life situations, connections to other curriculum areas are easily integrated. **Feeding the animals** (Num 7.2/1) and **Setting the table** (Num 4.5/2) are examples of activities which have spin-offs to art and health.

Estimation, Number Sense & Numeration, Numbers & Operations, Computation, Number Systems (Standards 5-7 [Grades K-4]; 8 [Grades 5-8]). One entire network of Skemp learning activities treats **Numbers and their properties** from 'sorting dots' to 'square numbers' and 'relations between numbers.' Another complete network, **The naming of numbers**, carefully builds everything one needs to know about place value for numerals of any number of digits. A network is devoted to each of the basic arithmetic operations, **Addition, Subtraction, Multiplication, and Division**. In all of these networks there is emphasis on number sense, operations sense, reasonable estimates, making and testing predictions, mental computation, thinking strategies, and relationships between concepts and between operations . . . all recommended in Standards 5 through 8. Calculators are used when appropriate, as they would be in real-life situations. Activities which use calculators include: **One tonne van drivers** (Num 3.10/3), **Cargo boats** (Num 5.7/3), **Treasure chest** (Num 5.10/2), **Cargo airships** (Num 6.8/4), **Division by calculator** (Num 6.9/1), **Predict, then press** (Num 7.8/2), **"Are calculators clever?"** (Num 7.8/3), and **Number targets by calculator** (Num 7.8/4).

Geometry, Measurement (Standards 9, 10 [Grades K-4]; 12, 13 [Grades 5-8]). The Shape networks cover properties of two- and three-dimensional shapes and components. The number track and the number line network develops basic linear measurement skills while providing useful embodiments for the activities developing number concepts and arithmetic operations. The Meas 1, 2, 3, 4, 5, and 6 networks did not appear in the **Structured Activities for Primary Mathematics** volumes published in England by Routledge in 1989. These networks in the *SAIL* volumes cover the concepts and relationships of Standards 9 & 10 for Grades K-4 and Standards 12 & 13 for Grades 5-8.

Statistics, Probability (Standards 11 [Grades K-4]; 10, 11 [Grades 5-8]). The pre-requisite skills have been well developed in the ten original networks where students are introduced to probability from real-life situations. As discussed on page 7, for example, in **Crossing** students roll a die to move their counters up a 10-square number track. A student whose counter is on the ninth square, just needs to roll a '1' to finish. The other child is at square five. Each child rolls the die; the one at square five gets a '5' and wins the game. The first child says, "How did that happen? I was closer to the end." The teacher discusses how probability works when you throw a die. One child may be working on addition: $5 + 5 = 10$ whereas the other child is thinking about probability.

Fractions and Decimals (Standard 12). The Fractions network begins with the development of real-object concepts of 'equal parts,' 'denominators,' and 'numerators,' building to fully-symbolic treatments of fractions, decimal-fractions, and operations with decimals. The whole development of fraction 'number sense' is based on the use of concrete and pictorial models. The relationships between fractions and decimal fractions are very well developed.

The *Standards* and Professional Development

Each of these activities embodies both a mathematical concept, and also one or more aspects of the theory. So by doing these with a group of children, both children and their teacher benefit. The children benefit by this approach to their learning of mathematics; and the teacher also has an opportunity to learn about the theory of intelligent learning by seeing it in action. Theoretical knowledge acquired in this way relates closely to classroom experience and to the needs of the classroom. It brings with it a bonus, since not only do the children benefit from this approach to mathematics, but it provides a good learning situation for teacher also. In this way we get 'two for the price of one', time-wise.³

As mentioned previously, a set of 39 videoclips of the *SAIL* activities and of the theory which they embody are available .

Courses, based on Skemp's learning theory, have been developed to give teachers opportunities to learn about the theory, to do a selection of activities together, to make them, and to discuss them. Follow-up staff development support is available as teachers incorporate the activities in their classrooms.

*One ship sails East, another West
by the self-same winds that blow.
It isn't the gales, but the trim of their sails
that determines the way they go.
Traditional*

Notes

¹ Skemp, R. R. (1989). *Mathematics in the Primary School*. London: Routledge. p. 166.

² National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: The Council.

³ Skemp, 1989, op. cit., p. 111.



INTRODUCTION

1 Why this book was needed and what it provides

The *Curriculum and Evaluation Standards for School Mathematics*¹ document produced by the National Council of Teachers of Mathematics stresses the importance of a *developmentally appropriate* curriculum. There is now a wide consensus that practical work is essential for ensuring developmentally appropriate concept formation throughout the elementary school years, and not just for younger children. There are now a number of these activities available, and individually many of them are attractive and worth-while. But collectively, they lack two essential requirements for long-term learning: structure, and clear stages of progression. The present volumes provide a fully structured collection of more than three hundred activities, covering a core curriculum for children aged from four to eleven years old, which uses practical work extensively at all stages.

This collection is not, however, confined to practical work. Mathematics is an abstract subject, and children will need in the future to be competent at written mathematics. Putting one's thoughts on paper can be a help in organizing them, as well as recording them for oneself and communicating them to others. What is important is that this should not come prematurely. It is their having had to memorize a collection of rules without understanding which has put so many generations of learners off mathematics for life, and destroyed their confidence in their ability to learn it. Practical, oral, and mental work can provide the foundation of understanding without which written work makes no sense. Starting with these, the present collection provides a careful transition from practical work to abstract thinking, and from oral to written work.

Activities for introducing new concepts often include teacher-led discussion. Many of the other activities take the form of games which children can play together without direct supervision, once they know how to play. These games give rise to discussion; and since the rules and strategies of the games are largely mathematical, this is a mathematical discussion. Children question each other's moves, and justify their own, thereby articulating and consolidating their own understanding. Often they explain things to each other, and when teaching I emphasize that "When we are learning it is good to help each other." Most of us have found that trying to explain something to someone else is one of the best ways to improve one's own understanding, and this works equally well for children.

This volume also provides the following:

- (a) A set of diagrams (concept maps, or networks) showing the overall mathematical structure, and how each topic and activity fits.
- (b) Clear statements of what is to be learned from each group of activities.
- (c) For each activity, a list of materials and step by step instructions. (In Volume 1a, photo-masters are also provided to simplify the preparation of materials.)
- (d) For each topic, discussion of the mathematical concept(s) involved, and of the learning processes used.

The last of these will, it is hoped, be useful not only for classroom teachers, but also for support teams, mathematics advisers, those involved in the pre-service and in-service education of teachers, and possibly also those whose main interests are at the research level.

2 The invisible components and how to perceive them

The activities in this collection contain a number of important components which are invisible, and can only be perceived by those who know what to look for. These include (i) real mathematics, (ii) structure, and (iii) a powerful theory.

(i) *Real mathematics*. I contend that children can and should enjoy learning real mathematics. You might ask: “What do you mean by this? Is it just a puff?” I say “Begin,” because a fuller answer depends on personal experience. If someone asks “What is a kumquat?” I can tell them that it is a small citrus fruit, but two of the most important things for them to know are what it tastes like, and whether they like it or not. This knowledge they can only acquire by personally tasting a kumquat.

Real mathematics is a kind of knowledge. I can describe it, and I hope you will find this a useful start. But some of the most important things about mathematics people cannot know until they have some of this kind of knowledge in their own minds; and those who acquired real mathematics when they were at school are, regrettably, in a minority. A simple preliminary test is whether you enjoy mathematics, and feel that you understand it. If the answers are “No,” then I have good news for you: what you learned was probably not real mathematics. More good news: you can acquire real mathematics yourself while using these activities with your children. You will then begin to perceive it in the activities themselves: more accurately, in your own thinking, and that of your children, while doing these activities. And you will begin to discover whether or not what you yourself learned as a child was real mathematics.

Mathematics (hereafter I will use ‘mathematics’ by itself to mean real mathematics) is a kind of knowledge which is highly adaptable. In the adult world, this adaptability can be seen in the great variety of uses to which it is put. Mathematics is used to make predictions about physical events, and greatly increases our ability to achieve our goals. Our daily comfort and convenience, sometimes our lives, depend on the predictive use of mathematics by engineers, scientists, technicians, doctors and nurses. At an everyday level, we use mathematics for purposes such as predicting approximately how long we should allow for a journey. Highly sophisticated mathematics is required to project communication satellites into orbits whereby they hang stationary relative to the earth; and also in the design of the satellites themselves, whose electronic equipment allows us to watch on our television screens events many thousands of miles away.

Mathematics has also an important social function, since many of the complex ways in which we co-operate in modern society would not be possible without mathematics. Nuts could not be made to fit bolts, clothes to fit persons, without the measurement function of mathematics. Businesses could not function without the mathematics of accountancy. If the person in charge of this gets his calculations wrong, his firm may go out of business: that is to say, others will no longer cooperate by trading with them.

Another feature of mathematics is creativity — the use of one’s existing knowledge to create new knowledge. Can you say what are ninety-nine sevens? Probably not, but if you think “A hundred sevens make seven hundred, so ninety-nine sevens will be one seven less: six hundred and ninety three,” then you are using your own

mental creativity. Creating new mathematics which nobody ever knew before is creativity at the level of the professional mathematician; but anyone who has some real mathematics is capable of creating knowledge which is new to them, and this way of using one's mind can give a kind of pleasure which those who have not experienced it may find hard to understand.

These are some of the adult uses of mathematics, which make it so important in today's world of advanced science, technology, and international commerce. At school, most children still learn a look-alike which is called by the same name, but whose uses have little in common with the uses of real mathematics. School mathematics as it is experienced by children is mostly for getting check marks, pleasing teachers, avoiding reproofs and sometimes also the humiliation of being made to feel stupid. It is also used for passing exams, and thereafter quickly forgotten. Yet real mathematics can be taught and learned at school. For an example of mathematics used predictively, try **Missing stairs** (Org 1.5/1). Success in most of the games also depends largely on making good predictions. Mathematics is used socially in all children's work together in groups; and in some, e.g., **Renovating a house** (Num 3.9/3), a social use is embodied in the activity itself. I hope that you will find pleasure in discovering examples of creativity in the thinking of your own children when they are learning real mathematics in contexts like these.

(ii) *Structure*. This is an essential feature of real mathematics. It is this which makes possible all the features described in (i), so for emphasis I am giving it a section to itself.

By structure we mean the way in which parts fit together to make a whole. Often this whole has qualities which go far beyond the sum of the separate properties of the parts. Connect together a collection of transistors, condensers, resistors, and the like, most of which will do very little on their own, and you have a radio by which you can hear sounds broadcast from hundreds of miles away. That is, if the connections are right: and this is what we mean by structure in the present example.

In the case of mathematics, the components are mathematical concepts, and the structure is a mental structure. This makes it much harder to know whether it is there or not in a learner's mind. But the difference between a mathematical structure and a collection of isolated facts is as great as the difference between a radio and a box of bits. There is the same difference between a radio set and a wrongly connected assortment of components, but this is harder to tell by looking. The important test of the presence or absence of structure, i.e. of the right set of connections, may best be inferred from performance. This is also true for mathematics, and for its look-alike which goes by the same name. Of these two, only real mathematics performs powerfully, enjoyably, and in a wide variety of ways.

Each individual learner has to put together these structures in his own mind. No one can do it for him. But this mental activity can be greatly helped by good teaching, an important part of which is providing good learning situations. The requirements of a good learning situation include full use of all of the three modes of building conceptual structures. Mode 1 is learning by the use of practical materials; Mode 2 is learning from exposition, and by discussion; and Mode 3 is expanding one's knowledge by creative thinking. These categories are expanded and discussed more fully in *Mathematics in the Primary School*.²

The activities in this book are intended to help teachers provide learning situations of the kind described. They are also fully structured, meaning that the concepts embodied in each fit together in ways which help learners to build good mathematical structures in their own minds. This also includes consolidation, and developing mathematical skills.

(iii) *A powerful theory.* In 1929, Dewey wrote “Theory is in the end . . . the most practical of all things”;³ and I have been saying the same for many years, even before I knew that Dewey had said it first. The activities in this book embody a new theory of intelligent learning. This had its origins in the present author’s researches into the psychology of learning of mathematics,⁴ and was subsequently expanded and generalized into a theory of intelligent learning which can be applied to the learning of all subjects.⁵ It is not essential to know the theory in order to use the activities. But readers who are professional teachers will want to know not only what to do but why. Mathematics advisers, or lecturers in mathematical education, will wish to satisfy themselves of the soundness of the underlying theory before recommending the activities.

This theoretical understanding is best acquired by a combination of first-hand experience, reading, and discussion. Each of the activities embodies some aspects of the general theory, so by doing the activities with children we can observe the theory in action. For school teachers, this is a very good way to begin, since the theoretical knowledge acquired in this way begins with classroom experience, and as it develops further will continue to relate to it. This also has the advantage that we get ‘two for the price of one,’ time-wise: what might be called a ‘happy hour’ in the classroom! These observations can then form the first part of the trio

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whose value for school-based inservice education will be mentioned again in Section 5.

Reading helps to organize our personal experience, and to extend our knowledge beyond what can be gathered first-hand. A companion book to the present volume is *Mathematics in the Primary School*,⁶ and this also offers suggestions for further reading.

3 How this book is organized and how to use it

The two *SAIL* volumes contain teaching materials for eight school years, together with explanations and discussions. This is a lot of information. Careful thought has therefore been given to its organization, to make it easy to find as much as is required at a given stage, and to avoid feeling overloaded with information. Mathematics is a highly concentrated kind of information, so it is wise to take one's time, and to go at a pace which allows comfortable time for assimilation. The amount eventually to be acquired in detail by a class teacher would be no more than one-eighth of the total, if all children in the class were of the same ability. In practice it will, of course, be more because of children's spread of ability.

The aim has been to provide first an overview; then a little more detail; and then a lot of detail, of which many readers will not need all, nor all at the same time. This has been done by organizing the subject matter at four levels, into THEMES, NETWORKS, TOPICS, and ACTIVITIES. The themes and networks are tabulated below.

THEMES	NETWORKS	CODES
Organizing	Set-based organization	Org 1
Number	Numbers and their properties	Num 1
	The naming and recording of numbers	Num 2
	Addition	Num 3
	Subtraction	Num 4
	Multiplication	Num 5
	Division	Num 6
	Fractions	Num 7
Space	Shape	Space 1
	Movement and location in space (Including motion geometry and symmetry)	Space 2
Synthesis of Number and Space	The number track and the number line	NuSp 1
Measurement	Length	Meas 1
	Area	Meas 2
	Volume and capacity	Meas 3
	Weight and mass	Meas 4
	Temperature	Meas 5
	Time	Meas 6

The five main themes run in parallel, not sequentially, though some will be started later than others. Within the theme of Number, there are seven networks. For some themes, there is at present just one network each; but I have kept to the same arrangement for consistency, and to allow for possible future expansion. By ‘network,’ I mean a structure of inter-related mathematical ideas. It can well be argued that all mathematical ideas are inter-related in some way or other, but the networks help to prevent information overload by letting us concentrate on one area at a time.

Greater detail for each network is provided by a concept map and a list of activities. These are on pages 27 through 61. It will be useful to look at a pair of these as illustrations for what follows. Each concept map shows how the ideas of that network relate to each other, and in particular, which ones need to be understood before later ones can be acquired with understanding. These interdependencies are shown by arrows. A suggested sequence through the network is shown by the numbers against each topic. Use of the concept map will help you to decide whether other sequences may successfully be followed, e. g., to take advantage of children’s current interests. The concept map is also useful diagnostically. Often a difficulty at a particular stage may be traced to a child’s not having properly understood one or more earlier concepts, in which case the concept map will help you to find out which these are. Another function of the concept maps is to help individual teachers to see where their own teaching fits into a long-term plan for children’s learning, throughout the primary years: what they are building on, and where it is leading.

Below each concept map is shown a list of the activities for each topic. (I call them ‘topics’ rather than ‘concepts’ because some topics do not introduce new concepts, but extend existing ones to larger areas of application.) Usually these activities should be used in the order shown. An alphabetical list of the activities is also given, at the end of the book.

To find activities for a particular topic, the best way is via the concept maps and the lists of activities opposite them. Suppose for example that you want activities for adding past ten. For this you naturally look at the concept map for addition, and find adding past ten here as topic 6. On the adjacent page, for this topic, you will find seven activities. Not all topics have so many activities, but this indicates the importance of this stage in children’s learning of addition. If on the other hand you want to find a particular activity by name, then the alphabetical index at the end of the book will enable you to do so.

The codes for each topic and activity are for convenience of reference. They show where each fits into the whole. Thus Num 3.8/2 refers to Network Num 3, Topic 8, Activity 2. If the packet for each activity has its code on it, this will help to keep them all in the right order, and to replace each in the right place after use.

4 Getting started as a school

Since schools vary greatly, what follows in this section and the rest of this introduction is offered as no more than a collection of suggestions based on the experience of a number of the schools where the materials have already been introduced.

It has been found useful to proceed in two main stages: getting acquainted, and full implementation. Since the latter will be spread over one or two years at least, the first stage is important for getting the feel of the new approach, and to help in deciding that it will be worth the effort.

For getting acquainted, a good way is for each teacher to choose an activity, make it up, and learn it by doing it with one or more other teachers. (Different activities are for different numbers of persons.) Teachers then use these activities with their own children, and afterwards they discuss together what they have learned from observation of their own children doing the activities. It is well worth while trying to see some of the activities in use, if this can be arranged. Initially, this will convey the new approach more easily than the printed page.

When you are ready to move towards a full implementation, it will be necessary to decide the overall approach. One way is to introduce the activities fully into the first and second years, while other teachers gradually introduce them into later years as support for the work they are already doing. This has the advantage that the full implementation gradually moves up the school, children being used to this way of learning from the beginning. Alternatively, activities may be introduced gradually throughout the school, individual teachers choosing which activities they use alongside existing text-based materials while they gain confidence in the new approach. As another alternative, a school may wish to begin with one network (perhaps, for example, multiplication), arranging for all the teachers to meet together to do all of the activities in the network and then to implement them at the appropriate grade levels.

Arrangements for preparation of the materials need to be planned well in advance. This is discussed in greater detail in section 5. A detail which needs to be checked in good time is whether the commercial materials needed, such as Multilink or Unifix and base ten materials, are already in the school in the quantities needed.

In considering the approach to be used, it is important to realize that while benefit is likely to be gained from even a limited use of the activities, a major part of their value is in the underlying structure. The full benefit, which is considerable, will therefore only be gained from a full implementation.

5 Organization within the school

Overall, the organization of the new approach is very much a matter for the principal and staff of each individual school to work out for themselves; so, as has already been emphasized, what follows is offered only in the form of suggestions, based on what has been found successful in schools where this approach has been introduced.

Whatever organization is adopted, it is desirable to designate an organizer who will coordinate individual efforts, and keep things going. It is a great help if this teacher can have some free time for planning, organizing, advising, and supporting teachers as need arises.

One approach which has been found successful is as follows. The mathematics organizer holds regular meetings with the staff in each one- or two-year group, according to their number. Each teacher chooses an activity, makes it up if necessary, and teaches it to the other teachers in the group. They discuss the mathematics involved, and any difficulties. Subsequently they discuss their observations of their own children doing these activities, and what they have learned by reflecting on these. This combination may be summarized as

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and is an important contribution to school-based inservice education. (So much so, that several leading mathematics educators have said that it should be printed on every page. As a compromise, I have printed it at the end of every topic.)

It is good if the mathematics organizer is also in a position to help individual teachers, since it is only to be expected for them to sometimes feel insecure when teaching in a style which may be very different from that to which they are accustomed. Two useful ways to help are by looking after the rest of a teacher's class for a while, so that this teacher is free to concentrate entirely on working with a small group; and by demonstrating an activity with a small group while the class teacher observes, the rest of the class being otherwise occupied.

6 Getting started as an individual teacher

The most important thing is actually to do one or more activities with one's own children, as early on as possible. This is the best way to get the feel of what the new approach is about. After that, one has a much better idea of where one is going. If there are particular topics where the children need help, suitable activities may be found via the concept maps and their corresponding lists of activities. Alternatively, here is a list of activities which have been found useful as 'starters'. The stages correspond roughly to years at school.

			vol	page
Stage 1	Lucky dip	Org 1.3/1	1	72
	'Can I fool you?'	Org 1.3/2	1	73
	Missing stairs	Org 1.5/1	1	76
Stage 2	Stepping stones	Num 3.2/3	1	162
	Crossing	Num 3.2/4	1	164
	Sequences on the number track	NuSp 1.2/1	1	309
Stage 3	The handkerchief game	Num 3.5/1	1	175
	"Please may I have?" (complements)	Num 3.5/2	1	176
	Number targets	Num 2.8/1	1	141
Stage 4	Slippery slope	Num 3.7/3	1	184
	Slow bicycle race	NuSp 1.5/1	1	319
	Doubles and halves rummy	Num 1.10/3	1	121
Stage 5	Place value bingo	Num 2.10/3	1	146
	Renovating a house	Num 3.9/3	1	199
	Constructing rectangular numbers	Num 6.4/1	2	
	The rectangular numbers game	Num 6.4/2	2	
Stage 6	Cycle camping	Num 3.10/2	2	
	One tonne van drivers	Num 3.10/3	2	
	Multiples rummy	Num 5.6/6	2	
Stage 7	Cargo boats	Num 5.7/3	2	
	Classifying polygons	Space 1.12/1	2	
	Match and mix: polygons	Space 1.12/3	2	

When the children are doing an activity, think about the amount of mathematics which the children are doing, including the mental and oral activity as well as the written work, and compare it with the amount of mathematics they would do in the same time if they were doing written work out of a textbook.

So far, I have interpreted the heading of this section as meaning that the reader is an individual teacher within a school where most, or at least some, colleagues are also introducing the new approach. But what if this is not the case? When talking with teachers at conferences, I have met some who are the only ones in their school who are using this kind of approach to the teaching of mathematics.

This is a much more difficult situation to be in. We all need support and encouragement, especially when we are leaving behind methods with which we are familiar — even though they have not worked well for many children. We need to discuss difficulties, and to share ways we have found for overcoming them. So my suggestion here is that you try to find some colleague with whom you can do this. At the very least, you need someone with whom to do the activities before introducing them to the children; and further discussions may arise from this.

7 Parents

Parents are naturally interested in their children's progress at school. Written work is something they can see — what they cannot see is the lack of understanding which so often underlies children's performances of these 'rules without reasons.' Sometimes also they try to help children at home with their mathematics. Unfortunately, this often takes the form of drill-and-practice at multiplication tables, and pages of mechanical arithmetic. This is the way they were taught themselves, and some parents have been known to respond unfavourably when their children come home and say that they have spent their mathematics lessons playing games. As one teacher reported, "Games are for wet Friday afternoons. Mathematics is hard work. They aren't meant to enjoy it."

How you deal with problems of this kind will, of course, depend partly on the nature of existing parent-teacher relationships in your own school. When explaining to parents who may be critical of what you are doing, it also helps if you are confident in your own professional understanding, and if there is a good consensus within the school. These are areas in which sections 2 and 5 offer suggestions. Some approaches which have been used with success are described here. They may be used separately or together.

A parents' evening may be arranged, in which parents play some of the games together. Teachers help to bring out the amount of mathematics which children are using in order to decide what move they will make, or what card to play. To do this well, teachers need to be confident in their own knowledge of the underlying mathematics, and this can be built up by doing the activities together, and discussing with each other the mathematics involved.

A small group of parents may be invited to come regularly and help to make up activities. This needs careful organizing initially, but over a period it can be a great saving in teacher's time. When they have made up some activities, parents naturally want to do them in order to find out what they are for; so this combines well with the first approach described.

Some parents may also be invited to come into classrooms and supervise consolidation activities. (It has already been mentioned that activities which introduce new concepts should be supervised by a teacher.)

For parents who wish to help children at home with their mathematics, the games provide an ideal way to do this. Many of these can be played at varying levels of sophistication, which makes them suitable as family games; and there is usually also an element of chance, which means that it is not always the cleverest player who wins. None of the games depends entirely on chance, however. Good play consists in making the best of one's opportunities. Parents who help their children in this way will also have the benefit of knowing that what they are doing fits in with the ways in which their children are learning at school.

8 Some questions and answers

Q. How long will it take to introduce these materials?

A. The materials embody many ideas which are likely to be new to many teachers, in two areas: mathematics, and children's conceptual learning. So it is important to go at a pace with which one feels comfortable, and which gives time to assimilate these ideas into one's personal thinking and teaching. It took me twenty-five years to develop the underlying theory, three to find out how to embody it in ways of teaching primary mathematics, and another five to devise and test the integrated set of curriculum materials in this volume. If a school can have the scheme fully implemented and running well in about two years, I would regard this as good going. But of course, you don't have to wait that long to enjoy some of the rewards.

Q. Isn't it a lot of work?

A. Changing over to this new approach does require quite a lot of work, especially in the preparation of the materials. The initial and on-going planning and organization are important, so that this work can go forward smoothly. As I see it, nearly all worth-while enterprises, including teaching well, involve a lot of work. If you are making progress, this work is experienced as satisfying and worth the effort. As one teacher said, 'Once you get started, it creates its own momentum.' And once it is established, the fact that children are learning more efficiently makes teaching easier.

Q. How were the concept maps constructed?

A. First, by a careful analysis of the concepts themselves, and how they relate to each other in the accepted body of mathematical knowledge.⁷ This was then tested by using it as a basis for teaching. If an activity was unsuccessful in helping children to develop their understanding, this was discussed in detail at the next meeting with teachers. Sometimes we decided that the activity needed modification; but sometimes we decided that the children did not have certain other concepts which they needed for understanding the one we had been introducing — that is, the concept map itself needed revision. So the process of construction was a combination of mathematics, applied learning theory, and teaching experiment.

Q. Is it all right to use the activities in a different order?

A. Within a topic, the first activity is usually for introducing a new concept, and clearly this should stay first. Those which follow are sometimes for consolidation, in which case the order may not matter too much. Sometimes, however, they are for developing thinking at a more abstract level, in which case the order does matter. However, once the activities have been used in the order suggested, it is often good to return to earlier ones, for further consolidation and to develop connections in both directions: from concrete to abstract, and from abstract to concrete.

Whether it is wise to teach topics in a different order you can decide by looking at the concepts map itself. These, and the teaching experiments described, show that the order of topics is very important. They also show that for building up a given knowledge structure, there are several orders which are likely to be successful – and many which are likely to be unsuccessful.

Q. Is it all right to modify the activities?

A. Yes, when you are confident that you understand the purpose of the activity and where it fits into the long-term learning plan. The details of every activity have been tested and often re-written several times, both from the point of view of the mathematics and to help them to go smoothly in the classroom; so I recommend that you begin by using them as written. When you have a good understanding of the mathematics underlying an activity, you will be in a position to use your own creativity to develop it further.

Sometimes children suggest their own variation of a game. My usual answer is that they should discuss this among themselves, and if they agree, try it together next time: but that rules should not be changed in the middle of a game. They may then discuss the advantages and disadvantages of the variation.

Q. Can I use the materials alongside an existing scheme? And what about written work, in general?

A. Especially with the older children, I would expect you to begin by introducing these activities alongside the scheme you are already using and familiar with. With younger children, the activities introduce as much written work as I think is necessary. Thereafter, existing text-book schemes can be put to good use for gradually introducing more written work of the conventional kind. But these should come after the activities, rather than before. Children will then get much more benefit from the written work because they come to it with greater understanding.

Q. Doesn't all this sound too good to be true?

A. Frankly, yes. Only personal experience will enable you to decide whether it can become true for you yourself, and in your school. Each school has its own microclimate, within which some kinds of learning can thrive and others not. Where the microclimate is favourable to this kind of learning, what I have been describing can become true, and I think you will find it professionally very rewarding. Where this is not at present the case, the problems lie beyond what can be discussed here. I have discussed them at length elsewhere.⁸

Q. What do you see as the most important points when implementing this approach?

A. Good organization; personal experience of using the activities; observation, reflection and discussion.

Notes

- ¹ National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: The Council.
- ² Skemp, R. R. (1989). *Mathematics in the Primary School*. London: Routledge.
- ³ Dewey, J. (1929). *Sources of a Science of Education*. New York: Liveright.
- ⁴ Skemp, R. R. (1971, 1986 2nd ed.). *The Psychology of Learning Mathematics*. Harmondsworth: Penguin.
- ⁵ Skemp, R. R. (1979). *Intelligence, Learning, and Action*. Chichester: Wiley.
- ⁶ Skemp, 1989, op. cit.
- ⁷ Skemp, 1971 and 1978, op. cit. See Chapter 2 of the first edition, or Chapter 1 of the second edition of Skemp, 1971.
- ⁸ Skemp, 1979, op. cit., Chapter 15.



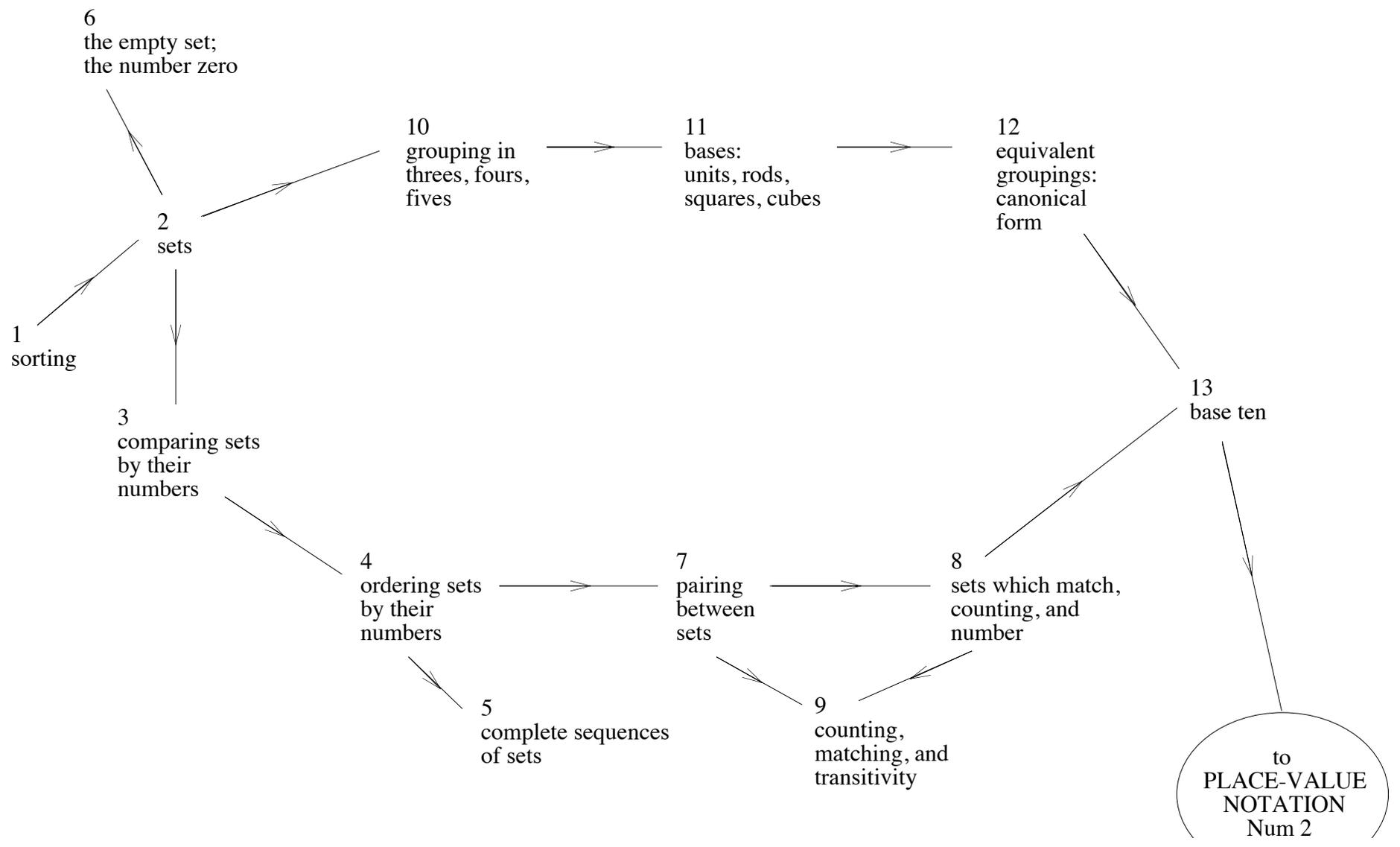
*'Stepping stones' [Num 3.2/3] **

* A frame electronically extracted from
NUMERATION AND ADDITION,
a colour video produced by the
Department of Communications Media,
The University of Calgary

CONCEPT MAPS and LISTS OF ACTIVITIES

Activities in **bold** are those found in this volume.

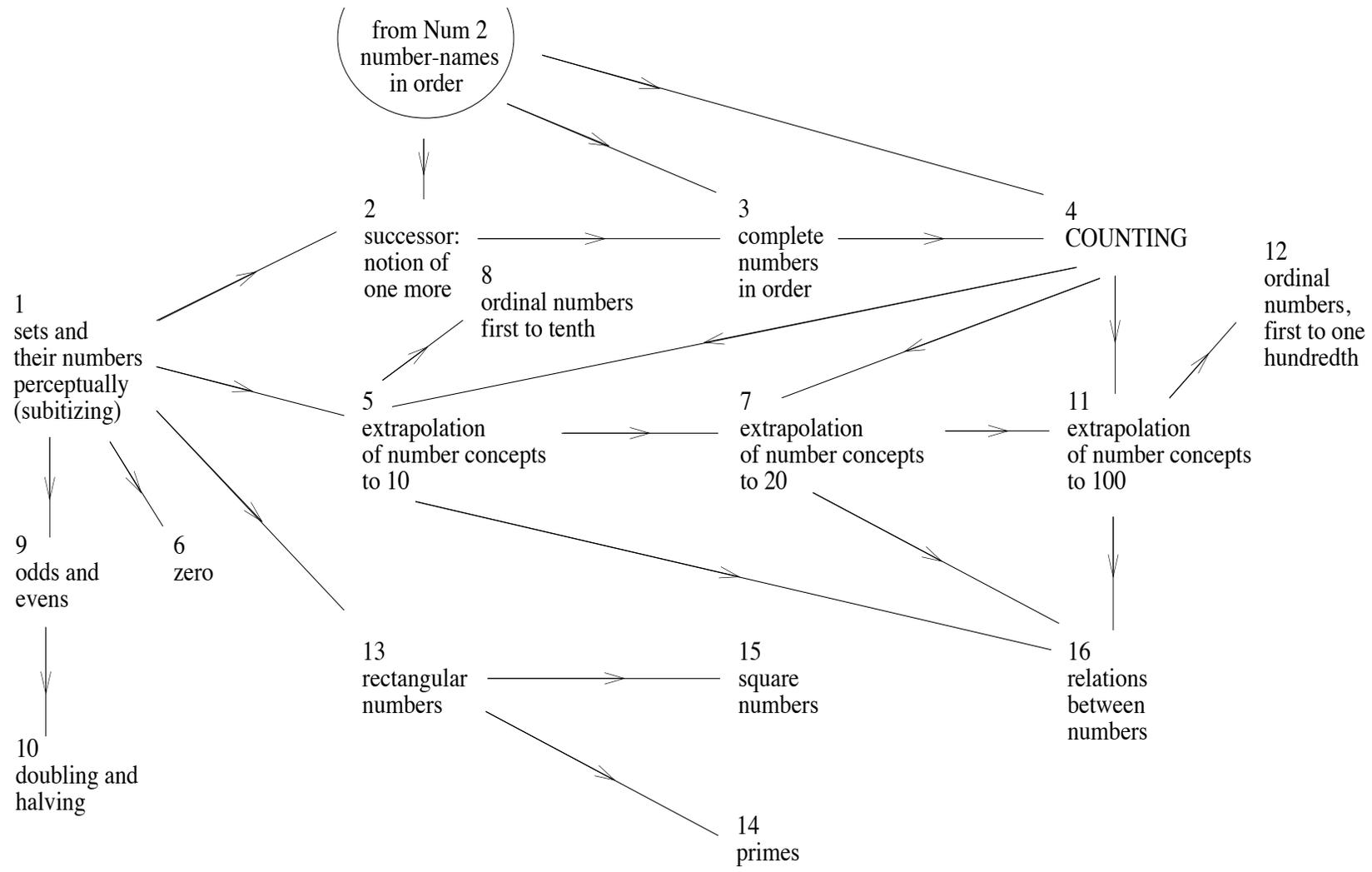
Org 1 Set-based organization



Org 1 Set-based organization

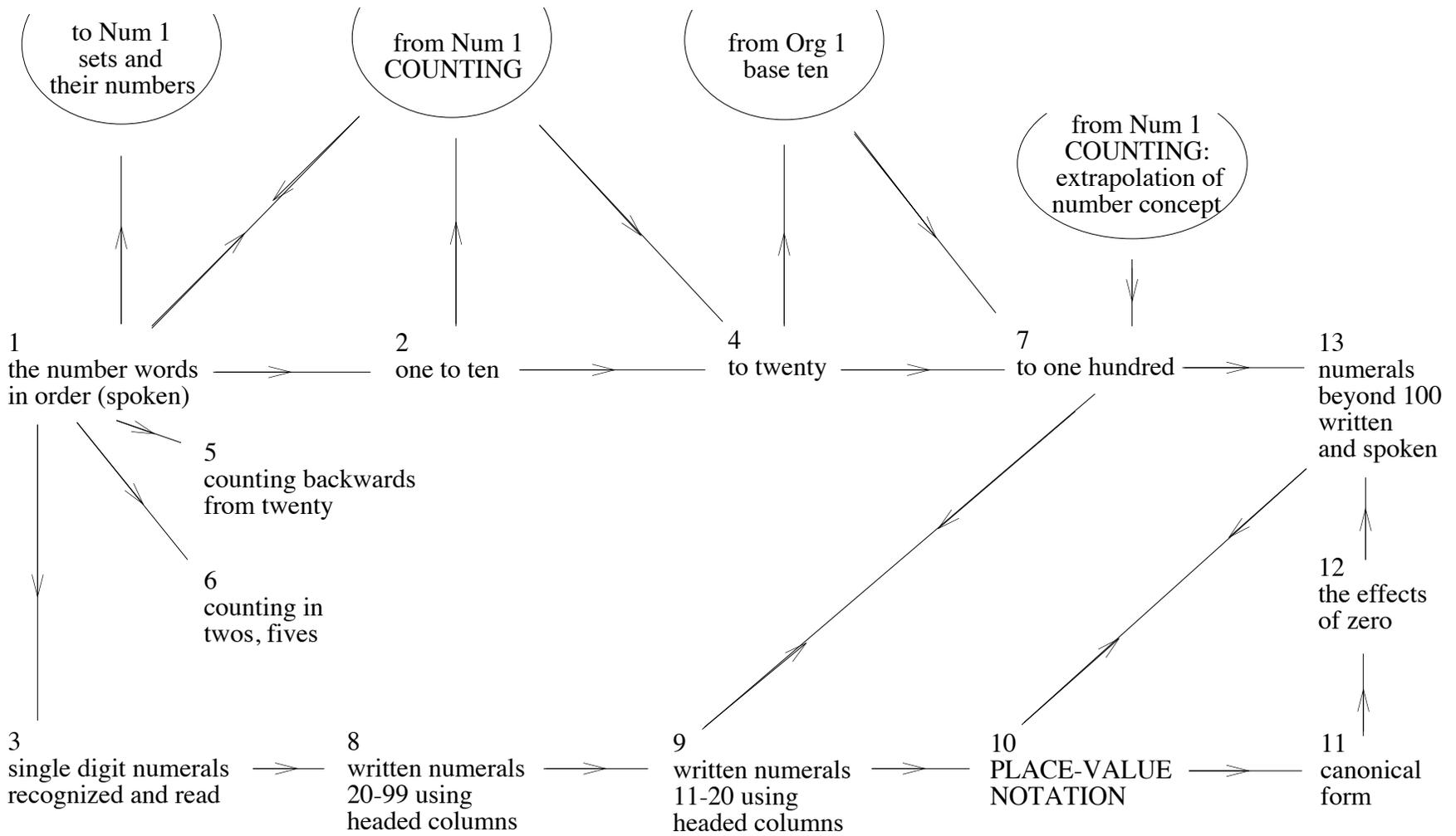
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4/2	Combining order of number, length, and position	74	12/2	Exchanging small coins for larger 95
5	<i>Complete sequences of sets</i>	76	13	<i>Base ten</i> 96
5/1	Missing stairs	76	13/1	Tens and hundreds of cubes 96
6	<i>The empty set; the number zero</i>	78	13/2	Tens and hundreds of milk straws 97
6/1	The empty set	78	13/3	Thousands

Num 1 Numbers and their properties



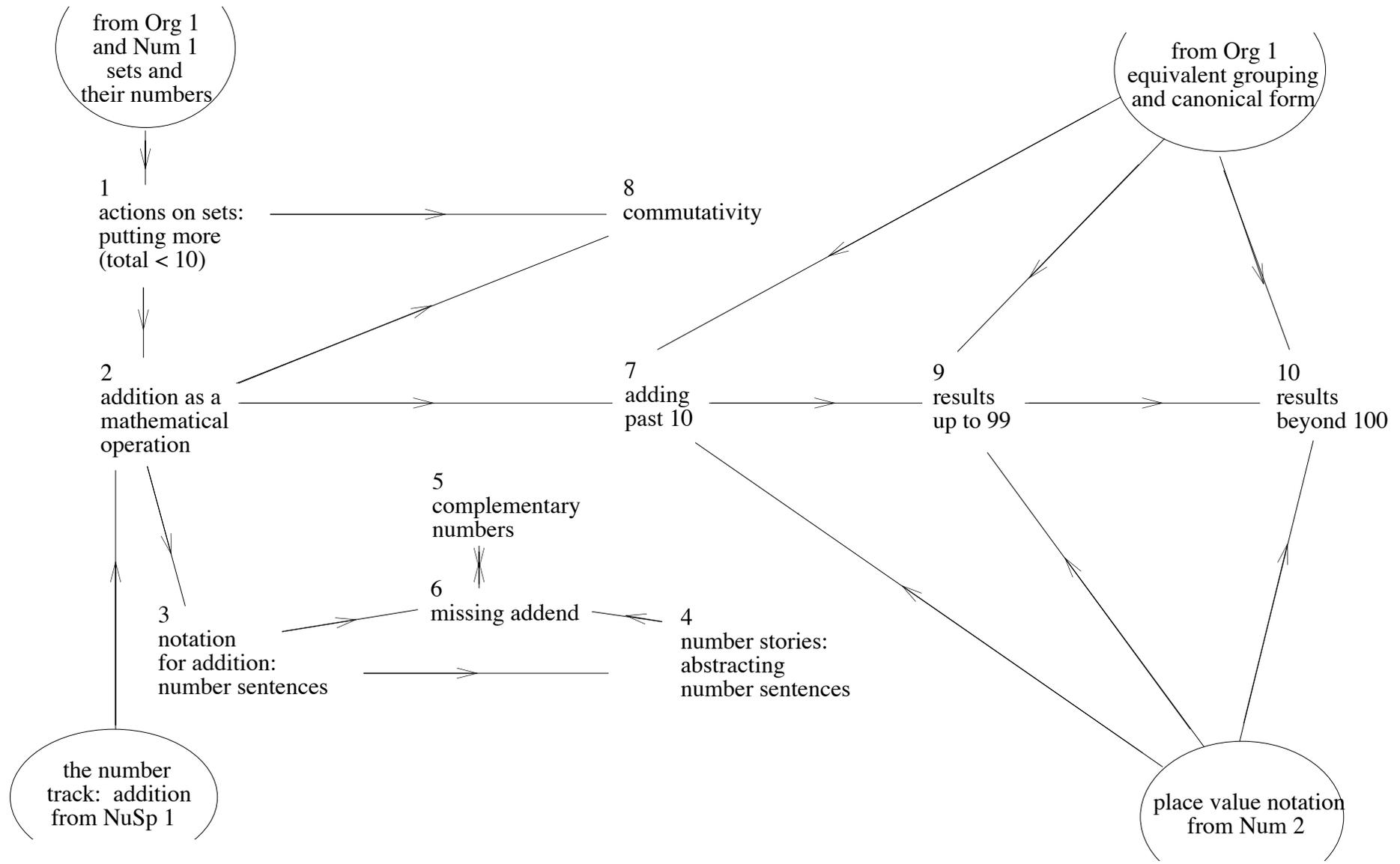
Num 1 Numbers and their properties

1	<i>Sets and their numbers perceptually (subitizing)</i>	99		9/1 “Yes or no?”	116
1/1	Sorting dot sets and picture sets	99		9/2 “Can they all find partners?”	117
1/2	Picture matching game using dot sets and picture sets	100		9/3 “Odd or even?”	117
2	<i>Successor: notion of one more</i>	101		10 <i>Doubling and halving</i>	119
2/1	Making successive sets	101		10/1 “Double this and what will we get?”	119
2/2	Putting one more	101		10/2 “Break into halves, and what will we get?”	120
				10/3 Doubles and halves rummy	121
3	<i>Complete numbers in order</i>	103		11 <i>Extrapolation of number concepts to 100</i>	122
3/1	“Which card is missing?”	103		11/1 Throwing for a target	122
4	<i>Counting</i>	104		11/2 Putting and taking	124
4/1	Finger counting to 5	104		12 Ordinal numbers first to one hundredth	
4/2	Planting potatoes	105		13 <i>Rectangular numbers</i>	
5	<i>Extrapolation of number concepts to 10</i>	106		13/1 Constructing rectangular numbers	} These activities will be found in Num 6 (Division) in topics 6.4 and 6.5
5/1	Finger counting to 10	106		13/2 The rectangular numbers game	
5/2	Missing stairs, 1 to 10	106		14 <i>Primes</i>	
5/3	“I predict-here” on the number track (Same as NuSp 1.1/1)	107		14/1 Alias prime	
5/4	Sequences on the number track (Same as NuSp 1.2/1)	109		14/2 The sieve of Eratosthenes	
6	<i>Zero</i>	110		14/3 Sum of two primes	
6/1	“Which card is missing?” (Including zero)	110		15 <i>Square numbers</i>	
6/2	Finger counting from 5 to zero	110		15/1 Square numbers	
7	<i>Extrapolation of number concepts to 20</i>	112		15/2 An odd property of square numbers	
7/1	Finger counting to 20: “Ten in my head”	112		16 <i>Relations between numbers</i>	
8	<i>Ordinal numbers first to tenth</i>	113		16/1 “Tell us something new”	
8/1	“There are . . . animals coming along the track”	113		16/2 “How are these related?”	
8/2	“I’m thinking of a word with this number of letters.”	114			
8/3	“I think that your word is . . .”	115			
9	<i>Odds and evens</i>	116			



Num 2 The naming of numbers

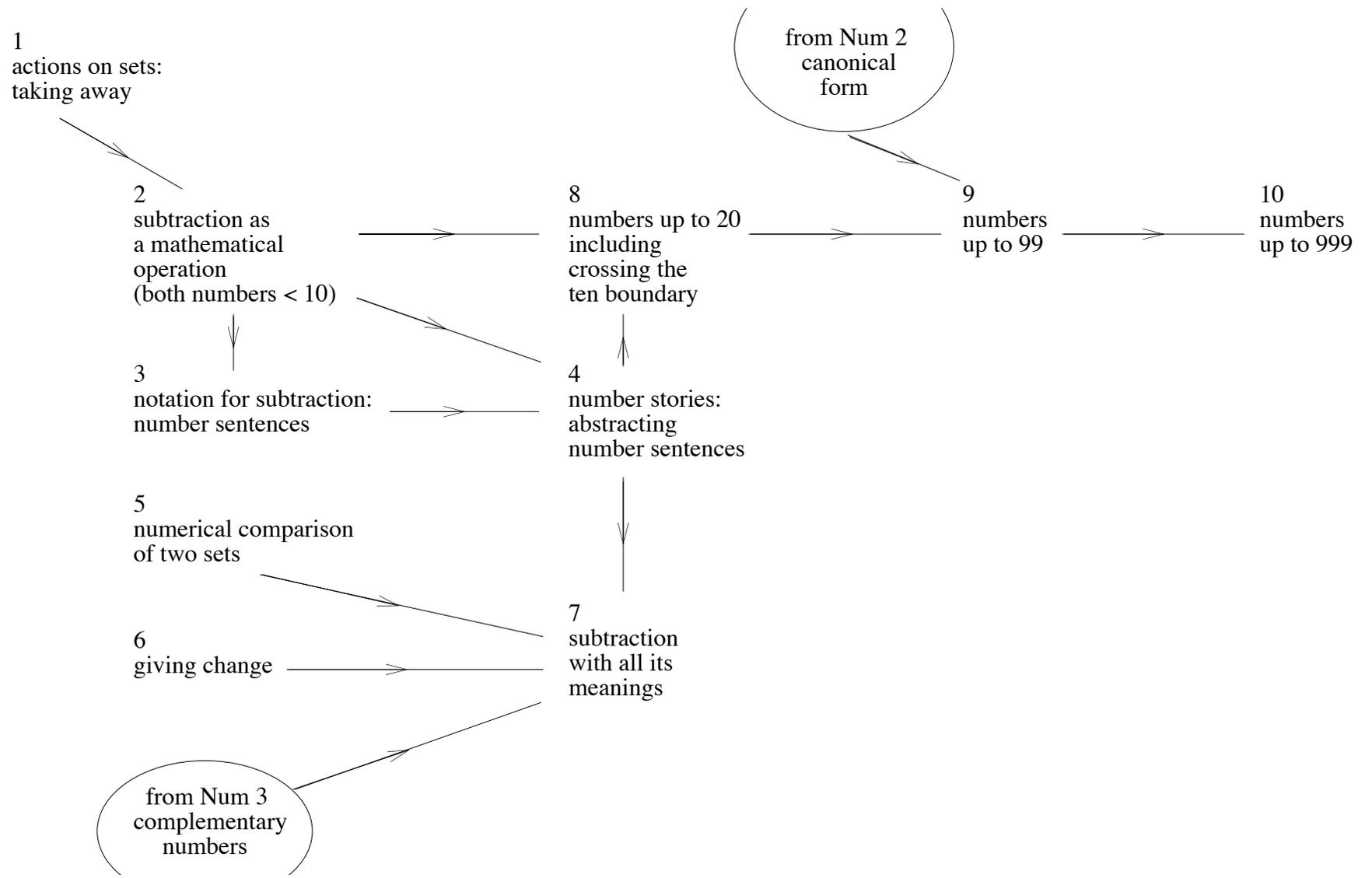
1	<i>The number words in order (spoken)</i>	126			
1/1	Number rhymes	126		8/1	Number targets 141
				8/2	Number targets beyond 100 142
2	<i>Number words from one to ten</i>	128		9	<i>Written numerals from 11 to 20</i> 143
2/1	Number rhymes to ten	128		9/1	Seeing, speaking, writing 11-19 143
				9/2	Number targets in the teens 144
3	<i>Single-digit numerals recongnized and read</i>	129		10	<i>Place-value notation</i> 145
3/1	Saying and pointing	129		10/1	"We don't need headings any more." 145
3/2	"Please may I have . . .?"	129		10/2	Number targets using place-value notation 146
3/3	Joining dots in order, to make pictures	130		10/3	Place-value bingo 146
3/4	Sets with their numbers	130			
3/5	Sequencing numerals 1 to 10	131		11	<i>Canonical form</i> 149
				11/1	Cashier giving fewest coins 150
4	<i>Continuation of counting: 1 to 20</i>	132		11/2	"How would you like it?" 152
4/1	Number rhymes to twenty	132		12	<i>The effects of zero</i> 154
				12/1	"Same number, or different?" 154
5	<i>Counting backwards from 20</i>	133		12/2	Less than, greater than 155
5/1	Backward number rhymes	133		13	<i>Numerals beyond 100, written and spoken</i>
5/2	Numbers backwards	134		13/1	Big numerals
				13/2	Naming big numbers
6	<i>Counting in twos, fives</i>	136			
6/1	Counting with hand clapping	136			
6/2	Counting 2-rods and 5-rods	136			
6/3	Counting money, nickels	137			
6/4	Counting sets in twos and fives	137			
7	<i>Extrapolation of counting pattern to one hundred</i>	138			
7/1	Counting in tens	138			
7/2	Counting two ways on a number square	138			
7/3	Tens and ones chart	139			
8	<i>Written numerals 20 to 99 using headed columns</i>	140			



Num 3 Addition

1	<i>Actions on sets: putting more (Total <10)</i>	158		7	<i>Adding past 10</i>	181
1/1	Start, Action, Result (do and say)	158		7/1	Start, Action, Result over ten	181
1/2	Putting more on the number track (verbal) [NuSp 1.3/1]	160		7/2	Adding past 10 on the number track (Same as NuSp 1.3/4)	183
2	<i>Addition as a mathematical operation</i>	161		7/3	Slippery slope	184
2/1	Predicting the result (addition)	161		7/4	Addfacts practice	185
2/2	“Where will it come?” (Same as NuSp 1.3/2)	162		7/5	Addfacts at speed	185
2/3	Stepping stones	162		7/6	Predictive number sentences past 10	187
2/4	Crossing (Same as NuSp 1.3/3)	164		7/7	Explorers	187
3	<i>Notation for addition: number sentences</i>	167		8	<i>Commutativity</i>	190
3/1	Writing number sentences for addition	167		8/1	Introducing commutativity	191
3/2	Write your prediction (addition)	168		8/2	Introducing non-commutativity	192
4	<i>Number stories: abstracting number sentences</i>	170		8/3	Using commutativity for counting on	192
4/1	Personalized number stories	170		8/4	Commutativity means less to remember	193
4/2	Abstracting number sentences	171		9	<i>Adding, results up to 99</i>	195
4/3	Personalized number stories - predictive	172		9/1	Start, Action, Result up to 99	195
5	<i>Complementary numbers</i>	175		9/2	Odd sums for odd jobs	198
5/1	The handkerchief game	175		9/3	Renovating a house	199
5/2	“Please may I have?” (complements)	176		9/4	Planning our purchases	200
6	<i>Missing addend</i>	177		9/5	Air freight	201
6/1	“How many more must you put?”	177		10	<i>Adding, results beyond 100</i>	
6/2	Secret adder	178		10/1	Start, Action, Result beyond 100	
6/3	Personalized number stories: what happened?	179		10/2	Cycle campling	
				10/3	One tonne van drivers	
				10/4	Catalogue shopping	

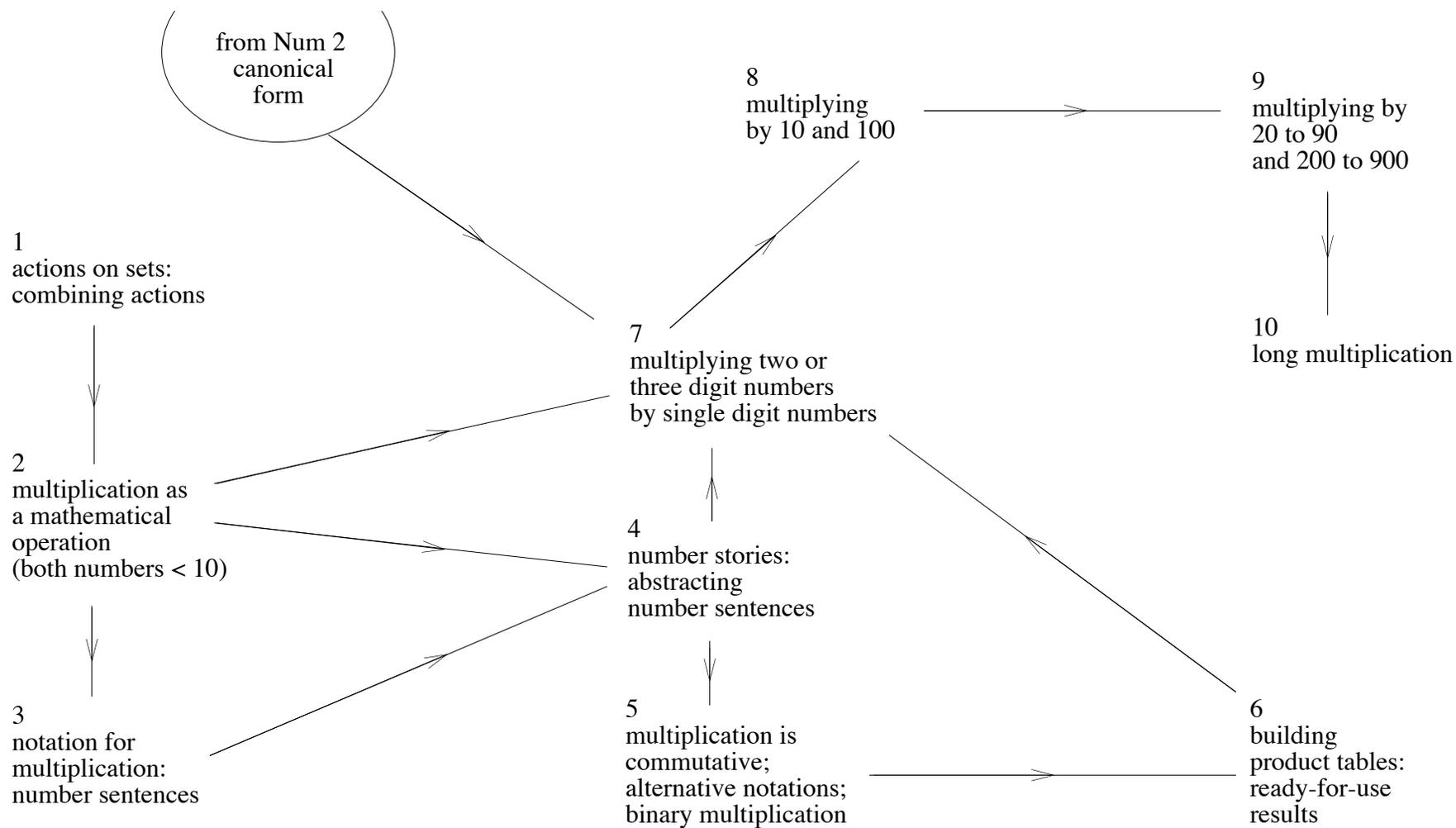
Num 4 Subtraction



Num 4 Subtraction

1	<i>Actions on sets: taking away</i>	204			
1/1	Start, Action, Result (do and say)	204	7	<i>Subtraction with all its meanings</i>	226
1/2	Taking away on the number track (do and say) [NuSp 1.4/1]	205	7/1	Using set diagrams for taking away	226
2	<i>Subtraction as a mathematical operation</i>	206	7/2	Using set diagrams for comparison	227
2/1	Predicting the result (subtraction)	206	7/3	Using set diagrams for finding complements	228
2/2	What will be left? (NuSp 1.4/2)	207	7/4	Using set diagrams for giving change	228
2/3	Returning over the stepping stones	207	7/5	Unpacking the parcel (subtraction)	229
2/4	Crossing back (NuSp 1.4/3)	208	8	<i>Subtraction up to 20, including crossing the 10 boundary</i>	231
3	<i>Notation for subtraction: number sentences</i>	209	8/1	Subtracting from teens: choose your method	231
3/1	Number sentences for subtraction	209	8/2	Subtracting from teens: “Check!”	233
3/2	Predicting from number sentences (subtraction)	211	8/3	Till receipts up to 20¢	233
4	<i>Number stories: abstracting number sentences</i>	212	8/4	Gift shop	234
4/1	Personalized number stories	212	9	<i>Subtraction up to 99</i>	235
4/2	Abstracting number sentences	213	9/1	“Can we subtract?”	236
4/3	Personalized number stories - predictive	214	9/2	Subtracting two-digit numbers	237
5	<i>Numerical comparison of two sets</i>	216	9/3	Front window, rear window	239
5/1	Capture (NuSp 1.4/4)	216	9/4	Front window, rear window - make your own	241
5/2	Setting the table	217	10	<i>Subtraction up to 999</i>	
5/3	Diver and winner	217	10/1	Race from 500 to 0	
5/4	Number comparison sentences	219	10/2	Subtracting three-digit numbers	
5/5	Subtraction sentences for comparisons	221	10/3	Airliner	
6	<i>Giving change</i>	223	10/4	Candy shop: selling and stocktaking	
6/1	Change by exchange	223			
6/2	Change by counting on	224			
6/3	Till receipts	224			

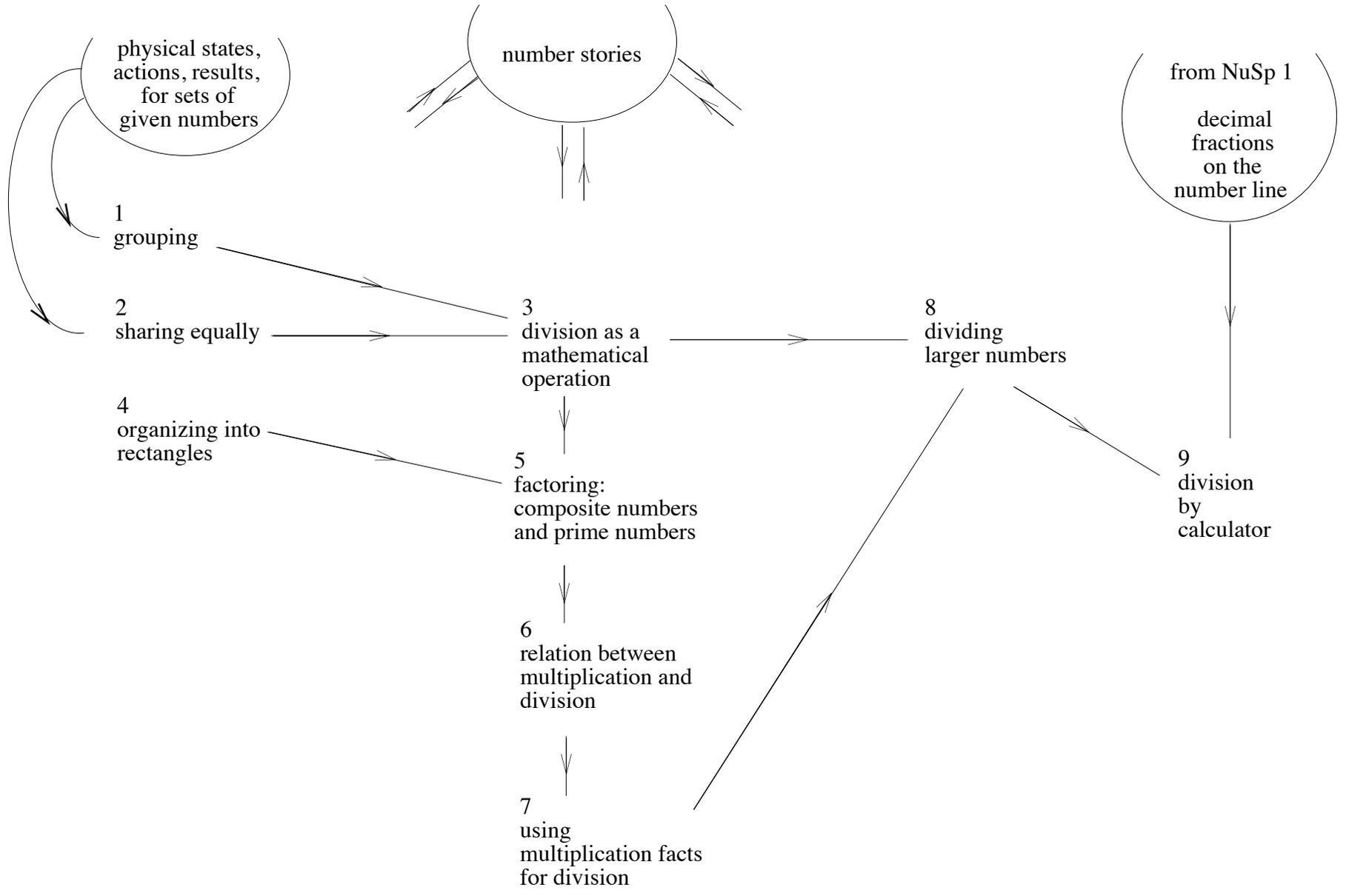
Num 5 Multiplication



Num 5 Multiplication

1	<i>Actions on sets: combining actions</i>	242		
1/1	Make a set. Make others which match	243	6	<i>Building product tables: ready-for-use results</i>
1/2	Multiplying on a number track	244	6/1	Building sets of products
1/3	Giant strides on a number track	244	6/2	“I know another way”
			6/3	Completing the products table
2	<i>Multiplication as a mathematical operation</i>	248	6/4	Cards on the table
2/1	“I predict - here” using rods	248	6/5	Products practice
2/2	Sets under our hands	249	6/6	Multiples rummy
			7	<i>Multiplying 2- or 3-digit numbers by single-digit numbers</i>
3	<i>Notation for multiplication: number sentences</i>	250	7/1	Using multiplication facts for larger numbers
3/1	Number sentences for multiplication	251	7/2	Multiplying 3-digit numbers
3/2	Predicting from number sentences	252	7/3	Cargo boats
			8	<i>Multiplying by 10 and 100</i>
4	<i>Number stories: abstracting number sentences</i>	254	8/1	Multiplying by 10 or 100
4/1	Number stories (multiplication)	254	8/2	Explaining the shorthand
4/2	Abstracting number sentences	255	8/3	Multiplying by hundreds and thousands
4/3	Number stories, and predicting from number sentences	256		
			9	<i>Multiplying by 20 to 90 and by 200 to 900</i>
5	<i>Multiplication is commutative; alternative notations: binary multiplication</i>		9/1	“How many cubes in this brick?” (Alternative paths)
5/1	Big Giant and Little Giant		9/2	Multiplying by n-ty and any hundred
5/2	Little Giant explains why			
5/3	Binary multiplication		10	<i>Long multiplication</i>
5/4	Unpacking the parcel (binary multiplication)		10/1	Long multiplication
	Alternative notations		10/2	Treasure chest

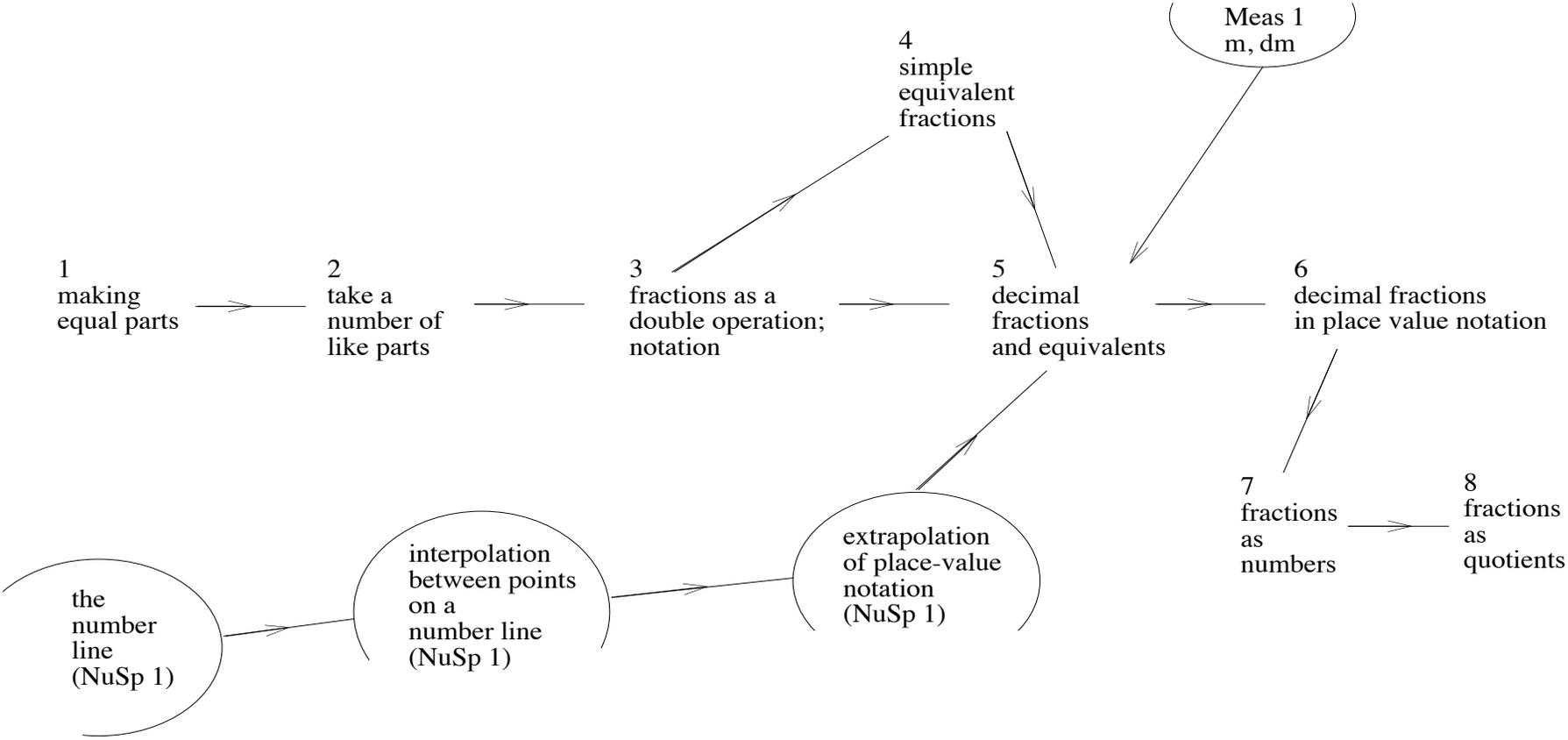
Num 6 Division



Num 6 Division

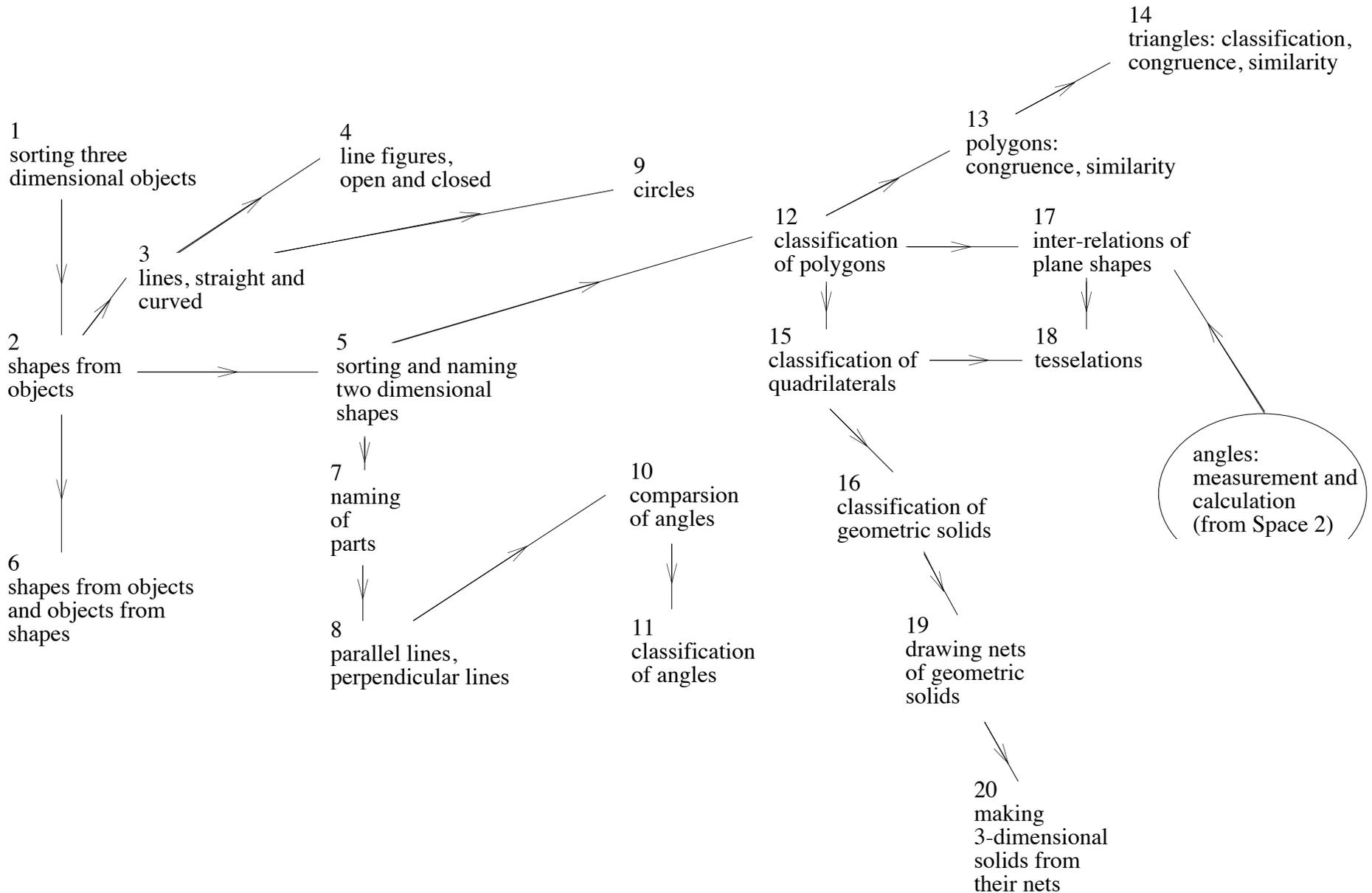
1	<i>Grouping</i>	258	5	<i>Factoring: composite numbers and prime numbers</i>
1/1	Start, Action, Result: grouping	258	5/1	Factors bingo
1/2	Predictive number sentences (grouping)	260	5/2	Factors rummy
1/3	Word problems (grouping)	260	5/3	Alias prime
			5/4	The sieve of Eratosthenes (Also in Num 1.12)
2	<i>Sharing equally</i>	264	6	<i>Relation between multiplication and division</i>
2/1	Sharing equally	264	6/1	Parcels within parcels
2/2	“My share is . . .”	266		
2/3	“My share is . . . and I also know the remainder, which is . . .”	266	7	<i>Using multiplication results for division</i>
2/4	Word problems (sharing)	268	7/1	A new use for the multiplication square
			7/2	Quotients and remainders
3	<i>Division as a mathematical operation</i>	270	7/3	Village Post Office
3/1	Different questions, same answer. Why?	270		
3/2	Combining the number sentences	273	8	<i>Dividing larger numbers</i>
3/3	Unpacking the parcel (division)	273	8/1	“I’m thinking in hundreds . . .”
3/4	Mr. Taylor’s game	274	8/2	“I’ll take over your remainder”
			8/3	Q and R ladders
4	<i>Organizing into rectangles</i>		8/4	Cargo Airships
4/1	Constructing rectangular numbers (Also in Num 1.11)			
4/2	The rectangular numbers game (Also in Num 1.11)		9	<i>Division by calculator</i>
			9/1	Number targets: division by calculator

Num 7 Fractions
as a double operation
as numbers
as quotients



Num 7 Fractions

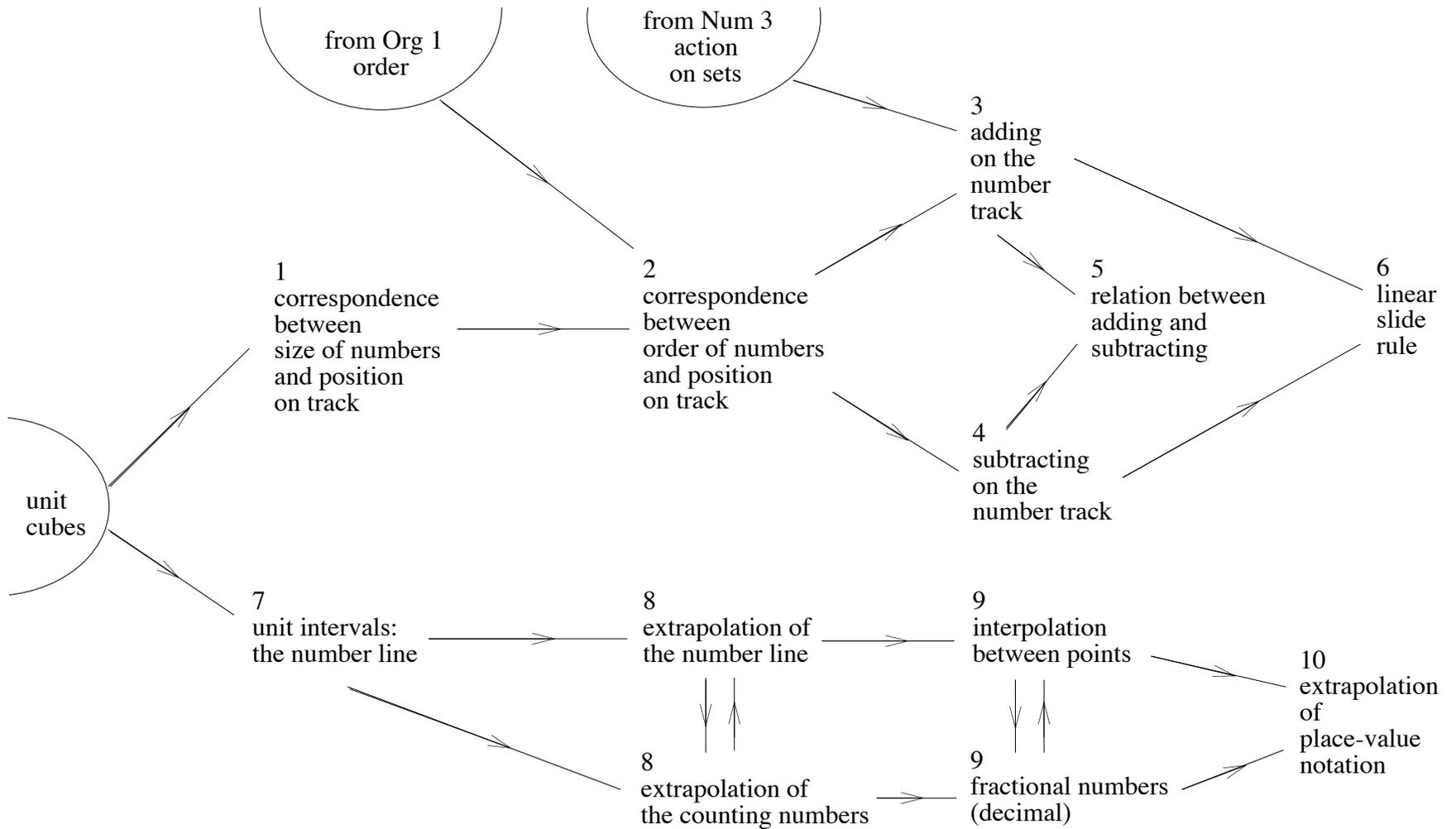
1	<i>Making equal parts</i>	276	5	<i>Decimal fractions and equivalents</i>	
1/1	Making equal parts	276	5/1	Making jewellery to order	
1/2	Same kind, different shapes	278	5/2	Equivalent fraction diagrams (decimal)	
1/3	Parts and bits	280	5/3	Pair, and explain	
1/4	Sorting parts	280	5/4	Match and mix: equivalent decimal fractions	
1/5	Match and mix: parts	282			
2	<i>Take a number of like parts</i>		6	<i>Decimal fractions in place-value notation</i>	
2/1	Feeding the animals		6/1	Reading headed columns in two ways	
2/2	Trainee keepers, qualified keepers		6/2	Same number, or different?	
2/3	Head keepers		6/3	Claiming and naming	
3	<i>Fractions as a double operation: notation</i>		7	<i>Fractions as numbers. Addition of decimal fractions in place-value notation</i>	
3/1	Expanding the diagram		7/1	Target, 1	
3/2	“Please may I have?” (Diagrams and notation)		7/2	“How do we know that our method is still correct?”	
4	<i>Simple equivalent fractions</i>		8	<i>Fractions as quotients</i>	
4/1	“Will this do instead?”		8/1	Fractions for sharing	
4/2	Sorting equivalent fractions		8/2	Predict, then press	
4/3	Match and mix: equivalent fractions		8/3	“Are calculators clever?”	
			8/4	Number targets by calculator	



Space 1 Shape

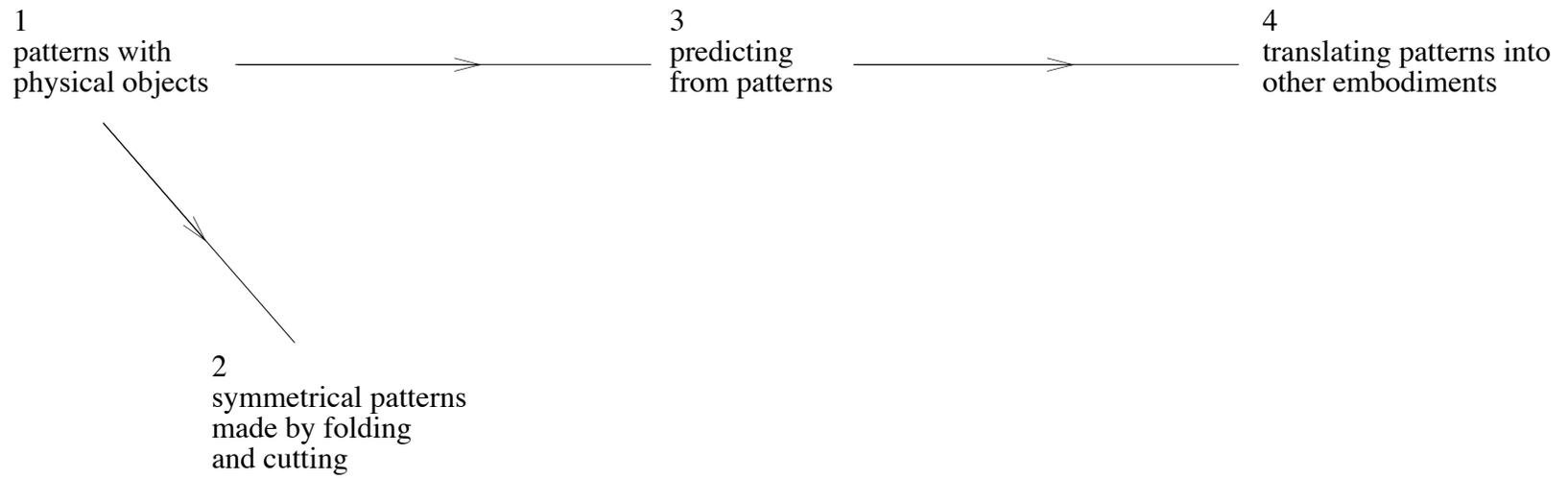
1	<i>Sorting three dimensional objects</i>	284	9	<i>Circles</i>
1/1	Sorting by shape	284	9/1	Circles in the environment
1/2	Do they roll? Will they stack?	285	9/2	Constructing circles
			9/3	Parts of a circle
2	<i>Shapes from objects</i>	286	9/4	Circles and their parts in the environment
2/1	Matching objects to outlines	286	9/5	Patterns with circles
			10	<i>Comparison of angles</i>
3	<i>Lines, straight and curved</i>	287	10/1	“All make an angle like mine”
3/1	Drawing pictures with straight and curved lines	287	10/2	“Which angle is bigger?”
3/2	“I have a straight/curved line, like . . .”	288	10/3	Largest angle takes all
3/3	“Please may I have . . . ?” (Straight and curved lines)	288	10/4	Angles in the environment
			11	<i>Classification of angles</i>
4	<i>Line figures, open and closed</i>	290	11/1	Right angles, acute angles, obtuse angles
4/1	“Can they meet?”	290	11/2	Angle dominoes
4/2	Escaping pig	293	11/3	“Mine is the different kind”
4/3	Pig puzzle	293	11/4	“Can’t cross, will fit, must cross”
4/4	Inside and outside	293	12	<i>Classification of polygons</i>
			12/1	Classifying polygons
5	<i>Sorting and naming two dimensional shapes</i>	295	12/2	Polygon dominoes
5/1	Sorting and naming geometric shapes	295	12/3	Match and mix: polygons
5/2	Sorting and naming two dimensional figures	296	13	<i>Polygons: congruence, similarity</i>
5/3	I spy (shapes)	296	14	<i>Triangles: classification, congruence, similarity</i>
5/4	Claim and name (shapes)	296	14/1	Classifying triangles
			14/2	Triangles dominoes
6	<i>Shapes from objects and objects from shapes</i>	298	14/3	Match and mix: triangles
6/1	“This reminds me of . . .”	298	14/4	Congruent and similar triangles
			15	<i>Classification of quadrilaterals</i>
7	<i>Naming of parts</i>	299	15/1	Classifying quadrilaterals
7/1	“I am touching . . .” (three dimensions)	299	15/2	Relations between quadrilaterals
7/2	“Everyone touch . . .” (three dimensions)	300	15/3	“And what else is this?”
7/3	“I am pointing to . . . (two dimensions)	300	15/4	“I think you mean . . .”
7/4	“Everyone point to . . .” (two dimensions)	300	16	<i>Classification of geometric solids</i>
7/5	“My pyramid has one square face . . .”	301	17	<i>Inter-relations of plane shapes</i>
7/6	Does its face fit?	301	17/1	Triangles and polygons
			17/2	Circles and polygons
8	<i>Parallel lines, perpendicular lines</i>	303	17/3	“I can see”
8/1	“My rods are parallel/perpendicular”	303	17/4	Triangles and larger shapes
8/2	“All put your rods parallel/perpendicular to the big rod”	304	18	<i>Tessellations</i>
8/3	Colouring pictures	304	18/1	Tessellating regular polygons
			18/2	Tessellating other shapes
			18/3	Inventing tessellations
			18/4	Tessellating any quadrilateral
			19	<i>Drawing nets of geometric solids</i>
			20	<i>Making 3-dimensional solids from their nets (if they can!)</i>

NuSp 1 The number track and the number line



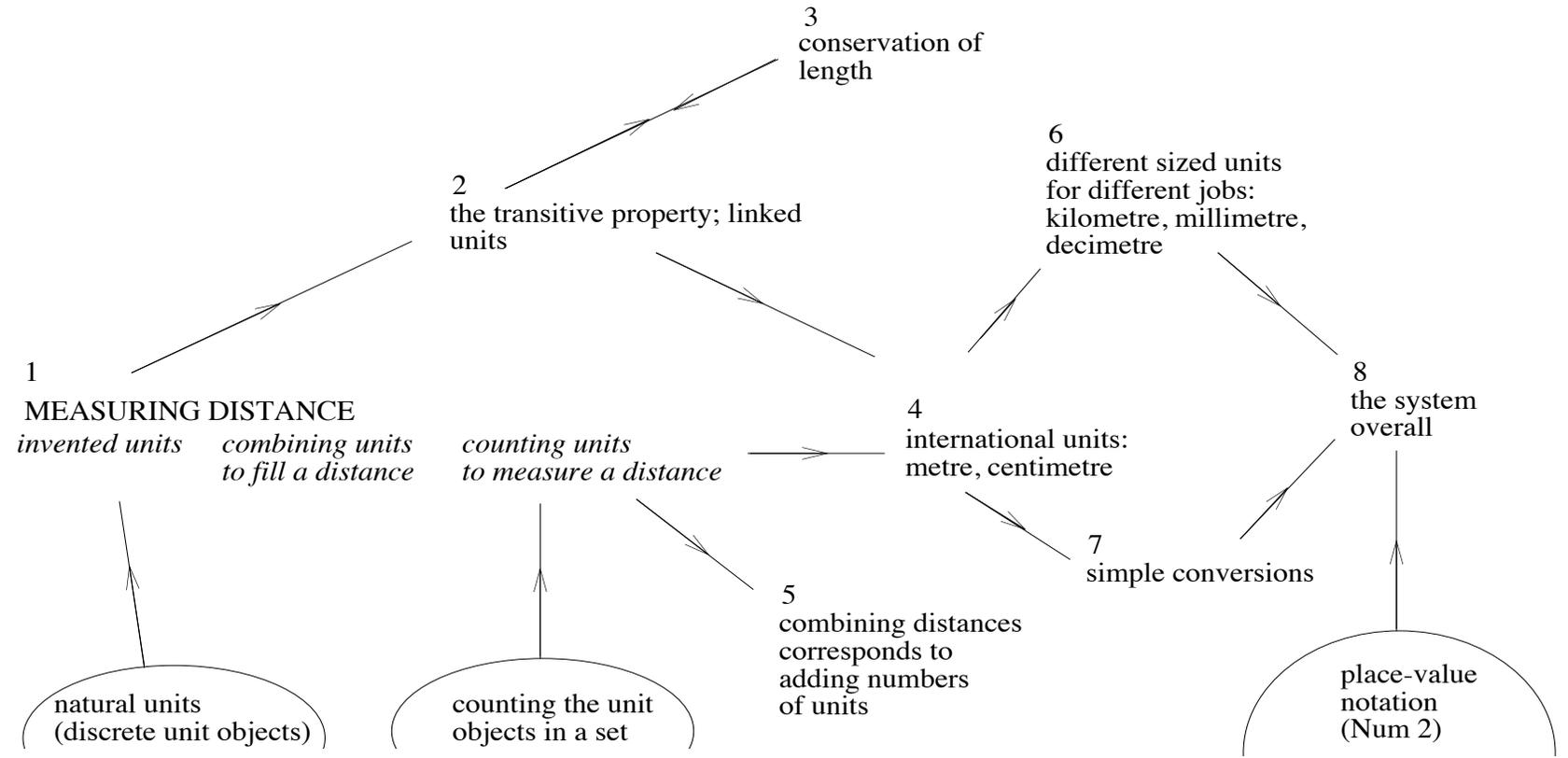
NuSp 1 The number track and the number line

1	<i>Correspondence between size of number and position on track</i>	306			
1/1	“I predict - here” on the number track	306	7	<i>Unit intervals: the number line</i>	324
2	<i>Correspondence between order of numbers and position on track</i>	309	7/1	Drawing the number line	324
2/1	Sequences on the number track	309	7/2	Sequences on the number line	325
3	<i>Adding on the number track</i>	311	7/3	Where must the frog land?	325
3/1	Putting more on the number track (verbal) [Same as Num 3.1/2]	311	7/4	Hopping backwards	325
3/2	“Where will it come?” (Same as Num 3.2/2)	313	7/5	Taking	325
3/3	Crossing (Same as Num 3.2/4)	313	7/6	A race through a maze	326
3/4	“Where will it come?” (Through 10) (Same as Num 3.7/2)	315	8	<i>Extrapolation of the number line</i>	
4	<i>Subtracting on the number track</i>	316		<i>Extrapolation of the counting numbers</i>	
4/1	Taking away on the number track (verbal) [Same as Num 4.1/2]	316	8/1	What number is this? (Single starter)	
4/2	What will be left? (Same as Num 4.2/2)	316	8/2	What number is this? (Double starter)	
4/3	Crossing back (Same as Num 4.4/4)	317	8/3	Is there a limit?	
4/4	Capture (Same as Num 4.5/1)	317	8/4	Can you think of . . . ?	
5	<i>Relation between adding and subtracting</i>	319	9	<i>Interpolation between pairs</i>	
5/1	Slow bicycle race	319		<i>Fractional numbers (decimal)</i>	
5/2	Ups and downs	320	9/1	What number is this? (Decimal fractions)	
6	<i>Linear slide rule</i>	322	9/2	Snail race	
6/1	Add and check	322	9/3	Snails and frogs	
6/2	Adding past 20	323	10	<i>Extrapolation of place-value notation</i>	
			10/1	“How can we write this number?” (Headed columns)	
			10/2	Introducing the decimal point	
			10/3	Pointing and writing	
			10/4	Shrinking and growing	



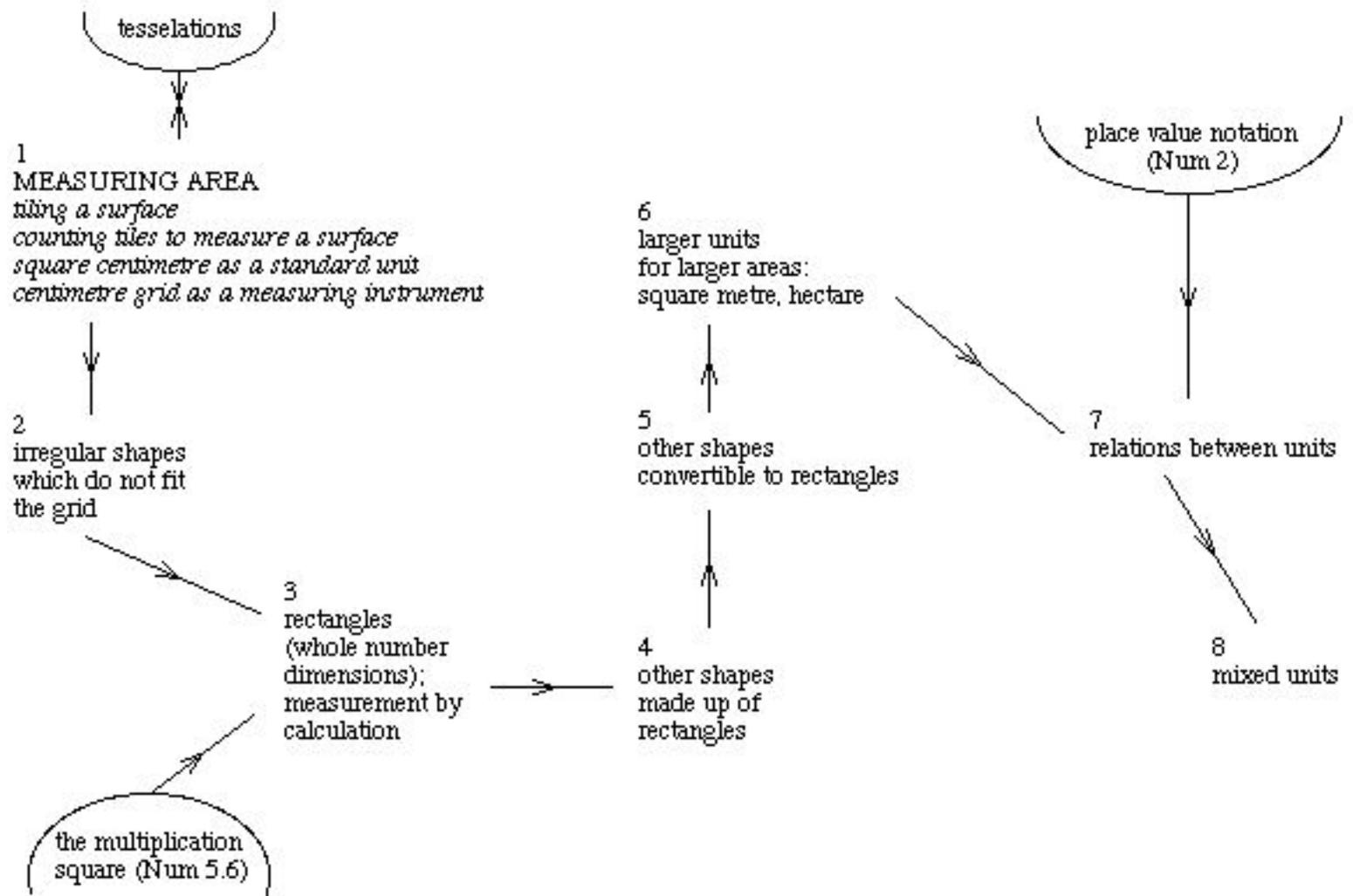
Patt 1 Patterns

1	<i>Patterns with physical objects</i>	328
1/1	Copying patterns	328
1/2	Patterns with a variety of objects	329
1/3	Making patterns on paper	329
2	<i>Symmetrical patterns made by folding and cutting</i>	331
2/1	Making paper mats	331
2/2	Bowls, vases, and other objects	332
2/3	Symmetrical or not symmetrical?	333
3	<i>Predicting from patterns</i>	334
3/1	What comes next?	334
3/2	Predicting from patterns on paper	334
4	<i>Translating patterns into other embodiments</i>	336
4/1	Different objects, same pattern	336
4/2	Patterns which match	337
4/3	Patterns in sound	337
4/4	Similarities and differences between patterns	338
4/5	Alike because . . . and different because . . .	338



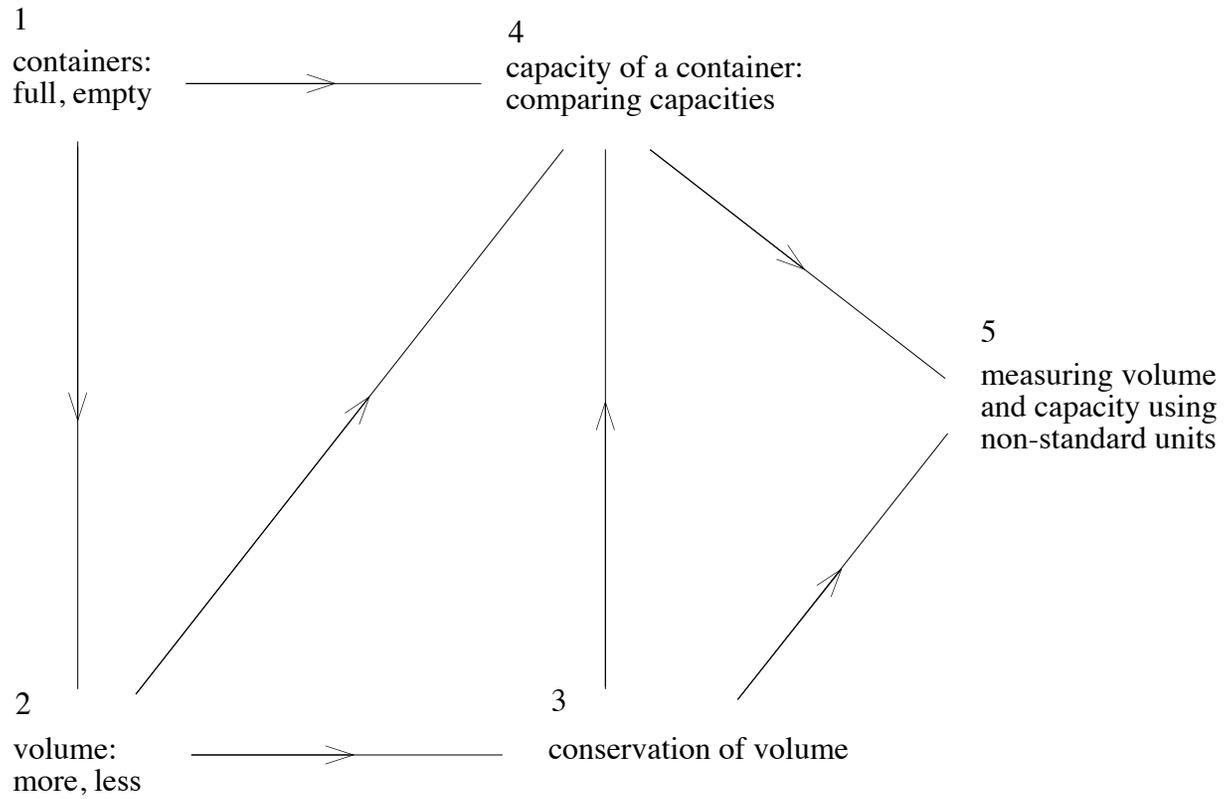
Meas 1 Length

1	<i>Measuring distance</i>	341	5	<i>Combining lengths corresponds to adding numbers of units</i>
1/1	From counting to measuring	342	5/1	Model bridges
1/2	Tricky Micky	342	5/2	How long will the frieze be?
1/3	Different names for different kinds of distance	344	6	<i>Different sized units for different jobs: kilometre, millimetre, decimetre</i>
2	<i>The transitive property; linked units</i>	346	6/1	“Please be more exact” (Telephone shopping)
2/1	Building a bridge	346	6/2	Blind picture puzzle
3	<i>Conservation of length</i>	348	6/3	“That is too exact” (Power lines)
3/1	“Can I fool you?” (length)	348	6/4	Using the short lengths (Power lines)
3/2	Grazing goat	349	6/5	“That is too exact” (Car rental)
4	<i>International units: metre, centimetre</i>	352	6/6	Chairs in a row
4/1	The need for standard units	352	7	<i>Simple conversions</i>
4/2	Counting centimetres with a ruler	353	7/1	Equivalent measures, cm and mm
4/3	Mountain road	354	7/2	Buyer, beware
4/4	Decorating the classroom	356	7/3	A computer-controlled train
			8	<i>The system overall</i>
			8/1	Relating different units
			8/2	“Please may I have?” (Metre and related units)



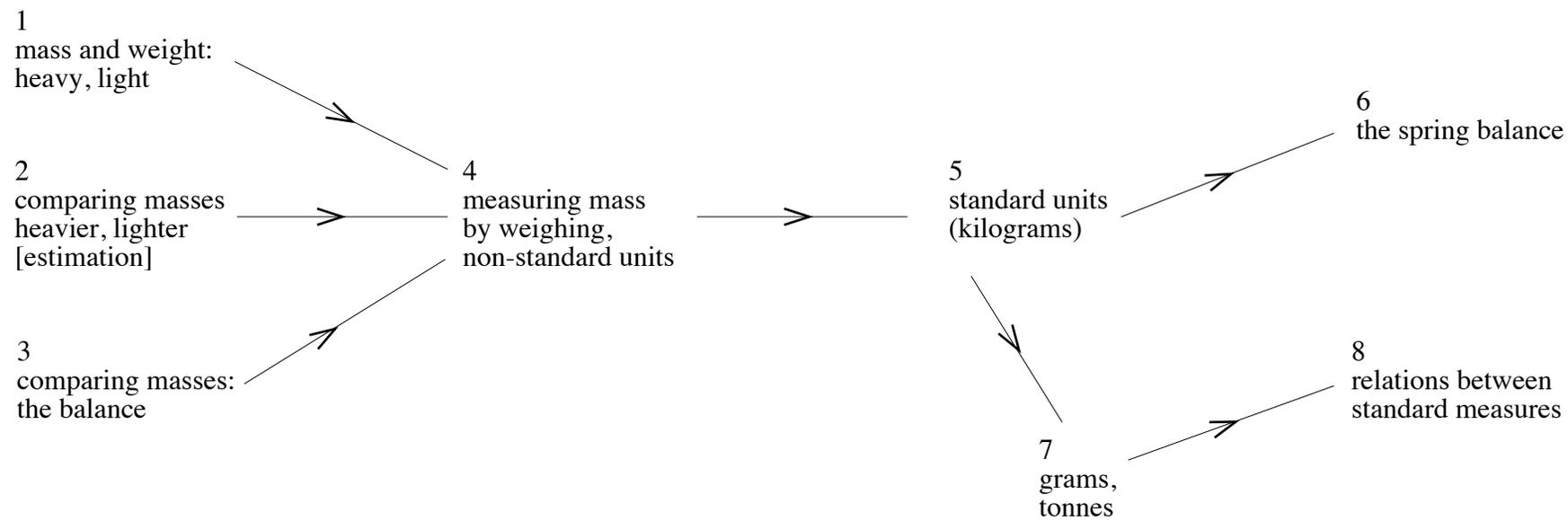
Meas 2 Area

- 1 *Measuring area*
 - tiling a surface*
 - counting tiles to measure a surface*
 - square centimetre as a standard unit*
 - centimetre grid as a measuring instrument*
- 1/1 “Will it, won’t it?”
- 1/2 Measuring by counting tiles
- 1/3 Advantages and disadvantages
- 1/4 Instant tiling
- 2 *Irregular shapes which do not fit the grid*
 - 2/1 Shapes and sizes
 - 2/2 “Hard to tell until we measure”
 - 2/3 Gold rush
- 3 *Rectangles (whole number dimensions)*
 - Measurement by calculation*
- 3/1 “I know a short cut”
- 3/2 Claim and explain
- 3/3 Carpeting with remnants
- 4 *Other shapes made up of rectangles*
 - 4/1 “Do it yourself” in a doll’s house
 - 4/2 Claim and explain (harder)
 - 4/3 Net of a box
 - 4/4 Tiling the floors in a home
- 5 *Other shapes convertible to rectangles*
 - 5/1 Area of a parallelogram
 - 5/2 Area of a triangle
 - 5/3 Area of a circle
- 6 *Larger units for larger areas: square metre, hectare*
 - 6/1 Calculating in square metres
 - 6/2 Renting exhibition floor space
 - 6/3 Buying grass seed for the children’s stately garden
 - 6/4 Calculating in hectares
 - 6/5 Buying small holdings
- 7 *Relations between units*
 - 7/1 What could stand inside this?
 - 7/2 Completing the table
 - 7/3 “It has to be this one”
- 8 *Mixed units*
 - 8/1 Mixed units



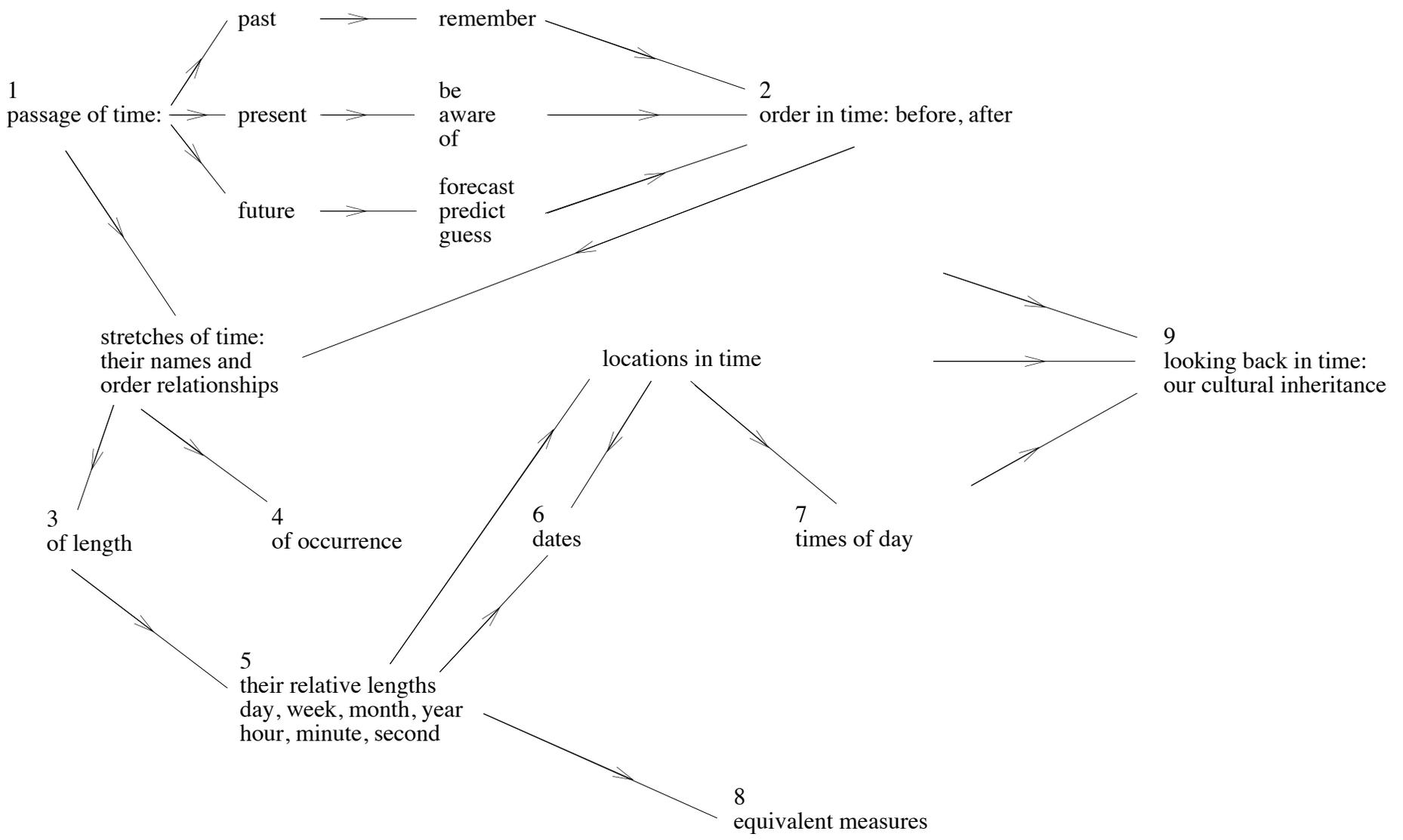
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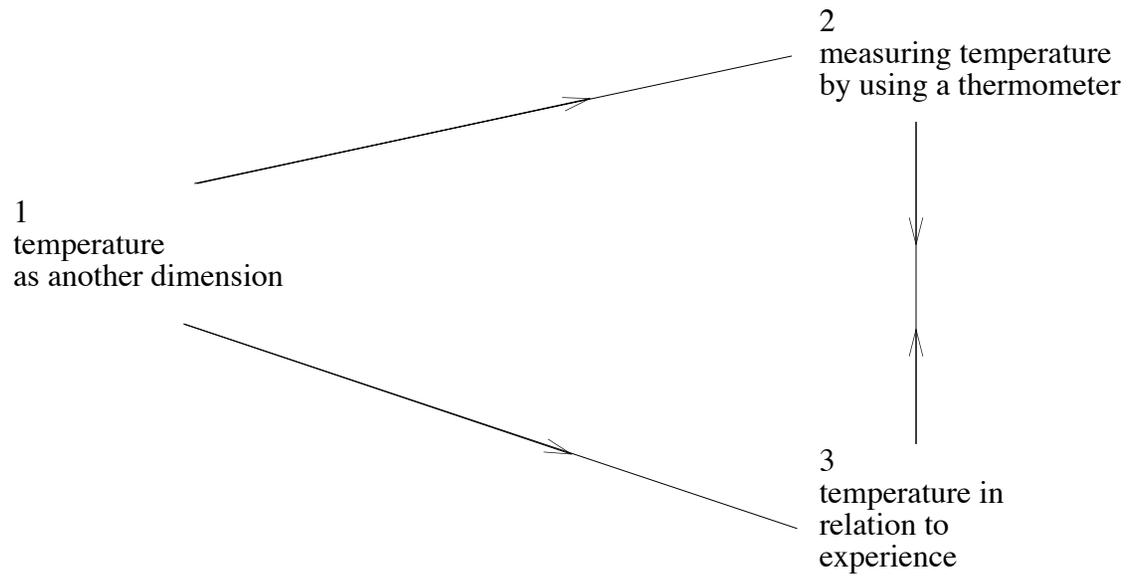
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THE NETWORKS AND ACTIVITIES



*'Slippery slope' [Num 3.7/3] **

* A frame electronically extracted from
NUMERATION AND ADDITION,
a colour video produced by the
Department of Communications Media,
The University of Calgary



Perceptual matching of objects [Org 1.1/1]

[Org 1] SET BASED ORGANIZATION

Organizing in ways which lay foundations for concepts relating to number

Org 1.1 SORTING

Concept Likeness between objects.

Ability Sorting a mixture of objects into sets whose members are alike in some way. This common property may be perceptual or conceptual.

Discussion of concept Children will do this with little prompting, given suitable materials . . . sometimes spontaneously. This is because sorting is closely linked with intelligence. Until we have classified an object, i.e., until we know what it is, we do not know in what ways we can use it to help us achieve our goals.

Activity 1 Perceptual matching of objects [Org 1.1/1]

A teacher-led activity for a small group of children. Its purpose is to help them to become aware of the likenesses and differences in the objects.

- Materials*
- Boxes of different kinds of assorted small objects, several of each kind. E.g., sea shells, crayons, empty matchboxes, screws.
 - Shallow trays for sorting into. E.g., box lids.
 - Large box into which the sorted trays can be put.
 - For later stages, with older children: objects of the same kind, but not identical. E.g., all screws, or all sea shells, etc.

- What they do*
1. Tip the assorted objects onto the table and say, “These are all mixed up. Can you sort them?” If necessary demonstrate: “Like this,” placing the objects into the trays.
 2. When the objects have been sorted, place all the separate trays in the box. We don’t want to undo the children’s sorting before their eyes, as soon as they have finished.

Activity 2 Matching pictures [Org 1.1/2]

An activity for a small group. The purpose is to progress from sorting the actual objects to sorting pictures which represent them.

- Materials*
- A pack of picture cards, which can be sorted into sets.*
*The pictures should not be identical, but different objects of the same kind. Pictures cut from mail order catalogues, stuck on plain cards, are a useful source for these.

Org 1.1 Sorting (cont.)

- What they do*
1. Place the cards face up on the table and say, “These are all mixed up. Can you sort them?” If necessary, demonstrate: “Like this,” placing pictures of similar objects in one pile.
 2. *Encourage naming the sets* which result from the sorting.

Activity 3 A picture matching game [Org 1.1/3]

For a small group. This is to give more practice in sorting.

- Materials*
- As in Activity 2.

- What they do*
1. The cards are shuffled and dealt face down.
 2. The player on the left of the dealer turns over his top card and puts it face upwards on the table.
 3. Each player in turn does likewise. If his card is like a card or pile already there, it is added to that pile and the player has a second turn. (But not more, or the game ends too quickly.)
 4. If it does not match one of the existing piles, a new pile is started.
 5. The winner is the first to have put down all his cards, after which the others may play out their hands until all the cards are down.
 6. *Encourage naming the sets* which result from the sorting.

Activity 4 Dominoes [Org 1.1/4]

This is a very useful game for two players, since once the rules have been learned (and they are not difficult), the game can be played with the same rules but using increasingly advanced concepts of matching.

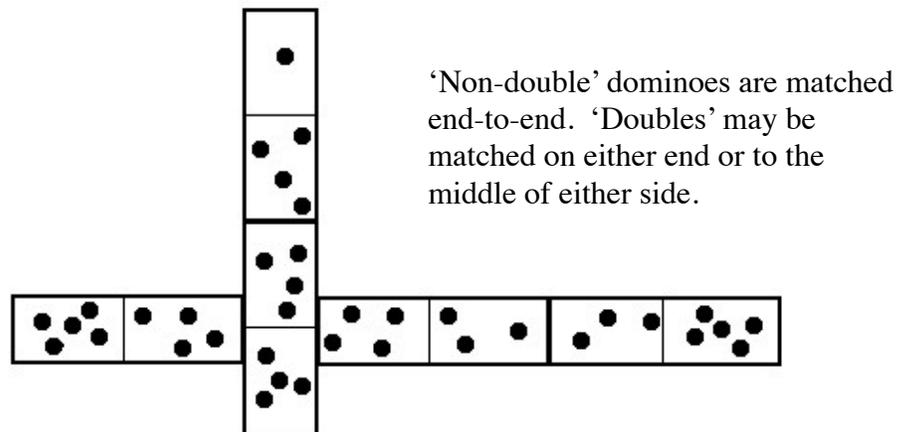
- Materials*
- 3 sets matching by
 - a) colour,
 - b) shape, or
 - c) number of dots*

*These should be positioned irregularly, so that children match by number and not by patterns. The set may go up to 5 dots, but for beginners only up to 3, then 4. These are numbers which can be recognized perceptually, without counting.

- What they do*
- Given the appropriate set of dominoes, the children will match them by colour, by shape, or by number of dots while playing dominoes as follows:
1. All of the dominoes are placed face down and mixed.
 2. Each player draws 4 dominoes, looking at them but keeping them hidden. The remaining dominoes are left face down as a pool to draw from.
 3. The game is started when one player places a ‘double’ in the centre of the playing area. [If no ‘doubles’ have been drawn, all of the dominoes are returned face-down to the pool, reshuffled, and redrawn.]
 4. Play proceeds to the left. The second player tries to match a domino to one end or to a side of the ‘double’ placed by the first player. If no match can be made, dominoes are drawn from the pool, one-at-a-time, until one can be made. If the

last domino has been drawn and a match cannot be made, play continues to the left. Only one domino may be played at each turn by being placed length-wise at any open end (not at right angles, except in the case of a 'double,' as shown below, or to avoid the edge of the playing area). A player must play a domino if able to do so. 'Doubles' are placed crosswise to the end matched, giving two new directions in which to place dominoes.

5. The next player tries to match a domino to one end of a 'non-double' or 'double' or to either side of a 'double.'
6. Play continues until one player has no dominoes left or until no one can play. When no further plays can be made, the player with no dominoes, or the least number of dominoes, is the winner.



Activities 5 and 6 Conceptual matching [Org 1.1/5,6]

The aim in these activities is to begin sorting by properties which are conceptual, not perceptual. This means that the likenesses are not visible but in our thinking.

- Materials*
- A pack of picture cards which can be sorted into sets. E.g., vehicles, tools, garments, dwellings.

Functional concepts (what we use things for) are a fruitful source of ideas here.

What they do **Activity 5**

1. Place the cards face up on the table and say, "These are all mixed up. Can you sort them?" If necessary, demonstrate: "Like this," placing similar pictures in one pile.
2. *Encourage naming the sets* which result from the sorting.

Activity 6

1. The cards are shuffled and dealt face down.
2. The player on the left of the dealer turns over his top card and puts it face upwards on the table.
3. Each player in turn does likewise. If his card is like a card or pile already there, it is added to that pile and the player has a second turn. (But not more, or the game ends too quickly.)
4. If it does not match one of the existing piles, a new pile is started.
5. The winner is the first to have put down all his cards, after which the others may play out their hands until all the cards are down.
6. *Encourage naming the sets* which result from the sorting.

Org 1.1 Sorting (cont.)

Activity 7 Attribute cards [Org 1.1/7]

An activity for a small group. Its purpose is to extend what has gone before to matching by two attributes simultaneously.

- Materials*
- A pack of attribute cards.*
 - 3 reminder cards on which are written respectively ‘Number’, ‘Colour’, ‘Shape’.**

*Each card has one, two, three, or four shapes, all alike. These shapes may be squares, circles, triangles, oblongs, all of the same colour. The colours may be red, blue, yellow, or green. This gives 64 cards, leading to 16 piles.

**The word on each card should be written four times so that every player can see it the right way up.

- What they do*
1. Two of the reminder cards are placed on the table to show which two attributes are currently to be used for sorting: e.g., colour and shape.
 2. The game is then played as for activities 3 and 5, except that matching is by two attributes simultaneously. E.g., if matching by colour and shape, a card with 2 red triangles would be matched with cards having any number of red triangles.
 3. The game may be introduced using only one attribute, and a small number of cards.

Discussion of activities

In Activities 1, 2, 3, 5, 6, children learn to sort by qualities which are increasingly abstract. This progression may need plenty of time and repetition, and should not be hurried. Activity 4 provides variety, and this may also be given by different boxes of objects to be sorted or different packs of cards. Activity 7 introduces greater complexity. The later stage of Activity 1 (e.g., all shells, all buttons) requires careful observation and thought in order to decide on qualities by which they can be sorted. This heightens consciousness of what they are doing, and is valuable for older children.

In all cases, discussion should be encouraged for reaching agreement, or explaining, which objects or cards are put together. This, together with naming the sets, encourages Mode 2 schema construction.

These activities can only be done by the use of conceptual learning – they would be impossible by rote learning. So sorting is of even greater importance as a foundation for mathematical learning than is generally realized, since it ‘switches in’ the right kind of learning from the beginning.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.2 SETS

Concept A set, as the result of the process of sorting.

Abilities

- (i) To be able to make the mental decision as to what goes into a set and what does not, with physical objects.
- (ii) Also with mental objects.
- (iii) To be able to sort the same mental or physical objects differently according to choice.

Discussion of concept The result of sorting is a collection of objects (physical or mental) called a *set*. So we can think of sorting as a process, and a set as the result. What goes into a set, or does not, is the result of a mental decision, so the same objects may be sorted differently according to choice, and the resulting sets will be different.

Activity 1 Introduction to Multilink or Unifix [Org 1.2/1]

Its purpose is to introduce the cubes to the children and to give them experience in manipulating them. Also, to give further experience with making sets.

Materials

- Multilink or Unifix cubes in a variety of colours for each child.

What they do

1. Sort these by colour.
2. Join into rods, to form 'the blue set', 'the red set', etc.

Activity 2 Making picture sets [Org 1.2/2]

An activity for any number of children. The purpose is to give practice in naming sets.

Materials

- Envelopes containing pictures cut from mail order catalogues.
- Sheets of paper.
- Glue.

What they do

1. The children sort the pictures into sets.
2. They stick each of these onto a separate sheet of paper.
3. Name these sets, e.g., 'a set of flowers', 'a set of milk bottles'.
4. These can then form permanent sets for display.

Activity 3 “Which set am I making?” [Org 1.2/3]

An activity for a small group. Its purpose is to give the children experience of sorting the same objects by different attributes.

- Materials*
- A set of attribute blocks, which can be sorted by colour, shape, size, or thickness.
 - A set loop.

What they do Stage (a)

1. The children familiarize themselves with the attribute blocks. Let them take one each, and ask in turn, “What can you say about your block?” E.g., “It’s red. It’s a circle.” And, comparing it with others, “It’s small and thick.” (Or, smaller and thicker).
2. Tip all the blocks onto the table.
3. Put out the set loop and discuss how we could make a set by choosing, e.g., all the blue blocks, or all the squares, or all the thin ones.
4. Let the children together pick out the blocks which belong to the chosen set and put them into the loop.
5. Ask them to describe this set, e.g., “The set of oblongs.”
6. Repeat until the children are confident, using different kinds of attribute, e.g., colour, or shape, or size, or thickness.

Stage (b)

1. As before, the blocks are spread out on the table and a set loop is put down.
2. One player decides on an attribute.
3. She begins to sort by this attribute, putting inside the loop all blocks which belong to the set she is thinking of.
4. She should also put some non-examples outside the loop.
5. The others watch and try to discover which set she is making.
6. If one of the players thinks that she knows, she demonstrates this by sorting the next two or three blocks. In this way the others are not told prematurely.
7. Finally the set is named, e.g., “The set of red blocks.”
8. Repeat, using different kinds of attribute.

Activity 4 “Which two sets am I making?” [Org 1.2/4]

An activity for a small group. This activity extends the concept of a set to two attributes.

- Materials*
- A set of attribute blocks.
 - 2 set loops.

- What they do*
1. This is played as in Activity 3, except that two loops are put down, not overlapping.
 2. Initially, the player doing the sorting is likely to choose two attributes of the same kind, e.g., red, green.
 3. With experience, attributes of different kinds may be used, e.g., red, circle.
 4. So all the red blocks will be put in one set loop, and all the circular blocks in the other loop. But what about the red circles?



“Which two sets am I making?” [Org 1.2/4]

5. Given time, children will usually arrive at the idea of overlapping the two set loops. It is good to allow them time to think of this themselves.

**Discussion
of activities**

The purpose of these activities is to separate out the idea of a set from the sorting activity which leads to it. In Activity 4, both these ideas are used in a more sophisticated way.

As in Topic 1, conceptual learning is the only way in which these activities can be done successfully.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.3 COMPARING SETS BY THEIR NUMBERS

- Concepts*
- (i) A set as being composed of units.
 - (ii) Larger or smaller sets as referring to the number of units, not the size or location of the objects.

- Ability*
- (i) To say which of two sets has more units, or whether they both have the same number.

Discussion of concepts By "Comparing sets by numbers" I only mean "Which set has the larger number? Or have they both the same number?" This is the preliminary stage only for Num 4.5, in the subtraction network, where the further question is asked "How many more, or less?"

A set of three large objects, widely spaced, looks much larger than one of five small objects close together. To realize that the latter has the larger number of objects therefore requires the ability to think of the elements just as units, regardless of size, location, or anything else except the fact that they are separate objects. This isn't easy, and we must give children time to form this concept starting with unit objects which are all the same size.

Activity 1 Lucky dip [Org 1.3/1]

An activity for a small group. Its purpose is to introduce the comparisons 'larger' and 'smaller' in relation to sets.

- Materials*
- Cubes: equal numbers of each colour.*
 - An opaque bag.
 - A box or container for discarded cubes.
- * Hereafter used to mean Multilink or Unifix.

- What they do*
1. Each player chooses a colour.
 2. The cubes are mixed up in the bag.
 3. In turn they reach in and take out a cube.
 4. If a player gets a cube of the colour he is collecting he joins it to those he already has to form a rod.
 5. If the wrong colour, he discards it.
 6. The winner is the one who finishes with the longest rod.
 7. Discarded cubes should be put into the container, not returned to the bag.

Variation

If a player gets a cube of the wrong colour for himself, he may either discard it, or give it to another player who is collecting that colour and take out two more cubes for himself. These he uses or discards.

Activity 2 “Can I fool you?” [Org 1.3/2]

A group game, for a teacher and children together. Its purpose is to emphasize that the number of units in a set does not depend on their positions.

- Materials*
- Cubes of different colours for each child.
 - Some counters.

- What they do*
1. Each child in turn puts down some cubes. The cubes must be separate, not joined.
 2. The teacher also puts down some cubes for the child to compare with his own set, and predict which will make the longer rod.
 3. If his prediction is correct, the child receives a counter.
 4. The teacher tries to ‘fool’ the children by the way he arranges his cubes. E.g., he might put a smaller number of cubes, more widely spaced; or a larger number, closer together.
 5. The winner is the child who has the most counters.



“Can I fool you?” [Org 1.3/2]

Discussion of activities In these activities, the alternation between a set of scattered cubes and the set joined into a rod is of great value in developing the two aspects of the set concept: a single entity, made up of a number of units.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.4 ORDERING SETS BY THEIR NUMBERS

Concepts (i) Order in space (left to right).
(ii) Order of length.
(iii) Order of number of units. } These three concepts are closely connected.

Abilities (i) To put several sets in order of number, ascending from left to right.
(ii) The same descending.

Discussion of concepts The difficulty of doing this should not be under-estimated, since it involves not just a single comparison as in Org 1.3, but combining the result of several comparisons. We therefore take it quite slowly.

Activity 1 Ordering several rods by their lengths [Org 1.4/1]

An activity for a small group of children with their teacher. Its purpose is to teach them to combine ordering by length with ordering in space.

Materials For each pair:

- Cubes of different colours.
- A shallow tray or box lid.

What they do

1. Each child makes three rods, all of different lengths and colours.
2. They put these on the tray in order of length, shortest on the left, tallest on the right.
3. They then mix up the rods, while leaving them on the tray; and exchange with another child, who puts them back in order again.
4. Steps 1, 2, 3 are repeated with different lengths and colours. This may conveniently be done by exchange between different children.
5. When they can do this easily, repeat with four rods, then five.

Activity 2 Combining order of number, length, and position [Org 1.4/2]

A harder version of Activity 1. This makes a start with ordering by number, independently of length. This is then tested as in Activity 1.

Materials • As for Activity 1.

What they do

- Steps 1 and 2 are the same as in Activity 1, starting with three rods.
3. They then break the rods into single cubes. These are left in groups of the same colour, not in a row.
4. They exchange with another child. This child first says which group she thinks will make the longest rod. A spare cube of the same colour is put down to record her prediction.

Org 1.4 Ordering sets by their numbers (cont.)

5. The cubes are then re-formed into rods, and put in order of length. The prediction made in step 4 is tested.
6. Steps 3, 4, and 5 are repeated. This time, the other child has to predict which group will make the shortest rod.
7. When they can do this easily, repeat with four rods, then five.

Variation

After step 2, a colour record is made of the order by forming cubes of appropriate colours into a rod. After step 5, this colour record is compared with the reconstituted collection.



Combining order of number, length, and position [Org 1.4/2]

Discussion
of activities

We are asking the children to combine three orderings: by length, position, and number. Moreover, each of these orderings involves a number of comparisons. E.g., when ordering by length, a given rod has to come after every shorter rod, and before every longer rod. So we should allow them as much time as they need to work through the stages described. This varies greatly between children.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.5 COMPLETE SEQUENCES OF SETS

Concept An ordered sequence of sets with ‘no gaps’, i.e., in which each set is of number one more than the one before, and one less than the one after (except for the first and last sets).

Abilities

- (i) To construct a sequence of this kind.
- (ii) To extrapolate such a sequence.
- (iii) To tell whether a sequence is complete or not; and if it is not, to locate and fill the gaps.

Discussion of concept When we have ordered a given tray-full of rods, we have a sequence of a kind. But there is no regularity about it. If we take a rod away, there is nothing to show which one is missing. And between any two rods, as likely as not we can insert another which will conform to the same ordering.

A sequence of counting numbers, however, has a special property. Provided we know where it starts and finishes, we can tell whether or not it is complete; and if not, which ones are missing and where they should go in order.

A staircase of rods shows this property very clearly.

Activity 1 Missing stairs [Org 1.5/1]

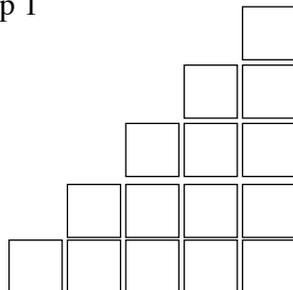
An activity for children to play in pairs. Its purpose is to introduce them to the concepts described, in a way which allows testable predictions.

Materials • Cubes. A different colour for each of the pair.

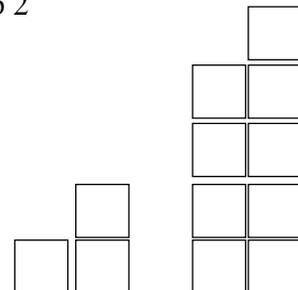
What they do Stage (a)

1. Child A makes a staircase from rods made up of one to five cubes, all the same colour, as illustrated.
2. B removes one rod, hiding the missing rod from sight.
3. A then makes from loose cubes of a different colour a rod which he predicts will fit the gap.
4. This prediction is tested in two ways: by insertion into the gap in the staircase, and by comparison with the rod which was removed.

Stage (a) Step 1



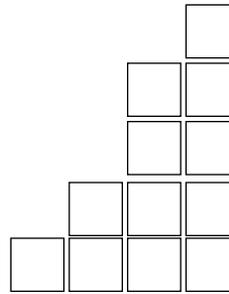
Step 2



Stage (b)

As in stage (a), except that in step 2, child A closes the gap.

Stage (b) Step 2



Stage (c)

As in stage (b), except that now child B closes his eyes during step 2. So he now has to decide where there is a missing rod, as well as make a matching replacement.

Stage (d)

The number of rods may gradually be increased to 10. Children will often make this extrapolation spontaneously.

The first three stages may usually be taken in fairly rapid succession.

Discussion of activity This was one of the earliest activities I devised, and it was an important learning experience for me. I realized that even such a simple mathematical model as the first five counting numbers, in order, could be used to make testable predictions; and I observed the pleasure of five year old children when their predictions were correct.

 With further reflection, I came to realize that the simplicity of this mathematical model is only apparent. On closer examination, one begins to appreciate how condensed and sophisticated it is.

OBSERVE AND LISTEN

REFLECT

DISCUSS



*'Missing stairs' [Org 1.5/1] **

* A frame from
 NUMERATION AND ADDITION,
 a colour video produced by the
 Department of Communications Media,
 The University of Calgary

Org 1.6 THE EMPTY SET; THE NUMBER ZERO

- Concepts*
- (i) The set which contains no element,
 - (ii) and whose number is therefore zero.

- Abilities*
- (i) To recognize the empty set when it occurs.
 - (ii) To state its number, zero.

Discussion of concepts The mathematical name for this is 'the null set', but I think that 'the empty set' is a better name for beginners. It is the set which has no member, like an empty box. Why 'the'? Because mathematicians argue that there is only one null set, but we do not need to stress this point.

The number of this set is zero. "Nothing," or "None," are reasonable answers to the questions "What is in the loop?", or "How many objects in the loop?" But a number is a property of a set, and the question we are asking is "What is the number of this (the empty) set?"

From the foregoing, we see that the concept of zero is more subtle than usually realized. It is the only number in all these networks which has a topic all to itself.

Activity 1 The empty set [Org 1.6/1]

A teacher-led discussion for a small group. Its purpose is to introduce the concept of the empty set, together with that of its number zero.

- Materials*
- Cubes in 3 different colours.
 - 4 set loops.

- What they do*
1. Ask the children to sort the mixture of colours (say) red, blue, yellow, using the 4 set loops (see Org 1.2 / 3).
 2. One of the loops will remain empty.
 3. Point to one of the loops with cubes in it and ask "What do we call this set?" "And what is its number?"
 4. Answers, e.g., "The blue set"; "Three." Likewise for the red and yellow sets.
 5. Then point to the empty set loop and ask "What can we call this set?"
 6. Accept any reasonable suggestions, and explain that many people call it "The empty set" because the loop is empty.
 7. "Does anyone know what we call the number of this set?" "Nothing," or "None," are perfectly sensible replies, but these are not names of any number. "Oh" is incorrect, since this is the name of a letter. Explain that the correct word for the number of this set is "zero".

Variations

Other mixtures to be sorted; other containers (e.g., lids, shallow boxes). We shall return to the empty set in Num 1.6.



The empty set [Org 1.6/1]

Discussion
of activity

We are here extrapolating the idea of a number, which until now has applied only to sets of objects which we can see. So the concept of zero is more abstract than the number concepts which children have encountered so far.

The children are thus dependent on their imaginations to form this new number concept. It is mode 3 reality building, with help from mode 2 (communication and discussion). Vocabulary is important here, since it helps to give precision to the new idea.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.7 PAIRING BETWEEN SETS

Concept That of pairing between objects in different sets.

- Abilities*
- (i) To be able to make mental and physical pairs of single objects in one set with single objects in another set.
 - (ii) To understand and use the associated vocabulary.

Discussion of concepts When we collect objects into sets, physically or mentally, we are putting together objects which in some way 'go together'. (If we reflect on this further, we find that the important connection is the mental one. Putting things within set loops, or into the same box, are helpful ways of representing this; but a set is a mental entity, not a physical one. (Don't bother if this is not clear right away).

Another important kind of connection is pairing: putting together, physically or mentally, single objects in one set with single objects in another set. Everyday examples abound. Given a set of children and a set of milk bottles, we (physically) pair one child with one milk bottle. Cups and saucers, socks and feet, car wheels and tires, are other examples of 'natural' pairings.

At this stage we are not concerned about whether all the objects in both sets can be paired, or whether there are some left over. This comes in the next topic. The present topic is about the pairing activity itself.

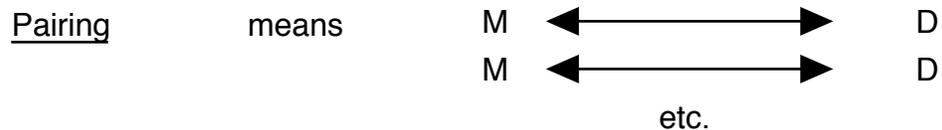
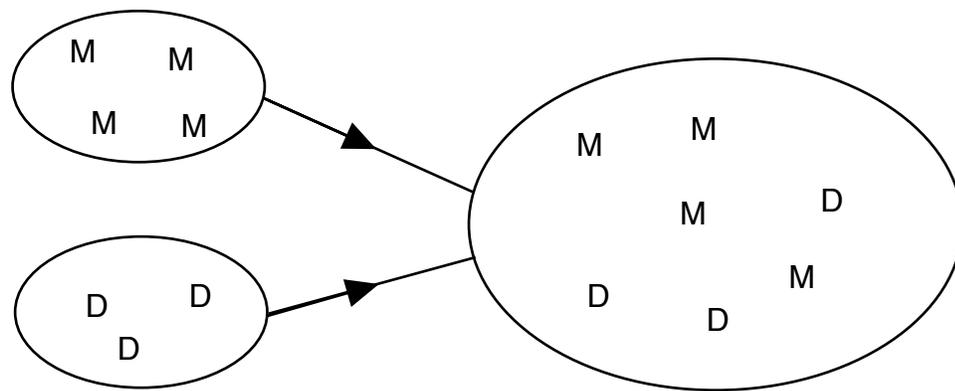
Activity 1 Physical pairing [Org 1.7/1]

An activity for a small group of children. Its purpose is to introduce the concept and vocabulary of pairing.

- Materials*
- A box of physical objects, not necessarily equal numbers of each kind, e.g., model dogs, men, sheep, lambs.
 - Likewise, in pictures: one picture to a card.
 - Cubes in different colours.

- What they do*
1. Take from the box two sets of objects which it makes sense to pair, e.g., men and dogs.
 2. Put together a man and a dog, explaining "I'm *pairing* this man and this dog. The two together make a pair."
 3. Let the children continue until no more pairs can be made. If there are objects of one set left over, point this out. E.g., "We can't pair these because there aren't any more dogs to pair the men with."
 4. Repeat with other objects. In some cases there should be the same number of articles of each kind (we feel that each lamb should have an ewe, and vice versa); and in other cases, there should be unequal numbers.
 5. Repeat with materials (b) and (c).
 6. Also use naturally arising situations, e.g., coats and pegs, children and chairs, foot and shoe.

Note that we are not “putting the men with the dogs,” which would mean doing this:



Activity 2 Mentally pairing [Org 1.7/2]

A continuation of activity 1. Its purpose is to move them on to pairing mentally.

Materials The same as for Activity 1.

What they do Get children to discuss. Move on to other natural pairings which are not perceptually present, e.g., cup and saucer, car and driver.

Discussion of activities In Activity 1, the children build the concept of pairing (verb) and that of a pair (noun) from physical examples. In Activity 2 they use these at an abstract level of thinking, seeking in their memories for other examples of these concepts. They are thus moving around ideas, in their minds, instead of physical objects, on a table. In ways such as these we help them to progress in the abstract way of thinking which is characteristic of mathematics.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.8 SETS WHICH MATCH, COUNTING, AND NUMBER

- Concepts* (i) One-to-one correspondence. (The concept, not the vocabulary yet.)
(ii) Sets which match, as described below.

Ability To state correctly whether two sets match.

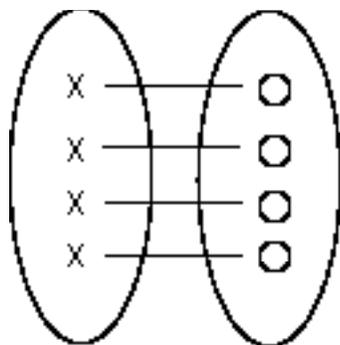
Discussion of concepts

'Match' is used with more than one meaning, as an adjective and a verb. To avoid confusion, I shall use it to describe objects which do match, i.e., are alike in some particular way; and not to mean comparing them to see whether they match. E.g., 'matching sewing cotton and material' means cotton and material of the same colour, not that we are comparing cotton and material to see whether we have colours which match.

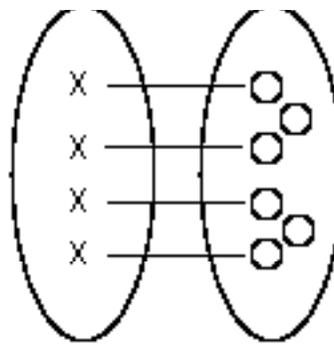
Objects may match according to a variety of qualities, such as colour, shape, kind of fruit (sorting apples, oranges, . . .), what they are used for. A set may be also thought of as a single entity, so the possibility arises of having sets which do or do not match.

Mathematicians centre on a particular quality for matching sets, which is their number. The test for whether two sets match, i.e., have the same number, is to pair the objects in one set with the objects in the other set. If this can be done with no objects left over in either set, the sets match, and the number of each set is the same. To put it in other words, two sets match if and only if all the objects in either set can be put in one-to-one correspondence with all the objects in the other set. Mathematicians use the term 'equivalent sets' with the same meaning as 'sets which match.'

A diagram shows this relation between sets better than words, though it does not so clearly put it in the context just described.



These sets match.
They have the same number.



These sets do not match.
They have different numbers.

Activity 1 Sets which match [Org 1.8/1]

An activity for two children, or a small group. Its purpose is (i) to develop the idea of pairing objects into that of making sets which match, i.e. are in one-to-one correspondence. (ii) To relate the concept of matching sets with the process of counting.

Materials For each child:

- A set loop.
- A container with 10 objects of the same kind, e.g., bottle tops, pebbles, sea shells, cubes (not more than one per child).

What they do Stage (a)

1. One child lays out some of his objects, roughly in a line.
2. Another child pairs one of his own objects with each of the objects on the table.
3. The pairs are then separated, and put into different set loops.
4. Explain that these sets *match*, because every object makes a pair with an object in the other set.
5. Each of the two sets is then counted.
6. Steps 1 to 4 are repeated with other objects and children.

Stage (b) (Predictive)

1. One child puts some of his objects into his set loop.
2. One (or more) of the others tries to make a matching set, of his own objects in his own set loop.
3. For small numbers this can be done visually, but for large numbers counting is the only reliable way. I suggest that children be left to use this of their own initiative. (See discussion of activities.)

Discussion
of activities

Piaget has demonstrated that children can count a set of objects correctly while failing to realize its full significance - that all sets which have the same count must match each other in the manner described. Without this realization, counting remains just a verbal activity, not truly related to number. (For a fuller discussion, see *The Psychology of Learning Mathematics* Chapter 8.)

Stage (a) is for building the concept of matching sets, and relating it to counting and number. Stage (b) uses this relationship to make sets which, it is predicted, will match. This prediction is then tested. If children do not spontaneously use counting, I think it is better to go back to Stage (a) than to tell them. The latter will give them the method, but with the risk that it is no more than a method, without the underlying conceptual understanding.

OBSERVE AND LISTEN

REFLECT

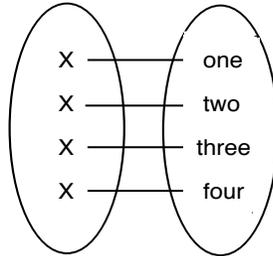
DISCUSS

Org 1.9 COUNTING, MATCHING, AND TRANSITIVITY

Concept The relationships between these concepts.

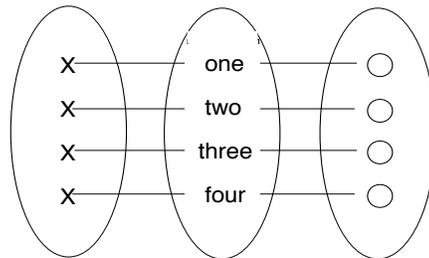
Ability To use these relationships: for children, only at the intuitive level.

Discussion of concept When we use counting to find the number of a set, we are pairing the objects in the set with the words in a set of number-names.



The last word, in this case 'four' is the number of words and also the number of the other set.

All sets of objects which match this set of number-names will have the same number as each other.



(Imagine joining these lines, through the words in the middle set.)

This is an example of **transitivity**. Transitivity is a property of a relationship, in this case of matching between sets. Some relationships have this property, some do not. E.g., if a person A has the same name as B, and B has the same name as C, then A has the same name as C. So the relationship 'has the same name as' is transitive. But the relationship 'is the parent of' is not transitive.

The following three mathematical relationships, which we have already been working with, are all transitive.

'Matches'			
'Has smaller number than'	}		relations between two sets
'Has larger number than'			
'is equal to', written	=	}	corresponding relations between their numbers
'is less than', written	<		
'is greater than', written	>		

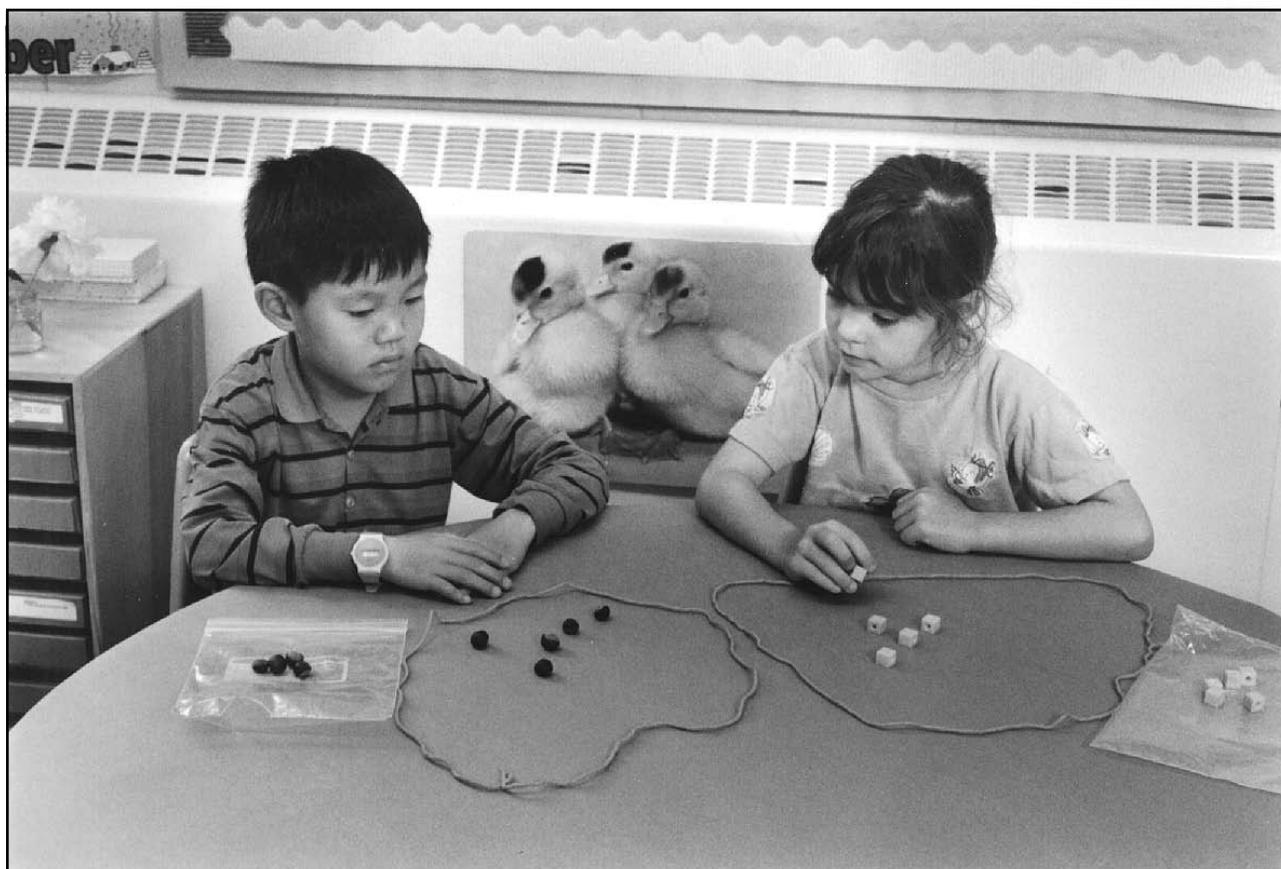
In the case of counting, we use a set of number words as a 'portable' set to see whether two sets have the same number, without physically pairing. This is valid only because the relationship 'matches' is transitive.

Children use this concept intuitively in the present context, and also in that of ordering. For them, I think this is sufficient, so I have not provided activities to make it explicit. In view of how much depends on this property, however, I suggest that we ourselves need to use it more consciously, and to know its name. Hence the foregoing discussion.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Sets which match [Org 1.8/1, Stage (b) Predictive]

Org 1.10 GROUPING IN THREES, FOURS, FIVES

- Concepts*
- (i) Threes, fours, fives (meaning groups of these numbers).
 - (ii) A group regarded as a single object.

- Abilities*
- (i) To group (physically rearrange) a set of physical objects into threes, fours, fives.
 - (ii) To count how many groups there are in a set, and how many single objects (ones).

Discussion of concepts We are now beginning on the path towards decimal notation. ('Decimal' simply means 'relating to ten': its meaning is not confined to decimal fractions). We start by grouping objects in tens and then treating these groups-of-ten as single objects with which we repeat the process to get new groups, each of ten groups-of-ten. And so on. In combination with place-value notation, it is a powerful and sophisticated technique for dealing with large numbers: representing them, thinking about them, and calculating with them.

Place-value notation is a critical stage in children's mathematical development. It can be either a major stride forward in their powers of mental organization, using clever arrangements of symbols which carry rich meaning; or downwards on the road towards (mathematical) ruin, the rote manipulation of near-meaningless symbols. So it is very important to build up this conceptual structure gradually and methodically, and this we do over topics 10 to 13 in Org 1 and the whole of Num 2.

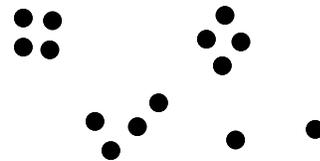
We start with the concept of grouping, using groups small enough to be subitized. (This means that we can perceive their numbers without counting.)

Regarding these groups as single entities in themselves is the next step. It is quite a large step. We are now going to describe (e.g.)

this:

as 'three fours and two ones'.

This implies that we are counting each of the groups as if it were a single object.



For those who are familiar with the term 'group' as used in higher mathematics, I should explain that I am using it here with a different meaning. This is closer to the everyday meaning, with the addition that all the groups have the same number.

Activity 1 Making sets in groups and ones [Org 1.10/1]

An activity for a small group of children. Its purpose is to introduce them to the concepts of groups and ones.

Materials For each player:

- A set loop.
- Not fewer than 36 cubes.
- A container to keep separate those not in use.

For each group:

- 2 dice of different colours.
- A shaker.
- 3 cards as in the diagram below.* One card is marked THREES, one FOURS, and one FIVES.

* Provided in the photomasters

threes	ones
[colour ₂]	[colour ₁]

- What they do*
1. One of the cards is chosen and put on the table.
 2. The first player throws the dice, and puts them on the card according to colour.
 3. All the players then construct sets as determined by the fall of the dice, making all their groups of the number shown on the card.
 4. Explain that ONES means single objects which have not been grouped. These must be spaced apart from each other to show this.
 5. All spares must be kept in the containers.
 6. Children check each other's sets for match with their own, and discuss if there is disagreement. The ultimate test is physical pairing, and it is good review to use this occasionally. Groups should be paired with groups, ones with ones.
 7. When agreement has been reached, the thrower says "Clear set loops," and all the players return their cubes to their stores.
 8. The dice and shaker are passed to the next player, and steps 1 to 7 are repeated, using the same card or a different one as desired.

Notes

- (i) There is nothing against forming the groups of cubes into rods, but this takes quite a time, and is not necessary conceptually at this stage.
- (ii) If the number of ones is larger than the number of a group, children may spontaneously make another group. This is a sensible thing to do. It anticipates Topic 12, canonical form.

Activity 2 Comparing larger sets [Org 1.10/2]

An activity for a small group of children. Its purpose is to show the usefulness of grouping for dealing with larger numbers of objects than they are yet able to count. (We assume that they can count up to about 10.)

Materials For each player:

- A set loop.
- About 30 cubes.
- A container.

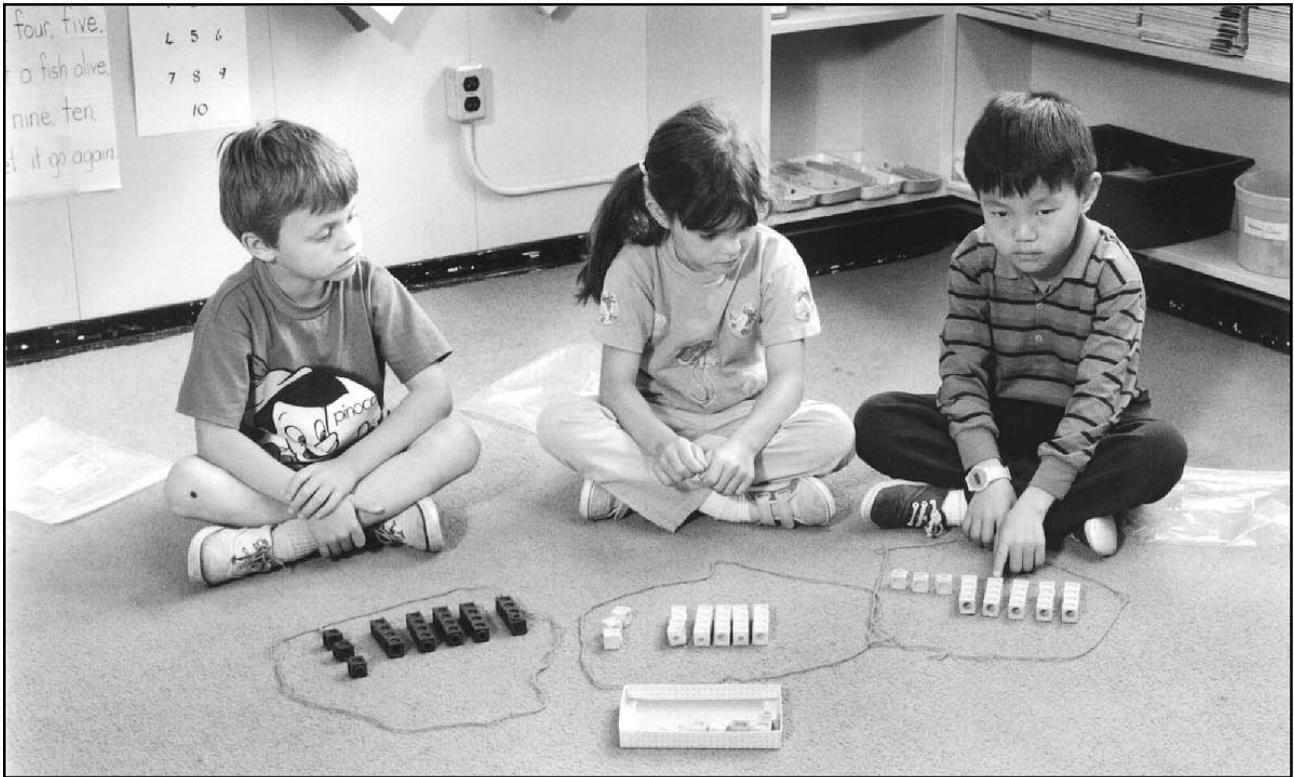
- What they do*
1. One of the players puts a largish number (over 20) of cubes into his loop.
 2. Ask the others if they can make sets which match this one. Since the number is too large for them to count, they'll have to use a different way.
 3. Matching them singly (as in Org 1.7) would work, but this would be rather slow. Can we think of a better way?
 4. Let's ask the first player to group his set in threes, making as many of these groups as he can.
 5. He does this. Result (e.g.) 7 threes and 2 ones.
 6. Now we are back to something which we can all count. We can count the groups, and we can count the ones.
 7. The children are now able to make matching sets. They check each other's. If there is disagreement which cannot be resolved by discussion, the ultimate test is physical pairing: group with group, one with one.
 8. If they like, they can break up the groups and check that the sets do match one-to-one.
 9. Steps 1 to 7 (or 8) are repeated, starting with a different player and using a different group number.

Activity 3 Conservation of number [Org 1.10/3]

An activity for a small group of children. This follows on from Activity 2. Its purpose is to help children to realize that the number of a set is not changed by the way in which it is grouped.

Materials As for Activity 2.

- What they do*
1. One of the players makes a set in groups and ones, choosing 3, 4, or 5 as group number, whichever she likes. (Suppose she chooses 4.)
 2. The others make matching sets.
 3. Ask: "If everyone broke up their groups, and made groups of 3 (or 5) instead, would all the sets still match?"



Conservation of number [Org 1.10/3, Step 4]

4. The children check their predictions by re-grouping.
5. Repeat steps 1, 2, 3, 4 until the children decide if the sets will match in any particular grouping (or singly), they will still match in any other grouping.
6. Ask why they think this is. A good answer would be "Because the number of objects is always the same."

Discussion of activities

Physically grouping the objects together makes it easier to think of the result as a single entity. So also does naming these groups, e.g., "a THREE." Choosing small group numbers takes away the need for counting to get the correct numbers in each group.

In Activity 2, grouping is used to change a set which was (for them) too large to count into one which is countable. The simplicity of this technique makes it easy to overlook how powerful it is in extending the organizing ability of our thinking.

In Activity 3, the assumption which has been explicit in Activity 2 is made conscious. This is, that the number of objects in a set is not changed by changes in position of the objects, such as grouping. If this were not true, place-value notation (and much else in mathematics) would be invalid.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.11 BASES: UNITS, RODS, SQUARES, AND CUBES

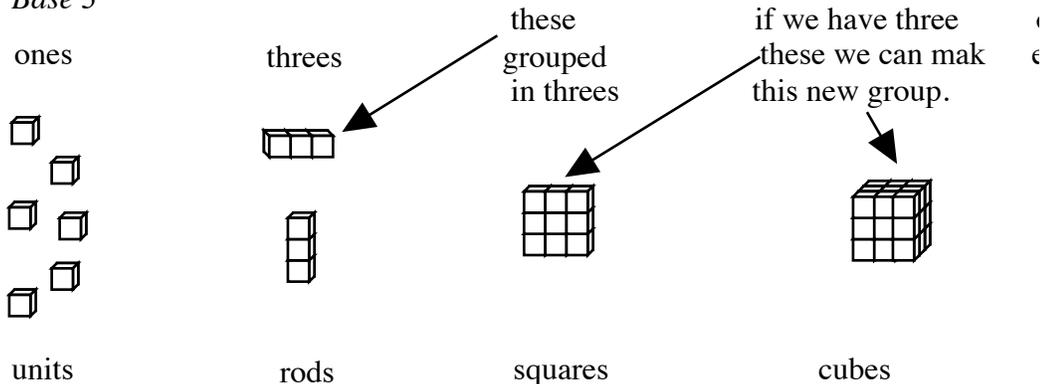
- Concepts* (i) Groups of groups which make larger groups.
(ii) A base: i.e., a number used for each repeated grouping.

Ability To make these, using suitable objects.

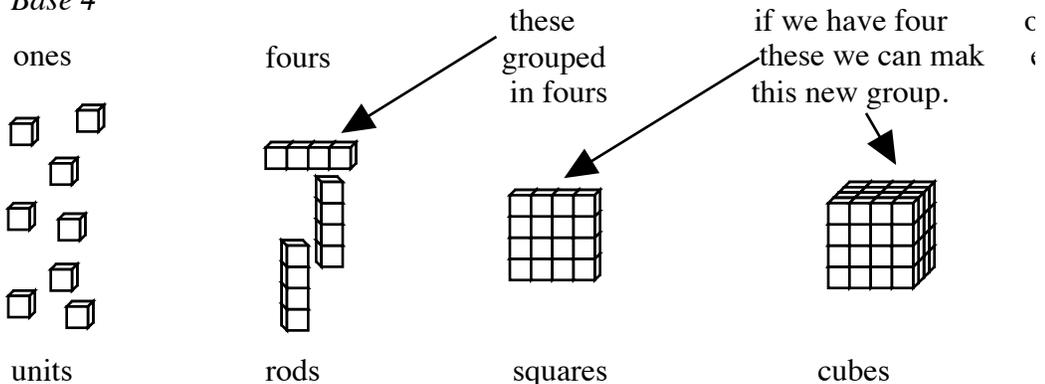
Discussion of concepts The key step in place-value notation is to repeat this grouping process, treating as units the groups already formed. We have been preparing the children for this step in two ways: (i) by the use of cubes such that a group of these can be joined into a single rod; (ii) by activities such as Org 1.10/1 in which groups are treated as single entities which can be counted and matched.

The same number is used each time the grouping process is repeated. This number is called the *base*. For example:

Base 3



Base 4



In this topic we use bases 3, 4, 5. These numbers are small enough to be subitized (perceived without counting), and allow attention to be focused on the process and products of grouping. The reason for using several bases is to provide several examples from which to form the concept. I see no point at all in teaching children to *calculate* in these bases. The only bases in which they need to calculate are base 10, and possibly in the future (for working with computers) bases 2 and 16.

Activity 1 Units, rods, and squares [Org 1.11/1]

An activity for a small group of children. Its purpose is to introduce the two concepts described above. This follows on from Org 1.10/1 (Making sets in groups and units).

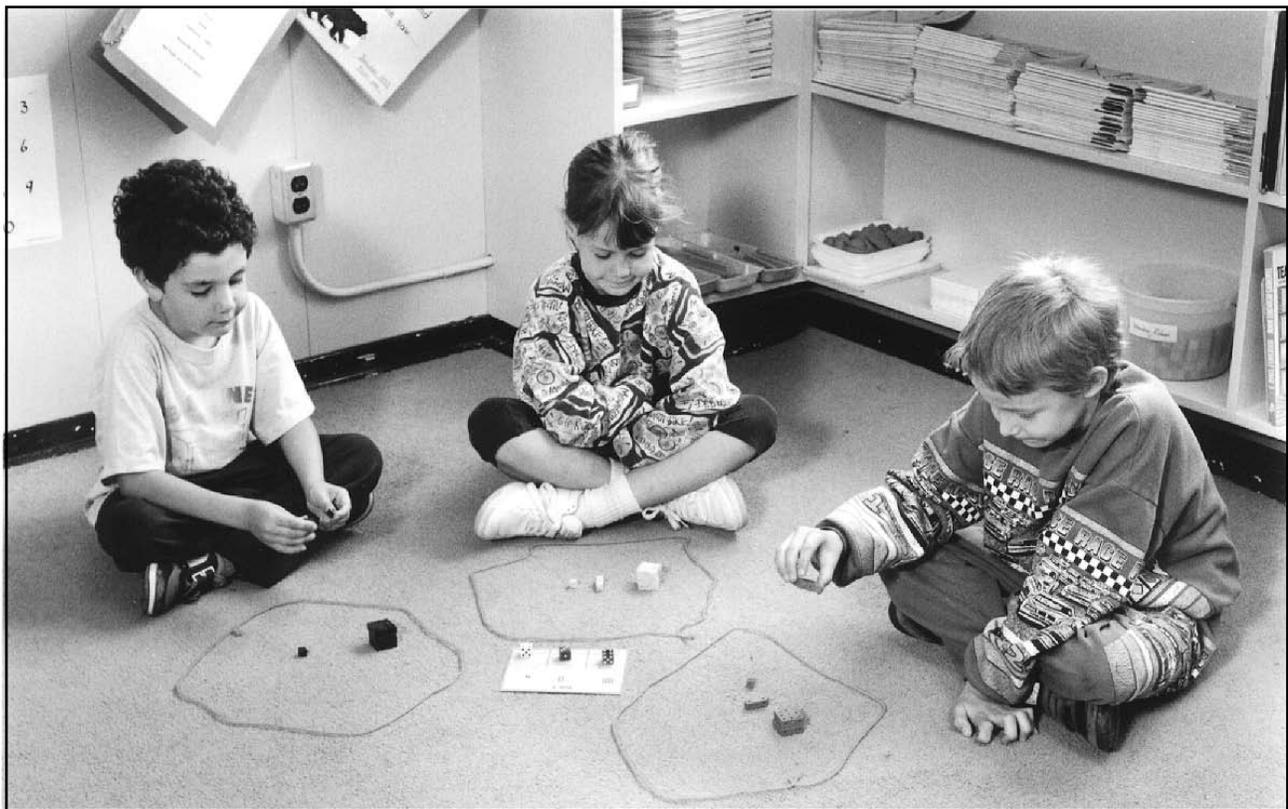
- Materials*
- Base card for base 3, 4, or 5 * (see figure below showing base card 3).
 - 2 dice of different colours for rods and units, to match the base card.
 - A shaker.
 - A set loop.
 - Interlocking cubes.**
 - A container for the cubes.

* Provided in the photomasters

** Although ready-made squares and cubes for various bases are available, and useful perhaps for later work, there is nothing so good as having the children themselves physically put together single cubes into rods, rods into squares, squares into cubes. Ten cubed would need a lot of cubes, but we are at present using much smaller bases. It is worth going to some trouble to make sure that enough cubes are available for the activities in this section to be done properly.

<u>Base 3</u>		
		
squares	rods	units
[colour ₃]	[colour ₂]	[colour ₁]

- What they do*
1. The dice are thrown and put on the base board according to colour.
 2. A set is made in the set loop, according to the numbers shown by the dice.
 3. Some situations will occur which result in a number of ones equal to or greater than the base.
 4. Suppose that we are using base 3 and there are 5 ones. Ask “Could we do something with these?”
 5. The card suggests the answer: make three-rods.
 6. If there are now 3 or more three-rods, the same question arises. This time the answer is, make three-squares.
 7. Repeat steps 1 to 6. Emphasize that if we are working in base 3, only groupings based on 3 are allowed. We may have units, three-rods, three-squares and later three-cubes; but not two-rods, or three-by-two rectangles.
 8. Repeat with other bases.



On to cubes [Org 1.11/2]

Activity 2 On to cubes [Org 1.11/2]

Also for a small group of children. It follows on from Activity 1, and is played in the same way but with a 3rd die for the squares column. There will now be occasions when there are enough squares to be grouped according to the base currently in use, resulting in a cube. In this case the card does not cue the children. When they have decided to group squares to make a cube, they also have to decide where to put it.

Discussion of activities	In this topic, we concentrate on physical activities for building a good understanding of the concepts underlying place-value notation. These provide foundations which are essential if later work using this notation is to be meaningful.
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OBSERVE AND LISTEN

REFLECT

DISCUSS

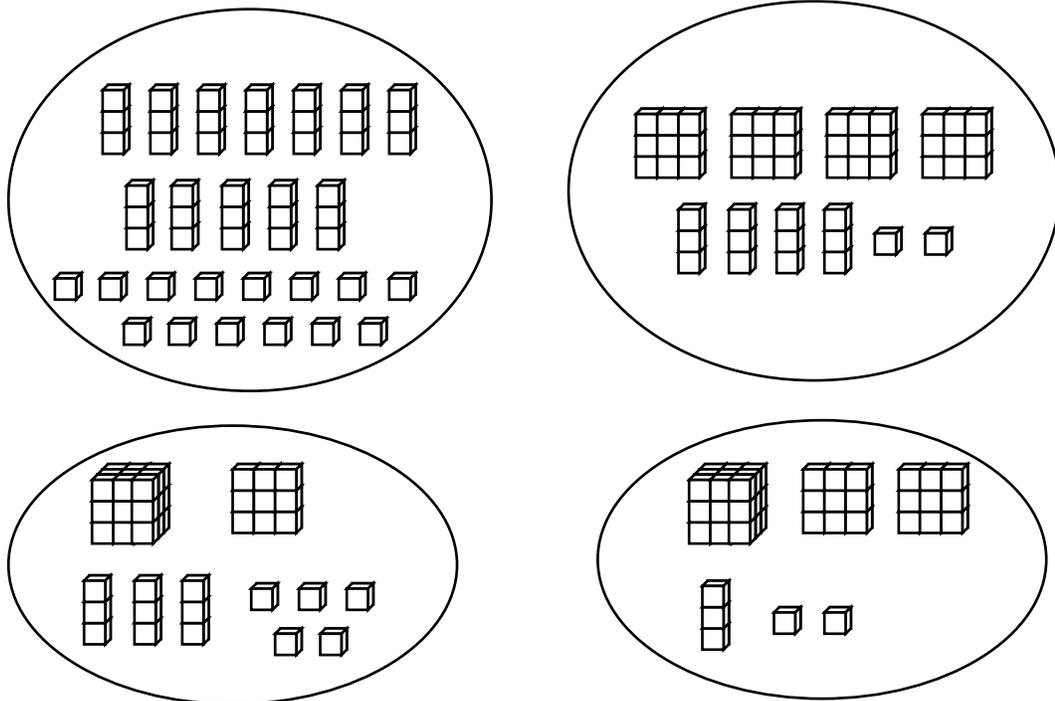
Org 1.12 EQUIVALENT GROUPINGS: CANONICAL FORM

- Concepts*
- (i) Equivalent groupings: that is, different groupings of a set which do not change its number.
 - (ii) Canonical form: that is, the equivalent grouping which uses only the largest possible groups (and hence the smallest possible number of groups).

- Abilities*
- (i) To re-arrange a given set in several equivalent groupings.
 - (ii) To recognize whether or not a set is in canonical form.

Discussion of concepts Below are shown several equivalent groupings of the same set. Only the last is in canonical form.

Base 3



The concept of canonical form is implicit in both of the activities of the previous topic. In this topic we make it explicit, and bring out some of its usefulness for showing the number of a set unambiguously.

Changes to and from canonical form are an important part of the techniques used for adding and subtracting, multiplying and dividing. These would not be valid if the number of a set was changed by grouping it differently. Activities 2 and 3 of Org 1.10 have taken care of this requirement in advance.

Activity 1 “Can I fool you?” (Canonical form) [Org 1.12/1]

An activity for a small group, in two teams. Its purpose is to teach children the name for canonical form, and introduce them to its use.

Materials For each team:

- Plenty of cubes.
- A set loop.
- A container for the cubes.

For the group:

- A pack of 9 cards.* ‘SAME’ is written on 3 of these, ‘LARGER’ on another 3, ‘SMALLER’ on the rest.
- 3 dice and a shaker.**
- Base card for 3, 4, or 5.**

* Provided in the photomasters.

** The same as for Org 1.11 / 1 and 2.

What they do Stage (a)

They play with 2 dice only, using the colours for ones and tens.

Stage (b)

They play with 3 dice.

1. Team A has the dice and shaker. Team B has the pack of cards, which are shuffled and put face down.
2. Team A throws the dice, and puts them on the card according to colour.
3. They construct a set according to the fall of the die.
4. Team B takes the top card of the pile, looks at it, but does not let team A see it.
5. Team B then constructs a set which is the same in number, or smaller, or larger, than A’s set, as written on their card. They make their set look different by using different numbers of units, rods, and squares, even if their card says ‘SAME’.
6. Team A now has to decide whether team B’s set is in fact the same in number, or larger or smaller.
7. Team B show their card.
8. The correctness of team A’s answer, and also team B’s set construction, are tested by changing both sets into canonical form. That is, they make as many rods as possible having regard to the base in use; and then as many squares as possible. The sets can then easily be compared.
9. Repeat steps 1 to 8 with teams interchanged.
10. Repeat, with a different base.

Activity 2 Exchanging small coins for larger [Org 1.12/2]

A game for a small group, of not more than 4 players. Its purpose is to practise canonical form in a different embodiment, and to introduce the concept of exchanging instead of grouping.

- Materials*
- Single die marked 1 to 6
 - Shaker.
 - Box of pennies.
 - Box of nickels.

- What they do*
1. The players in turn throw the die, and take that amount of money in cents. E.g., a player who threw 2 would take two 1 cent coins; one who threw 6 would take a 5 cent and a 1 cent coin.
 2. Whenever possible, they exchange five 1 cent coins for one 5 cent coin.
 3. They agree in advance how many rounds are to be played. (Say, 3 to begin with.)
 4. The winner is the player who has the smallest total number of coins.
 5. If at the end there is a tie, those concerned may agree to play a given number of extra rounds.

Discussion of activities In Activity 1, both teams have to think equally hard. If B's card reads 'LARGER' they have to put out a set which is larger in number, but looks smaller; and vice versa. And team A has to try not to be 'fooled' by appearance. But of course the purpose of this activity does not depend on whether or not children are fooled, since the purpose is to practice converting to canonical form and see how much easier this makes comparison. Nor is the purpose defeated even if they use counting in base 10 (though for 3 dice this might involve counting up to 186, if 3 sixes are thrown in base 5). In that case they are converting mentally to canonical form in base 10.

Activity 2 is somewhat easier than Stage (b) of Activity 1, so it might be used after Stage (a). This uses base 5, with exchange instead of physical grouping. They may well have encountered this in everyday experience. In this case, the present activity will relate that experience to the present context.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Org 1.13 BASE TEN

Concept Tens, hundreds, thousands.

- Abilities*
- (i) To form and recognize sets of these numbers.
 - (ii) To recognize and match different physical representations of them, embodying different levels of abstraction.

Discussion of concepts Base 10 involves the same concepts as those used in bases 3, 4, 5. Practically, however, there are 3 major differences: (i) Our monetary system, and the measures used in commerce, technology and sciences, all use base 10; (ii) This base enables us to represent larger numbers with fewer figures; (iii) But 10 is too big to subitize. So manipulations which can be done perceptually for bases up to five depend on counting when working with base ten. Whether, using hindsight, it might have been better to choose some other base, may be for some an interesting subject for discussion. Many of us can remember the mixture of bases 4, 12, and 20 used in England's former monetary system. Our measurement of time still uses bases 60, 12, 24, 7, 30, 31, and 365! Computers work in base 2. This is not a good one for humans, who convert to base 16 (hexadecimal). But for most practical and theoretical purposes, base ten is the established one, and it is an important part of our job as teachers to help children acquire understanding, confidence, and fluency (in that order of importance) in working with base ten in decimal notation. It is with this aim that the foundations have been so carefully prepared in topics 1 to 12 of this network.

Activity 1 Tens and hundreds of cubes [Org 1.13/1]

A teacher-led activity for a small group. Its purpose is to help children to transfer to base ten the same thinking as they have developed for bases three, four, and five in earlier topics.

Materials A large box of 1 cm cubes

- What they do*
1. Introduce this activity by asking if they know what base is most used in everyday life. Relate this to the fact that we have ten fingers.
 2. Ask them to make some ten-rods.
 3. Ask them to join these up to make ten-squares.
 4. Very likely they will run out of cubes: certainly they will not be able to make many ten-squares. This may surprise them! Tell them that the number of single cubes in a ten-square is called a hundred, which (as they have discovered) is quite a large number. Tell them that we're now on our way to big numbers.

Activity 2 Tens and hundreds of milk straws [Org 1.13/2]

An activity for a small group. Its purpose is to repeat grouping into tens and hundreds with a different embodiment.

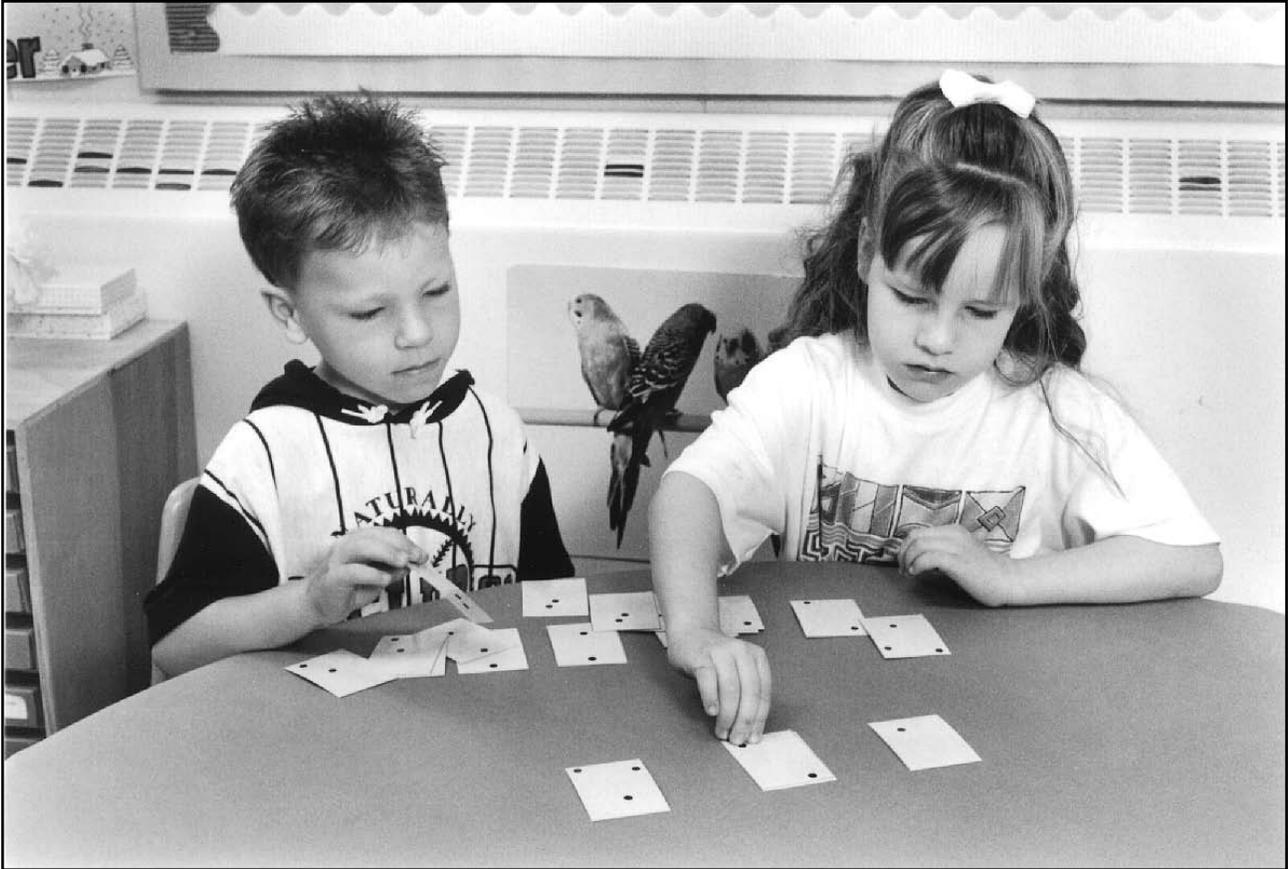
- Materials*
- A large number of milk straws cut into halves.
 - Rubber bands.

- What they do*
1. Ask the children to find out how many hundreds, tens, and ones there are.
 2. Working together they group the straws first into tens with a rubber band round each ten.
 3. They then group the tens into bundles of ten tens, with a rubber band around each big bundle.

Discussion of activities In the preceding topics, children developed the concepts of a base, and of rods, squares, and cubes, using bases small enough for all the grouping to be done physically. With base ten, this becomes laborious; and when it comes to the cube of base ten, impractical.

However, provided that the earlier concepts have been well established, they can be combined with children's concept of the number ten in such a way that the concepts which were learned by using bases three, four, five expand to include base ten. This is what we have been doing in this topic. We have thus gradually been reducing children's dependence on physical objects for representing numbers and moving them towards representing them in other ways. We have also been extending children's concepts of numbers into the thousands. They have now come a long way, and should be allowed to return to the support of physical materials at any time when they feel the need. This will help to keep their concepts of numbers strong and active, and reduce the danger that when symbols become the main method for handling numbers, the concepts fade away.

OBSERVE AND LISTEN**REFLECT****DISCUSS**



Sorting dot sets and picture sets [Num 1.1/1]

[Num 1] NUMBERS AND THEIR PROPERTIES

Numbers as mental objects which, like physical objects, have particular properties

Num 1.1 SETS AND THEIR NUMBERS PERCEPTUALLY (SUBITIZING)

Concept Number as a way in which different sets may be alike.

Ability To sort sets by number, ignoring all else. At this stage this ability is only involved at the perceptual level, using only numbers from 1 to 5. These numbers can be subitized – perceived without counting.

Discussion of concept In everyday life, we often treat collections of objects or persons as single entities: e.g., a football team, a grade at school. Here we do the same for the mathematical concept of a set, and begin sorting and matching sets in the same way as we sorted single objects in Org 1.1. For example, in Org 1.1/5 and 6 (conceptual matching), we put together hammer, saw, screwdriver, . . . because although they are different objects we can think of them as alike in a particular way – they are all tools. The most important likeness between different sets is that of having the same number.

Activity 1 Sorting dot sets and picture sets [Num 1.1/1]

An activity for a small group. Its purpose is to start children thinking about the number of a set as something independent of what the objects in the set are, their nature or position. This activity should come after Org 1.2, so that the children are already used to sorting cards and looking for likenesses between them by which to sort.

- Materials*
- A pack of cards having on them various numbers of dots from 1 to 5, all of the same size and colour. There should be several cards of each number, with the dots differently positioned.*
 - A harder pack in which the dots are of different sizes and colours.
 - A pack in which the dots are replaced by ‘blobs’ or ‘patches’ of irregular shape.*
 - A pack of cards on which are 1 to 5 little pictures of the same or similar objects. (Many schools have rubber stamps which would be useful for this.)
- * Suitable sets of these cards are provided in the photomasters.

- What they do*
1. Start with the easiest pack, using only cards with numbers 1 to 3 to begin with. Mix the cards, and let the children sort them.
 2. When the children can sort these cards, introduce 4, then 5.

Num 1.1 Sets and their numbers perceptually (cont.)

3. When they are ready, introduce the harder packs.
4. If the children start to attach numbers to sets, good. If they count spontaneously, of course do not discourage this. But do not deliberately introduce counting at this stage. It is a more sophisticated concept than is usually realized, and we want to give children the means for constructing it correctly. This involves several earlier stages.

Activity 2 Picture matching game using dot sets and picture sets [Num1.1/2]

A game for up to six children. Its purpose is to consolidate the concept and ability learned in Activity 1.

Materials • Dot sets and picture sets from Activity 1.

- What they do*
1. The cards are shuffled and dealt face down.
 2. The player on the left of the dealer turns over his top card and puts it face upwards on the table.
 3. Each player in turn does likewise. If his card is like a card or pile already there, it is added to that pile and the player has a second turn. (But not more, or the game ends too quickly.)
 4. If it does not match one of the existing piles, a new pile is started.
 5. The winner is the first to have put down all his cards, after which the others may play out their hands until all the cards are down.
 6. Encourage naming the sets which result from the sorting.

Discussion of activities A considerable feat of abstraction is required here, since (e.g.) a set of two red dots and a set of three red dots *look* more alike than a set of three red dots and a set of three pictures of cows. Attaching the same number-word helps, and this is a good way of relating the informal knowledge which most children acquire before they come to school to the more organized learning which they are now beginning. In this topic, we are helping children to expand their concepts of likeness and matching from single objects to sets, and to think of a set as a single entity.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.2 SUCCESSOR: NOTION OF ONE MORE

- Concepts*
- (i) The successor of a number, i.e., the number which is one more.
 - (ii) The predecessor of a number, i.e., the number which is one less.

- Abilities*
- (i) To construct and recognize the successor of a number.
 - (ii) To construct and recognize the predecessor of a number.

Discussion of concepts Easier words for the children to use are “next number” and “number before.” These concepts only apply to whole numbers, but for the present these are the only numbers children know.

In this context, the next number, the number which is one more, and the next number-name in the counting sequence all belong together. The close connection between these ideas is an important part of the foundations we are building for counting with understanding, as against counting mechanically.

Activity 1 Making successive sets [Num 1.2/1]

An activity for a small group. Its purpose is to start children forming the concepts and connections described above.

- Materials*
- A sorting tray.
 - Some objects for sorting.

- What they do*
1. Starting with the tray empty, the first child puts (e.g.) one button into a compartment.
 2. He says “We have one button on the tray,” and passes it on.
 3. The next child puts two objects into the next compartment, says “We have one button and two sea shells on the tray,” and passes it on.
 4. The next child puts three objects into the next compartment, says “We have one button, two sea shells, and three pebbles on the tray.”
 5. And so on, as far as they know the number names.

Activity 2 Putting one more [Num 1.2/2]

An activity for a small group.

- Materials*
- A single container.
 - A set of small objects such as buttons or beans.

- What they do*
1. The first child puts in one button (or whatever it is) and says “One button in the cup.”
 2. He then passes the container to the next child.



Putting one more [Num 1.2/2, Step 3]

3. The next child puts in another button, saying as he does so: “One in the cup, put one more, two in the cup.”
4. It is then passed to the next child, who puts in another button and describes what happens as before.
5. Continue as long as they know the number-names.
6. They can then pass the container around in the reverse direction, taking out one object at a time and making the appropriate statements.
7. What the children say may be varied according to their language ability. The important ideas to be put into words are the actions, putting in one more or taking out one object, and the result, a different number of objects.

Discussion of activities

In these activities, children are both learning new mathematical ideas from physical activities, and linking these activities and ideas with spoken language. This will help them, in the future, to use language to manipulate mathematical ideas mentally, without the need for physical activities. Here we are particularly concerned with the ideas which connect counting and number; but the principle is one which is widely used in our present approach.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.3 COMPLETE NUMBERS IN ORDER

Concept A complete sequence of numbers in order, i.e., in which every number has a successor and every number except the first has a predecessor.

Abilities (i) To put numbers in order.
(ii) To recognize whether the sequence is complete.
(iii) If it is not, to provide the missing one(s).

Discussion of concept	In Missing Stairs (Org 1.5/1) we were working with successive sets. Now we move on to successive numbers. The connection is a close one, but the thinking involved is one level more abstract, since numbers (unlike 'stairs') cannot be seen.
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Activity 1 "Which card is missing?" [Num 1.3/1]

A game for children to play in pairs. Its purpose is to develop children's thinking to the more abstract level described above.

Materials • Dot-set cards 1 to 5, later 1 to 10 and then 0 to 10.*
*These are like the cards used in sorting dot-sets (Num 1.1/1), but there is only one of each. Zero is represented by a blank card. Note that the dots are random, not in patterns, and the cards do not have numerals. A suitable set of cards is provided in the photomasters.

Rules of the game

1. Before beginning, the children lay the cards out in order, and observe the sequence.
2. Player A shuffles the cards and holds them out face downwards to player B.
3. Player B takes a card without letting A see which card it is.
4. Player A then has to say which card is missing.
5. Player B shows the missing card, thereby checking A's prediction.
6. They then interchange as A and B.

Stages (a) With cards 1 to 5.
(b) After Num 1.5, with cards 1 to 10.
(c) After Num 1.6, with cards 0 to 10.

Discussion of activity	This activity cannot be done by rote, and ensures the use of an abstract sequence of numbers. It thus continues the emphasis on conceptual learning which has been present from the beginning.
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OBSERVE AND LISTEN

REFLECT

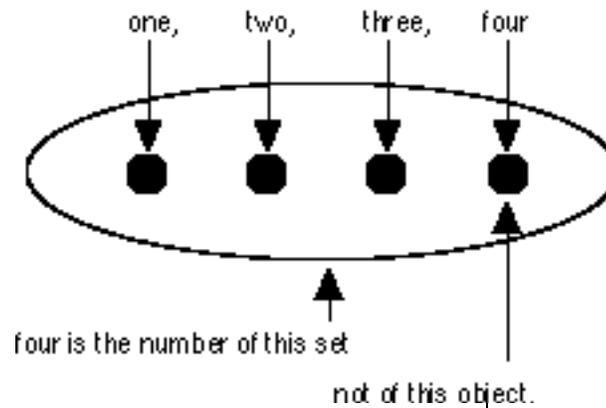
DISCUSS

Num 1.4 COUNTING

Concept Counting, as a technique for finding the number of a set.

Ability To count how many objects there are in a set.

Discussion of concept Counting is a more sophisticated technique than is usually realized. When we say the number-names in order, the last name we say is always the total number of words we have said, e.g., “One, two, three, *four*.” We have said four words, so if we pair each word with an object, this also tells us how many objects we have done this with. All the time we are talking about the latest total of objects, not the last object.



Counting is a multi-purpose technique. It can be used for finding the (cardinal) number of a given set; for making a set of a given number; for adding (by counting on); for subtracting (by counting back); for multiplying (by counting groups treated as units); and for dividing (using a similar method in reverse). So it is important that children get this right conceptually, from the start.

An additional difficulty arises from the fact that counting may also be used to find the ordinal number of an element in a set. To prevent confusion I think it is better omitted at this stage, until cardinal number is well established. When in due course ordinal numbers are introduced, they should be clearly distinguished from cardinals by using the words “first,” “second,” . . .

Activity 1 Finger counting to 5 [Num 1.4/1]

An activity for a small group. Its purpose is to introduce finger counting with a good conceptual foundation. (See ‘Discussion of activities.’)

What they do Begin with the left hand, palm downward. Start with the left little finger.

1. Touch the table with the little finger only. Say “One.”
2. Lift hand, and touch the table with the little and fourth fingers together. Say “Two.”

- | | |
|---|--------------|
| 3. Lift hand, and touch the table with the little, fourth, and middle fingers together. | Say “Three.” |
| 4. Lift hand, and touch the table with all four fingers together. | Say “Four.” |
| 5. Lift hand, and touch the table with all four fingers and thumb together. | Say “Five.” |

Activity 2 Planting potatoes [Num 1.4/2]

An activity for a small group. Its purpose is to link finger counting with a number rhyme.

- What they do*
1. They learn the following number rhyme, which you will recognize as adapted from a children’s counting-out rhyme. “One potato, two potatoes, three potatoes, four. Five potatoes, that’s enough, so we will plant no more.”
 2. They then link it with finger counting, as in Activity 1. From “five potatoes . . .” on, they keep all five fingers on the table.

Discussion of activities Counting on our fingers is so natural a technique that we almost certainly derive our use of base 10 from the fact that we have 10 fingers (more accurately, digits). The way children often finger count, however, there is a danger that they will attach numbers to single fingers rather than to sets of fingers. (See ‘Discussion of concept.’) The purpose of these activities is to provide a good conceptual foundation to finger counting.

At this age, some children will have difficulties of co-ordination. But finger counting in this way is so useful for adding by counting on, subtracting by counting back, that it is worth introducing early and practicing regularly as it recurs in later networks. Other number rhymes will be found in Num 2.1.

The counting activities in this network relate closely to those in network Num 2, ‘The Naming of Numbers,’ and I was for a while undecided whether to put them into the latter network. In the end I decided to put into this network counting activities in which the greater emphasis is on the numbers themselves, their properties and interrelationships; and into Num 2 those in which the greater emphasis is on the symbols (number-words and numerals). Counting thus keeps a place in both networks, and helps to link them together.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.5 EXTRAPOLATION OF NUMBER CONCEPTS TO 10

- Concepts*
- (i) The complete numbers from 1 to 10,
 - (ii) These numbers linked to their names.
 - (iii) Correspondence between size of number and position on a number track.

- Abilities*
- (i) To count how many objects there are in a set.
 - (ii) To relate the number of a set to length on a number track.

Discussion of concepts The concepts of the numbers from 1 to 5 have been constructed and used so far mainly at a perceptual level. They have also been systematically related to the concepts of order, successor, and counting in preparation for the use of counting to extend number concepts to larger numbers.

Activity 1 Finger counting to 10 [Num 1.5/1]

An activity for a small group. Its purpose is to extend the ideas and method of finger counting up to 10.

What they do The method just described extrapolates nicely to 10.

1. Touch the table with the left hand as for five, and with the right *thumb*. Say "Six."
2. Left hand as before, right thumb and forefinger. Say "Seven."
3. And so on, up to 10.

Activity 2 Missing stairs, 1 to 10 [Num 1.5/2]

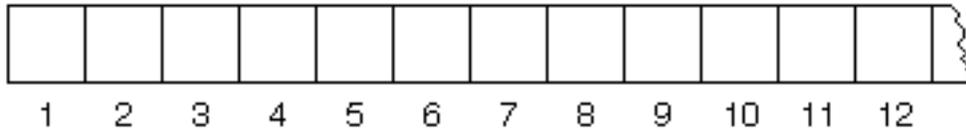
An activity for children to play in pairs. Its purpose is to help in extrapolating the ideas of a complete sequence of numbers in order to 10.

- Materials* For each pair:
- 55 cubes of one colour.
 - 10 cubes of a different colour.

What they do This game is played in the same way as 'Missing stairs' in Org 1.5/1, but to 10.

Note **Activities 3 and 4**, which follow, also appear in NuSp 1, 'The Number Track and the Number Line' network. They are included here to provide another representation of number.

Discussion of concept

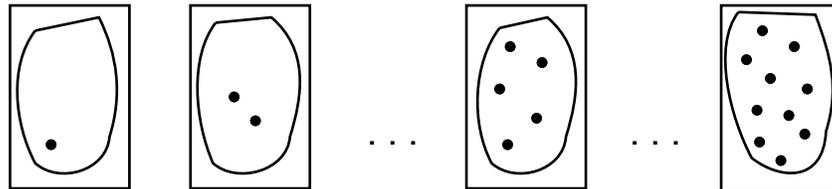


The number track is a spatial representation of number. It leads on to the number line, and these together are of importance throughout mathematics, from the kindergarten class to mathematics at university level. They provide a valuable support for our thinking about numbers in the form of a pictorial representation.

Activity 3 “I predict - here” on the number track [Num 1.5/3]

A game for children to play in pairs. Its purpose is to establish the relation between size of number and position on the track. This activity also appears as NuSp 1.1/1 in ‘The Number Track and the Number Line’ network.

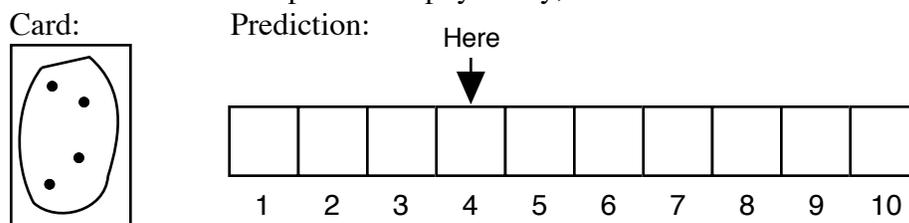
- Materials*
- Number track up to ten.*
 - 10 small objects such as bottle tops, pebbles, small shells.
 - 10 cubes for number track.
 - Set cards as illustrated (1 to 10).*
- * Provided in the NuSp 1.1/1 photomasters.



Note The set cards are not numbered, to avoid short-circuiting the conceptual activity by simply matching the numerals on card and track. The only numerals are on the track, and these are written outside the squares.

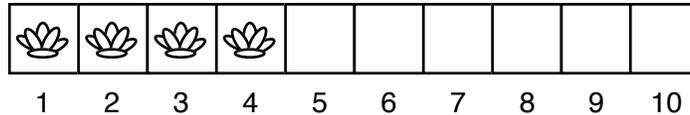
What they do Form (a)

1. The set cards are shuffled and put face down.
2. Child A turns over the top card and puts it face up on the table.
3. Together the children put one of the small objects on each dot in the set loop, to make a physical set matching the set of dots on the card.
4. Child A then predicts how far these objects will come on the number track when one of them is put in each space.
5. He says “I predict - here” and marks his prediction in some way. (If the number track is covered in plastic film, a blob of plasticine has the advantage that it will stay put.)
6. Child A then tests his prediction physically, as illustrated below.



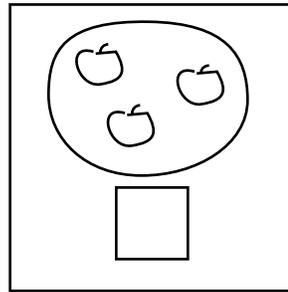
Num 1.5 Extrapolation of number concepts to 10 (cont.)

Test of prediction (using shells):



Form (b) This is played exactly as in *Form (a)*, except that cubes are used and a number track. Please use both forms. We do not want the concept linked only with cubes.

Variation.*
This uses pictures instead of dots, with a box under the set loop.



1. The child counts the pictures and writes their number in the box.
2. He puts one cube on each picture.
3. These are then transferred to the number track. The last square reached should have the same numeral as has been written in the box.

* Suggested by Mrs. Marion Jones, of Lady Katherine Leveson's School, Solihull.

Discussion of activity

The number track and the number line have much in common. In the number line, however, numbers are associated with points on a line, not spaces on a track. The number line begins at zero, the number track begins at one. The number line schema, moreover, is developed further in a variety of ways. It is extrapolated backwards to represent negative numbers; points are interpolated to represent fractional numbers, and later on irrational numbers. The number track is much less abstract, and lends itself more readily to activities with physical embodiments of the concepts we want children to acquire.

Even here, however, abstraction has begun. When in Activity 3, Form (a) we let the size of the space used up on the track be independent of the size of the object, we are already moving towards the idea of a unit object. The introduction of unit cubes takes this idea a step further.

Activity 4 Sequences on the number track *[Num 1.5/4]

A game for children to play in pairs. Its purpose is to teach the concepts and abilities described above.

* Based on an idea from Mrs. Yvonne Selah, advisory teacher with the Inner London Education Authority. This activity also appears as NuSp 1.2/1 in 'The Number Track and the Number Line' network.

- Materials*
- Number track 1 to 10 (later, 1 to 20).*
 - 4 cubes for each player (for 1 to 10), different colours for different players . [Later, 7 cubes each for 1 to 20]
 - Number cards 1 to 10 (later, 1 to 20).*
- *Provided in the NuSp 1.2/1 photomasters.

- What they do*
1. The pack of number cards is shuffled and put face down. The top card is then turned face up, starting a separate pile.
 2. Each player in turn may
 - (i) either use the card showing, if the other player did not, or turn over another card;
 - (ii) put down one of his cubes in the corresponding position on the number track, or not. Both players use the same track, and only one cube is allowed in each space.
 3. The aim is to get as many cubes next to each other as possible.
 4. The game finishes when one player gets 3 in a row [for 1 to 10] (5 in a row for 1 to 20), or when both players have put down all their cubes.
 5. If before this the pack has been finished it is shuffled and used again as in step 1.
 6. Scoring is as follows. 1 by itself scores zero; 2, 3, 4, 5 cubes in a row score respectively 2, 3, 4, 5.

Discussion of activities Physical activities, and mathematical language, are used in combination (Modes 1 and 2) to expand the children's existing ideas of number. The connections with order, successor, and number-names are carefully preserved.

The earlier (1 to 5) version of 'Missing Stairs' could be played largely at a perceptual level, though counting was certainly a help. Here in Activity 2, counting becomes essential.

Activities 3 and 4 combine both the first and the second kinds of use of mathematics described in the Introduction. Cooperation in playing the games depends on a shared mathematical schema; but choosing the best alternative involves prediction, not in this case at a level of certainty, but based on a variety of possibilities.

Though the concepts of predecessor and successor are strongly involved in these activities, and are not difficult ideas, these terms may well be thought unsuitable for children of this age. Though we ourselves need names for these concepts, they are not particularly required by the children.

What are being matched in these activities are not objects, but orders of two kinds. A relation (matching) is involved between two different order relationships, size of number and spatial order. Once again we see how even at so elementary a level as this, mathematics is a really abstract subject. Yet young children master it without difficulty if it is presented right. If we can help them to use their intelligence to the full, they show themselves much more clever than they are usually given credit for.

Num 1.6 ZERO

- Concepts*
- (i) Zero as the number of the null set.
 - (ii) Zero as the number before one.

- Abilities*
- (i) To attach this number to any example of the null set which they encounter.
 - (ii) To extrapolate the counting sequence backwards past one to zero.

Discussion of concepts Compare these questions and answers. (i) “How many objects in this set?” “None.” This is appropriate in the context of Org 1.6. But if we reword the question as (ii) “*What is the number of this set?*”, the correct answer is “Zero.” To understand zero as a number involves more sophisticated thinking than any of the number concepts which the children have encountered so far, since it refers to the absence of any object in a set. The simplest way in which it can be presented is as described in Org 1.6 (The empty set; the number zero). As soon as children begin counting backwards, zero can also be introduced as the number before one (i.e., after one when counting backwards). As well as being a number in its own right (albeit an unusual kind of number), zero acquires extra importance in place-value notation. When we write 30, the zero not only tells us that there are no ones, but it changes the meaning of the 3 from 3 ones (if written alone) to 3 tens (if followed by a zero). It is the only number in the present curriculum which is treated as a topic all by itself.

Activity 1 “Which card is missing?” (Including zero) [Num 1.6/1]

As already noted in Num 1.3/1, this game may be played with the inclusion of a blank card as soon as children have encountered the null set (Org 1.6). Its purpose is to introduce zero as the number of the null set.

Activity 2 Finger counting from 5 to zero [Num 1.6/2]

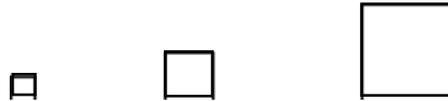
An activity for a small group. Its purpose is to introduce zero as the number before one.

- What they do*
1. Reverse Num 1.4/1, ‘Finger counting to 5,’ starting with five and finishing with one.
 2. Repeat as in step 1, but after touching the table with little finger only, palm downward and saying “one,” turn the hand palm upwards showing an empty hand and saying “zero.”

Discussion of activities These activities are all helping children to form the concept of zero as a number by extrapolating their existing concept, derived from and until now restricted to, numbers of objects which they can see. This is Mode 3 schema building: creativity. They are creating a new concept out of an existing schema.

The first activity is an easy one, since the blank card stands out from the others. Nevertheless, it comes in the context of cards with varying numbers of dots on them, so it prepares the way for the idea of the null set as having a number also. The concept of zero involves thinking, not “This card doesn’t have a number,” but “The number of dots is zero.” After this simple beginning, Activity 2 makes the extrapolation explicit by putting zero into a sequence of counting numbers, and associating it simultaneously with no fingers touching the table, and an empty hand.

The next point is a subtler one. When we are extrapolating, the next objects have to be more objects of the same kind; e.g., what comes after these:



has to be another square, not a triangle, still less a letter of the alphabet. So when we extrapolate the sequence of numbers 5, 4, 3, 2, 1 . . . , whatever comes next has to be a number of some kind.

In these activities we have extrapolation of two distinct and complementary kinds. There is more here than meets the eye!

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.7 EXTRAPOLATION OF NUMBER CONCEPTS TO 20

Concepts The complete numbers from 11 to 20 in order, linked with their names.

- Abilities*
- (i) To put numbers from 11 to 20 in order.
 - (ii) To recognize whether the sequence is complete.
 - (iii) If it is not, to provide the missing one(s).

Discussion of concepts This topic involves the formation of new concepts of what is now a familiar kind.

Activity 1 Finger counting to 20: “Ten in my head” [Num 1.7/1]

An activity for a small group. Its purpose is to use the familiar activity of finger counting to continue the counting sequence to 20.

- What they do*
1. They begin by finger counting from 1 to 10, as in Num 1.5/1.
 2. After reaching 10, they raise both their hands from the table and say “Ten in my head.”
 3. They then continue counting, “Eleven” (left little finger), “Twelve” (two fingers of left hand) and so on.
 4. When the children have practiced this, they form pairs and ask each other questions of two kinds.
 - (a) Question: “Show fourteen.”
Answer: “Ten in my head” and touches table with four fingers of left hand.
Question: “Show four.”
Answer: “None in my head,” fingers as above.
 - (b) Question: “What number is this? I have ten in my head.” and touches table with all fingers and thumb of left hand, thumb of right hand.
Answer: “Sixteen.”
Question: “What number is this? None in my head” and touches table with all digits of left hand, thumb and first finger of right hand.
Answer: “Seven.”

Discussion of activity This activity relates the new number concepts to physical materials and activities with which the children are already familiar. Activity 1 also relates the numbers 11 to 20 to a repetition of the same pattern as 1 to 10, which is especially useful since the spoken number-words do not follow a regular pattern. Fortunately the written numerals do.

OBSERVE AND LISTEN

REFLECT

DISCUSS

NUM 1.8 ORDINAL NUMBERS FIRST TO TENTH

Concepts The ordinal numbers first to tenth, as representing positions in a sequence rather than numbers of a set.

- Abilities*
- (i) To use the ordinal number-words correctly to describe a position in a sequence.
 - (ii) To say these words in sequence, independently of physical objects.

Discussion of concepts When children were learning to use the counting numbers for finding the number of a set, we were careful to ensure they understood that the last number gave the number of the set as a whole, and did not refer only to the last number counted. However, when we apply the words ‘first,’ ‘second,’ ‘third,’ . . . in turn to the objects in a set, the opposite is the case. These words refer to the positions of the separate objects in a sequence.

We therefore need to make sure that the introduction of ordinal numbers does not introduce any confusion. Counting numbers apply to all sets, and are independent of the positions of the objects in the set. Ordinal numbers apply only to sets which are ordered in some way, and are dependent (e.g.) on the locations of the separate objects or on their succession in time. The activities in this topic aim to keep the distinction clear by using both the counting numbers (which are also called cardinal numbers when we want to contrast them with ordinals) and the ordinal numbers, with their correct and contrasting meanings.

Activity 1 “There are . . . animals coming along the track” [Num 1.8/1]

An activity for a small group of children. Its purpose is to introduce the concepts described above. Most children of this age will already know the earlier ones, but this activity will begin to extend them and clarify the distinction.

- Materials.*
- Ten or more small model animals.
For stage (a) only five of these are used.
 - A strip of paper to represent a track.

What they do Stage (a)

1. Begin by putting three animals on the track, one behind the other and all facing in the same direction. Explain that numbers like 1, 2, 3 say how many there are, but there is another kind of number which says what order they are in. We can all see that there are three animals coming along the track. But we can also use the new kind of numbers to say that the elk is first, the sheep is second, and the goat is third (or whatever the animals are). To describe what we see, ask them all to repeat “There are three animals coming along the track. The elk is first, the sheep is second, and the goat is third.”
2. The child who is to start puts another animal on the track, either in front or inserted between two of the others. (Preferably not last, since this would leave unchanged the order of the others.)

Num 1.8 Ordinal numbers first to tenth (cont.)

3. She then says (e.g) “Now there are four animals coming along the track. The elk is first, the deer is second, the sheep is third, and the goat is fourth.”
4. The others continue likewise. After ‘third,’ the names of the ordinal numbers are formed directly from the cardinals, so when they have seen the pattern the children will be able to provide each new number-name as required.
5. At this stage they continue up to five, and then start again. this time with perhaps just one or two animals.
6. Instead of putting down another animal, a child may if she wishes turn them all to face in the opposite direction. In this case, the child who follows step 4 above would say something like “Now they are all going in the opposite direction. There are still four animals, but now the goat is first, the sheep is second, the deer is third, and the elk is fourth.” This emphasizes that the cardinal number of the set remains unchanged, but the ordinal numbers of all its members are different.

Stage (b)

As for stage (a) except that the first child puts down at least 5 objects to start with, and they continue until there are ten animals on the track.

Activity 2 “I’m thinking of a word with this number of letters.” [Num 1.8/2]

An activity for a small group of children. Its purpose is to consolidate the concepts introduced in Activity 1 by using them in a different context.

- Materials*
- A set of cards numbered 2 to 10.
 - Any suitable book, preferably with large print.
 - Pencil and paper for each child. Squared paper is helpful.

- What they do*
1. The cards are shuffled and put face down in the middle of the table.
 2. The child whose turn it is to begin turns over the top card, and puts it where all can see. Suppose that it is 6.
 3. She opens the book at random, and looks for a word with this number of letters. Suppose that this is ‘cannot.’ She writes this down, but does not let the others see it.
 4. She says “I’m thinking of a word with six letters. The second letter is ‘a’.” The others all write this down.
 5. She might then say, “The sixth letter is ‘t’,” and the others would all write this in the appropriate position on their papers. (This is where squared paper helps.) In this and the previous step, she may give the letters in any order she likes.
 6. This continues until everyone has written down the whole word.
 7. Steps 2 - 6 are repeated with another child acting as ‘caller.’ The numbers are not re-used, so the word chosen will be of a different length each time.
 8. I see no reason why guessing should not be allowed, but it is in fact introduced specifically in the next activity.

Note The purpose of the book is simply to make it easy to choose a word of the right number of letters in a random way.

Activity 3 “I think that your word is . . .” [Num 1.8/3]

An activity for a small group of children, in which ordinal position is used to ask for information to help in guessing. It is an extension of Activity 2, but appreciably harder.

Materials As for Activity 2, except that it is better to use only the cards with larger numbers, say from five upwards.

- What they do*
1. One of the children chooses a word as in step 2 of the previous activity. She tells the others how many letters it has, and puts out the number card to remind them. Players are reminded that plurals are possible.
 2. The others in turn try to guess the word by asking for one of the letters by its ordinal number. Only the player whose turn it is may guess, and she is allowed only one guess. She may do so before or after a letter, but clearly it is better to do so after.
 3. Example. The word has five letters, and the first player has asked for the first letter. This is ‘p.’ The next child asks for the last letter. This is disallowed, since she must use its ordinal number. She loses her turn. Next player: “What is the third letter?” Answer “Another ‘p’.” This player guesses “piper.” Word-holder: “Wrong.” This gives the next player the idea that it might be ‘pipes,’ and asks for the fifth letter. Answer ‘r,’ so it cannot be ‘pipes.’ ‘Piper’ was clearly a near miss. Perhaps this player will guess correctly?
 4. The player who guesses correctly becomes the next word-holder.
 5. If no one guesses correctly, it does not really matter since the learning goal here is ordinal numbers, not vocabulary and spelling, which come as valuable extras. In this case, the next player clockwise acts as word-holder.

Variation This game could also be played with words in a sentence.

Discussion of activities All these activities provide physical embodiments of ordinal numbers, the order being spatial in every case. Turning the animals around in Activity 1 is an important feature, since although left-to-right is more widely used, children should not think that it is the only one. In Activity 3, they are using ordinal numbers to test hypotheses. These are all group activities in which the children check each other for correct use of the terms.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.9 ODDS AND EVENS

Concepts Even numbers as
(i) numbers which can be made from twos,
(ii) numbers which can be divided into two equal parts.

Odd numbers as
(i) numbers which cannot be made from twos,
(ii) numbers which cannot be divided into two equal parts.

Abilities (i) To say whether any given number is odd or even.
(ii) To test the accuracy of a statement of this kind.

Discussion of concepts Physical objects have a variety of properties such as being hard, or red, or soluble in water. Numbers are mental objects, and these too have a variety of properties which we now begin to investigate.

The concepts of even and odd numbers described under (i) and (ii) above indicate the two different beginnings which these concepts have. The final concepts are formed when a child recognizes that these are two aspects of the same property.

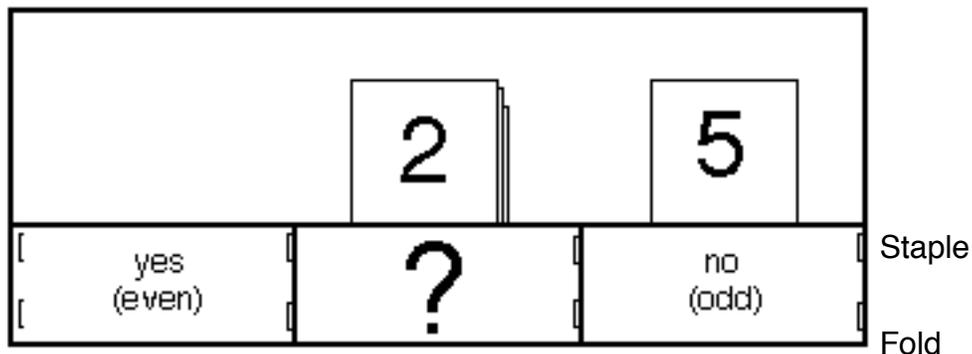
Activity 1 “Yes or no?” [Num 1.9/1]

An activity for 2 children. Its purpose is to introduce the concepts of even/odd numbers as described in (i) above. (This activity was invented by a working group of teachers at a conference organized by the Inner London Education Authority.)

Materials

- 3 pockets joined together (made from thin card, and stapled as shown below).*
- Number cards 1 to 10.*
- 20 cubes of at least 5 different colours.

* Photomasters provided.



What they do

1. The number cards are shuffled and put in the centre pocket.
2. Player A has 10 single cubes.
3. Player B has 10 cubes joined in twos, each two of a different colour.
4. A takes a card from the ? pocket, and puts out that number of singles.

5. B then says either “Yes, this number can be made with twos,” and puts the correct number of twos; or “No, this number cannot be made with twos.” At this stage, they may decide to check physically, by joining their cubes to form rods and seeing whether these are the same length. If A agrees with B’s answer, they may think that this is not necessary.
6. The card is then put into the appropriate pocket.
7. When all the cards have been used, the children take them out of their respective pockets and look for a pattern.

- Notes*
- (i) When B’s twos are made into a rod, it is desirable for them to be still distinguishable. This is why B should have two’s in several different colours.
 - (ii) At least for the first few times, children should test physically, so that they know exactly what is involved.
 - (iii) After a while, children can replace “Yes” or “No” with “Even” or “Odd.”
 - (iv) When children are confident with numbers from 1 to 10, they can play from 1 to 20.

Activity 2 “Can they all find partners?” [Num 1.9/2]

An activity for children to do by themselves. This uses the even/odd concepts predictively.

- Materials*
- Picture sets of children, varying between 5 and 20 in number.*
 - Non-permanent markers.

*These should be on card covered in plastic film. At the top of each, print “Can they all find partners?”

- What they do*
1. A child receives one of these.
 2. He writes “Yes” or “No” after the question.
 3. He then tests his prediction by drawing lines to join the children into pairs.
 4. If his prediction was wrong, he corrects it.

Activity 3 “Odd or even?” [Num 1.9/3]

A game for children to play in pairs. Its purpose is to introduce aspects (ii) of the concepts listed at the beginning of Num 1.8, ‘Odds and evens.’

- Materials*
- 20 cubes.

- What they do* Begin with the following preliminary explanation. Make two rods, say 6 and 7 cubes respectively. Break the 6 rod into two 3s, put these side by side. Say “This is a different meaning for even. We can make a 6 into two rods of even length. This is another way of showing that 6 is an even number.” Break the 7 rod into a 3 and a 4, put these side by side, and say “Now there’s an odd one left over. 7 is an odd number.” When they have grasped this, the children play as follows:

Num 1.9 Odds and evens (cont.)

1. A makes a rod, puts it on the table and says to B, e.g., “Seven: odd or even?”



“Odd or even?” [Num 1.9/3, Step 2]

2. When B has answered, A gives him the rod so that he can test as described above.
 3. If correct, B gets a point; if incorrect, he doesn't.
 4. A and B change roles, and steps 1 to 3 are repeated.
- Play initially with 10 cubes, then extend to 20.

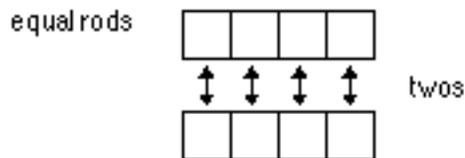
Discussion of activities

“Yes or No” begins with Mode 1 schema building (physical experience), using materials which embody the concept of an even/odd number as one which can/cannot be made physically from pairs. At first they should test by Mode 1 (prediction). If they have played the game long enough to have the concepts well established, they can test by Mode 2: agreement based on a shared schema. If they do not agree, they will have to discuss, and probably appeal to Mode 1 testing.

Activity 2, “Can they all find partners?” embodies the same concept in different materials. Testing is by Mode 1 (prediction).

Activity 3, “Odd or even?” uses both Mode 1 building (physical experience) and Mode 1 testing (prediction).

The final stage is reached by Mode 3 building, when children realize that Activity 1 and Activity 3 are two different physical results of the same mathematical property. *Even* is a higher order concept combining both aspects. If necessary you can help by this demonstration.



OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.10 DOUBLING AND HALVING

- Concepts* (i) Doubling a number.
(ii) Halving a number.

Abilities Given an even number, to double or to halve it.

Discussion of concepts Here we continue the parallel described in Num 1.8 between physical objects, and numbers regarded as mental objects. We can do things to physical objects, and what we can do depends on their nature. Similarly we can do things to numbers, again depending on their nature. All numbers can be doubled; only even numbers can be halved, so long as we are talking about whole numbers. To remind us simultaneously of the parallel between these, and also the difference, it is useful to talk about physical *actions* and mathematical *operations*. So doubling and halving are mathematical operations. Adding and subtracting, multiplying and dividing, are other examples of mathematical operations.

Activity 1 “Double this, and what will we get?” [Num 1.10/1]

A game for two children to play in pairs. It introduces children to the concept of doubling and to a simple method of multiplication.

- Materials*
- Cubes, 55 each. (See step 1)
 - A pack of cards 1 to 10.*
 - A number track 1 to 20.*
- * Provided in photomasters

- What they do*
1. Each makes a staircase from 1 to 10. It helps if the rods from 6 to 10 are made with 5 cubes of one colour and the rest of a different colour.
 2. The pack of cards is shuffled and put face downwards.
 3. A turns over the top card and puts it face upward, starting a new pile.
 4. She then takes out the rod of that number from her staircase, and predicts the number of the rod which will result from joining this to the rod of the same number from B’s staircase. Let them devise their own methods if possible. If they are stuck, one good way to do this is by working in base 5, assisted by the colours of the rods, counting singles twice. E.g., if the rod is 7, she says “Double five is ten, eleven, twelve, thirteen, fourteen.”
 5. Her prediction is then tested by using the number track.
 6. If correct, A scores a point.
 7. They then change about.

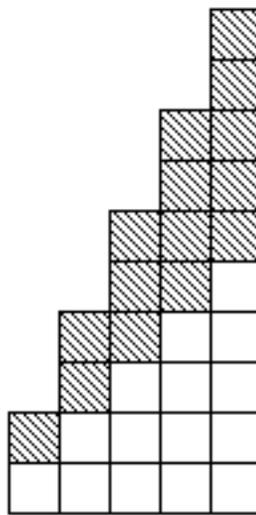
Num 1.10 Doubling and halving (cont.)

Activity 2 “Break into halves, and what will we get?” [Num 1.10/2]

A game for children to play in pairs. Its purpose is to show the inverse relation between doubling and halving.

- Materials*
- 30 cubes in two different colours.
 - Even number cards from 2 to 10.*
- * Provided in photomasters

- What they do*
1. Each child makes a 1 to 5 staircase of a different colour.
 2. These are then joined to make a 2 to 10 staircase of even numbers, which shows clearly the doubles/halves relationship.
 3. Explain that “Break into halves” means “Break into two matching parts,” and demonstrate.



A doubles and halves staircase.

Stage (a)

4. Shuffle the cards and place face downwards.
5. Player A turns over the top card and puts it face upwards, starting a new pile.
6. If the number is (say) 6, she points to the 6 rod and says “Half of 6 is 3.”
7. If B does not agree, they test physically by breaking the rod.
8. If B does agree, she awards A a point.
9. They then continue as above with B turning over the next card.

Stage (b)

As above, except that the staircase is covered with a sheet of paper, so A does not point to the rod, the paper is removed and they check visually or physically.

Activity 3 Doubles and halves rummy [Num 1.10/3]

A card game for up to four players. Up to six may play if a third pack of cards is introduced. Its purpose is to practise the concepts of halves and doubles of a given number, independently of physical materials.

- Materials*
- 2 double-headed number packs 1 - 20, without the odd numbers over 10.*
 - * Provided in photomasters.

- Rules of the game*
1. The packs are put together and shuffled. 5 cards are dealt to each player.
 2. The rest of the pack is put face down on the table, with the top card turned over to start a face upwards pile.
 3. The object is to get rid of one's cards by putting down pairs of cards in which one is the half or double of the other.
 4. Players begin by looking at their cards and putting down any pairs they can. They check each other's pairs.
 5. The first player then picks up a card from either the face down or the face up pile, whichever she prefers. If she now has a pair, she puts it down. Finally she discards one of her cards onto the face-up pile.
 6. In turn the other players pick up, put down a pair if they can, discard.
 7. The winner is the first to put down all her cards. Play then ceases.
 8. The others score the number of pairs she has made. The winner will thus score 3, the others 2, 1, or 0.
 9. Another round may then be played, and the scores added to those of the previous round.

Discussion of activities

Activity 1, "Double this and what will we get?", uses physical experience and prediction for building and testing the concept of doubling.

Activity 2, "Break into halves, and what will we get?", uses physical experience and prediction for building and testing the concept of halving. Stage (b) requires this to be done entirely mentally. Children who can play this game have developed the mathematical operation to a stage where it is quite independent of physical action.

Activity 3 is a game in which players have to realize that one number is double or half another. Whereas being odd or even is a property of a number itself, being double or half is a relationship with another number; so this activity takes children into a new area. This game will later be played with the same rules, but different mathematical relationships. An earlier game used here was 'Halves and doubles snap.' I replaced it by the present Activity 3 because I wanted one which did not involve speed of response.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 1.11 EXTRAPOLATION OF NUMBER CONCEPTS TO 100

Concepts The complete counting numbers in order to 100, grouped in tens.

- Abilities*
- (i) To state the number of a given set from 1 to 100.
 - (ii) To make a set of a given number from 1 to 100.

Discussion of concept Here we are concerned, not with a totally new concept (such as being odd or even) but with increasing the examples which a child has of his existing concept of number. Order and completeness provide a framework to ensure that these new examples fit the established pattern.

The key feature of the extrapolation is the idea that we can apply the process of counting, not only to single objects but to groups of objects, treating each group as an entity. So this topic links with all the topics in Org 1 (Set-based organization) which are shown in the upper part of the network as leading to topic 13 (Org 1.13, 'Base ten').

Activity 1 Throwing for a target [Num 1.11/1]

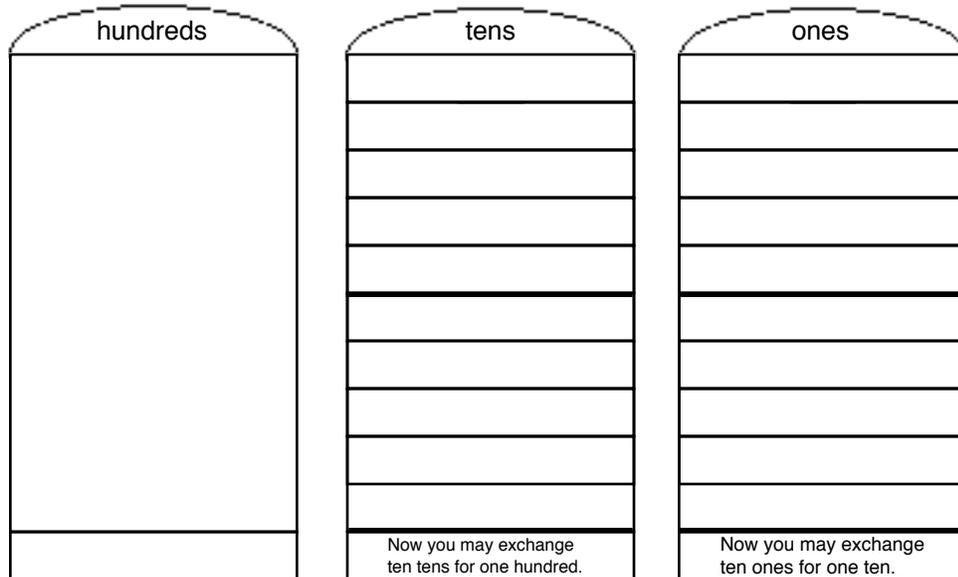
An activity for one player, two working together, or it may be played as a race between two players throwing alternately. Its purpose is to help children to extrapolate their number concepts up to 100, and to consolidate their use of grouping in tens.

Note Before this activity, children should have completed Org 1.

- Materials*
- A game board, see Figure 1.*
 - For stage (a), base 10 material.
 - For stage (b), a variety of other materials as described below.
 - 2 dice.
 - Slips of paper on which are written target numbers, e.g., 137, 285. (It is best not to go above 300 at most, or the game takes too long.)
- * Provided in photomasters

What they do Stage (a)

1. The player throws the dice and adds the two numbers.
2. He puts down that number of ones on the game board, one per line and starting from the top.
3. Each time he reaches 10 ones he exchanges them for one 10, placing it on a line in the 'tens' column. Similarly, 10 tens are exchanged for one 100 to go in the 'hundreds' column.
4. He must finish with a throw of the exact number to reach the target number.
5. If the number is 6 or less he uses one die only.

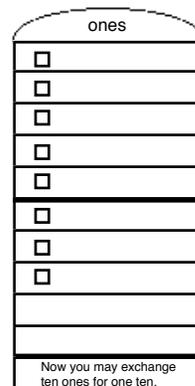


Stage (b)

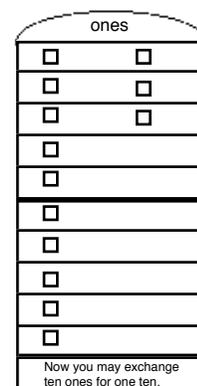
To prevent children becoming too attached to a particular embodiment, this game should also be played with other suitable materials, such as popsicle sticks for ones, bundles of ten with a rubber band around, and ten bundles of ten. Drinking straws cut in half are good. Also: pennies, dimes, and \$1 coins.

Notes

- (i) At stage (a), children will often put more than ten ones in the 'ones' column, and these should be available. E.g., starting with the state on the left, and throwing 5, they put down 5 and get the state on the right. Then they remove the line of ten cubes, exchanging these for a ten-rod which they put in the ten column. This is a good way to begin.



State 1



State 2

- (ii) However, at stage (b) provide only 10 ones. This leads to a variety of strategies, which children should be given time to discover for themselves. It is important to restrain one's urge to tell them, or they may acquire the method but not its interiority. I recommend that you say something like this: "When you find short cuts, as soon as you are sure they are correct you may use them."

Num 1.11 Extrapolation of number concepts to 100 (cont.)

Activity 2 Putting and taking [Num 1.11/2]

A game for two players. It is a simple variant of Activity 1, with the same purpose.

- Materials*
- 50 (halved) milk straws in bundles of ten, for each player.
 - ‘Exchange pool’: 5 more tens and 20 ones.
 - 2 dice.
 - A box to put spares in.

- What they do*
1. Each player starts with 50 straws, in bundles of 10. They agree which will put, and which will take, each working separately with his own set of straws.
 2. They throw the dice alternately.
 3. The ‘putter’ begins and puts down at his place the number that is the total shown on the dice.
 4. The ‘taker’ plays next and takes away from the bundles at his place the total shown when he throws the dice.
 5. The ‘putter’ wins by reaching 100.
 6. The ‘taker’ wins by reaching 0.
 7. Exchanging of 1 ten for 10 ones will be necessary whenever they cross a ten boundary upwards or downwards.
 8. If the game is played with the same requirement as in Activity 1, that the exact number must be thrown to win, this could prolong the game unduly. It is therefore probably best to agree that a throw which would take the number past 100 or 0 is also acceptable.

Discussion of activities Activities 1 and 2 both use the now-familiar concepts of grouping in tens, and canonical form (see Org 1.12) to lead children on to 100. Although they are extrapolating their number concepts, which is schema building by Mode 3 (creativity), this extrapolation is also strongly based on physical experience (Mode 1 building). This is an excellent combination.

An interesting feature here is that the physical experience by itself would not be sufficient to lead to the formation of these new concepts. A suitable schema is also needed which can organize this experience, and contains a pattern ready to be extrapolated by this experience. This has never been put better than by Louis Pasteur, when he said: “Discoveries come to the prepared mind.”

OBSERVE AND LISTEN

REFLECT

DISCUSS



*'Number Targets' [Num 2.8/1] **

* A frame electronically extracted from
NUMERATION AND ADDITION,
a colour video produced by the
Department of Communications Media,
The University of Calgary

[Num 2] **THE NAMING OF NUMBERS**
in ways which help us to organize them and use them effectively

Num 2.1 THE NUMBER WORDS IN ORDER (SPOKEN)

Concept At this stage, mathematical concepts are not necessarily involved.

Ability To speak the number words in order.

Discussion of concept Counting involves both saying (or thinking) the number words in order, and also making each word correspond to an object in the set to be counted, without omitting any and without counting any object twice. We begin by learning the number words, ready for use in this way.

Activity 1 Number rhymes [Num 2.1/1]

Activities for a small group. Their purpose is for children to memorize the number words in order, ready for their application in counting.

Materials You will already know a number of these. Useful sources for other ones are:

One, Two, Three, Four compiled by Mary Grice, published by Frederick Warne.
Number Rhymes by Dorothy and John Taylor, published by Ladybird Books.
Counting Rhymes, edited by Clive Samsom, published by A. & C. Black.

What they do 1. Everyone says the number rhymes together. This one uses the first five number-words only.

One, two, three, four, five,
Once I caught a fish alive,
Why did you let it go?
Because it bit my finger so.

2. When these are well established, they can be linked with finger counting (see Num 1.4/1). For this they need something which goes more slowly. Here is one from Mary Grice's book.

Peter taps with one hammer,
One hammer, one hammer,
Peter taps with one hammer,
This fine day.

And so on up to five.

Num 2.1 The number words in order (spoken)

3. Here is a speedier one, adapted from the traditional counting-out rhyme.

One potato, two potatoes, three potatoes, four.
Five potatoes, that's enough, so we will plant no more.

4. Note that the relation between these and the finger movements is a truly counting one. E.g., 'Peter taps with three hammers' means three fingers tapping; 'three potatoes' means a total of three potatoes planted, corresponding to three fingers touching the table.

Discussion of activities

No new mathematical concept is being learned here. This topic and the one which follows are concerned with memorizing the number words, so that they are available for attachment to the number concepts acquired in other topics, e.g., Num 1.1. However, the correspondence between the memorized order, and the structural order of the complete sequence of numbers, is an important concept, and an invaluable tool for manipulating the number concepts. Much the easiest way to arrive at (e.g.) 'the number one more than seven' is by knowing that it is given by the next word after seven, i.e., eight. This leads to adding by counting on, and subtraction by counting back. These topics thus link closely with the concepts learned in Num 1.4, Counting.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Saying and pointing [Num 2.3/1]

Num 2.2 NUMBER WORDS FROM ONE TO TEN

Concept } As already discussed in Num 2.1, continued to ten.
Ability }

Discussion of concepts These are a continuation of the same kind as in Num 2.1.

Activity 1 Number rhymes to ten [Num 2.2/1]

One, two, three, four, five,
Once I caught a fish alive.
Six, seven, eight, nine, ten,
And then I let it go again.

This continues the rhyme already learned. When the children are good enough at finger counting (see Num 1.4), it can be linked with finger counting to ten. Here is one from *Number Rhymes and Finger Plays* by E.R. Boyce and Kathleen Bartlett, reproduced here by permission of Pitman Publishing, London. I hope that Kathleen Bartlett will forgive me for changing 'housewives' to 'husbands' on alternate lines!

One busy housewife sweeping up the floor,
Two busy husbands polishing the door,
Three busy housewives washing baby's socks,
Four busy husbands winding up the clocks,
Five busy housewives washing out the broom,
Six busy husbands tidying the room,
Seven busy housewives cleaning out the sink,
Eight busy husbands giving puss a drink,
Nine busy housewives stirring up the stew,
Ten busy spouses with nothing else to do.

Discussion of activity The discussion at the end of the previous topic applies equally well here.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.3 SINGLE-DIGIT NUMERALS RECOGNIZED AND READ

Concept Written digits as having the same meanings as the spoken number words.

- Abilities*
- (i) Seeing a single digit, to say the corresponding number words.
 - (ii) Hearing a number word, to identify the corresponding written or printed numeral.
 - (iii) Later, to write these themselves.

Discussion of concept A numeral is a symbol for a number. The digits are the single-figure numerals 0, 1, 2, . . . 9. So 574 is a three-digit numeral, standing for a single number. This distinction between numbers and numerals is not one which I would make explicit to the children, who use these ideas quite well intuitively, but it is as well to be clear about it ourselves. It is a particular case of the difference between the name of an object, and the object itself, which is hardly to be regarded as trivial.

Myself, I regard any symbol for a number as a numeral. E.g., I would say that 4, Roman IV, the written word 'four,' and also the spoken word 'four,' are all numerals for the same number concept. Some may not agree with all of these, and it will be sufficient if we agree to use the term 'numeral' for written symbols such as 4, and 7053, and, later on, $0.325\frac{3}{4}$, . . .

We are so used to writing numbers (or rather, numerals) that we do not realize what a major step this is for children. Spoken words are naturally attached to ideas; doing the same with written symbols is like learning a new language.

Activity 1 Saying and pointing [Num 2.3/1]

An activity for a small group. Its purpose is to link spoken and written numerals.

- Materials*
- Numerals 1 to 10 written large on a card.

What they do During the written number rhymes, the teacher or a child points to the numerals as they are spoken.

Activity 2 "Please may I have . . . ?" [Num 2.3/2]

A game for 4 to 6 children. Its purpose is to consolidate the connections between written numerals and spoken number-words.

- Materials*
- An even number of packs of number cards 1-5, at least as many packs as players.*

* Provided in the photomasters.

- Rules of the game*
1. The cards are shuffled, and all are dealt to the players.
 2. The object is to get rid of one's cards by putting down pairs with the same number (the word we use with the children).
 3. After the deal, players look at their cards. If they have any pairs of cards with the same numeral, they put these down, face up.
 4. They then play in turn, asking whoever they like for cards they need to make further pairs. E.g., "Please, Sally, may I have a three?"
 5. If Sally has a 3, she gives it. The asking player can then make a pair and put it down. If Sally doesn't have a 3, she says "Sorry."
 6. This continues in turns.
 7. The winner is the first to put down all his cards, but the others continue and play out their hands.

Activity 3 Joining dots in order, to make pictures [Num 2.3/3]

This well-known activity, current in children's play materials for over a half a century, is excellent practice in using the written numerals in order.

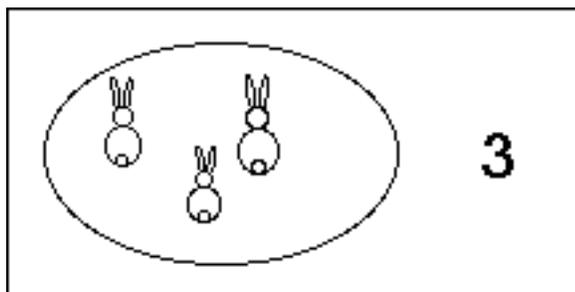
Activity 4 Sets with their numbers [Num 2.3/4]

For children working on their own.

- Materials*
- Little rubber stamps of animals, or other pictures, that children can quickly put on paper.
 - Pieces of paper cut up ready to make into pictures.
 - Pencils.

- What they do*
1. The children make a set of a chosen number at one side of the paper.
 2. Enclose them in a drawn set loop.
 3. They then write the correct numeral against it, as shown below.

This might be allowed as a reward for having neatly copied a line of this numeral, on the grounds that before making a picture of this kind it is necessary to be able to write the numeral nicely.



Num 2.3 Single-digit numerals recognized and read (cont.)

Activity 5 Sequencing numerals 1 to 10 [Num 2.3/5]

A game for children to play in pairs. Its purpose is to give further practice in putting the first ten written numerals in order.

- Materials*
- A set of double headed number cards, with numerals from 1 to 10.*
- * Provided in the photomasters

What they do Stage (a)

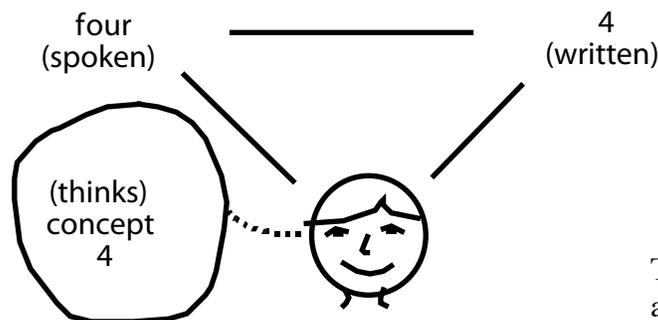
1. The cards are spread out on the table face up.
2. The children cooperate in putting these in order.

Stage (b)

1. The cards are shuffled and dealt, five to each player. They look at their cards, and play whichever they choose.
2. One player puts down a card.
3. The other player, if she can, puts down a card in sequence.
4. The first player does likewise.
5. The winner is the player who first puts down all her cards.

Discussion of activities

The learning in this topic is more associative than conceptual. In Num 1 the spoken number words will have become associated with their number concepts at the same time as the latter were formed. The present topic is concerned with associating the written digits, the spoken words, and the concepts, all with each other. In the process, children will come to realize that the spoken sound and the mark on paper have the same meaning. E.g.:



The number concept 4 is a mental object and we cannot see or hear it.

Just as the words needed to be memorized in Num 1.1 and 1.2, so in the present topic the written symbols have to be memorized. But whereas with spoken words, the abilities to speak them and to recognize them are acquired simultaneously, it is different with the written symbols. Since learning to write the digits is so much more laborious than learning to speak them, the physical skill of writing them needs to be established independently of using them for mathematical purposes.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.4 CONTINUATION OF COUNTING: 1 TO 20

Concepts The number names one to twenty.

Ability To speak the number words in order, from one to twenty.

Discussion of concept	By now the children know that the words in counting rhymes also stand for numbers. So in this topic, they are also beginning to extend their range of number concepts, particularly if finger counting is linked with the number rhyme.
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Activity 1 Number rhymes to twenty [Num 2.4/1]

An activity for a small group. Its purpose is to extend children's range of number names up to twenty. This can be linked with finger counting to twenty if the children work in pairs.

Materials This is a good rhyme to begin with. For others, see the sources already quoted.

One, two, buckle my shoe.
Three, four, knock at the door.
Five, six, pick up sticks.
Seven, eight, lay them straight.
Nine, ten, a big fat hen.
Eleven, twelve, dig and delve.
Thirteen, fourteen, maids a-courting.
Fifteen, sixteen, maids in the kitchen.
Seventeen, eighteen, maids in waiting.
Nineteen, twenty, my plate's empty.

- What they do*
1. The one on the left puts down fingers for the first decade.
 2. The one on the right continues from eleven to twenty. This links the words eleven, twelve . . . twenty, with ten and one finger, ten and two fingers . . . ten and ten fingers (= two tens).
 3. When this is well established, children can finger count to twenty on their own, lifting all their fingers after ten and starting again with left little finger at eleven. In this case the first ten are stored mentally. So they say: "Ten in my head, eleven, twelve . . ."

Discussion of activities	'Eleven' and 'twelve' follow no pattern; and the pattern from thirteen to twenty is not consistent with the pattern from twenty onwards, which is a fairly consistent one thereafter. Also the spoken number-words do not correspond with the written numerals (thirteen is written as ten, three and so on) unless like the Arabs we read from right to left. So the learning in this activity is largely associative rather than conceptual.
---------------------------------	--

Num 2.5 COUNTING BACKWARDS FROM 20

Concept Reversal of the counting sequence.

Ability To say the number words in order backwards, beginning with any number up to twenty and ending with zero.

Discussion of concepts As in several of the earlier topics, we are here mainly concerned with developing a verbal skill. This will later be linked with the concept of subtraction.
It is important to be able to start with any number.

Activity 1 Backward number rhymes [Num 2.5/1]

An activity for a small group. Its purpose is to teach children the number names in reverse.

- What they do* 1. Here is a nice one from Mary Grice's book. The name of a child goes in the space, a different child for each verse. Everyone says the number rhymes together.

Five currant buns in the baker's shop,
Big and round with sugar on top.
Along came . . . with a penny one day,
She/he bought a currant bun and took it away.

Four currant buns (etc.)

No currant bun in the baker's shop,
Big and round with sugar on top.
Along came . . . with a penny one day,
She/he couldn't buy a bun and take it away.

If you like, this could be used with objects on a 'plate' representing buns.

2. Here is a backwards rhyme from ten, also from Mary Grice's book.

Ten little school boys went out to dine;
One choked his little self, and then there were nine.

Nine little school boys sat up very late;
One overslept himself, and then there were eight.

Eight little school boys travelling in Devon;
One said he'd stay there, and then there were seven.

Num 2.5 Counting backwards from 20 (cont.)

Seven little school boys chopping up sticks;
One chopped himself in half, and then there were six.

Six little school boys playing with a hive;
A bumble-bee stung one, and then there were five.

Five little school boys going in for law;
One got in chancery, and then there were four.

Four little school boys going out to sea;
A red herring swallowed one, and then there were three.

Three little school boys walking in the zoo;
A big bear hugged one, and then there were two.

Two little school boys sitting in the sun;
One got frizzled up, and then there was one.

One little school boy living all alone;
He got married, and then there were none.

Other popular rhymes are ‘Ten green bottles,’ and ‘There were ten in a bed.’ I have seen these successfully used with the children miming while they sing, each holding a number card.

Activity 2 Numbers backwards [Num 2.5/2]

A game for up to 6 children. Its purpose is to practise the skill of counting backwards.

- Materials*
- Number cards 0 to 20.*
 - Two or three lists of numbers 0 to 20.*
- * Provided in the photomasters

- Rules of the game*
- Stage (a)* is played with the visual help of the number lists.
1. The number lists are put so that every player can see one of these right way up.
 2. The cards are shuffled and put face down.
 3. One of the players turns over the top card, puts it down face up, and says its number-word aloud, e.g., “Eleven.”
 4. The others in turn count “Ten,” “Nine,” “Eight,” . . . “Zero.”
 5. The player who says “Zero” is the one to turn over the next card, after which steps 3 and 4 are repeated.



Numbers backwards [Num 2.5/2]

Stage (b) is played as above, but without the help of the number lists.

- Variation*
1. One of the players turns over the top card as before, and puts it face up. She now starts counting at zero.
 2. The others in turn clockwise count forwards until they reach the number shown.
 3. Then they go into reverse, and count anti-clockwise from this number down to zero.
 4. As before, the player who says “Zero” turns the next card.
 5. After a while they are likely to notice that this is always the same player. You might ask them whether this must always be so.

Discussion of activities As with the earlier number rhymes, the learning in this topic is largely associative. However, in Activity 1, some of the rhymes used do relate to diminishing the set by a one each time. In the future, counting backwards provides a valuable technique for subtraction. It can then be conceptualized in terms of taking away one object for each word spoken.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.6 COUNTING IN TWOS, FIVES

Concept Sequences of number-words by which a set can be counted two at a time, or five at a time.

Ability To match these sequences with groups of 2 or 5 objects, and thereby to find the number of a given set.

Discussion of concepts Although “one, two, three . . .” is the most basic counting sequence, by memorizing others derived from this we acquire the means to handle number concepts in other ways. E.g., learning to count backwards provides a technique for subtraction. The sequences “Two, four, six . . .” and “Five, ten, fifteen . . .” provide a quicker way of counting large sets. The latter is also useful when working in base ten.

Activity 1 Counting with hand clapping [Num 2.6/1]

A group activity for any number of children. Its purpose is to teach the word-sequences themselves, in preparation for their use for counting.

What they do Stage (a) Counting in twos.

1. They count aloud together, clapping hands on every multiple of 2, thus: “One, two, three, four, five, six . . .”.
2. When they can do this, they whisper the ‘in-between’ words.
3. Next, they ‘say in their heads’ the in-between words.
4. Finally, without hand-clapping, they say “Two, four, six, eight . . .”.

Stage (b) Counting in fives.

As in Stage (a), but for multiples of five, thus: “One, two, three, four, five, six, seven, eight, nine, ten . . .”.

Activity 2 Counting 2-rods and 5-rods [Num 2.6/2]

A group activity. Its purpose is to link the verbal skill with the number concepts.

Materials • Interlocking cubes.

What they do Stage (a)

1. The children first make quite a lot of 2-rods.
2. They then count them, using the pattern acquired in the first stage of Activity 1.

Stage (b)

As in stage (a), but using 5-rods. Return to this activity when the children are able to count beyond 20.

Num 2.6 Counting in twos, fives (cont.)

Activity 3 Counting money, nickels [Num 2.6/3]

An activity for children working in pairs. Its purpose is to consolidate the skills which have been learned, in a new and useful situation.

- Materials*
- Nickels (plastic or genuine).
 - Containers for these.
 - Pencil and paper for each child.

- What they do*
1. One child takes a random set of coins, and decides on its value by counting in fives.
 2. He writes down the result, which he does not show to the other child.
 3. He gives this set of coins to the other child, who does likewise.
 4. They compare results. If these are different, they check together until they reach a result which they agree is correct.
 5. Steps 1 to 4 are repeated, starting with the other child.

Activity 4 Counting sets in twos and fives [Num 2.6/4]

An activity for a small group. Its purpose is to give further practice in these skills. Return to this after Num 2.9, when children are able to count and record beyond 20.

- Materials* For each child:
- A container in which is a set of objects between 20 and 100 in number.
 - A slip of paper and a pencil.

- What they do*
1. The children each have a container of objects .
 2. They tip these onto the table, and find the number of the set by counting in twos.
 3. This they record on the slip of paper which they fold over and place on top of the objects which they return to the container.
 4. The children then exchange containers, and again find the number of their sets, but this time they count in fives.
 5. This result they write on the outside of the slip of paper.
 6. Finally they compare these two numbers.
 7. If these are different, then the children check together to try to reach a result which they agree is correct.
 8. The numbers of the sets used do not need to be exact multiples of 2 or 5. Singles left over are counted singly, e.g., “. . .thirty, thirty-five, thirty-six, thirty-seven.”

Discussion of activities Activity 1 is a way of deriving two new counting sequences (counting in twos, fives) from the basic sequence which children already know. Only associative learning is involved here; but in Activities 2, 3, and 4, the new counting sequences are related to number concepts which children have already.

Activities 3 and 4 both use Mode 2 testing (checking their own conclusions against each other's, with discussion if their results do not agree). In Activity 3, Mode 3 testing (internal consistency) is also involved: the number of a set is the same however we count it.

Num 2.7 EXTRAPOLATION OF COUNTING PATTERN TO ONE HUNDRED

- Concepts*
- (i) The pattern of spoken number-words and written numerals.
 - (ii) Their associated numbers, in physical embodiments.

Ability To count from 1 to 100.

Discussion of concepts From 20 onwards there is a clear pattern, so conceptual learning can now be brought in. This is a combination of two patterns. The first is that of the words for the sequence of tens: ten, twenty (meaning two-ty), thirty (three-ty), forty, fifty (five-ty), sixty, seventy, eighty, ninety. The second pattern links to this the existing sequence of number words from one to nine: twenty one, twenty two, twenty three, etc. Thus the whole pattern repeats, though not identically, for every new decade.

This topic relates to Org 1.13 (Grouping in tens) and Num 1.11 (Extrapolation of number concepts to one hundred).

Activity 1 Counting in tens [Num 2.7/1]

An activity for a small group. Its purpose is to apply the technique of counting to groups of 10, and to learn the new number-names used for this.

- Materials*
- Ready-made 10-rods such as base 10 Dienes material; or ready-made bundles of milk straws.
 - Tens and ones chart (see Activity 3).

- What they do*
1. One child puts these one at a time onto the tens side of the chart, and all count in unison “Ten, twenty, thirty . . .”. Initially they will need help from you.
 2. Afterwards they try to say the words themselves without being reminded.

Activity 2 Counting two ways on a number square [Num 2.7/2]

An activity for a small group. It could also be used as a class activity. Its purpose is to link the spoken words with the written numerals.

- Materials*
- A number square, 1 to 100, for each child, when working in groups.*
 - A number square on the chalkboard when taken as a class activity.
- * Provided in the photomasters

- What they do*
- Stage (a)*
Children count in unison down the right-hand column, saying, “ten,” “twenty,” “thirty” . . . at the same time pointing to the corresponding numerals.

Stage (b)
Children count vertically down a column, e.g., “three, thirteen, twenty-three . . .”

Num 2.7 Extrapolation of counting pattern to one hundred (cont.)

This could also be done in turn round a group, either with each child saying a whole column, or with one number-name per child.

Stage (c)

One child points while the others say the number. The pointer starts at the top edge and zig-zags (without jumping) *as she likes* until she reaches the bottom or the right-hand edge, e.g., 14, 24, 35, 45, 46, 47, 57, 67, 68, 69.

This could also be used for larger groups or for the class as a whole. In this case either the teacher, or one of the children, points to the numerals on the chalkboard.

Activity 3 Tens and ones chart [Num 2.7/3]

An activity for a small group. They must all be able to see the chart the right way round, with tens on the left. Its purpose is to consolidate children's understanding of the two counting patterns in combination, that in ones and that in tens.

- Materials*
- Base 10 material, ones and tens.
 - Tens and ones chart, as illustrated below.

Tens	Ones

- What they do*
1. Each child in turn puts in either a one or a ten, whichever she likes.
 2. At the same time she continues the counting sequence, either one more or ten more.
 3. Around the group, a typical sequence might be: "ten," "eleven," "twenty-one," "thirty-one," "thirty-two," "thirty-three," "forty-three" . . .

Discussion of activities Activity 1 uses physical experience (Mode 1) combined with communication from you (Mode 2) to link the numbers of ten in the chart with their corresponding number-words.

The number square used in Activity 2 is a good way of showing how the two patterns, in ones from 1 to 10 and in tens from 10 to 100, are combined to provide a counting pattern from 1 to 100. By counting in various ways based on this square, children will begin to establish this double pattern in their own minds.

Activity 3 connects this two-way counting pattern with the underlying mathematical concepts, these concepts being embodied in physical materials (ten-rods or ten-bundles, units or ones) and physical actions. Putting one more ten-rod corresponds to saying the next ten-word, putting one more unit corresponds to saying the next word used for ones.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.8 WRITTEN NUMERALS 20 TO 99 USING HEADED COLUMNS

Concept That a particular digit can represent a number of ones, tens, (and later hundreds . . .) according to where it is written.

- Abilities* (i) To match numerals of more than one digit with physical representations of ones, tens, (and later hundreds . . .).
(ii) To speak the corresponding number-words for numbers 20 to 99.

Discussion of concept

First, let us be clear about what a digit is. It is any of the single-figure numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (corresponding to the numbers we can count on our fingers). Just as we can have words of one letter (such as a), two letters (such as an), three letters (such as ant), and more, so also we can have written numerals of one digit (such as 7), two digits (such as 72), three digits (such as 702), and more.

The same numeral, say 3, can be used to represent 3 buttons, or shells, or cubes, or single objects of any kind. If we want to show which objects, we can do so in two ways. We can either write '3 buttons, 5 sea shells, and 8 cubes,' or we can tabulate:

buttons	sea shells	cubes
3	5	8

Likewise the same numeral, say 3, can be used to represent 3 single objects, or three groups of ten, or 3 groups of ten groups of ten (which we call hundreds for short). We could write '3 hundreds, 5 tens, and 8 units'; or we could tabulate:

hundreds	tens	units
3	5	8

We are so used to thinking about (e.g.) 3 hundreds that we tend not to realize what a major step has been taken in doing this. We are first regarding a group of ten objects as a single entity, so that if we have several of these we can count "One, two, three, four, five . . . *groups of ten*." Then we are regarding a *group of ten groups of ten* as another entity, which can likewise be counted "One, two, three . . ." And by the end of this network, we shall no longer be regarding these as groups of physical objects, but as abstract mental entities which we can arrange and re-arrange. We shall also have introduced a condensed and abstract notation (place value).

These two steps need to be taken one at a time. While the first, described above, is being taken, we need to use a notation which states clearly and explicitly what is meant: i.e., headed column notation.

Also, because the correspondence between written numerals and number words only becomes regular from 20 onwards, we start children's thinking about written numerals here where the pattern is clear. The written numerals 11 - 19 are also regular, but their spoken words are not, so these are postponed until the next topic.

Activity 1 Number targets [Num 2.8/1]

A game for as many children as can sit so that they can all see the chart right way up; minimum 3. It follows on from 'Tens and ones chart' (Num 2.7/3), which was the last activity described. Its purpose is to link the spoken number words, just learned, with the corresponding written numerals.

- Materials.*
- Target cards.*
 - Tens and ones chart.**
 - Pencil and headed paper for each child.
 - Base 10 material, tens and ones.***

* Provided in the photomasters. See also note (iii), following.

** The same as for 'Tens and ones chart' (Num 2.7/3).

*** This game should be played with a variety of base ten material such as milk straws or popsicle sticks in ones and bundles of ten; multibase material in base ten.

- What they do*
1. The target cards are shuffled and put face down.
 2. In turn, each child takes the top card from the pile. He looks at this, but does not let the others see it.
 3. Before play begins, 2 tens are put onto the chart. (This is to start the game at 20)

Tens	Ones
4	7

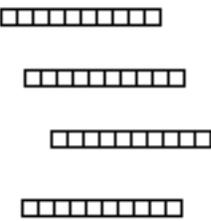
Tens	Ones
5	4

Target cards

Tens	Ones

Paper

4. The objective of each player is to have on the chart his target number of tens and ones.
5. Each player in turn may put in or take out a ten or a one.
6. Having done this, he writes on his paper the corresponding numerals and speaks them aloud in two ways. For example:

Tens	Ones
	

Chart

4	6
---	---

Writes

“four tens, six ones”;
“forty-six”

Speaks

Num 2.8 Written numerals 20 to 99 using headed columns (cont.)

7. In the above example, if a player holding a 47 target card had the next turn, he would win by putting down one more one. He would then show his target card to confirm that he had achieved his target.
8. Since players do not know each others' targets, they may unknowingly achieve someone else's target for them. In this case the lucky player may immediately reveal his target card, whether it is his turn next or not.
9. When a player has achieved a target, he then takes a new target card from the top of the pile, and play continues.
10. The winner is the player who finishes with the most target cards.

Notes (i) If one side of the tray is empty, a corresponding zero must be written and spoken: e.g.,



“four tens, zero ones”;
“forty”

and also



“zero tens, seven ones”
“seven”

- (ii) Players are only required to write the numbers they themselves make. It would be good practice for them to write every number, but we have found it hard to get them to do it.
- (iii) Several sets of target cards should be prepared in which the numbers are reasonably close together, both the tens and the ones. If they are too far apart, the game may never end.

Variation It makes the game more interesting if, at step 5, a player is allowed two moves. For example, he may put 2 tens, or put 2 ones, or put 1 ten and take 1 one, etc. This may also be used if no one is able to reach his target.

Activity 2 Number targets beyond 100 [Num 2.8/2]

When children are familiar with numbers greater than 100, they can play this game with suitable modifications using targets in hundreds, tens and units.

Discussion of activities In preparation for place-value notation, it is important for children to have plenty of practice in associating the written symbols and their locations with visible embodiments of hundreds, tens, units and in associating both of these with the spoken words. In this topic ‘location’ means ‘headed column’; in Num 2.10 it will mean ‘relative position.’

So that their conscious attention is free to concentrate on the meaning of what they are doing, children should before beginning this activity be able to write the digits 0, 1 . . . 9 without too much effort. This is a copying exercise, not in itself a mathematical one.

This activity uses concept building by physical experience (Mode 1). The social context provided by a game links these concepts with communication (Mode 2) using both written and spoken symbols.

Num 2.9 WRITTEN NUMERALS FROM 11 TO 20

Concepts The written numerals 11 - 19 as having the same meanings as the number-words with which they are already familiar.

Abilities (i) To match the numerals 11 - 19 with the spoken number-words.
 (ii) To match both with physical embodiments of these numbers.

Discussion of concepts The discussion of Num 2.8 applies equally here. However, the clear and regular correspondence which we find from 20 onwards, e.g.:

2 tens, 7 ones
 twenty seven
 27

does not apply from 11-19. E.g., although 1 ten, 2 units is written (as we would expect) 12, it is spoken not as “onety two” but as “twelve.” And 1 ten, 7 units is written (as we would expect) 17, but the spoken form is backwards, seventeen. So the present topic contains, implicitly, the notion of irregularity – departure from an expected pattern.

Activity 1 Seeing, speaking, writing 11 - 19 [Num 2.9/1]

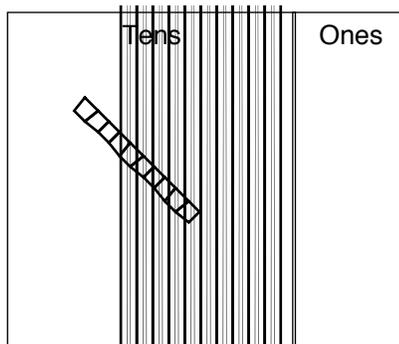
A teacher-led activity for a small group. Its purpose is to relate the number-words spoken but not written down in Num 1.7/1 to the written numerals 11-19.

Materials

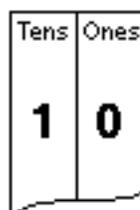
- Tens and ones chart.
- Base 10 material.
- Paper and pencil for yourself.

Suggested sequence for the discussion

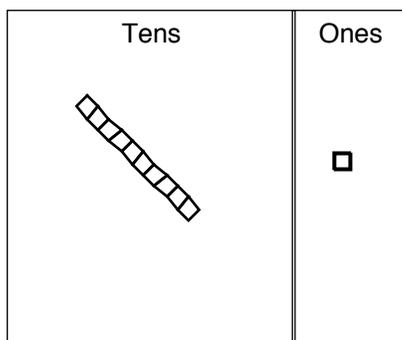
1. You put in



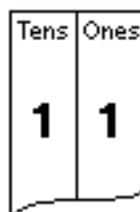
2. Write



3. Say: "One ten, zero ones. Ten."
4. Put in



5. Write



6. Say: "One ten, one one. Eleven."
7. Carry on through the teens. (So you don't put in another ten.)
8. Soon the children will join in. You might point out that if these followed the same pattern as 21, 31 . . . we would talk about onety-one, onety-two, etc., and explain that in olden times people hadn't thought about it carefully: so these names got attached, and have 'stuck.'

Activity 2 Number targets in the teens [Num 2.9/2]

The same number targets game as in the topic just before this (Num 2.8/1) should now be played, starting with the tens and ones chart empty.

Materials As for (Num 2.8/1) except:

- Target cards now 11 to 19 (provided in the photomasters).

Discussion of activities The written numerals 10 - 19 follow the same pattern as those from 20 on, so these concepts are acquired by extrapolation (Mode 3 schema building). However, the spoken number-words do not follow this pattern, and do not correspond well to the numerals, even from 13 on. ("Thirteen," "fourteen" . . . are read from right to left.) However, children will already be familiar with the spoken number-words, and the activity now links these to the numbers in physical embodiments (tens and ones chart), and to the written numerals.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.10 PLACE-VALUE NOTATION

Concept That a particular digit can represent a number of ones, tens, hundreds, according to whether it comes first, second, third in order reading from right to left.

Abilities (i) To identify separately which digits represent ones, tens, hundreds, by their positions relative to each other.
(ii) To read aloud two and three digit numerals.
(iii) To match these with physical representations.

Note Hundreds are not included until the second time round, after Num 2.13 in *Volume 2*.

Discussion of concept Provided that we have only one digit in each column (which may be a zero), we can leave out the headings and ruled columns and still know what each digit stands for. The result is a brilliantly simple notation, whose brilliance is easily overlooked just because it is so simple. The Greek mathematicians, excellent though they were in many ways, did not think of it; nor did the Romans, nor the earlier Egyptians nor the Babylonians. As a result, they made relatively slow progress in arithmetic and algebra.

It is also a condensed and abstract notation, and this is why it has been approached with such careful preparation in the preceding activities.

Activity 1 “We don’t need headings any more.” [Num 2.10/1]

A teacher-led activity for a small group. Its purpose is to introduce children to place-value notation, as described above.

Materials. • Pencil and paper.

Suggested sequence for the discussion 1. Write some number between 21 and 99, with headed columns, as below.

Tens	Ones
4	7

2. Ask, “Can you say this number?” Accept either “forty-seven” or “four tens, seven ones.”
3. Then say, “Can you say it another way?” to get the alternative reading. (At this stage we want both, every time.)
4. Repeat with other numbers, including ones between 11 and 19. Numbers such as 30 should be read as “Three tens, zero ones; thirty.”

5. Fold the top of the paper under so that the headings for tens and ones do not show, and write another number. Ask the children if they can still say the numbers as before. Practise this until they are confident, and then do the same without the dividing line.
6. If the children are fully proficient in the new notation, say, "So we don't always need the headings now, though they are still useful sometimes," and continue to Activity 3. Children who have progressed through the earlier topics in this network should have no difficulty at this stage. For those who do, it would be best to go back to topics Num 2.8 and 2.9 to ensure that they are fully prepared. They should then do Activity 2 of this topic.

Activity 2 Number targets using place-value notation [Num 2.10/2]

The children should now play again the number targets game, exactly as in Num 2.8/1 and Num 2.9/1, but with plain number cards* and plain paper (no headings).

* Provided in the photomasters.

Activity 3 Place-value bingo [Num 2.10/3]

A game for 5 or 6 players. Its purpose is to consolidate children's understanding of the relationships between written numerals in place value notation, the same read aloud, and physical embodiments of the numbers.

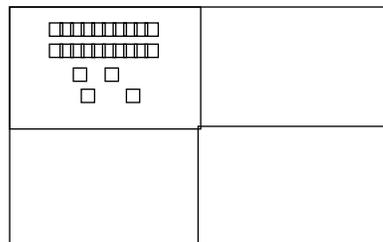
- Materials*
- Number cards 0 to 59.*
 - Base 10 material, tens and ones.
 - Die 0 to 9.**
 - Die 0 to 5.**
 - Shaker.
 - For each player, pencil and letter-size paper.

* Provided in the photomasters.

** Spinners may be used instead.

Rules of the game Stage (a) Preparation

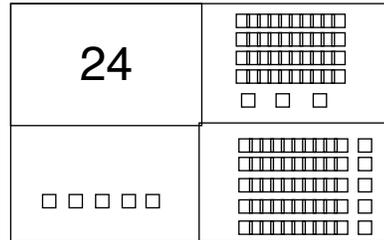
1. The players fold their papers once each way to make 4 rectangular spaces. These are used as bingo cards.
2. The first player throws the two dice. The 0 to 5 die gives the tens, the 0 to 9 die gives the ones.
3. Suppose that he gets 2 tens and 4 ones. He takes the corresponding ten-rods and ones from the box and puts these into the first space on his bingo card.
4. The other players do likewise in turn.
5. Steps 2, 3, 4 are repeated until each has filled all his spaces.



Num 2.10 Place-value notation (cont.)

The game

1. The pack of number cards is shuffled and put face downward on the table.
2. The players in turn turn the top card over, calling each two ways. E.g., “Three tens, five ones; thirty-five.” “Seven tens, zero ones; seventy.” “Zero tens,* four ones; four.”
3. It is important to speak each number in these two ways. After calling, the cards are put face down in another pile.
4. When a number is called corresponding to the ten-rods and units in one of a player’s spaces, he takes these off and writes the numeral for them.
5. The first player to replace all his ten-rods and units by numerals is the winner. He calls “Bingo.”
6. The game continues until all the players have done likewise.



Notes

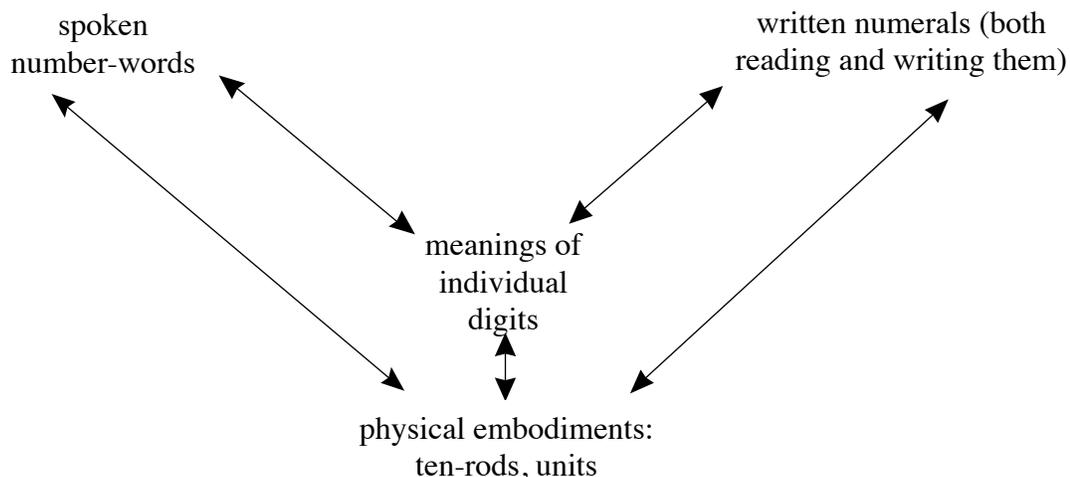
- * The question of whether zero may be omitted is considered later, in Num 2.12.
- (i) The game can be played with cards from 0 to 99, but this uses a lot of ten-rods and no new ideas are involved.
 - (ii) The paper may be folded to make 6 spaces if desired.

Stage (b)

This is played in the same way as Stage (a), except that instead of ten-rods and units, 10-cent and 1-cent coins (genuine or plastic) are used.

Discussion of activities Activity 1 introduces the final step into place-value notation, in which an important part of the meaning of each digit (namely whether it means that number of ones, tens, hundreds . . .) *is not written down at all*, but is implied by relative position. It is therefore very important that this meaning has been accurately and firmly established, which is the purpose of all the preparatory activities in earlier topics. Also, that the connection of the new notation with this meaning is established and maintained. Since the tens and ones no longer appear visibly, children now have to use their memory-images instead. These images are exercised and consolidated by having the children speak aloud (e.g., “four tens, seven ones”) every time. Activity 2 relates the new notation to the individual meanings of each digit both as expressed in words (five tens, three ones) and as embodied in physical materials (ten-rods and single cubes).

Activity 3 uses all these connections:



All three modes of schema building, and all three modes of testing, are brought into use for this final step into place-value notation.

- Mode 1 building* The new notation is related to physical embodiments.
- Mode 2 building* In Activity 1, the new notation is communicated, both verbally and in writing by a teacher. Activities 2 and 3 consolidate the shared meaning of this notation, by using it in a social context as basis for two games.
- Mode 3 building* The new notation is an extrapolation of the headed columns notation which they already know.
- Mode 1 testing* In Activity 1, children see column headings which confirm the accuracy of their readings in the ‘tens, ones’ form.
- Mode 2 testing* Activities 2 and 3 are games in which the players check the accuracy of each other’s actions.
- Mode 3 testing* This is implicit in the transition from headed columns to place value, since if the new notation were not consistent with the one they already know, children would not accept it so readily as they do.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.11 CANONICAL FORM

Concept Canonical form as being one of a variety of ways in which a number may be written.

- Abilities* (i) To recognize whether a number is or is not written in canonical form.
(ii) To re-write a number into or out of canonical form.

Discussion of concepts The interchangeability of headed column notation and place-value notation depends on there being one digit only in each column, so that the first, second, third . . . columns reading from right to left always correspond one-to-one with the first, second, third . . . digits reading from right to left. However, it is one thing to note the advantages of having one digit only per column, and another to tell children that “We must not have more than one digit in any column,” or “We must not have numbers greater than nine in any column.” This is incorrect, because when adding (e.g.) 57 and 85, we shall (temporarily) have numbers greater than nine in both columns; and when calculating (e.g.) $53 - 16$, children are told to take a ten from the tens to the ones column so that we can subtract 6.

The present approach recognizes that there are a variety of correct ways of writing numbers, one of these being called canonical form. This has only one digit per column (which may be a zero), and has the advantage that in this case, and not otherwise, headed column notation can be replaced by place-value notation. For this reason, canonical form is used unless there is a reason for using one of the alternatives – which we often need to do as a temporary measure.

Here are some examples of the same numbers written in non-canonical and canonical forms. For a given number there is only one canonical form, but many non-canonical.

Canonical			Non-canonical	
Tens	Ones		Tens	Ones
4	2	← same number →	3	12
			2	22
			1	32
Tens	Ones		Tens	Ones
5	0	← same number →	4	10

Canonical		Non-canonical																		
<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border: none; padding: 2px;">Hundreds</th> <th style="border: none; padding: 2px;">Tens</th> <th style="border: none; padding: 2px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="border: 1px solid black; text-align: center; padding: 5px;">8</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">4</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">6</td> </tr> </tbody> </table>	Hundreds	Tens	Ones	8	4	6	\longleftrightarrow same number \longleftrightarrow	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border: none; padding: 2px;">Hundreds</th> <th style="border: none; padding: 2px;">Tens</th> <th style="border: none; padding: 2px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="border: 1px solid black; text-align: center; padding: 5px;">7</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">13</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">16</td> </tr> <tr> <td style="border: 1px solid black; text-align: center; padding: 5px;">7</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">14</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">6</td> </tr> <tr> <td style="border: 1px solid black; text-align: center; padding: 5px;">8</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">3</td> <td style="border: 1px solid black; text-align: center; padding: 5px;">16</td> </tr> </tbody> </table>	Hundreds	Tens	Ones	7	13	16	7	14	6	8	3	16
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8	4	6																		
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7	0	5																		
Hundreds	Tens	Ones																		
6	9	15																		
6	4	65																		

Non-canonical forms like the above often arise during calculations. To give the final answers in place-value notation, facility in converting to canonical form is required. But place-value notation depends on there being only one digit per column, so I think that headed column notation should be used whenever changes to and from canonical form are involved, until the children are absolutely clear what they are doing. The conventional notation, involving little figures written diagonally above, is a further condensation which may be quicker to write but is not consistent with this requirement. And now that calculators are readily available, it is *understanding* that is most important in written calculations. For speed, calculators are better.

A further complication, usually ‘swept under the carpet,’ is that in non-canonical form, place-value notation makes an appearance within the headed columns. There is no mathematical inconsistency, but we are now using two notations in combination. I see no objection to this provided that we show both of them clearly. This is what I have tried to do in the activities which follow.

Activity 1 Cashier giving fewest coins [Num 2.11/1]

A game for up to five players. Its purpose is to present canonical form in a familiar embodiment. Children should first have done Org 1.12 and 13.

- Materials*
- Coins, 1-cent, 10-cent and, later, 100-cent (\$1).
 - Number cards 1 - 49, or 1 - 99 if preferred.*
 - Pieces of paper for cash slips.

* Provided in photomasters

What they do Stage (a)

1. One child acts as cashier, and has a supply of 1-cent and 10-cent coins. (She should organize these to suit herself.)
2. The number pack is put face down on the table.

Num 2.11 Canonical form (cont.)

3. Each player in turn, turns over the top card and records on her own cash slip what she gets, using headed columns.

Say she gets 47. This entitles her to 47 single pennies, or their equal in value. She first writes 47 in the single cent column.

Hundreds ¢	Tens ¢	Ones ¢
		47

4. She hands it to the cashier. The cashier replies, "I want to use the fewest coins," and returns the paper.

5. The player changes her request to 4 ten-cent pieces and 7 single pennies, which the cashier accepts and pays. She records her agreement with a tick, and the closing of the transaction with a line.

Hundreds ¢	Tens ¢	Ones ¢
		47
	4	7✓
<hr/>		

6. If a player turns a card showing less than 10, conversion is of course not required.
7. After three rounds, each player counts her money, and if she has more than nine coins of a kind, exchanges as appropriate with the cashier.
8. The winner is the one who has most, and she acts as cashier for the next round.

Extension

A bonus of 100¢ (\$1) may be earned by first adding the three underlined amounts, converting, and predicting the result before checking physically.

When the game is established, players may forestall the cashier's refusal by making the conversion before passing her their cash slip. Both forms should however always be recorded: first the figure on the number card, then the equivalent value in canonical form. The whole object of this game is to establish that these are two ways of writing the same number.

Stage (b)

(Return to this after topic Num 2.13, 'Numerals beyond 100, written and spoken,' which is in *Volume 2*.)

It is played exactly as for stage (a) except that two number packs of different colours are used, one signifying dimes (10¢ coins) and the other pennies (1¢ coins).

The conversion to canonical form may now require several steps. Suppose a player turns up 15 tens, 12 singles.

First we make separate conversions.

Then we combine their results.

Hundreds ¢	Tens ¢	Ones ¢
	15	12
	1	2
1	5	
1	6	2

To begin with every step should be written, as above. In this case it makes no difference whether the 12 singles or the 15 tens are converted first. With proficiency, however, the process of combining can be done mentally, and it is then easier to work from right to left.

A player who turns over (say) 46 tens, 71 singles will have further steps to take before conversion to canonical form is complete.

Hundreds ¢	Tens ¢	Ones ¢
	46	71
	7	1
4	6	
4	13	1
5	3	1

(One step has been done mentally here.)

Activity 2 “How would you like it?” [Num 2.11/2]

This is similar to Activity 1, except that the cashier behaves differently. Instead of paying out the amount in the smallest number of coins, he acts like a bank cashier who asks, “How would you like it?” when we cash a cheque.

Suppose, as before, a player turns over 47. She records this, as before, but she might then ask for it to be paid like this

or this

or even like this.

Hundreds ¢	Tens ¢	Ones ¢
		47
	3	17
	2	27
	0	47

Num 2.11 Canonical form (cont.)

So the cashier needs to keep plenty of piles of ten single pennies. If the cashier runs out of change before all have had three turns, she will have to ask the customers to help by giving her back some change in return for larger coins.

This game can be played at Stages 1 and 2, as in Activity 1. In both cases the final counting, and conversion to canonical form, will now be complicated, so Activity 2 should not be tackled until Activity 1 is well mastered. At any time when a player has made a mistake, the conversions should be done with coins to check the pencil-and-paper conversions.

Discussion of activities Children have already formed the concept of canonical form as it applies to the repeated grouping of physical materials, in Org 1.12 and the topics leading up to it; and, for base 10, in Org 1.13. The present topic is concerned with the corresponding regroupings done mentally, and recorded using the mathematical notation shown in Activities 1 and 2.

This notation conveys, but more clearly, the same meaning as the makeshift devices which most of us learned at school, such as this:

$$\begin{array}{r} 3 \quad 1 \\ \mathbf{47} \end{array}$$

Canonical form is concerned with different notations for representing the same numbers. The symbolic manipulations practiced here all represent regrouping exchanges such as those done in Org 1 with physical materials. Children now need to acquire facility at a purely symbolic level, so the base 10 physical material is replaced by coins. These provide an excellent intermediate material for the present stage, since they represent number values in a way which is partly physical, partly symbolic.

This topic forms cross-links with the various calculations which cause non-canonical forms to arise, and practice with these will be found in the appropriate networks. But canonical form is a major concept in its own right, and other examples of it occur in later mathematics. It needs to be presented in a way which allows children to concentrate on learning this important concept by itself, without having at the same time to cope with other mathematical operations.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 2.12 THE EFFECTS OF ZERO

Concept Zero as a place-holder.

Ability In place-value notation, to recognize when zero is necessary for giving their correct values to other digits, and when it is not.

Discussion of concept In place-value notation, it is the position first, or second, or third . . . in order from right to left which determines the value of a digit. So the numerals 04 and 4 both mean the same: 4 is in the ones place in each, and there are zero tens – explicitly in the first, implicitly in the second. However, the numerals 40 and 4 do not mean the same, since 4 is in the tens place in the first numeral and in the ones place in the second. In the numeral 40, zero acts as a place-holder. By occupying the ones position, zero determines the meaning of 4.

Though the zero in 04 is not necessary, it is not incorrect. Its use is becoming increasingly common, e.g., in digital watches, and in figures on cheques and elsewhere for computer processing.

Activity 1 “Same number, or different?” [Num 2.12/1]

A game for two or more players. They all need to see the cards the same way up. Its purpose is for children to learn when the presence or absence of a zero changes the meaning of a numeral, and when it does not.

- Materials*
- A double pack of single headed number cards from 0 to 9.*
 - A mixed pack of cards on which are the words ‘same number’ or ‘different number,’ about 5 of each.*
 - An extra zero, of a different colour.*
- * Provided in photomasters

Rules of the game Stage (a)

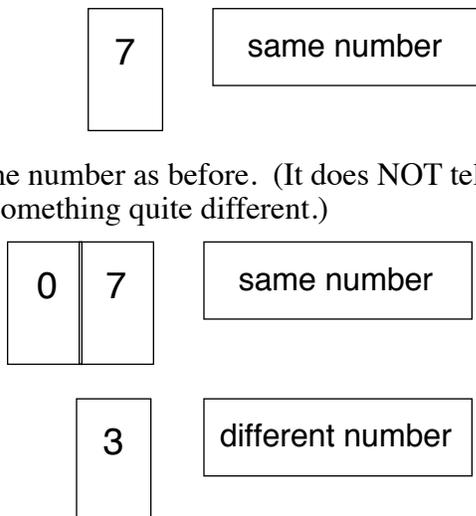
1. Both packs of cards are shuffled and put face down on the table.
2. The player whose turn it is has the extra zero.
3. He turns over one card from each pile.

Suppose he gets these.

He now has to put down a zero so that what

is on the table still means the same number as before. (It does NOT tell him to “add a zero.” This would mean something quite different.)

4. If he does this correctly, the others award him a point.



5. The circulating zero now goes to the next player, who might turn over these cards.

Num 2.12 The effects of zero (cont.)

6. This would be his correct response.

3	0
---	---

different number

7. A player who turns over 0 from the pile, together with 'different number,' still gets a point for saying, "It can't be done," and explaining why.
8. It is good for the players to speak aloud the numbers before and after putting down the zero. This makes it sound different if the numbers are different. E.g., in steps 3 and 4: "Seven ones." "Zero tens, seven ones." And in steps 5 and 6: "Three ones." "Three tens, zero ones."

Stage (b)

(Return to this after topic Num 2.13, 'Numerals beyond 100, written and spoken,' in *Volume 2*.) This is played as in Stage (a), except that they turn over *two* number cards from the pile and put them side by side.

Suppose a player gets this:

4	2
---	---

different number

In this case (though not always) there are two correct responses.

4	2	0
---	---	---

different number

If he gives both, and speaks both correctly, he is awarded 2 points.

4	0	2
---	---	---

different number

Activity 2 Less than, greater than [Num 2.12/2]

Stage (a)

A game for two players. Its purpose is to provide further examples of the effects of zero.

Materials For each player:

- A pack of single headed number cards 1-9.*

Shared between two:

- A single zero card.*
- An 'is less than' card.*
- An 'is greater than' card, both as illustrated on the following page.*

* Provided in the photomasters

What they do

1. The 'is less than' card is placed between the two players. Also on the table is the zero card.
2. Both players hold their own pack face down. Each takes the top card from his pack and lays it on his side of the 'is less than' card.
3. The first player to have a turn then picks up the zero and must place it next to his own card to make a true statement.

Suppose that here it is the left-hand player's turn.

5	$<$ is less than	7
---	---------------------	---

His correct response is:
For this he gets one point.

0	5	$<$ is less than	7
---	---	---------------------	---

Had it been the right-hand player's turn, one correct response would have been:
(What is the other?)

5	$<$ is less than	7	0
---	---------------------	---	---

In this case, only the right hand player can give a correct response.

5	$<$ is less than	3
---	---------------------	---

If it was the left-hand player's turn, he could if he wished take a chance with an incorrect response. In that case, the other may say "Challenge," and if the challenge is upheld the challenger gets two points.

When all cards have been used, the game is repeated using 'is greater than.'

Stage (b)

(Return to this after topic Num 2.13, 'Numerals beyond 100, written and spoken,' in *Volume 2*.)

As with Activity 1, this can be played with larger numbers. Each player now begins by putting down two cards. Several correct responses may then be possible, and players get a point for each which is both tabled and verbalized.

For example,
in this situation

5	3	$<$ is less than	7	2
---	---	---------------------	---	---

the right-hand player has three correct responses: 072, 702, and 720.

Discussion of activities

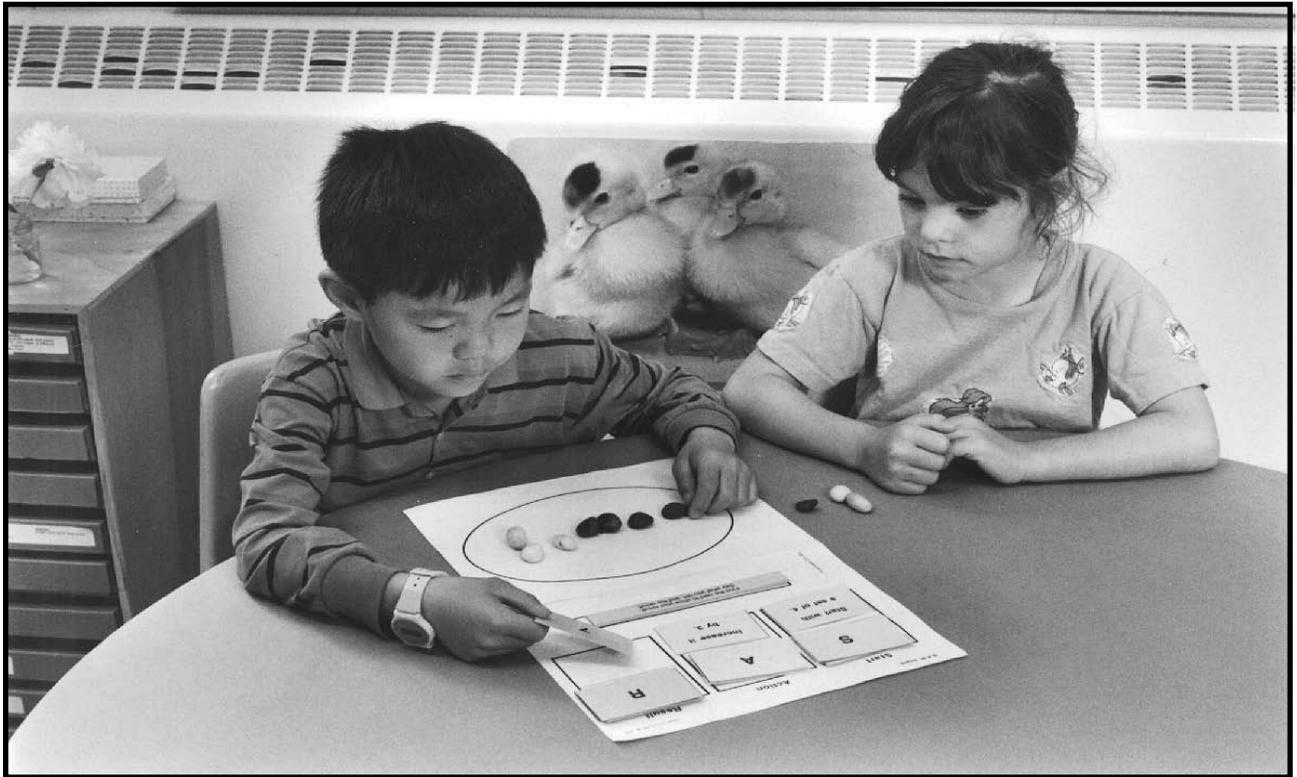
In this topic, children are now working at a purely symbolic level, without any support from physical materials. The activities involved are quite sophisticated, since they involve changing meanings for the same symbols. Agreement about these changing meanings depends on a shared schema for assigning values to symbols: this schema being (as we have noted) that of a condensed and sophisticated notation, in which much of the meaning is not explicitly written down at all, but is implied by relative positions.

Any difficulties will usually be best dealt with by relating these activities to the more explicit headed columns notation; and, if necessary, by providing further back-up in the form of base ten physical materials. These can be used to make very clear what a great difference in meaning can result from quite small changes in the positions of symbols.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Start, Action, Result (do and say) [Num 3.1/1]

[Num 3] **ADDITION**

A mathematical operation which corresponds with a variety of physical actions and events

Num 3.1 ACTIONS ON SETS: PUTTING MORE (Total < 10)

- Concepts*
- (i) Sets as operands.
 - (ii) Putting more as actions.
 - (iii) Starting and resulting numbers.

- Abilities*
- (i) To follow simple instructions for actions with sets.
 - (ii) To describe these actions and their results.

Discussion of concepts

In Org 1, from topic 2 onwards, we developed the idea of a set as something which, though made up of separate objects, could also be thought of as a single entity. In Org 1.3 for example, we compared sets to decide which had the larger number, and in Org 1.4 we ordered sets by their numbers.

In the present topic, we consider sets as something on which we can perform (physical) actions: for short, sets as operands. The actions in this case are putting more objects into the set; as a result of which, we finish with a set having a larger number. This is a very everyday affair – it happens whenever we put cookies on a plate. Nevertheless, as we shall see in topic 2, it is the foundation for two important mathematical concepts.

A general name for objects on which actions or operations are performed is ‘operand.’ We do not need to teach children this term, but it is convenient to have it available for our own use.

Activity 1 Start, Action, Result (do and say) [Num 3.1/1]

An activity for two to four children. Its purpose is to introduce the concepts described above.

- Materials*
- An SAR board (see Figure 2) *
 - ‘Start’ cards 1-5 (later 0-5) which say (e.g.) “Start with a set of 3.” *
 - ‘Action’ cards 1-5 (later 0-5) which say (e.g.) “Put 2 more,” “Increase it by 5,” “Make it 4 larger.” *
 - ‘Result’ cards numbered from 1-10 (later 0-10).*
 - Objects such as counters, shells, buttons, to put in the set loop.
 - A ‘reversible card.’*
 - On side one is written:
“Find the card to show your result. Say what you did, and the result.”
 - On side two [for Num 3.2/1] is written:
“Predict the result.”

* Provided in the photomasters

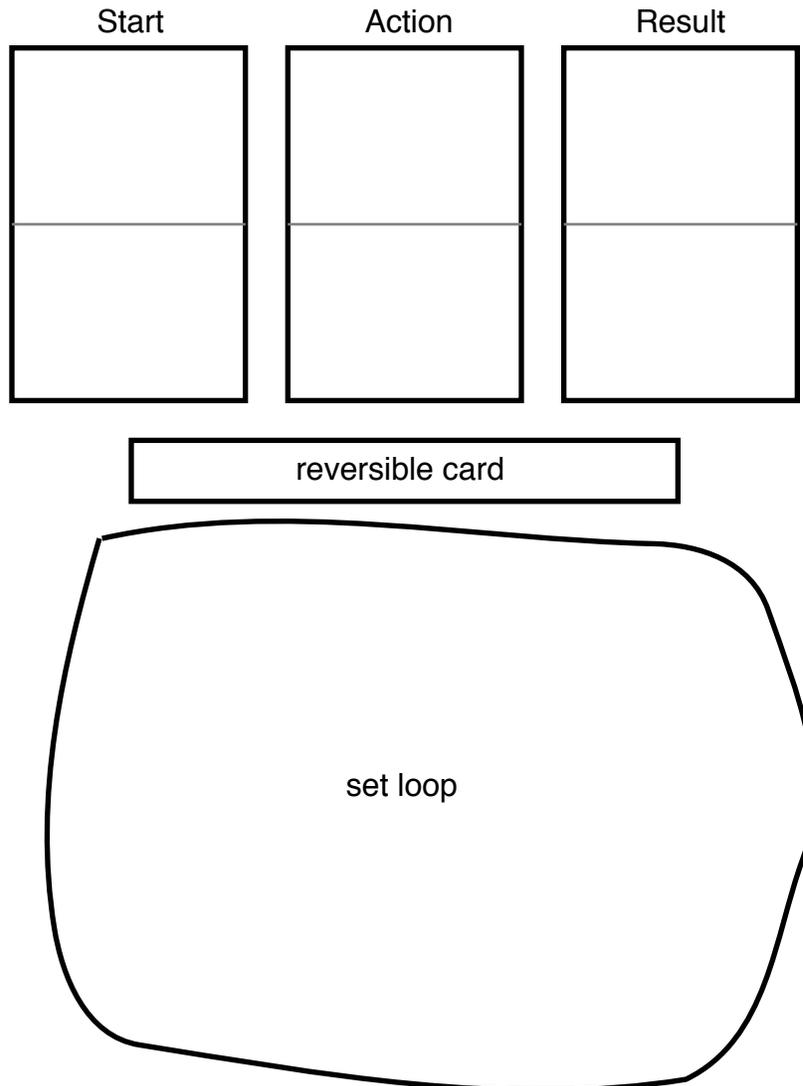


Figure 2 SAR Board

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces on the SAR board.
 2. The 'reversible card' is put in its space with side one showing.
 3. One child turns over the top 'Start' card into the space below, and puts a set of the required number into the set loop.
 4. Another child then turns over the 'Action' card, and puts more objects into the loop as instructed. Note that at this stage we do not talk about adding.
 5. He then finds the appropriate 'Result' card to show the number of the resulting set.
 6. Finally he describes to the others what he did, and the result.

Activity 2 Putting more on the number track (verbal) [Num 3.1/2]

An activity for two to four children. Its purpose is to introduce the use of the number track for adding. One group of children could be doing this while another is doing 'Start, Action, Result.' For these two activities, the order does not matter. (This is not usually the case.) A copy of this activity also appears as NuSp 1.3/1 in 'The number track and the number line' network.

- Materials*
- An SAR board, see Figure 2 (preceding page).*
 - A number track.**
 - Cubes to fit the track, in two colours.
 - 'Start' cards 1-5 (later 0-5), which say (e.g.) "Start with a set of 3."**
 - 'Action' cards 1-5 (later 0-5), which say (e.g.) 'Put 2 more,' 'Increase it by 5,' 'Make it 4 larger.'**
 - 'Result' cards numbered from 0-10.**
 - A 'reversible card.'**
 - On side one is written:
"Find the card to show your result. Say what you did, and the result."
 - On side two [for Num 3.2/2] is written:
"Predict the result."

* These are the same as used for Num 3.1/1.

** Provided in the photomasters for NuSp 1.3/1.

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces.
 2. The reversible card is put in its place with side one showing.
 3. One child turns over the top 'Start' card into the space below, and puts a set of the required number either singly into the number track (in which case the projections will have to be uppermost) or joined into rods.
 4. Next he turns over the 'Action' card, and puts more cubes into the track as instructed, using a different colour. Note that at this stage we do not talk about adding.
 5. Next he finds the appropriate 'Result' card to show the number of the resulting set.
 6. Finally he must describe to the others what he did, and the result.
 7. Steps 3 to 5 are then repeated by the next child.

Discussion of activities These two activities provide the physical experiences from which children will begin to abstract the concepts described in the next topic. The spoken description is an important part of the activities, since it links these experiences to the appropriate verbal symbols, in preparation for the more difficult written symbols to be introduced in Topic 3.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.2 ADDITION AS A MATHEMATICAL OPERATION

Concept Adding as something done mentally, with numbers.

Ability To predict the results of actions on sets which involve ‘putting more,’ by adding numbers mentally. (Initially, this is done with help from physical aids.)

Discussion of concept The word ‘adding’ is used with two different meanings, one everyday and the other mathematical. When we talk about “adding an egg,” or “adding to his stamp collection,” we are talking about physical actions with physical objects. When we are talking about “adding seven” or “adding eighty-two,” we are talking about mental actions on numbers. To avoid confusing these two distinct concepts, we shall hereafter use other words such as “putting more” for physical actions, and “adding” for what we do mentally with numbers. We shall also avoid using “action” for the latter, and use “operation” instead.

The distinction we are making is therefore between physical actions and mathematical (i.e., mental) operations. Adding is thus a mathematical operation. Other mathematical operations are subtracting, multiplying, dividing, factoring

Activity 1 Predicting the result (addition) [Num 3.2/1]

This is a direct continuation of Num 3.1/1, ‘Start, Action, Result (do and say).’ It can be played by two children, or two teams. Its purpose is to teach children to use the (mental) operation of addition for making simple predictions.

Materials

- As for Num 3.1/1, ‘Start, Action, Result (do and say).’
- Pencils.
- Stage (b) requires a handkerchief or bag for hiding the objects.

What they do Stage (a)

1. The SAR board and the ‘Start’ and ‘Action’ cards are put out as before. The ‘reversible card’ is put in its space with side two showing.
2. Player A turns over the top ‘Start’ card, and puts a set of the required number in the loop.
3. Player B turns over the top ‘Action’ card, after which she must predict the result and choose the appropriate card from the ‘Result’ pack. She puts this in the ‘Result’ space.
4. Finally she puts down more shells (or whatever) and physically checks her prediction.

Stage (b)

1. The first card is turned as in Stage (a), but this time the objects are put into a bag or under a handkerchief.
2. The second card is turned and the indicated number more are put with the first set.
3. A prediction is made as before, and tested physically by emptying the bag or lifting the handkerchief.

Num 3.2 Addition as a mathematical operation (cont.)

Activity 2 “Where will it come?” [Num 3.2/2]

An activity for two. This is a predictive form of Num 3.1/2. A copy of this activity also appears as NuSp 1.3/2 in ‘The number track and the number line’ network.

- Materials*
- The same as for Num 3.1/2.
 - Also, some plasticine.

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces on the SAR board. The ‘reversible card’ is put in its space with side two showing.
 2. The children make up a 1-5 staircase each, in different colours.
 3. Player A turns over the top ‘Start’ card, selects a rod of this number and puts it into the number track.
 4. Player B turns over the top ‘Action’ card, but does not yet take out a rod. First, he says “I predict that it’ll come to *here*,” pointing, and marking his prediction with a piece of plasticine. This will involve some form of counting on, which on the number track corresponds to movement to the right.
 5. Then he tests his prediction physically by joining a rod of the given number to the first on the number track.
 6. Steps 3, 4, 5 are repeated until all the cards are turned.
 7. Then the cards are shuffled, and they begin again with their roles interchanged.

Activity 3 Stepping stones [Num 3.2/3]

A board game for 2, 3, or 4. Its purpose is to give further practice at adding, in a predictive situation.

- Materials*
- Game board (see Figure 3).*
 - Die, 1 to 6.
 - Shaker.
 - Markers, one for each child. (Little figures are good, which do not hide the numbers.)

* Provided in the photomasters

- Rules for game*
1. Players start from the near bank, which corresponds to zero.
 2. Players in turn throw the die, add this number to that on the stone where they are, and move to the stone indicated. E.g., a player is on stone 3, throws 5, so moves to stone 8. When starting from the bank, they move to the stone with the number thrown.
 3. If that stone is occupied, she should not move since there is not room for two on the same stone.
 4. If a player touches her marker, she must move it. If this takes her to an occupied stone, she falls in the water and has to return to the bank.
 5. The exact number must be thrown to reach the island.

Note This will be used again in Num 4.2 as a subtraction game, to get back from the island.

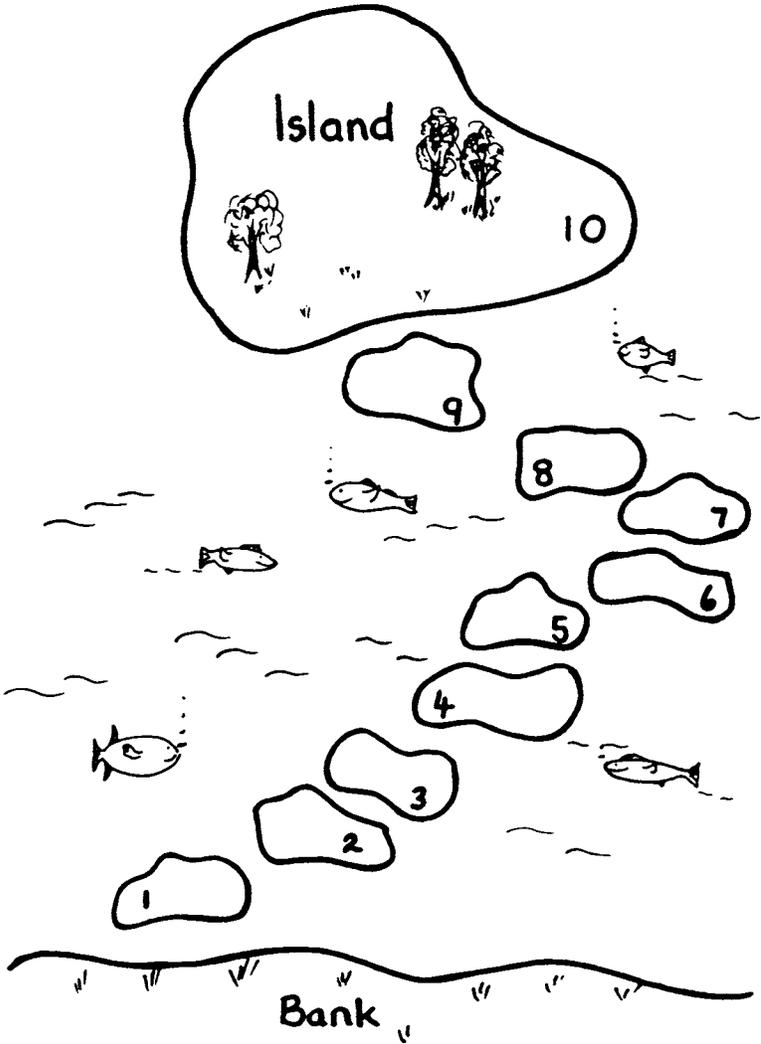


Figure 3 Stepping stones.

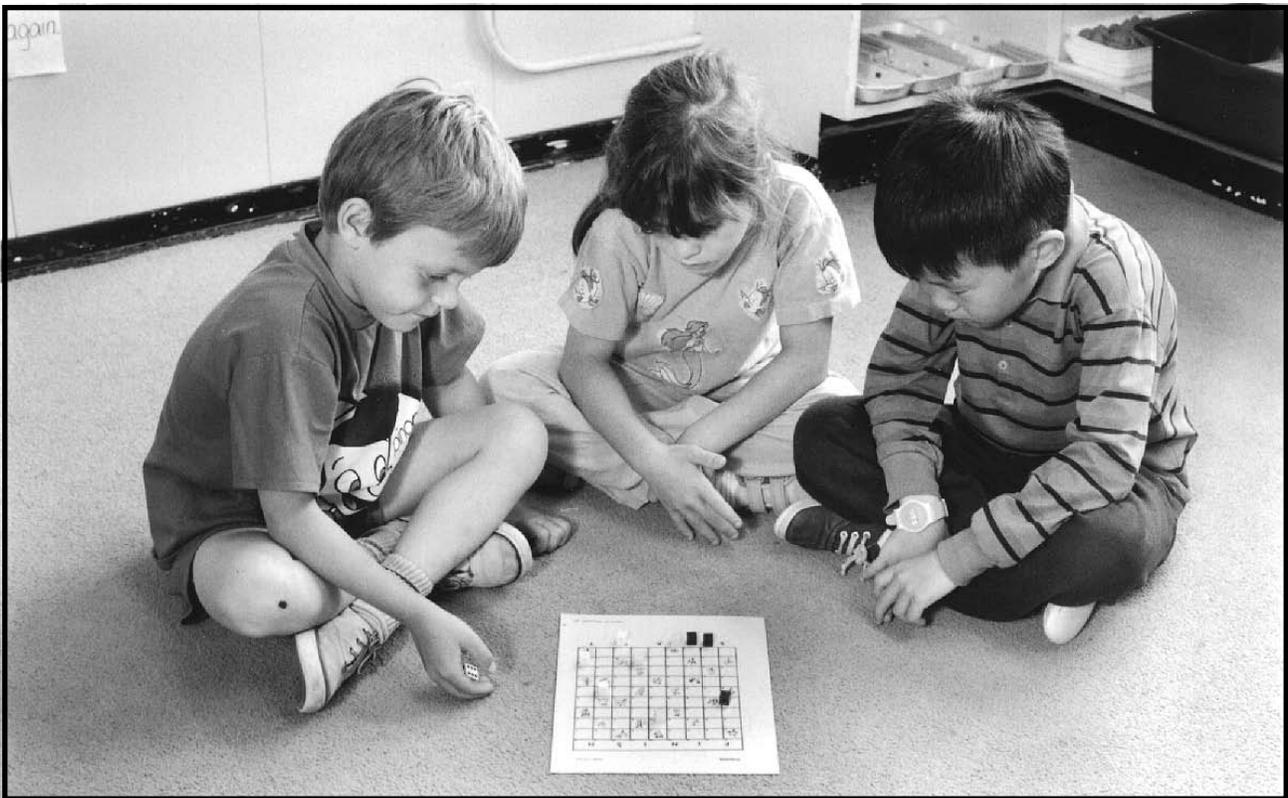
Num 3.2 Addition as a mathematical operation (cont.)

Activity 4 Crossing * [Num 3.2/4]

A board game for 2 or 3 children. Its purpose is to consolidate the abilities described above in a situation which requires several predictions to be made in order to choose the best action.

*This attractive improvement to the original version of the 'Crossing' game was suggested by Mrs. Mary Hamby of Leegomery County Infant School, Telford. 'Crossing' also appears as NuSp 1.3/3 in 'The number track and the number line' network.

- Materials*
- Game board, see Figure 4.**
 - 3 markers for each player, a different kind or colour for each player.
 - Die 1-6 and shaker.
- ** Provided in the photomasters



Crossing [Num 3.2/4]

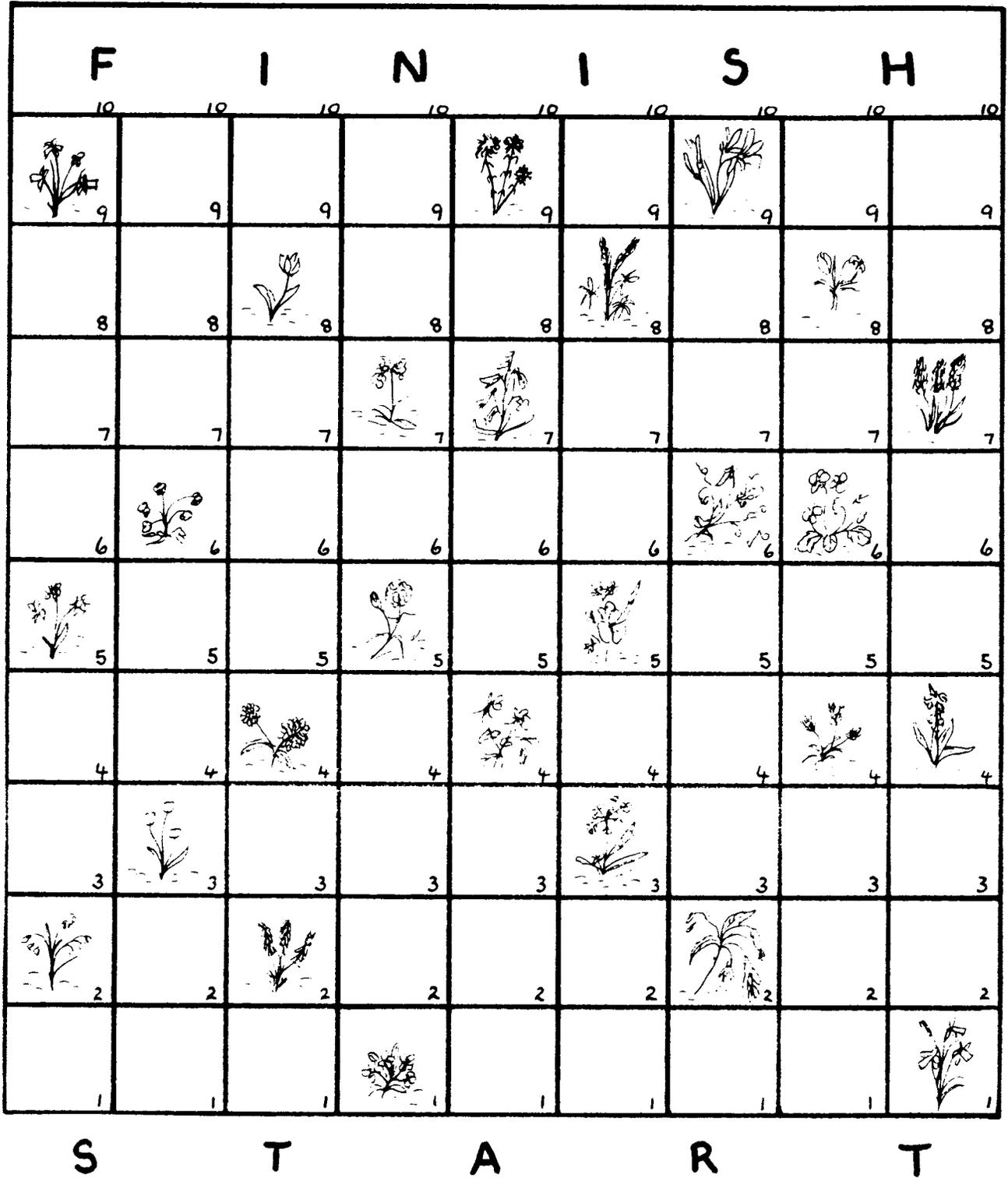


Figure 4 Crossing.

Num 3.2 Addition as a mathematical operation (cont.)

- What they do*
1. The blank squares on the board represent paving stones. Some of these have been removed to allow flowers to grow. The object is to get across the board from START to FINISH, treading only on the paving stones and not on the flowers.
 2. Each player starts with all 3 markers off the board, at the START.
 3. Players throw the die in turn. The number thrown shows how many steps they may take. This means that from START, they may put a marker on the board at the square with that number; and from a square on the board, they may move one of their markers forward that number of squares.
 4. They may move whichever of their markers they like. When starting, they may choose any vacant track. After that, they must keep moving straight forward along the same track.
 5. They may not land on a square marked with a flower. Players may move their markers over them normally, but if they make a move which stops on a square with a flower, that marker must go back to the start.
 6. If they touch a marker they must move it if they can, or go back to the START if they cannot. This rule may be relaxed when learning.
 7. The exact number must be thrown to finish. The first player to get all his markers to FINISH is the winner, but the others may continue playing until all are across.

Variation Players learning the game may start with just 2 markers.

Discussion of activities In Activity 1, there is an important progress to be made from *counting all* to *counting on*. At first, children need to count the first set, then the set added, and finally to count *all* the objects to get the total. Later, they may become able, after counting the first set to count *on* the number of the second set to get the total. At the simplest stage, this involves realizing that having already counted the first set, it is not necessary to do so again, since this number can be used as starting point for counting the enlarged set. This applies when the two sets are present physically and visibly. Counting on mentally is a much more sophisticated technique. To add (say) 5 and 3 now involves, first holding 'five' in one's head, and then saying "six, seven, eight" while also thinking "one, two, three." This would be very difficult without the help of some method such as finger counting to keep track of the "one, two, three," and children should be encouraged to use this. (Please see Num 1.4/2 and Num 1.5/1 for descriptions of the recommended method of finger counting.)

Another way of counting on is by using a number track or number line. With this, you find your starting number, and count on saying (or thinking) "one, two, three," while pointing to 6, 7, 8. This is different from the methods described earlier, since it uses the number track to show the running total. You don't speak the answer, but your finger ends up pointing to it.

Of these two, the number track is perhaps easier to do, but it is harder to interpret. Children often make mistakes because they say "one" while pointing to the starting number instead of its successor: in this example, pointing to 5 instead of 6. Or they may take the last word they speak as the total, rather than the number pointed to. Both techniques should be learned, but I suggest that finger counting be well established before using the number track.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.3 NOTATION FOR ADDITION: NUMBER SENTENCES

Concept The use of written number sentences of the form
 $3 + 2 = 5$
for representing the operation of addition, and its result.

- Abilities*
- (i) To write number sentences describing actions of ‘putting more’ with physical materials.
 - (ii) To use number sentences for making predictions about the results of physical actions, and to test these predictions.

Discussion of concept The notation in an addition number sentence corresponds well to its meaning. For example:

start	action	result
3	+ 2	= 5

may be read verbally as:

“three, add two equals five.”

I prefer ‘add’ to ‘plus,’ since ‘add’ is the name of the operation, while ‘plus’ is the name of the symbol.

Activity 1 Writing number sentences for addition [Num 3.3/1]

This is an activity for 2 to 4 children. Its purpose is to introduce written number sentences, used here for recording.

- Materials*
- SAR board and all other materials used in Num 3.1/1 .
 - Start cards and Action cards which include parts of a number sentence, as illustrated below. *
 - Reversible card.*
 - On one side is written:
“Write a number sentence to show what you did.”
 - On side two is written:
“First write a number sentence showing what you predict the result will be. Then test your prediction.”
 - Pencil and paper for each child.
- * Provided in the photomasters

Start with a
set of 3.

3

Put 2 more.

+2

What they do Form (a)

1, 2, 3, 4 are as in Num 3.1/1.

5. After doing the action, the instruction on the card is followed. In the above example, each child would therefore write:

$$3 + 2 = 5$$

6. They compare what they have written.

Form (b)

As in Num 3.1/2, the same activity may usefully be done here using cubes and a number track instead of a set loop.

Activity 2 Write your prediction (addition) [Num 3.3/2]

This is a predictive version of Activity 1.

Materials • As Form (a) for Activity 1.

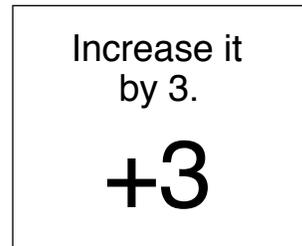
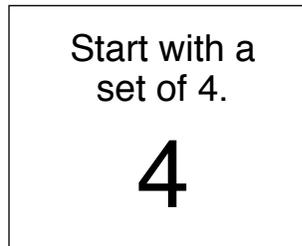
What they do Form (a)

1. The reversible card now shows side two.

2. Each start and action card is turned over as before.

3. Each child copies the first two parts of the number sentence, and completes it by doing the addition mentally (using aids such as finger counting or number track, as desired).

4. Finally they check their predictions by using the physical materials. E.g., if a child turns over these cards



he writes

$$4 + 3$$

Then he completes the number sentence

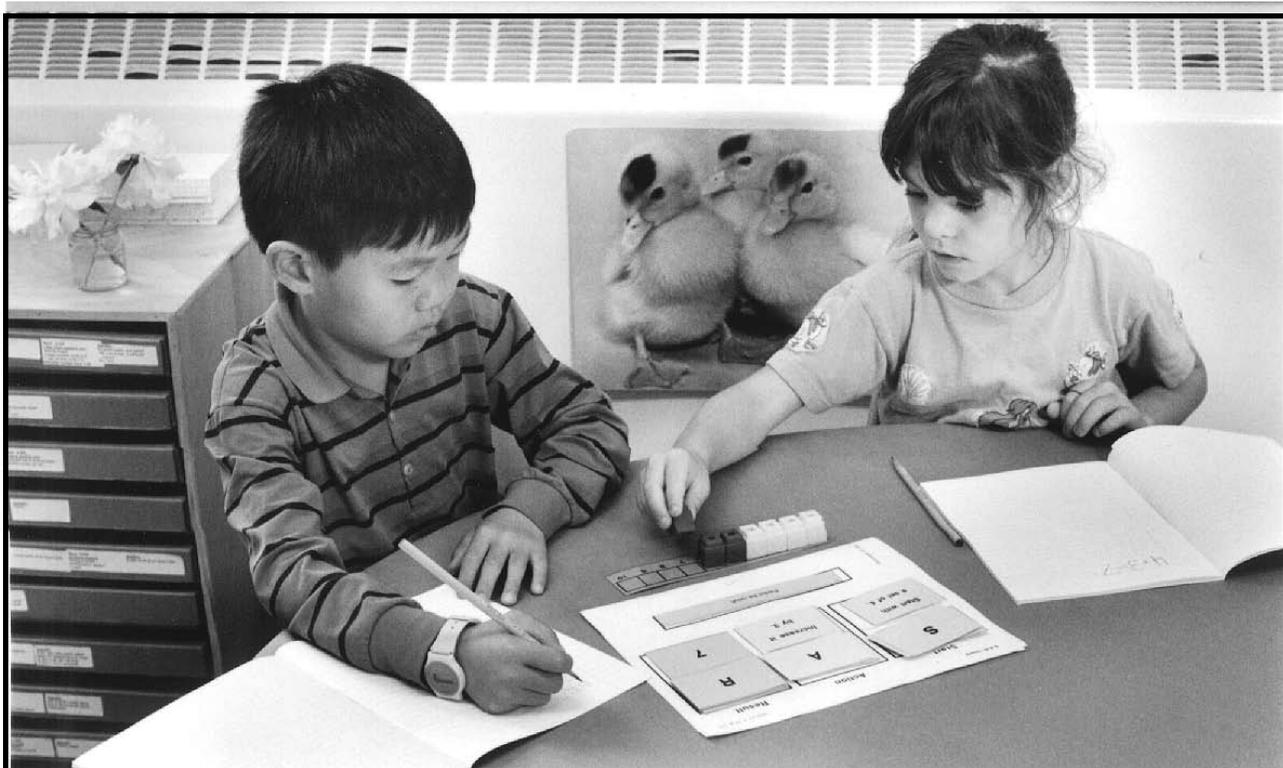
$$4 + 3 = 7$$

Finally he puts four objects into the set loop, then 3 more, and counts the total.

5. It might be of interest for another child to check the prediction.

Form (b)

The same activity should be done using cubes and a number track instead of a set loop. (See Num 3.2/ 2.)



Write your prediction (addition) [Num 3.3/2, Form (b)]

Discussion of activities In Activity 1, children are learning what is meant by “Write a number sentence,” by copying examples of these. The number sentences are written after the physical manipulations, so that the concepts are already there and ready to be attached to the written symbols. It is a concept-building and symbol-linking activity. The concepts involved here are not only the mathematical operation of addition, but the fact that the number sentence can represent both a physical (making a set, putting more) and a mental activity (adding).

In Activity 2, the combination of written symbols and associated concepts is used to make a prediction. In this case it is a simple one, but essentially this is the same kind of use that mathematics is put to in the adult world. Does our bridge stay up, does our aircraft find its destination? This is the crunch, and it depends partly on whether someone “got his figuring right.”

The notation is highly condensed, which is one of its strengths: by using it one can handle a lot of information. It also carries the risk that its meaning can easily become detached and lost. Hence the importance of continually relating it to physical experience.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.4 NUMBER STORIES: ABSTRACTING NUMBER SENTENCES

Concept Numbers and numerical operations as models for actual happenings, or for verbal descriptions of these.

Abilities

- (i) To produce numerical models in physical materials corresponding to given number stories, to manipulate these appropriately, and to interpret the result in the context of the number story.
- (ii) Later, to do this by using written symbols only.
- (iii) To use number sentences predictively, to solve verbally given problems.

Discussion of concept A model is something which represents in simpler form something else, e.g., a 'tube' map is a model on paper of the London Underground network. By reducing the amount of detail and leaving out non-essentials, it enables us to think and plan more easily and effectively. Mathematical models are mental models, though we use physical aids such as cubes and written symbols to help us to get hold of them. They are a very versatile and useful kind of model with applications as different as shopkeeping and communication satellites. Hence their importance in the world today.

The idea of a mathematical model is quite an abstract one, and will become clearer as more examples are encountered. The activities in this topic provide some examples to start with.

Activity 1 Personalized number stories [Num 3.4/1]

An activity for 2 to 6 children. Its purpose is to connect simple verbal problems with physical events, linked with the idea that we can use objects to represent other objects.

Materials

- Number stories of the kind in the example over the page.* For some stories you will need two versions, one with pronouns for girls, the other with pronouns for boys. These can be on different sides of the same card.
- The name of each child on a card or slip of paper.
- Three separate sets of number cards on different coloured cards:
 - Start cards 1-5 *
 - Action cards 1-5 (later 0-5) *
 - Result cards 1-10 *
- Objects such as bottle tops, pebbles, shells, counters.
- Paper and pencils.

* Provided in the photomasters

What they do (apportioned according to how many children there are).

1. A number story is chosen. The name, start, and action cards are shuffled and put face down.
2. The top name card is turned over and put in the number story.
3. Another child then turns over the top start card, and puts it in the appropriate space.
4. Another child then turns over the top action card, and puts it in the appropriate space.
5. The number story now looks like this. Depending on children's reading ability, it may be useful for you to read it aloud for them

Peter has **3** cookies.

Mother gives him **2** more.

How many cookies does he have now?

Answer: he now has cookies.

6. The named child then takes over, describing aloud what he is doing while the others check. (E.g.) "We haven't any cookies, so what can we use instead?" (Puts down bottle tops.) "These are the 3 I start with." (Puts down 2 pebbles.) "I've put 2 more. Now I have 5 altogether." It is a good idea to use different objects for the cookies mother gives him, to keep the two sets distinct. You could suggest that these are a different kind of cookie.
7. Finally another child puts the correct number card in the result space to complete the number story.

Activity 2 Abstracting number sentences [Num 3.4/2]

An extension to Activity 1 which may be included fairly soon. Its purpose is to teach children to abstract a number sentence from a verbal description.

- Materials*
- As for Activity 1, and also:
 - Number sentence card as illustrated below.

+ =

number sentence

Num 3.4 Number stories; abstracting number sentences (cont.)

What they do Steps 1 to 7 are as in Activity 1.

8. You, or a child, then say(s) “Now we make a number sentence recording what happened.” She puts the number sentence card below the story card, then moves the three number cards from the story card to the number sentence card as shown overleaf.

$$\boxed{3} + \boxed{2} = \boxed{5}$$

number sentence

She reads aloud “Three, add two, equals five.”

9. Finally every child copies this out to make a permanent record.

$$3 + 2 = 5$$

Activity 3 Personalized number stories – predictive [Num 3.4/3]

An activity for 2 to 6 children. It combines Activities 1 and 2, in predictive form.

- Materials*
- Number stories of the new kind shown in step 5, following. These now require a prediction.**
 - The name of each child on a card or a slip of paper.*
 - Two sets of number cards: start cards 1-5, and action cards 1-5 (later 0-5).*
 - Objects such as bottle tops, pebbles, shells, counters.*
 - Slips of paper to fit answer space.
 - Pencil and paper for each child.
- * These are the same as for Activity 1.
** Provided in the photomasters

What they do (*apportioned according to how many children there are*)

1. A number story is chosen. The name, start, and action cards are shuffled and put face down.
2. The top name card is turned over, and put in the appropriate space in the number story.
3. Another child then turns over the top start card, and puts it in the appropriate place.
4. Another child then turns over the top action card, and puts it in the appropriate space.
5. The number story now looks like the one on the next page. Depending on children’s reading ability, it may be useful for you to read it aloud for them.

Peter has **3** cookies.
 Mother says that she will give him **2** more if
 he can tell her how many he will have then.
 What should he say?
 Answer: he should say, "I will have cookies."

6. The named child now has to answer the question by writing and completing a number sentence, explaining as he does so, e.g.,

<i>Says</i>	<i>Writes</i>
"I have 3 cookies."	3
"I get 2 more."	3 + 2
"3, add 2, equals . . ."	3 + 2 = 5
"So I'll say, 'I will have 5 cookies.'"	

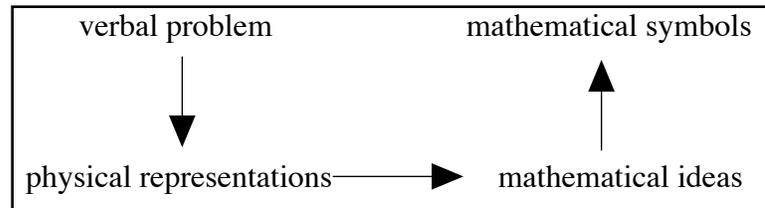
7. Peter then writes the resulting number on a slip of paper and puts it in the appropriate space to answer the question in the number story.
8. Meanwhile the other children write the same number sentence, as in step 9 of Activity 2.
9. Finally one of the other children tests the prediction by using the physical materials, describing what she is doing while the others check. E.g. (with appropriate actions), "I'm using these bottle tops and pebbles in place of cookies. These are the 3 he started with. Now he gets 2 more. There are now 5 altogether. So, Peter gave the correct answer."

Discussion of activities "We don't have any cookies here, so what can we use instead?" introduces children to *modelling*, already discussed at the beginning of this topic. This idea, that something can be used to represent something else, is a very important one. Also important is what we incorporate in the model, and what we leave out. So far as the model is concerned, it doesn't matter whether the story is about cookies, or marshmallows, or balloons; and the model would be the same for Jane and her auntie as for Peter and his mother. From this we see (i) how general a mathematical model is, and thus how versatile; (ii) how great is the abstraction involved when going from a number story to a mathematical model. It is because of the latter that children so often find it difficult to come up with the right model. "Please, teacher, is this add or multiply?"

Num 3.4 Number stories; abstracting number sentences (cont.)

This is why the process of modelling is here treated with such thoroughness. In Activities 2 and 3, the abstractive process is shown visibly when the numbers are removed from the story card and put out by themselves. The terms ‘number story’ and ‘number sentence’ also call attention to the relation between the described happening, and its model.

A very important feature of the present plan of development is having the children move first from verbal problems to physical representations of the objects, numbers, and actions described in the number story, and from the latter to the mathematical statement, not directly from words to mathematical symbols.

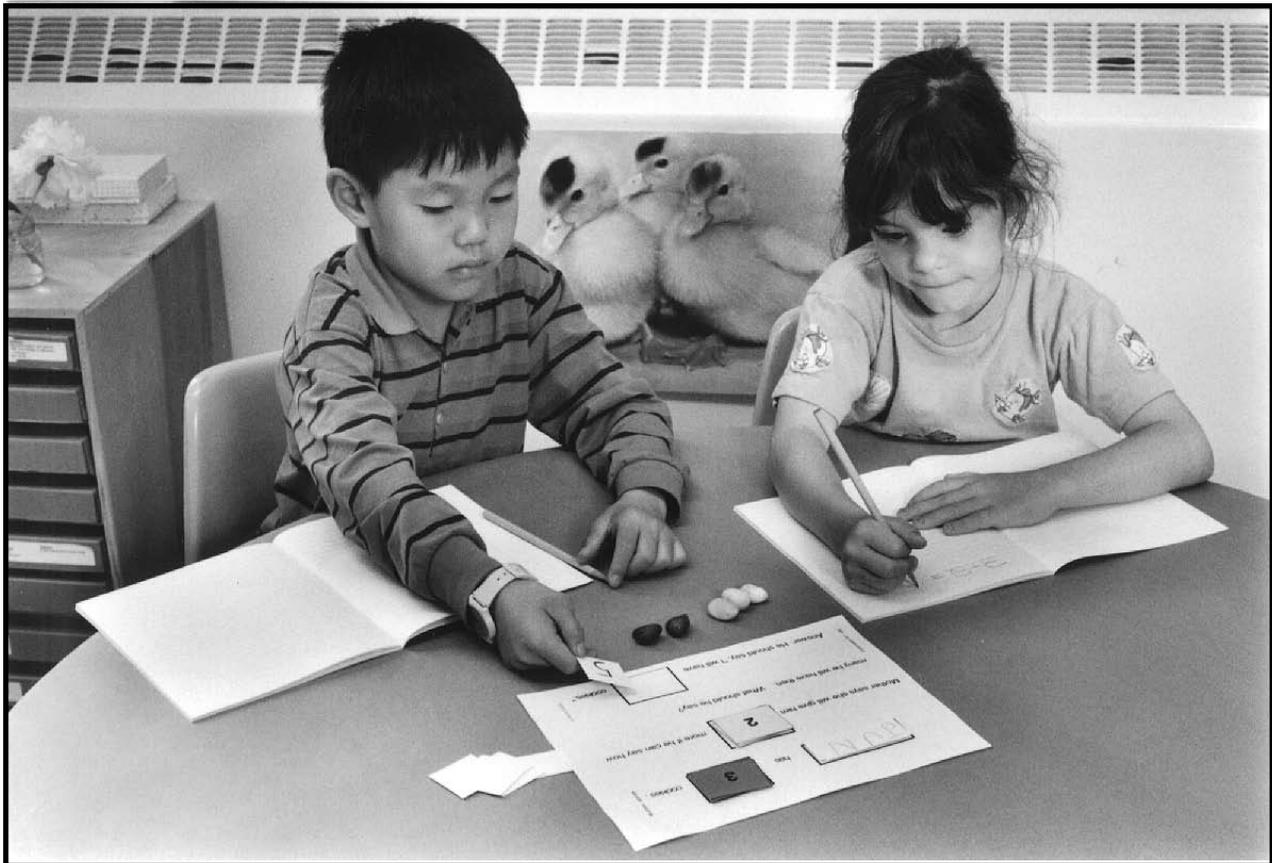


The reason is that there is a much closer match between physical representations and mathematical ideas than there is between the two kinds of symbolism. (This is discussed more deeply in *Mathematics in the Primary School*.)

OBSERVE AND LISTEN

REFLECT

DISCUSS



Personalized number stories – predictive [Num 3.4/3]

Num 3.5 COMPLEMENTARY NUMBERS

Concept The remaining part required to make a given whole.

Ability To state numerically the complement of any part relative to a given whole.

Discussion of concepts This concept forms a good bridge between the addition and subtraction networks. It fits into the present network if we call it missing addend: e.g.,

$$5 + \square = 8$$

It fits into the subtraction network, if we ask, e.g.,

“What is the difference between 5 and 8?”

Counting on is a good method for both of these. The second relates to the comparison aspect of subtraction rather than the ‘take away’ aspect.

Activity 1 The handkerchief game * [Num 3.5/1]

A game for children to play in pairs. (More can play together but there is more involvement with pairs.) Its purpose is to build the concept of complementary numbers in a physical situation which allows immediate testing.

* I first saw this game played at a school in Athens, Georgia. It was taught to the children by Dr. Leslie Steffe, University of Georgia.

- Materials*
- Handkerchief.
 - 10 small objects such as shells, bottle caps, beans, etc.
 - Number cards, 1-10 *

* Provided in the photomasters

- Rules of play*
1. The game is introduced by having one child put out ten small objects. (Suppose that shells are used.) The other children check the number.
 2. The children are asked to hide their eyes while a handkerchief is placed over some of the shells.
 3. The players are told to open their eyes and are asked, “How many shells are under the handkerchief?”
 4. They check by removing the handkerchief.
 5. The children then play in pairs, covering their eyes in turn.
 6. Repeat, using other numbers of objects. The children will need some kind of reminder of how many there are altogether, so, before putting down the handkerchief, a number card is put down for the total number.

Activity 2 “Please may I have?” (complements) [Num 3.5/2]

A game for four or six children. Its purpose is to take the concept to a mental and symbolic level.

- Materials*
- Four-way cards (see diagram) each showing a number from 6 to 10 (and each in a different colour).



- Double-headed number cards:
 - 6 pack: 4 sets (1, 2, 3, 3, 4, 5) same colour as four-way 6 card.
 - 7 pack: 4 sets (1, 2, 3, 4, 5, 6) same colour as four-way 7 card.
 - 8 pack: 3 sets (1, 2, 3, 4, 4, 5, 6, 7) same colour as four-way 8 card.
 - 9 pack: 3 sets (1, 2, 3, 4, 5, 6, 7, 8) same colour as four-way 9 card.
 - 10 pack: 3 sets (2, 3, 4, 6, 7, 8), 2 sets (1, 9), 2 sets (5), same colour as four-way card.

- Rules of play*
1. One of the four-way cards is put centrally, face up. Suppose this is the 7 card.
 2. All the cards from the pack matching the four-way 7 card are dealt to the players.
 3. The object is to put down pairs of complementary cards, i.e., which add up to (in this case) 7.
 4. Play begins with all players putting down pairs of complementary cards which they have been dealt. They put these down face up.
 5. Turns are taken clockwise, starting with the player on the left of the dealer.
 6. To collect more pairs of complementary cards, they ask other players for cards they want. E.g., a player who has a 2 might ask, “Please, Andrew, may I have a 5?”
 7. If Andrew has a 5 he must give it. Otherwise he says “Sorry,” and the turn passes to the next player.
 8. The cards which a player asks for are the complements of cards which they already hold. So when the player in the example in step 6 has asked for a 5, the others can then deduce that he holds the complement of 5, namely 2.
 9. The player who puts down all his cards first scores 2 bonus points. Play continues until all pairs of complementary numbers are put down.
 10. Players score 2 points for each pair.
 11. The game is then repeated using a different central card.

Discussion of activities Activity 1 introduces the idea of complement in a physical embodiment. Children first see the whole set, and then part of it, from which they have to deduce the part they cannot see. They are able immediately to test the correctness of their deduction.

Activity 2 may be played at two levels of sophistication. The first involves no more than using the new concept at a mental and symbolic level. The second is explained in step 8: the children should be allowed to discover this for themselves.

Counting on is a good way of doing both, and finger counting (see Num 1.5/1) should be encouraged and if necessary reviewed.

Num 3.6 MISSING ADDEND

Concept Missing addend.

Abilities To find missing addends
(i) in physical representations
(ii) in number sentences
(iii) in number stories.

Discussion of concept A missing addend is the answer to a question such as “Four and how many make seven?” Or,

$$4 + \square = 7$$

Or, “Peter had four cookies. His mother gave him some more, and then he had seven. How many did his mother give him?”

This concept relates closely to that of complement, in the preceding topic.

Activity 1 “How many more must you put?” [Num 3.6/1]

A teacher-led activity for a small group. Its purpose is to introduce the concept of missing addend in a physical representation.

- Materials*
- SAR board (as used in Num 3.1/1).*
 - Start cards 1-5 (later 0-5).*
 - Result cards 6-10 (ordinary number cards).*
 - Reversible instruction card.*

Side one reads:

“How many more did you put to get this result?”

Side two reads:

“How many more will you have to put to get this result?”

- Objects.
- Pencils, slips of paper.

* Provided in the photomasters

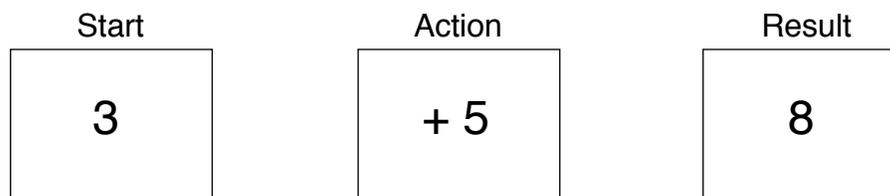
- What they do*
1. The SAR board is put out with ‘Start’ and ‘Result’ packs in position, as in previous activities. The reversible card is positioned with side one showing for continuation (a), side two for continuation (b).
 2. A ‘Start’ card and a ‘Result’ card are turned over.
 3. Objects are put into the set loop according to the number on the ‘Start’ card.

The activity may then be continued either practically or predictively, according to which side up the reversible instruction card is.

Num 3.6 Missing addend (cont.)

(a) *Practically*

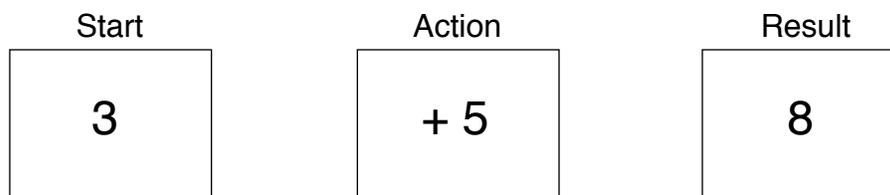
4. Additional objects are put until the required total is reached.
5. This action is then recorded on a slip of paper. This is used to fill the Action space on the SAR card. If the newly put objects are different from the starting ones, this makes it easier to distinguish them.
6. The SAR board will finally look like this.



(written by child)

(b) *Predictively*

4. Using finger counting on, or any other technique he likes, the child answers the question on the instruction.
5. He records his prediction on the slip of paper and puts it into the Action space, e.g.,



(written by child)

6. The prediction is then tested practically as in (a).

Activity 2 Secret adder [Num 3.6/2]

A game for a small group. It requires the same thinking as Activity 1(b), but without any support from physical materials.

Materials None.

Rules of the game

1. One child thinks of a number to be added, which he does not tell the others.
2. The others in turn speak a number, and the 'secret adder' responds by adding his number and giving the results.
3. The first to deduce the number being added takes over as 'secret adder.'
E.g.,
Player: "Three." Secret adder: "Seven."
Next player: "One." Secret adder: "Five."
Same player: "You're adding four." Secret adder: "Correct."
4. Using only numbers up to 5 (to give totals up to 10) this is quite an easy game – especially when one of the players thinks of saying "Zero." (They may then discuss whether this should, by agreement, be excluded.) When they can add beyond 10, the use of numbers below 5 can be excluded. This gives good practice in adding past 10.

Activity 3 Personalized number stories: what happened? [Num 3.6/3]

An activity for a small group. Its purpose is to apply what was learned in Activity 1(b) to the solution of number story problems.

- Materials*
- Story cards of the kind shown below.*
 - Start cards 1-5 (later 0-5) *
 - Result cards 6-10 *
 - Answer cards 0-5 (to fit spaces in story),* or a non-permanent marker
 - Number sentence card (as used in Num 3.4/2).*
 - Pencil and paper for each child.
- * Provided in photomasters

What they do This activity is like Activity 1, but a number sentence is used to record their predictions, as in the following example. The same number story is used here to make the variation stand out more clearly.

Peter had **3** cookies. While he was out of the room, his mother put some more on his plate. When he came back, he found he had **7**. How many more did his mother give him?
 Answer: his mother gave him more.

1. The story card is made ready as in earlier activities, with numbers in the Start and Action spaces and a name in the name space.
2. Each child writes a number sentence with an oblong to show where it is incomplete, thus:

$$3 + \square = 7$$

They may use the number sentence card as a guide to help them to do this.

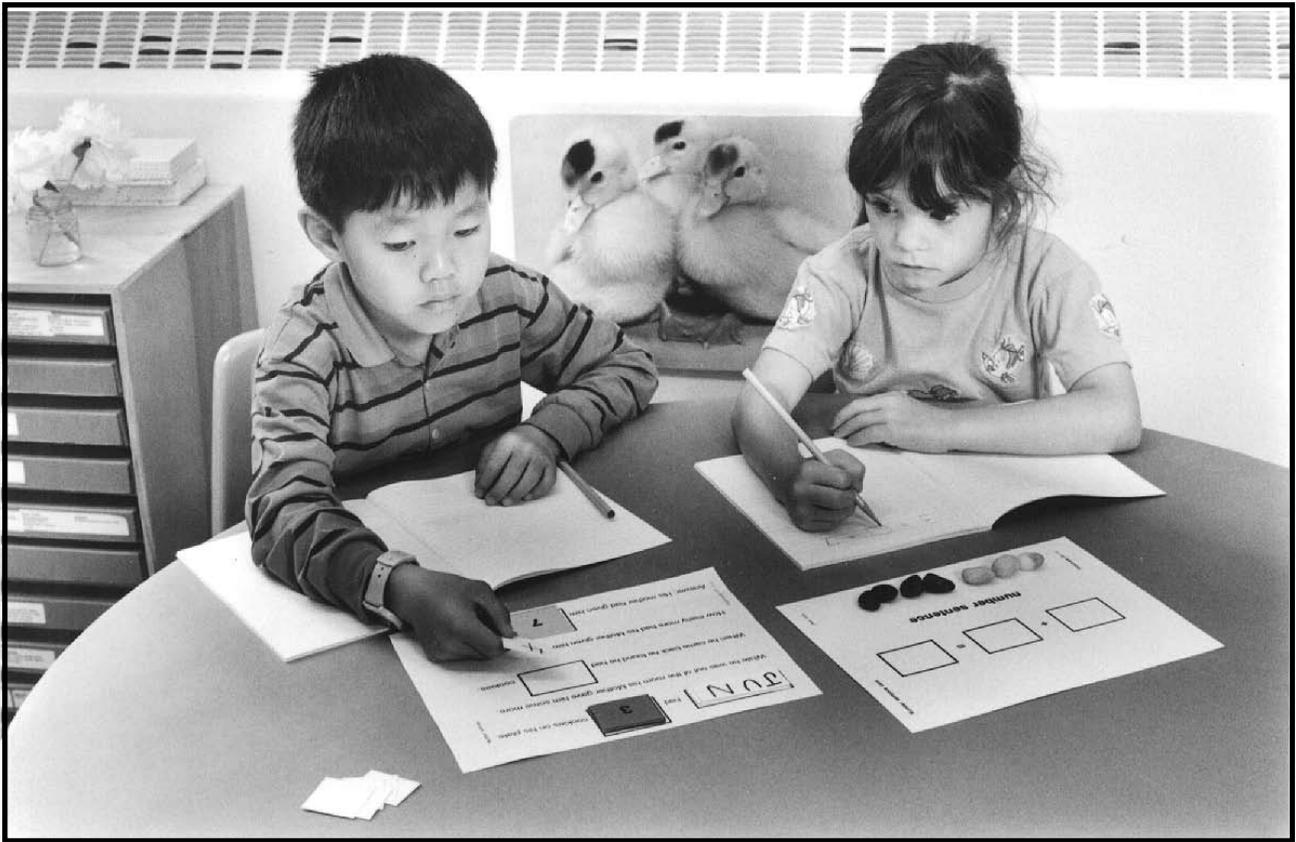
3. Each then writes a number in the oblong to complete the number sentence.

$$3 + \boxed{4} = 7$$

They use any method they like. Finger-counting is useful, but if they have any difficulty, the use of physical materials as in Activity 1 should be encouraged.

4. They compare results. If these disagree, there is discussion and if necessary the use of physical materials.
5. Finally, the agreed answer is put in the answer space on the story board, using either an answer-card or a non-permanent marker.

Note You may wish to introduce vertical notation for addition at this point.



Personalized number stories: what happened? [Num 3.6/3]

Discussion of activities Finding a missing addend is harder than finding a result. In the latter case our thinking goes forward: from start, through action, to result. This topic asks children to back-track: from start and result, back to discover what action would produce that result. (Here, at least, they know that the action was one of putting more, corresponding to adding. In later forms of this problem, it might correspond to any mathematical operation).

They are thus being asked to use their existing knowledge and abilities in a new way. This adaptability, which is a key feature of intelligence, is brought into use all the time in the present approach. The present topic exercises it to the full.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.7 ADDING PAST 10

Concept Adding when the sum is greater than 10, but not greater than 20.

Abilities To do everything in topics 1-6, with results greater than 10 but not greater than 20.

Discussion of concept This is the first step from quite small number operations, which can easily be handled in physical embodiments, towards operations with large numbers for which physical embodiments offer little or no help. For addition, base 10 material continues to be useful right into the thousands. But for multiplication, even of numbers no bigger than (say) 17×13 , children have to be able to think in abstract symbols. As a beginning for this transition this topic uses physical materials and symbols together.

Activity 1 Start, Action, Result over ten [Num 3.7/1]

For a small group. This activity is similar to the earlier version of Start, Action, Result (Num 3.1/1). By now, however, children are familiar with recording, so it does not need to be treated as a separate stage.

- Materials*
- SAR board (see the diagrams following).*
 - Start cards 5-9.**
 - Action cards 5-9.**
 - Base ten material, tens and ones.
 - Paper and pencil for each child.

* Provided in the photomasters. Note that SAR boards vary between different activities.

**If all the numbers from 1-10 are used, only a minority of results will be over 10.

- What they do*
1. The Start and Action cards are shuffled and put in position as usual, and the top two cards are turned over.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 10px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 5px;">S</div> <div style="margin: 0 5px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 5px;">7</div> </div> <div style="display: flex; align-items: center; margin-bottom: 10px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 5px;">A</div> <div style="margin: 0 5px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 5px;">+6</div> </div> <div style="margin-top: 20px;">Result</div> </div>	<table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <th colspan="2" style="padding: 5px;">Do and say</th> </tr> <tr> <td style="width: 50%; padding: 5px;">Tens</td> <td style="width: 50%; padding: 5px;">Ones</td> </tr> <tr> <td style="height: 100px;"></td> <td style="height: 100px;"></td> </tr> </table>	Do and say		Tens	Ones		
Do and say							
Tens	Ones						

Num 3.7 Adding past 10 (cont.)

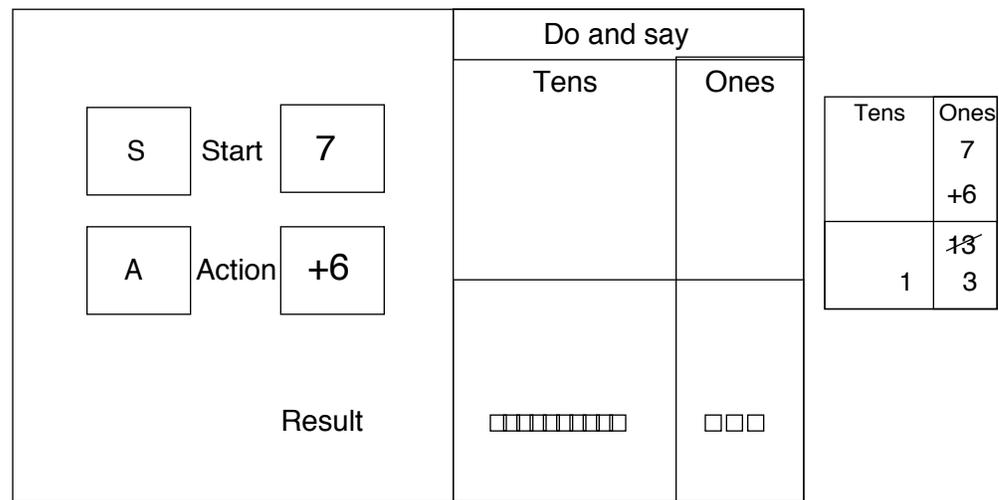
- Unit cubes are put down as shown on these cards by one of the children, who describes what she is doing and what it corresponds to on the SAR board. E.g., “The Action card means ‘Put 6 more.’” Each child records this on her own paper, which she first rules into headed columns. The board, and records, will now appear as below.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; gap: 10px;"> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">S</div> <div style="font-size: 1.2em;">Start</div> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">7</div> </div> <div style="margin-top: 20px; display: flex; align-items: center; gap: 10px;"> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">A</div> <div style="font-size: 1.2em;">Action</div> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">+6</div> </div> <div style="margin-top: 20px; font-size: 1.2em;">Result</div> </div>	Do and say	
	Tens	Ones
	<div style="display: flex; justify-content: space-around;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div> <div style="display: flex; justify-content: space-around; margin-top: 5px;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div> <div style="display: flex; justify-content: space-around; margin-top: 5px;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div> <div style="display: flex; justify-content: space-around; margin-top: 5px;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div>	

- The next stage is shown below.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; gap: 10px;"> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">S</div> <div style="font-size: 1.2em;">Start</div> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">7</div> </div> <div style="margin-top: 20px; display: flex; align-items: center; gap: 10px;"> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">A</div> <div style="font-size: 1.2em;">Action</div> <div style="border: 1px solid black; padding: 5px; width: 40px; text-align: center;">+6</div> </div> <div style="margin-top: 20px; font-size: 1.2em;">Result</div> </div>	Do and say									
	Tens	Ones								
		<table border="1" style="border-collapse: collapse; margin: auto;"> <tr> <td style="text-align: center; padding: 5px;">Tens</td> <td style="text-align: center; padding: 5px;">Ones</td> </tr> <tr> <td></td> <td style="text-align: center; padding: 5px;">7</td> </tr> <tr> <td></td> <td style="text-align: center; padding: 5px;">+6</td> </tr> <tr> <td></td> <td style="text-align: center; padding: 5px;">13</td> </tr> </table>	Tens	Ones		7		+6		13
Tens	Ones									
	7									
	+6									
	13									
		<div style="display: flex; justify-content: space-around;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div> <div style="display: flex; justify-content: space-around; margin-top: 5px;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div> <div style="display: flex; justify-content: space-around; margin-top: 5px;"> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> <div style="border: 1px solid black; width: 15px; height: 15px;"></div> </div>								

- Finally, ten of the ones are exchanged for a ten-rod. This is transferred to the tens column. The children again record this individually.



5. Children compare their final results. There should be no difference, but if there is, they will need to repeat the process and check each step together..
6. The board is cleared, and steps 1 to 5 are repeated with different numbers.

Activity 2 Adding past ten on the number track [Num 3.7/2]

This activity should be used as a preliminary to Activity 3. This is the same game as Num 3.2/2, but with results greater than 10. A copy of this activity also appears as NuSp 1.3/4 in ‘The number track and the number line’ network.

- Materials*
- SAR board as in Num 3.1/1.
 - Both players use staircases 1-9, the first 5 cubes being of a different colour from the remainder. E.g., an eight rod would be 5 blue, 3 white. The five rod would be all blue, and the one to four rods would all be white. These give the same grouping as finger counting.
 - Use ‘Start’ and ‘Action’ cards 4-9. These will give a mixture of results which do/do not cross the ten boundary.*
 - Result cards 1-20.*
 - Reversible card.*
 - Paper and pencil for each child.
 - Plasticine.
- * Provided in the photomasters (see NuSp 1.3/4)

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces on the SAR board.
 2. The children make up a 1-9 staircase each, in different colours.
 3. Player A turns over the top ‘Start’ card, selects a rod of this number and puts it into the number track.
 4. Player B turns over the top ‘Action’ card, but does not yet take out a rod. First, he says “I predict that it’ll come to *here*,” pointing, and marking his prediction with a piece of plasticine. This will involve some form of counting on, which on the number track corresponds to movement to the right.
 5. Then he tests his prediction physically by joining a rod of the given number to the first on the number track.
 6. Steps 3, 4, 5 are repeated until all the cards are turned.
 7. Then the cards are shuffled, and they begin again with their roles interchanged.

Activity 3 Slippery Slope [Num 3.7/3]

A board game for 2 or 3 children. Its purpose is to consolidate the skill of adding past 10 in a predictive situation.

- Materials*
- Game board (see Figure 5).*
 - Three small markers (three cubes) of a different colour for each player.
 - Die 1-6 and shaker.
- * Provided in the photomasters

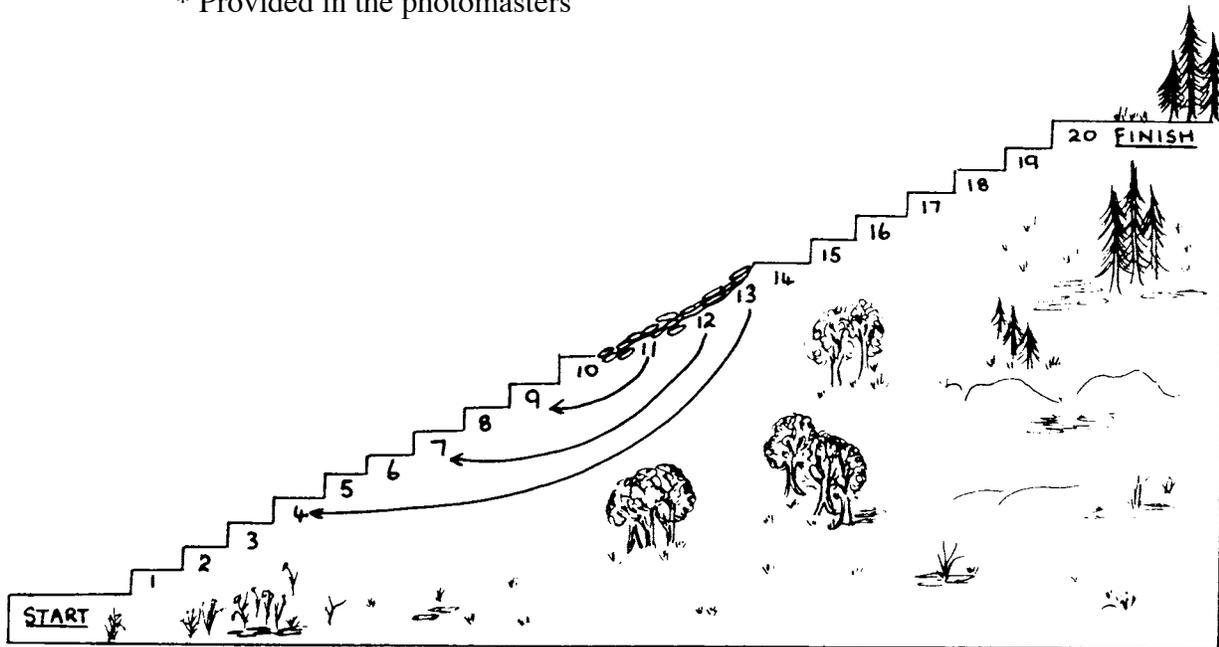


Figure 5 Slippery slope.

- What they do*
1. The board represents steps up a hillside. Steps 11, 12, 13 are missing. Here there is a slippery slope, and if a climber treads here she slides back to a lower step as shown by the arrows.
 2. The object is to reach the top. Each player manages three climbers, represented by markers.
 3. Players in turn throw the die, and move one of their climbers that number of steps up. They begin at START, which corresponds to zero.
 4. A climber may not move upwards to a step which is already occupied. Passing is allowed.
 5. Players may choose not to move. However, if a climber has been touched, it must be moved (but also see the following rule).
 6. If a climber is touched and the move would take her to an occupied step, she must return to the start.
 7. If a climber slides back to an occupied step, any climber already on that step is knocked off and must return to the start.
 8. The exact number must be thrown to finish.

Activity 4 Adfacts practice [Num 3.7/4]

An activity for children to play in pairs, as many as you have materials for. They can use this in odd times which might otherwise be wasted. Its purpose is to practise the recall of all their addition facts (often called ‘number bonds’).

- Materials*
- 10 sets of addition cards, with 10 cards in each set, from $1 + 1$ to $10 + 10$.*
 - One linear slide rule for each pair.* (This is described fully in NuSp 1.6.)
- * Provided in the photomasters

- What they do*
1. In each pair, one child has in her hand a single pack of cards, shuffled and face down. The other has on the table the linear slide rule.
 2. Child A looks at the top card in her hand and tries to recall this result. Child B then checks by using the linear slide rule.
 3. If A’s answer was correct, this card is put on the table. If incorrect, it is put at the bottom of the pile in her hand so that it will appear again later.
 4. A continues until all the cards are on the table. This method gives extra practice with the cards she got wrong.
 5. Steps 1 to 4 are repeated until A makes no mistake, and her hand is empty.
 6. The children then change roles, and repeat steps 1 to 5.
 7. Steps 1 to 5 are, then or at other times, repeated with a different pack until all the packs are known.
 8. Next, the game may be played with two packs mixed.
 9. The final stage is to mix all the packs together. Each child then takes from these a pack of 10 mixed cards, and repeats steps 1 to 5 with this pack.
 10. This activity should be continued over quite a long period, gradually introducing new packs. A good way to practise is little and often.

Activity 5 Adfacts at speed [Num 3.7/5]

A game for up to 6 children. Its purpose is further to consolidate children’s recall of addition results. This game may be introduced for variety before children have completed Activity 4, using the packs which they have learned so far.

- Materials*
- Addition cards: all the packs which they have learned, mixed together (as in Num 3.7/4).
 - A linear slide rule (as in Num 3.7/4).
 - Adfacts board * (see Figure 6).
- * Provided in the photomasters

- Rules of play*
1. All, or nearly all, the cards are dealt to the players. Each should have the same number, so when the remaining cards are not enough for a complete round, they are put aside and not used for the game. The adfacts board is put on the table between them.
 2. The players hold their cards face down. In turn they look at their top card (e.g., $7 + 4$) and put it in the appropriate space on the adfacts board (in this case 11). It does not matter if there is a card in that space already – the new card is then put on top.

Num 3.7 Adding past 10 (cont.)

= 1	= 2	= 3	= 4	= 5
= 6	= 7	= 8	= 9	= 10
= 11	= 12	= 13	= 14	= 15
= 16	= 17	= 18	= 19	= 20

4 + 3

Adfacts cards to fit spaces.

Figure 6 Adfacts board and sample card.

3. The others check. If it is wrong, they tell her the correct answer and she replaces the card at the bottom of the pack.
4. If she does not know, she asks and someone tells her. She then replaces the card at the bottom of the pack.
5. Any disagreements are settled by using the linear slide rule.
6. Play continues until all have put down all their cards. If there are no mistakes, all will finish in the same round. Those who do make mistakes, or do not know, will be left with cards in their hands to put down in subsequent rounds.
7. If one player finishes a clear round ahead of the others, she is the winner.

Variation If a stopwatch is available, this game may also be played as a race. To make it a fair race, each player needs to be using the same pack. This suggests various forms, e.g.,

Form (a) A single pack, of a table to be consolidated or revised.

Form (b) Several packs mixed.

Form (c) (For advanced players.) All packs from twos to tens, making 90 cards in all.

The rules for all forms are the same:

1. One player acts as starter and timekeeper.
2. The others in turn (one at a time) see how quickly and accurately they can put down all their cards.
3. Those not otherwise involved check for accuracy, after all the cards have been put down.
4. For each incorrect result, 5 seconds are added to the time. (This figure may be varied according to the skill of the players.)
5. The winner is the player with the fastest time after correction for errors.

Activity 6 Predictive number sentences past 10 [Num 3.7/6]

An activity for a small group. It develops the skills which will be used in Activity 7 and gives practice in all combinations of numbers, with a majority crossing past ten. It may also be played as a group game.

- Materials*
- Start cards 5-9.*
 - Action cards 1-9.*
 - Some cubes with which they can check their calculations if needed.
 - Pencil and paper for each child.
- * Provided in the photomasters

- What they do*
1. The two packs are shuffled and put face down on the table.
 2. The top 'Start' and 'Action' cards are turned and put side by side. Each child then writes a number sentence and completes it, as in the example below.

Start card	Action card	Number sentence
Start with a set of 8 8	Increase it by 5.	$\mathbf{8 + 5 = 13}$

3. Finally, they compare results. If there is disagreement they should use physical material to verify.

Activity 7 Explorers [Num 3.7/7]

Another board game, for two or three players. Its purpose is to consolidate addition skills, especially those past ten and in the teens.

- Materials*
- Game Board* (see Figure 7)
 - Die 1-6 and shaker
 - Markers: up to three per player

- Rules of the game*
1. Each player manages one, two, or three explorers, according to number and experience of players.
 2. The explorers have to find their way from the start, through the forest and desert, to the Lost City.
 3. They go forward according to the number thrown on the die by the players in turn.
 4. In the forest, positions 1, 2, 3, 4, 6, 7 may be occupied by only one explorer. The rest camp (5) has room for all who come.
 5. The oases likewise have room for as many as arrive. The missing numbers are in the trackless desert, where there is no landmark to show where you are. Explorers may only move to the numbers shown on the board. Thus, an

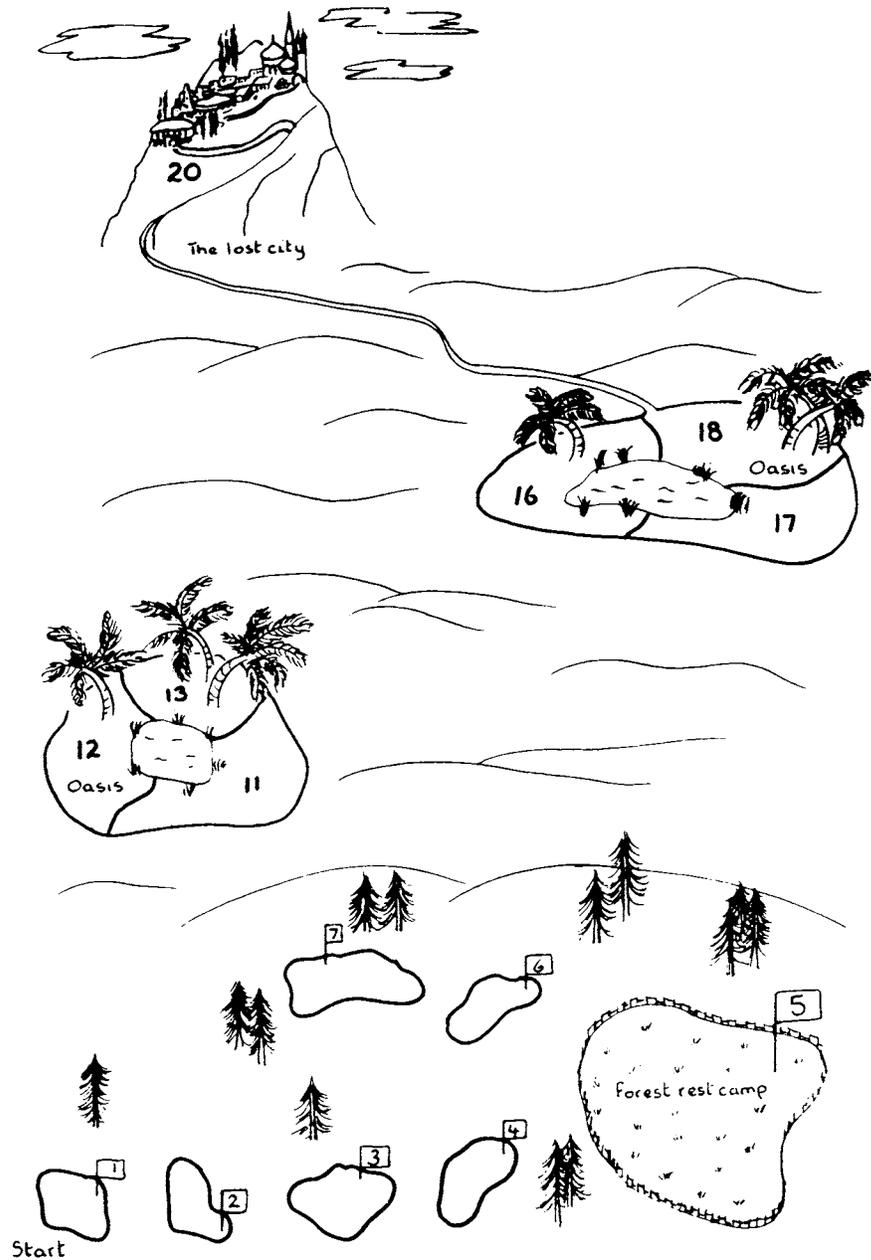
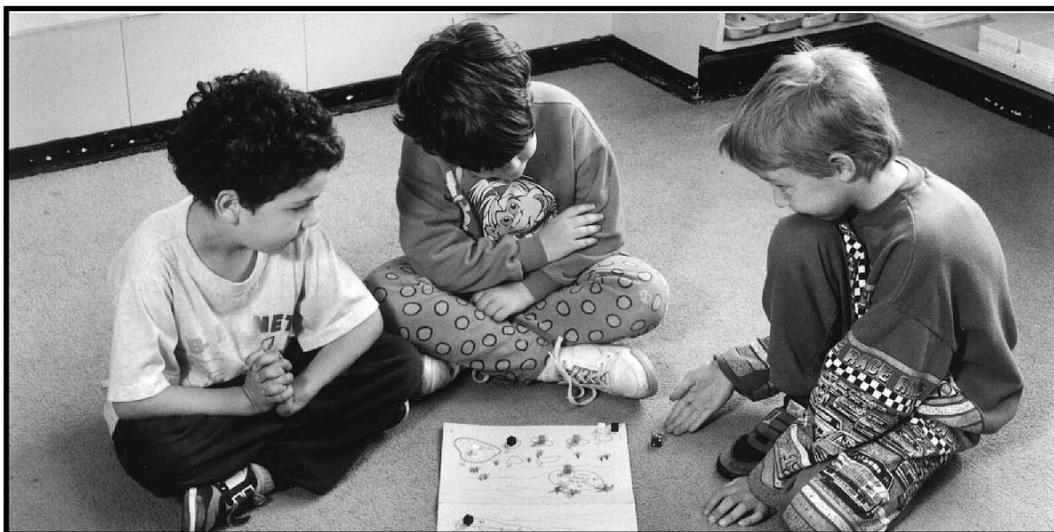


Figure 7 Explorers

explorer at 6 could only move to 11 (by a throw of 5) or 12 (by a throw of 6). A player at 13 could move to 16, 17, 18 (throw of 3, 4, or 5). The exact number must be thrown to reach the Lost City (20).

6. A player may choose not to move. However, a piece must be moved if touched. The penalty for a false move (to an occupied location in the forest, or to a number in the trackless desert) is to return to START if before 5, and otherwise to be rescued and taken back to the forest rest camp to recuperate. (This rule may be relaxed while learning the game.)
7. The winner is the first player to get all his explorers to the Lost City. Play may however continue until all have arrived.



Explorers [Num 3.7/7]

Discussion of activities Activities 1-5 in this topic provide a similar progression to that in earlier topics, but more rapidly since children are now extrapolating concepts which should already be well established.

Activity 1 makes a beginning with what will become the conventional way of recording, with a close parallel shown visually between the physical materials and the symbols. This is Mode 1 concept building. Activity 2 also provides strong physical support, in this case from the number track. Adding by use of the number track is probably easier than the method used in Activity 1. However, it does not easily extrapolate, whereas the base 10 material provides very well for extrapolation to hundreds and thousands. In this activity, the 'teens' notation is used since children will certainly have encountered it, albeit without fully understanding its rationale.

Activity 3, 'Slippery Slope,' also involves adding with the help of a number track. Visual support is still provided, but a predictive element has been introduced, with the purpose of planning which is the better piece to move. Steps 5, 6, 7 are bad ones to linger on, whereas 10 maximizes the number of throws which will take one past the slippery slope. This game can be played at different levels of sophistication.

Activity 4 gives practice in the newly developed skills, and relates these to the notation which children already know.

In Activities 3 and 5 we see in microcosm a key activity of intelligence: comparing alternative plans before deciding which to put into action.

The numbers in Activity 5 require plenty of calculations past ten and in the teens. A false move in the desert leads to additional calculations. There is now less visual support. Finger-counting, using 'Ten in my head' (Num 1.7/1) should be freely used as long as it is needed.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.8 COMMUTATIVITY

- Concepts* (i) Commutativity.
(ii) Non-commutativity.

Ability To recognize whether a given action or operation is commutative or non-commutative.

Discussion of concepts Although these two additions are different, the result is the same.

<i>Start</i>	<i>Operation</i>	<i>Result</i>
3	+ 5	= 8
5	+ 3	= 8

Children are also familiar with the vertical notation.

$$\begin{array}{r} 3 \\ + 5 \\ \hline 8 \end{array} \qquad \begin{array}{r} 5 \\ + 3 \\ \hline 8 \end{array}$$

Whichever way we write it, if the numbers to be added are interchanged, the result is the same. This is expressed by saying: *addition is commutative*. To us this may seem obvious. But commutativity does not always hold: it is not true for subtraction. When it does hold it can be very useful. If we know that addition is commutative for all pairs of numbers, we have only half as many addition facts to remember.

This distinction between commutative and non-commutative operations also applies to many physical actions. Here are some examples.

Commutative

Put sock on left foot, put sock on right foot.

Open textbook, open exercise book.

Put apple into bowl, put orange into bowl.

Non-commutative

Put on shirt, put on tie.

Undress, get into bath.

Pick up telephone, dial the number.

Activity 1 Introducing commutativity [Num 3.8/1]

A teacher-led discussion for 2, 4, or 6 children working in pairs. A suggested sequence is given below, which you can adapt to follow up leads given by the children themselves.

- Materials*
- For each pair:
 - Interlocking cubes, ten each of two colours.
 - A card number track 1-20.*
 - Paper and pencil.
- * Provided in the photomasters

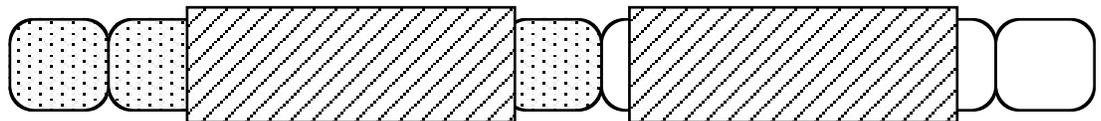
- Suggested sequence for the discussion*
1. Children work in pairs side-by-side, so that they see everything the same way up.
 2. Write a pair of open number sentences such as

$$5 + 2 =$$

$$2 + 5 =$$

3. Tell the children to show what these mean with cubes on the number track. They should use one of their colours for the 'start' number, the other for the 'action' number. One child in each pair does the first sentence, the other does the second.
4. Ask "What do you notice?" There are two points which need to be put into words, either now or at subsequent repetitions of steps 3 and 4.
 - (i) The numbers are the same in both sentences, but interchanged.
 - (ii) The result is the same in both cases.

"It doesn't make any difference if you change the numbers about" is a reasonable beginning, but doesn't bring out that the *same* result is being obtained by a *different* path. So it is better to say "It doesn't make any difference *to the result* if the two numbers are interchanged."
5. Tell each pair of children to write another pair of number sentences like the first, but using different numbers.
6. Repeat steps 3 and 4.
7. Repeat steps 5 and 6 until they decide that this will always be so, whatever the numbers.
8. This can be confirmed quite nicely by using two paper sleeves over the rods.



Say, "Now we can't see what the numbers are."

Turn it around. Ask "What can we say about this?"

9. Say that there is a shorter way of saying that (whatever their formulation was). We say "Adding is *commutative*."

Num 3.8 Commutativity (cont.)

Activity 2 Introducing non-commutativity [Num 3.8/2]

A continuation of the discussion in Activity 1. Its purpose is to prevent children from thinking that all operations are commutative.

Materials • As for Activity 1, as and if needed.

Suggested sequence for the discussion

1. Explain that this idea also applies to everyday actions. You can use the examples given in the discussion at the beginning of this topic, or invent your own.
2. Ask the children for further examples, and discuss these.
3. Ask whether they think that all pairs of actions are commutative. Use the given examples of non-commutative actions, or some of your own, until they can recognize whether an example is commutative or non-commutative.
4. Ask the children for further examples, and discuss these.
5. Ask whether they think that subtraction is commutative or non-commutative. Suggest that they try some particular cases, e.g.,

$$5 - 2 =$$

$$2 - 5 =$$

(*Note* If the two numbers are the same, then of course we can interchange them and the result is the same. If this arises, explain that we only say that an operation is commutative if it is always true whatever the numbers are.)

Activity 3 Using commutativity for counting on [Num 3.8/3]

A continuation of the discussion in Activity 2. Its purpose is to show children one of the uses of commutativity.

Materials • Pencil and paper for each child and for the teacher.
• Number track 1-10 for each pair of children.

Suggested sequence for the discussion

1. Write an incomplete sentence in which the first number is much smaller than the other, such as $2 + 7 =$
2. Ask them all to copy this and obtain the result by counting on from 2. In each pair one should use the number track, the other finger counting.
3. Write the same sentence the other way round, in this case $7 + 2 =$
4. Ask them all to copy this, and obtain the result by counting on from 7.
5. Ask them which was easier.
6. Repeat steps 1 and 2 with other numbers, but this time just ask them to get the result by counting on.
7. Did they make it easier for themselves by counting on from the larger number? If so, the point has been taken. If not, steps 3, 4, 5 may be repeated.
8. Continue until all have realized the advantage of starting with the larger number when counting on.
9. Consolidate by returning to the points made in Activities 1 and 2: that this only works because addition is commutative. Check that they remember what this means, expanded into a sentence.

Activity 4 Commutativity means less to remember [Num 3.8/4]

A continuation of the discussion in Activity 3. Its purpose is to show children another of the applications of commutativity.

Materials • Pencil and paper for each child, and for the teacher.

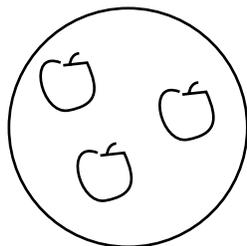
- Suggested sequence for the discussion*
1. Write an incomplete number sentence, such as $9 + 5 =$. Ask them all to copy this and complete it.
 2. Write the same sentence the other way round, in this case $5 + 9 =$. Ask them to copy and complete this.
 3. Ask how many worked this out again. Some may have realized that they didn't need to.
 4. Repeat steps 2 and 3 until all have realized that if they know one result, they know the other.
 5. Consolidate by making the point that this means that they only have about half as many addition facts to remember. (Slightly more than half, since there is no saving when the two numbers to be added are equal.)

Discussion of activities Though commutativity is quite an abstract idea for children at this stage, I have included it for several reasons. First, it is useful, as Activities 3 and 4 make clear. Second, children seem to grasp it fairly easily if given concrete examples. Third, unless it is made explicit and discussed, we cannot also make explicit that subtraction is not commutative. While the idea remains at an intuitive level, children are liable to generalize it incorrectly.

Discussion of related concepts There are related topics which I did not include in the discussion at the beginning of this topic, since I did not wish to make it too heavy. One of these is the distinction between unary and binary addition. Although these are different mathematically, I think that from the children's point of view it is better to treat them as the same. However, it may be as well for teachers to be aware of the distinction in case of need; here is a short explanation.

At a practical level, there is not a lot of difference between these two.

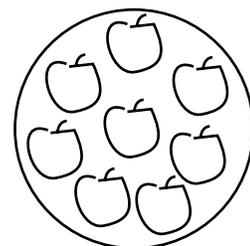
A. Start with a set of 3 apples in a bowl.



Put 5 more.

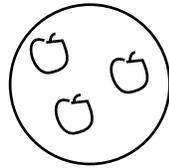


Result a set of 8 apples in a bowl.



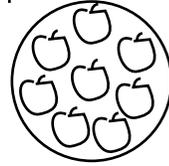
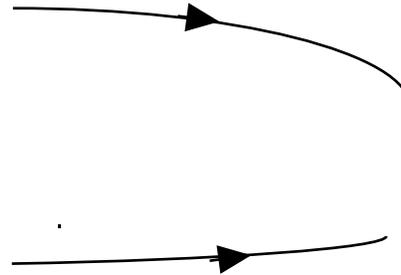
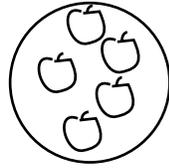
Num 3.8 Commutativity (cont.)

B. Start with a set of 3 apples in a bowl,
 into
 another bowl.
 and a set of 5 apples in a different bowl.



Put these all

Result, a set of 8 apples in the bowl.



Mathematically, A corresponds to unary addition, in which

we start with a number,

do an operation on it

and the result is another number.

3

+ 5

8

B corresponds to binary addition, in which

we start with a pair of numbers,

do an operation on this pair

and the result is another number.

3, 5

+

8

Number sentences for the first are written like this, as we already know.

$$3 + 5 = 8$$

When 'modern mathematics' came into vogue, a number of school texts introduced this notation

$$(3,5) \xrightarrow{+} 8$$

for the second. This is entirely correct mathematically, and a good notation at more advanced levels. But I do *not* recommend its use for young children, to whom parentheses () mean multiplication. The first, a widely accepted notation, saves having to decide whether, in a particular case, we mean unary or binary addition; and it lends itself readily to vertical addition and to adding in columns, when we come to hundreds, thousands, etc.

3	362
<u>+5</u>	<u>+574</u>
8	

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 3.9 ADDING, RESULTS UP TO 99

Concept Adding when the results are between 20 and 99.

Ability To add 2-digit numbers, the results still being 2-digit numbers.

Discussion of concept What is new here is not the concept of addition, but extension of the ability to do this with larger numbers.

Activity 1 Start, Action, Result up to 99 [Num 3.9/1]

This is a continuation of Num 3.7/1 (Start, Action, Result over 10) with larger numbers.

- Materials*
- SAR board as illustrated below in step 1.*
 - Start and Action cards with assorted numbers from 1 to 49.*
 - Base ten material, tens and ones.
 - Paper and pencil for each child.
- * Provided in the photomasters

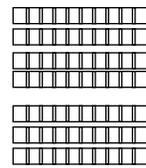
What they do *Stage (a) Headed columns and materials*
(The steps for this are the same as in Num 3.7/1, and are repeated here for convenience.)

- What they do* 1. The Start and Action cards are shuffled and put in position as usual, and the top two cards turned over.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 10px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 5px;">S</div> <div style="margin-right: 5px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 5px;">A</div> <div style="margin-right: 5px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say	
	Tens	Ones

Num 3.9 Adding, results up to 99 (cont.)

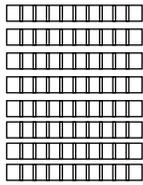
- Tens and single cubes are put down as shown on these cards by one of the children, who describes what she is doing and what it corresponds to on the SAR board. E.g., “The Start card means ‘Put down four tens and six ones.’” “The Action card means ‘Put three tens and eight ones more.’” Each child records this on her own paper, which she first rules into headed columns. The board, and records, will now appear as below.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 20px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">S</div> <div style="margin-right: 10px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">A</div> <div style="margin-right: 10px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say									
	Tens 	Ones 	<table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="padding: 5px;">Tens</th> <th style="padding: 5px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="text-align: center; padding: 5px;">4</td> <td style="text-align: center; padding: 5px;">6</td> </tr> <tr> <td style="text-align: center; padding: 5px;">+ 3</td> <td style="text-align: center; padding: 5px;">8</td> </tr> <tr> <td style="height: 20px;"></td> <td style="height: 20px;"></td> </tr> </tbody> </table>	Tens	Ones	4	6	+ 3	8	
Tens	Ones									
4	6									
+ 3	8									

- The next stage is shown below.

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 20px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">S</div> <div style="margin-right: 10px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">A</div> <div style="margin-right: 10px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say									
	Tens	Ones	<table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="padding: 5px;">Tens</th> <th style="padding: 5px;">Ones</th> </tr> </thead> <tbody> <tr> <td style="text-align: center; padding: 5px;">4</td> <td style="text-align: center; padding: 5px;">6</td> </tr> <tr> <td style="text-align: center; padding: 5px;">+ 3</td> <td style="text-align: center; padding: 5px;">8</td> </tr> <tr> <td style="text-align: center; padding: 5px;">7</td> <td style="text-align: center; padding: 5px;">14</td> </tr> </tbody> </table>	Tens	Ones	4	6	+ 3	8	7
Tens	Ones									
4	6									
+ 3	8									
7	14									

- Finally, if there are (as in this case) more than ten ones, ten of them are exchanged for a ten-rod. This is transferred to the tens column. The children again record this individually (as illustrated on the following page).

<div style="display: flex; flex-direction: column; align-items: center;"> <div style="display: flex; align-items: center; margin-bottom: 20px;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">S</div> <div style="margin-right: 10px;">Start</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">46</div> </div> <div style="display: flex; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin-right: 10px;">A</div> <div style="margin-right: 10px;">Action</div> <div style="border: 1px solid black; padding: 5px; margin-left: 10px;">+38</div> </div> <div style="margin-top: 20px;">Result</div> </div>	Do and say	
	Tens	Ones
		□□□□

Tens	Ones
4	6
+ 3	8
7	14
8	4

5. Children compare their final results. There should be no difference, but if there is, they will need to repeat the process and check each step together.
6. The board is cleared, and steps 1 to 5 are repeated with different numbers.

Stage (b) Headed columns, no materials

1. The SAR board and base ten materials are no longer used.
2. Start and Action cards are turned over as before, and all the stages of addition are written in headed column notation as in stage 1.

Stage (c) Place-value notation

1. Start and Action cards are turned as before. If the addition would result in 10 or more in the units position, a streamlined form of headed column notation is used.

2. Here is a simple example.

First the addition
to be done
is written.

$$\begin{array}{r} 3 \quad 7 \\ + 2 \quad 5 \\ \hline \end{array}$$

3. Adding 7 and 5 gives a result in 2 digits, so we need headed columns.

$$\begin{array}{r|l} \text{T} & \text{U} \\ 3 & 7 \\ + 2 & 5 \\ \hline 5 & 12 \end{array}$$

4. Regroup

$$\begin{array}{r|l} \text{T} & \text{U} \\ 3 & 7 \\ + 2 & 5 \\ \hline 5 & 12 \\ 6 & 2 \end{array}$$

Num 3.9 Adding, results up to 99 (cont.)

5. Finally write the result in place value notation

$$\begin{array}{r} 37 \\ + 25 \\ \hline 62 \end{array}$$

6. The whole may be set out like this.

$$\begin{array}{r} 37 \\ + 25 \\ \hline 62 \end{array}$$

↙

T	U
3	7
+ 2	5
5	12
6	2

This result is written last.

Activity 2 Odd sums for odd jobs [Num 3.9/2]

A game for about four children. Its purpose is to use the skills learned in Activity 1.

Materials

- 1 die bearing only 1s and 2s [for tens].
 - 1 normal 1-6 die (or 0-9, if you have one) [for ones].
 - 1 set of ‘odd job’ cards.*
 - 1 set of ‘target’ cards.**
 - Plastic or real money: 10¢, 5¢, 1¢ coins.
 - Money boxes with a slit in the lid for all but one of the players.
- *Written on these are jobs they might do such as “Wash car,” “Clean all downstairs windows.” (provided in the photomasters)
- **Written on these are the kind of things they might be working for, such as “Swimming, 60¢ for two,” “Present for sister, 78¢.” (provided in the photomasters)

What they do

1. One child acts as ‘Mom’ or ‘Dad.’
2. Each child picks a target from the target cards which are held in a fan or spread out face downwards on the table.
3. Then, starting with the child on the left of ‘Mom’ or ‘Dad,’ the first ‘does a job’ by turning over a card, and since there is no fixed rate, throws the dice to see what she is paid.
4. The player acting as ‘Mom’ gives her the money in coins.
5. She puts these in her money box, which she may not open without permission.
6. Each takes her turn, putting the money in her box each time.
7. To know when she has enough money, each child keeps a record, adding on her earnings each time.
8. The first who thinks she has enough asks permission to open her money box, and her recorded total is checked by her ‘parent’ against the money in the box.
9. If correct she gets to be ‘Mom’ next time.
10. If not she has to go on doing jobs!

Activity 3 Renovating a house [Num 3.9/3]

A co-operative game for 4 children. Its purpose is to practise the skills developed in Activities 1 and 2 in another play situation.

- Materials*
- Gameboard (see Figure 8).
 - On cards:*
 - house
 - chimney
 - windows
 - doors
 - Tens die, numbered 1 & 2 only.
 - Ones die, numbered 0 to 9.
 - Play money: \$10 bills, \$5 bills, and \$1 coins.
 - Slips of paper, pencils.

* Provided in the photomasters

It is a good idea to introduce the game with costs in round numbers, e.g., window \$30, door \$50, and progress to harder numbers as shown. For this, two houses will be needed or adhesive labels used, after laminating, on which there are various sets of prices.

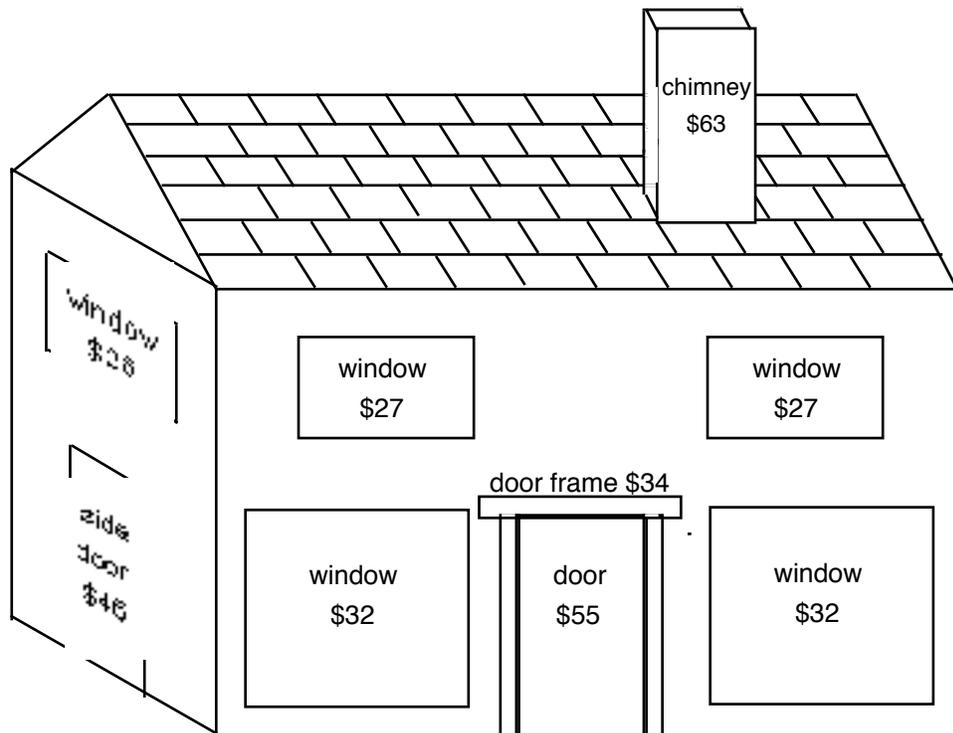


Figure 8 Renovating a house

- What they do*
1. One child acts as banker, one as building supplies dealer, and two as a young couple who are saving what they can each week and putting the money towards parts for their house. This is an old house which they have bought cheaply and are renovating.
 2. Each of the couple throws the dice to determine how much they have saved from their earnings that month.
 3. They record these amounts on a slip of paper, and add them together.
 4. They take their slip to the banker, who checks their total and gives them cash in exchange, keeping the slip.
 5. When they have enough, they go to the building supplies dealer and buy a door, window, or chimney. The banker may be asked to exchange larger bills for smaller.
 6. When the house is built, they may play another game, exchanging roles.

$\begin{array}{r} 24 \\ + 17 \\ \hline 41 \end{array}$
--

Activity 4 Planning our purchases [Num 3.9/4]

For a small group. This activity uses the same mathematics for a situation which is the reverse of Activity 2. Instead of accumulating money for a predetermined single target (which may be exceeded), they have a given amount of money which they plan to spend on a variety of purchases, keeping to a total cost within the given amount.

- Materials*
- A variety of labelled articles for the shop.*
 - Slips of paper or card.**
 - Plastic money.
 - Pieces of paper and pencils for each child.

*Varied prices in the range of 48¢, 19¢, 7¢ and a few penny objects.

**On these are varying sums from 50¢ to 90¢, in tens.

- What they do*
1. One child first acts as banker, and has charge of the money; then as shopkeeper.
 2. The other players draw slips of paper to discover how much they have to spend.
 3. Each child takes her slip to the banker to get cash.
 4. She then makes a shopping list which she totals.
 5. When ready, she goes to the shop with her money and shopping list.
 6. The shopkeeper then sells her the goods, naturally making sure that she receives the correct payment.

Shoppers can use penny objects to make up an exact amount, thereby avoiding the necessity for giving change. Children could also ask the banker to change a 10¢ coin into 5¢ and/or single pennies, which is good practice for mentally changing between tens and singles. Receiving change is however something the children will already have experienced outside school, so you may prefer to leave out the penny objects and let the shopkeeper give change instead.

Activity 5 Air freight [Num 3.9/5]

A more difficult addition activity, for children playing in pairs.

- Materials*
- Cards with pictures of objects together with their weights.*
 - Cards representing containers.

* Suitable pictures might be bags, suitcases, small items of furniture, domestic items like tape recorders, televisions, etc.

It is good to have several assorted sets of these. For example, a set having 19 cards, each with a picture from a mail order catalogue, marked with the following weights:

64kg,	54kg,	52kg,	48kg,	46kg,	42kg,	
39kg,	38kg,	35kg,	31kg,	28kg,	22kg,	
20kg,	19kg,	17kg,	15kg,	12kg,	9kg,	8kg

This makes a total of 599kg, which it is possible to pack into 6 containers (5 holding 100kg each, and 1 holding 99kg). You could use other combinations of weights. It is probably easiest to choose them by breaking down 100kg's

e.g.,	62,	27,	11.		
	53,	17,	22,	8,	
	49,	33,	18,		
	35,	31,	19,	10,	5 etc.



Air freight [Num 3.9/5]

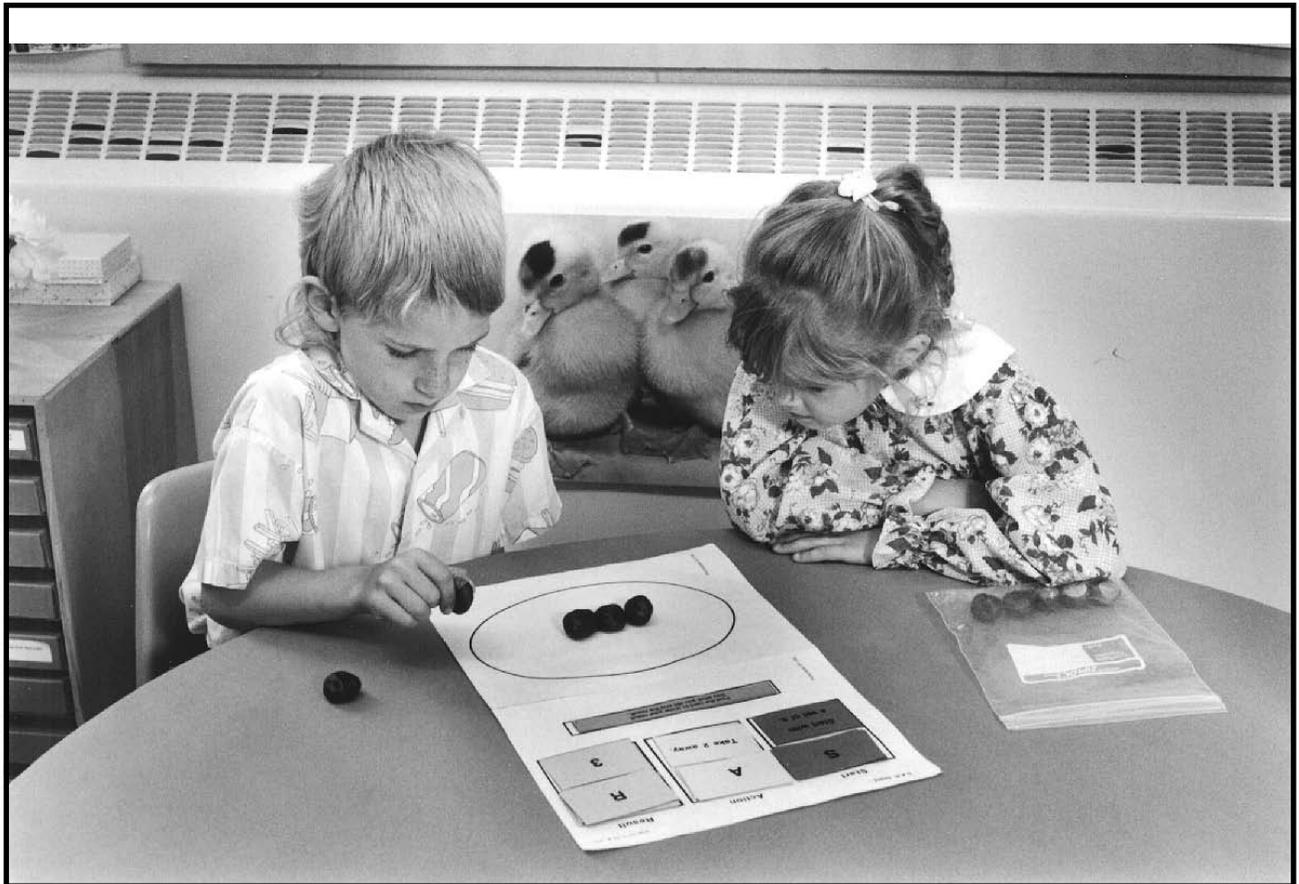
- What they do*
1. The object is to pack all the objects into the smallest possible number of containers. No single container may weigh more than 100 kilograms.
 2. They may work however they choose. One way is for one to act as packer, and the other as checker who makes sure that no container exceeds 100 kg.
 3. If there are several pairs doing this activity at the same time, they may compare results to see which pair is the best at packing the containers.

Discussion of activities These involve the extrapolation of techniques already learned to larger numbers, and the combining of concepts already formed. So the schema building involved in this topic is a good example of Mode 3: creativity. Grouping and re-grouping in tens from Org 1, adding past ten from the topic just before this, and notations for tens and singles from Num 2, are the chief concepts to be synthesized.

In Activity 1, stage (a) renews the connection between written addition and its meaning as embodied in place-value notation. Stage (b) is transitional to addition using place-value notation in stage (c). In all these stages, place-value notation sometimes occurs in combination with headed columns. This hybrid was only adopted after careful analysis and discussion with teachers. It is implicit in the conventional notation, when small 'carrying' figures are used; so is it not better to show clearly what we are doing, especially since it represents such an important part of the calculation? The layout suggested is very little slower than the traditional one – and if speed is the main object, then calculators are the best means to achieve it.

Activity 2 has been devised in the form shown to introduce a predictive element – checking the total on paper against coins in the money box. This you will recognize as an example of Mode 1 testing.

Activities 2, 4 and 5 introduce multiple addends. In Activity 2, this is done by stages, each total being recorded before the next one is added. This is how we add mentally, so it is a good preparation for Activity 4 where the running totals need not be recorded unless children find it helpful. Activity 3, and the more difficult Activity 5, have been included to give a further choice of activities in this section, since it is good for children to get plenty of assorted practice at addition with re-grouping.



'Start, Action, Result (do and say)' [Num 4.1/1]

[Num 4] SUBTRACTION

Taking away, Comparison, Complement, Giving change

Num 4.1 ACTIONS ON SETS: TAKING AWAY

- Concepts*
- (i) Sets as operands.
 - (ii) Taking away as actions.
 - (iii) Starting and resulting numbers.

Ability To relate the physical action of taking away to the starting and finishing numbers of a set.

Discussion of concepts This topic closely parallels Num 3.1, and it will be useful to re-read the discussion of concepts given there.

Activity 1 Start, Action, Result (do and say) [Num 4.1/1]

An activity for 2, 3, or 4 children (parallels Num 3.1/1). Its purpose is to introduce the 'take-away' kind of subtraction in a physical form.

- Materials*
- SAR board (as used in Num 3.1/1).*
 - Reversible card (as used in Num 3.1/1).*
 - Start cards 5-10; e.g., "Start with a set of 8."*
 - Action cards 1-5 (later, 0-5); e.g., "Take 2 away."*
 - Result cards 0-10. Just the numerals, 0-10 (as used in Num 3.1/1).*
 - Objects to put in the set loop: e.g., counters, shells, buttons (as used in Num 3.1/1).

* Provided in the photomasters

- What they do*
1. The cards are shuffled and put face down in the upper parts of their spaces.
 2. The reversible card is put in its place with side one showing.
 3. One child then turns over the top start card, and puts a set of the required number into the set loop.
 4. Another child then turns over the top action card, and takes away the indicated number of objects.
 5. Then she finds the appropriate result card to show the number of the resulting set.
 6. Finally she must describe to the other children what she did, and the result. E.g., "I started with a set of seven shells. I took away three. The result is a set of four shells."
 7. Simplified wording is acceptable to start with, but the aim should be something near the above.

Activity 2 Taking away on the number track (do and say) [Num 4.1/2]

An activity for a small group. Some children can be doing this while the others do Activity 1. In this case the order does not matter, although usually it does. Its purpose is to introduce the use of the number track for subtracting. This activity also appears as NuSp 1.4/1 in 'The Number Track and the Number Line' network.

- Materials*
- An SAR board.*
 - A number track.*
 - Cubes to fit the track.
 - Start cards 5-10; e.g., "Start with a set of 5."*
 - Action cards 1-5 (later 0-5); e.g., "Make it 2 less," "Take 3 away," "Make it 4 smaller."*
 - Result cards numbered from 0-10.*
 - A reversible card.*
 - On side one is written:
"Find the card to show your result. Say what you did, and the result."
 - On side two is written
"Predict the result."

* Provided in the photomasters

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces.
 2. The reversible card is put in its place with side one showing.
 3. One child turns over the top start card into the space below, and puts a set of the required number either singly into the number track (in which case the projections will have to be uppermost) or joined into rods.
 4. Next he turns over the action card, and removes cubes from the track as instructed. Note that at this stage we do not talk about subtracting.
 5. Next he finds the appropriate result card to show the number of the resulting set.
 6. Finally he must describe to the others what he did, and the result.
 7. Steps 3 to 5 are then repeated by the next child.

Reminder The children should verbalize their results each time.

Discussion of activities These two activities provide the physical experiences from which children will begin to expand the important concept of mathematical operation to include subtraction. Again, the spoken description of what they have done is an important part of the activities, since it links these experiences to the appropriate language. This not only serves as preparation for the more condensed written symbols which come in Topic 3, but makes a link with the word problems which children will encounter later on.

These activities closely parallel those at the beginning of the addition network, Num 3, and use the same materials. There, children were learning new activities, but the concept (addition) is an easier one than here. Now that they are learning the more difficult concept of subtraction, we start with a familiar situation.

Num 4.2 SUBTRACTION AS A MATHEMATICAL OPERATION

Concept Subtraction as a mental operation.

Abilities To predict the result of ‘taking away’ actions on sets, by mentally subtracting (using physical aids initially).

Discussion of concept As in the case of addition, we need to make a distinction between the actions we do in the physical world, and their corresponding mental operations. (It will be useful here to re-read the discussion in Num 3.2.)

Subtraction is, however, a more difficult concept than addition. In its fully developed form (topic Num 4.7) it is a mathematical model for not one but four physical counterparts: taking away, comparing two sets numerically, giving change, and complement. (See the dependency network for Num 4.) This has two consequences for teaching. One is that we should use the word ‘subtract,’ not ‘take away,’ for the mathematical operation. ‘Take away’ is all right when we are in fact taking away, but to talk about ‘taking away’ when what we are doing is comparing is bound to cause confusion. The other is that subtraction should be introduced rather later than is often done, so that they arrive at this topic when they already have good support from their work on the number line. In this topic, subtraction is a unary operation, done on a single number, e.g.,

we start with a number,	do an operation on it	and the result is is another number
7	– 2	5
operand	operation	result

Activity 1 Predicting the result (subtraction) [Num 4.2/1]

An activity for 2 children, or 2 teams. Its purpose is to use the new concept to predict physical outcomes.

- Materials*
- SAR board.*
 - Reversible card.*
 - Start cards 5-10.*
 - Action cards 1-5 (later 0-5).*
 - Result cards 0-10.*
 - Objects to put in the set loop; e.g., counters, shells, buttons.*

*These are the same as for Num 4.1/1.

What they do Version (a)

1. The cards are shuffled and put face down in the upper part of their spaces. The reversible card now shows side two.
2. Player A (or a player from team A) turns over the top Start card and puts a set of the required number into the loop.

3. Player B (or a player from team B) turns over the top Action card, after which he must predict the result and put the correct result card in the Result space.
4. Player A then physically checks this prediction by taking away the indicated number of objects and counting the resulting set.

Version (b)

1. The cards are shuffled and put face down in the upper part of their spaces.
2. In this version, the Start card is turned as in (a), but this time the objects are put into a bag or under a handkerchief.
3. Then the Action card is turned, and the indicated number of objects is taken out.
4. Finally the prediction is tested physically by emptying the bag or lifting the handkerchief.

Activity 2 What will be left? [Num 4.2/2]

This is an activity for two. It is a predictive form of Num 4.1/2. This activity also appears as NuSp 1.4/2 in 'The Number Track and the Number Line' network.

- Materials*
- As in Num 4.1/2, except that:
 - Player A has Start cards 5-10.
 - Player B has Action cards 1-5.
 - Plasticine.

- What they do*
1. Player A makes a staircase 1-10.
 2. The cards are shuffled and put face down in the upper part of their spaces.
 3. Player A turns over the Start card, selects the rod of this number and puts it onto the number track.
 4. Player B turns over the top Action card. She now has to predict what will be left on the track when a rod of that number is taken away. She marks her prediction with a piece of plasticine.
 5. She then tests it by physically removing the appropriate number of cubes.
 6. The length left in the track is checked against the marker, and the number taken away against the Action card.
 7. Steps 2 to 5 are then repeated with roles interchanged.

Activity 3 Returning over the stepping stones [Num 4.2/3]

A game for 2, 3, or 4 children. Its purpose is to introduce the idea of subtraction as counting back.

- Materials*
- Game board.*
 - An ordinary 1-6 die and shaker.*
 - A marker for each player.*
- *These are the same as for Num 3.2/3.

Num 4.2 Subtraction as a mathematical operation (cont.)

- Rules of play*
1. Each player is represented by a marker. He starts on the island, and tries to get back to the bank.
 2. Players in turn throw the die, and may then move back that number of stepping stones.
 3. A player may not move to a stone which is occupied.
 4. If a player touches his marker, he must move it. If it turns out that this would take him to an occupied stone, he falls in the water, misses his next turn, and returns to the island.
 5. A player may decide not to move.
 6. The exact number must be thrown to reach the bank.
 7. The winner is the first player to return to the bank.

Activity 4 Crossing back [Num 4.2/4]

A number track activity for 2, 3, or 4 children. It will also be found as NuSp 1.4/3 in 'The number track and the number line' network..

What they do This is played on the same board as 'Crossing' (Num 3.2/4 in the 'Addition' network), and in a similar way. The difference is that they all begin in the finishing space, corresponding to 10; and end where it says 'Start.'

- Notes*
- (i) Since this game is a little harder than its predecessor, 'Crossing,' it may be wiser not to use the 'Move if touched' rule initially. But it should come in when the children have sufficient experience in order to make the game a predictive one.
 - (ii) If children find it confusing to begin where 'Finish' is written, you could make another board. But using the same board helps to show the inverse relation between adding and subtracting.

Discussion of activities As in the previous topic, these activities closely parallel those in the corresponding topic of Num 3; and for the same reason.

This is the next stage of developing subtraction as a mathematical operation: that is, as a mental activity which is done independently of physical action, and may be used to predict the results of action. A good way to learn this is by counting back with the help of finger counting. (Please see Num 1.4/1 and Num 1.5/1 for the recommended method of finger counting.) With practice, this is replaced by the use of known number facts.

Activity 3, 'Returning over the stepping stones,' is intended to emphasize the need to 'Look before you leap,' or calculate before you move. In the adult world, the result of miscalculation may be more serious.

The discussion in Num 3.2 may usefully be re-read at this stage.

OBSERVE AND LISTEN

REFLECT

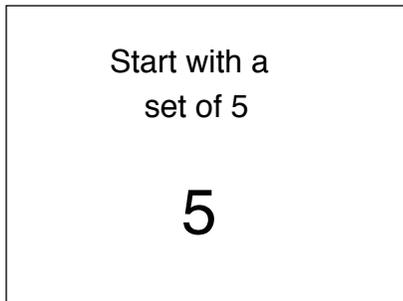
DISCUSS

Num 4.3 Notation for subtraction: number sentences (cont.)

*These are the same as for Num 4.1/1 and Num 4.2 /1.

**On side one is written “Write a number sentence to show what you did.” On side two is written “First write a number sentence showing what you predict the result will be. Then test your prediction.” (provided in the photomasters)

***The start and action cards are different. They now include the components of a number sentence, as shown here. (provided in the photomasters)



What they do (apportioned according to how many children there are)

1. The cards are shuffled and put face down in the upper parts of their spaces.
2. One child then turns over the top start card, and puts a set of the required number into the set loop.
3. Another child then turns over the top action card, and takes away the indicated number of objects. Then he finds the appropriate result card to show the number of the resulting set.
4. During steps 2 and 3, someone records the three stages (start, action, result) in the vertical notation for subtraction, as shown here. Initially this might be yourself, and afterwards one of the children.
5. Finally the number sentence which has been constructed is also written, and read aloud.

$$\begin{array}{r} 5 \\ - 2 \\ \hline 3 \end{array}$$

5	- 2	= 3
five	subtract two	result three

The vertical number sentence is also read aloud, and you explain that both mean the same.

five	5
subtract two	- 2
result three	3

6. The cards used are replaced at the bottom of the piles, and steps 2 to 5 are repeated.

Activity 2 Predicting from number sentences (subtraction) [Num 4.3/2]

An activity for 2 children, or 2 teams. Its purpose is to teach them to use written number sentences predictively.

Materials The same as for Activity 1, except that the result cards are replaced by pencil and paper for each child.

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces.
 2. Player A (or a player from team A) turns over the top Start card and the top Action card.
 3. Player A (or another from team A) then copies out the incomplete number sentence shown on the cards, e.g.,

$$7 \quad - \quad 4$$

and completes it, in this case

$$7 \quad - \quad 4 \quad = \quad 3$$

4. Player B (or a player from team B) meanwhile does the same calculation in the vertical notation; e.g.,

first
$$\begin{array}{r} 7 \\ -4 \\ \hline \end{array}$$
 and then
$$\begin{array}{r} 7 \\ -4 \\ \hline 3 \end{array}$$

5. These predictions are then compared. If they are different, no alteration is made at this stage.
6. Finally the prediction is tested by putting the required number of objects in the set loop, taking away the number of objects shown by the action card, and counting the result. At this stage if one of the written predictions was incorrect, it is corrected.

Discussion of activities

In Activity 1, the children are making the transition from ‘Do and say’ to ‘Do and record.’ At this stage purely mental calculations are as easy as (or easier than) those using pencil and paper. When the numbers get more difficult, written notation is an important help in keeping track of what we are doing. Initially, however, the recording itself is an additional task rather than a help, so it needs to be learned in a situation where the rest of what they have to do is familiar.

In Activity 2, recording is replaced by calculation. That is, the result of the pencil and paper work is used to predict a physical result. This is a major use of mathematics in the adult world, and at any age it is satisfying to find one’s prediction to be correct.

In both activities, horizontal and vertical notation are used alongside each other. Both have their advantages, and this helps children to learn another very general concept, that the same meaning can be expressed in more ways than one. Also, to acquire an important general ability – not to be put off if they see the same idea written in different ways.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 4.4 NUMBER STORIES: ABSTRACTING NUMBER SENTENCES

Concept Numbers and numerical operations as models for actual happenings, or for verbal descriptions of these.

- Abilities*
- (i) To produce numerical models in physical materials corresponding to given number stories, to manipulate these appropriately, and to interpret the result in the context of the number story: first verbally, then recording in the form of a number sentence.
 - (ii) To use number sentences predictively, to solve verbally given problems.

Discussion of concept The concept is that already discussed in Num 3.4, now being expanded to include subtraction. Since the idea of a mathematical model is of central importance, it will be well worth re-reading the discussion at the beginning of Num 3.4.

Activity 1 Personalized number stories [Num 4.4/1]

An activity for 2 to 6 children. Its purpose is to connect simple verbal problems with physical events, linked with the idea that we can use objects to represent other objects.

- Materials*
- Number stories of the kind in the example following. For some stories you will need two versions, one with pronouns for girls, the other with pronouns for boys. These can be on different sides of the same card.*
 - The name of each child on a card or slip of paper.
 - Two separate sets of number cards on different coloured cards:
 - Start cards 5-10,*
 - Action cards 1-5 (later 0-5).*
 - Slips of paper on which to write the results (to fit the spaces on the story board).
 - Objects such as bottle tops, pebbles, shells, counters.
 - Paper and pencils.
- * Provided in the photomasters

What they do (apportioned according to how many children there are).

1. A number story is chosen. The name, start, and action cards are shuffled and put face down.
2. The top name card is turned over and put in the number story.
3. Another child then turns over the top start card, and puts it in the appropriate space.

Num 4.4 Number stories: abstracting number sentences (cont.)

- Another child then turns over the top action card, and puts it in the appropriate space.
- The number story now looks like this.

Peter has 7 cherries on his plate.
He eats 3 of them.
Now he has <input type="text"/> cherries left on his plate.

As with Num 3.4, it may be helpful if you read this over for them.

- The named child then takes over, describing aloud what he is doing while the others check. (E.g.) “We haven’t any cherries, so we’ll use these beans instead.” (Puts down 7 beans.) “These are the 7 I start with.” (Takes away 3 beans.) “Now I’ve eaten 3. There are 4 left.”
- Finally another child writes the result on a slip of paper and puts it in place to complete the number story.

Activity 2 Abstracting number sentences [Num 4.4/2]

An extension to Activity 1 which may be included fairly soon. Its purpose is to teach children to abstract a number sentence from a verbal description.

- Materials*
- As for Activity 1, and also:
 - ‘Number sentence’ card as illustrated (provided in the photomasters).

<input type="text"/>	-	<input type="text"/>	=	<input type="text"/>
number sentence				

What they do 1-7 as for Activity 1.

- A child then says “Now we make a number sentence recording what happened.” He puts the number sentence card below the story card, then moves the three number cards from the story card to the ‘number sentence’ card as shown:

7	-	3	=	4
number sentence				

He reads aloud “Seven, subtract three, result four.”

Num 4.4 Number stories: abstracting number sentences (cont.)

9. Finally every child writes this in both horizontal and vertical notations.

$$7 - 3 = 4$$

$$\begin{array}{r} 7 \\ -3 \\ \hline 4 \end{array}$$

Note The word ‘subtract’ is used in step 8 because we are here referring to the mathematical operation, not the physical action.

Activity 3 Personalized number stories – predictive [Num 4.4/3]

An activity for 2 to 6 children. It combines Activities 1 and 2, in predictive form.

- Materials*
- Number stories of the new kind in the example below. These now require a prediction.**
 - The name of each child on a card or slip of paper.*
 - Two sets of number cards: start cards 5-10, and action cards 1-5 (later 0-5).*
 - Objects such as bottle tops, pebbles, shells, counters . . . *
 - Slips of paper on which to write the result.
 - Slips of paper to fit answer space.*
 - Pencil and paper for each child.

*These are the same as for Activity 1.

** Provided in the photomasters

What they do (apportioned according to how many children there are)

1. A number story is chosen. The name, start, and action cards are shuffled and put face down.
2. The top name card is turned over, and put in the appropriate space in the number story.
3. Another child then turns over the top start card, and puts it in the appropriate place.
4. Another child then turns over the top action card, and puts it in the appropriate space.
5. The number story now looks like this.

<p>Frances has 7 cherries on her plate. She eats 3 of them. How many cherries will be left on her plate? Answer: <input type="text"/> cherries.</p>
--

6. The named child now has to answer the question by writing and completing a number sentence, explaining as she does so. E.g.,

<i>Says</i>	<i>Writes</i>
“I have 7 cherries.”	7
“I eat 3.”	7 – 3
“7, subtract three, result . . .”	7 – 3 = 4

“So I’ll have 4 cherries left on my plate.”

7. The named child then writes the resulting number on a slip of paper and puts it in the appropriate space to answer the question in the number story.
8. Meanwhile the other children write the same number sentence in vertical notation, as in step 9 of Activity 2.
9. Finally one of the other children tests the prediction by using the physical materials, describing what she is doing while the others check. E.g., (with appropriate actions). “I’m using these bottle tops in place of cherries. These are the 7 she started with. Now she eats 3. These are the ones left: 4. So, Frances gave the correct answer.”

Discussion of activities

The activities in this topic parallel those in Num 3.4. The differences are (i) that subtraction replaces addition; that the third and fourth activities in Num 3.4 are here condensed into a single activity, Num 4.4/3. It will therefore be useful at this stage to re-read the discussion of activities at the end of Num 3.4.

Activity 1 is for building the concept of subtraction by mode 1, physical experience. It also uses again the idea that something can be used to represent something else – the concept of modelling.

Activity 2 also uses mode 1, in this case for building the concept of abstracting – in this case physically ‘drawing out’ the number sentence from the number story.

Activity 3 combines what has been learned in Activities 1 and 2, and also the following further steps:

- (i) From schema building to schema testing. A number sentence is used to make a prediction, which is then tested.
- (ii) The mathematical operation, in this case subtraction, is now done *before* the action, and so becomes independent of action.
- (iii) This involves a change from recording something which has just been done physically to putting thoughts on paper. These thoughts are the operation of subtraction.

Once again, there is more here than is immediately apparent. And when we do look below the surface and analyze what is involved, once again we realize how much it is that we are expecting children to learn; and hence, the importance of providing them with the right learning situations and materials.

Num 4.5 NUMERICAL COMPARISON OF TWO SETS

- Concepts.*
- (i) Numerical difference between two sets, combined with the relationship ‘is greater than’ or ‘is smaller than’: e.g., “This set is 3 greater than that set.”
 - (ii) Difference between two numbers, as in (i).

- Abilities.*
- (i) To be able to say which of two sets, or two numbers, is the larger; and by how many.
 - (ii) To express this comparison in the alternative way, i.e., which is the smaller, and by how many.
 - (iii) To be aware of the equivalence of these two statements.
 - (iv) To use the general notation for subtraction.

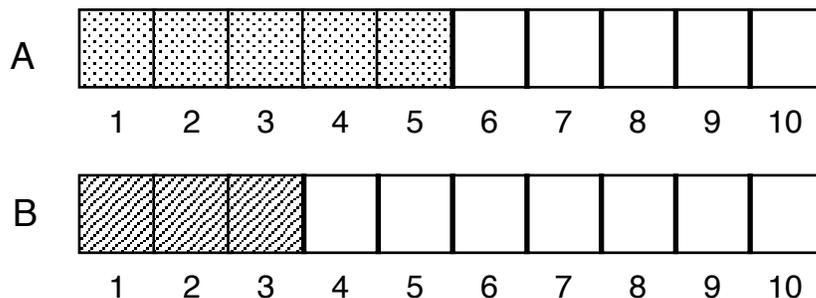
Discussion of concepts In this topic and the two which follow, we introduce contributors to the overall concept of subtraction which are quite different from that derived from ‘taking away.’ These do not involve any kind of taking away, but are varieties of comparison, of which the present one is the simplest.

Activity 1 Capture [Num 4.5/1]

This is a game for two. Its purpose is to introduce the comparison aspect of subtraction. This will also be found as NuSp 1.4/4 in ‘The number track and the number line’ network.

- Materials*
- 2 number tracks 1-10.*
 - 1 die 1-6.
 - 10 cubes for each player, a different colour for each.
- * Provided in the photomasters

- What they do*
1. The two players sit side by side and their number tracks are placed in front of them, parallel and with the 1s lined up.
 2. Each player throws the die, and puts the number of cubes indicated on the track. The result might look like this:



Since A has filled two more spaces than B, B must give A two cubes.

3. The cubes are taken off the track.

4. Both players throw again, and the process is repeated. Captured cubes may not be used to put on the track, but may be used if cubes have to be given to the other player.
5. The game finishes when either player has had all his cubes captured, or cannot put down what is required by the throw of the die.
6. The other player is then the winner.

Variation This game could be played without number tracks. Their use however has the advantage of showing numerals alongside the rods. This links the activity with the symbols used in number sentences.

Activity 2 Setting the table [Num 4.5/2]

An activity for six children. Its purpose is to introduce numerical comparison of two sets in a practical everyday situation.

- Materials*
- Plastic knives, forks, spoons, plates, cups, saucers. There should be at least 8 of each, and not all the same number.
 - Six cards, on which are written '4 knives,' '2 plates,' etc. Every number from 1 to 6 should be used.*
- * Provided in the photomasters

- What they do*
1. The cards are shuffled and put face down.
 2. The first child turns over the top card, e.g., '4 knives,' and takes 4 knives which she puts in front of her.
 3. The other children in turn do likewise, turning over the top card and taking the given number of utensils. This concludes the first round.
 4. To begin the second round, the first child says (e.g.) "We are 6 children and there are only 4 knives, so we need 2 more." So he takes 2 more knives.
 5. The other children in turn do likewise.
 6. Finally they set the table. If their subtractions have been correct, each child should have one utensil of each kind, six in all.

Note The description here is for six children. Fewer may play, in which case each takes a card, but some cards will not be used.

Activity 3 Diver and winner [Num 4.5/3]

A game for two children. Its purpose is to use numerical comparison at a mental level, to achieve goals in a physical situation.

- Materials*
- A model of a salvage boat as shown in Figure 8, complete with diver. The diver and his rope* are made so that he can be pulled up or lowered down. The depth scale at the side is in fathoms [double-metres], and can be hidden by a flap. A harder version may also be provided which covers all the water, including the diver. This corresponds more closely to the actual situation.
 - A 1-9 die and shaker.
- * The diver and rope may be conveniently drawn using a permanent marker on a strip of acetate cut from an overhead projector transparency.

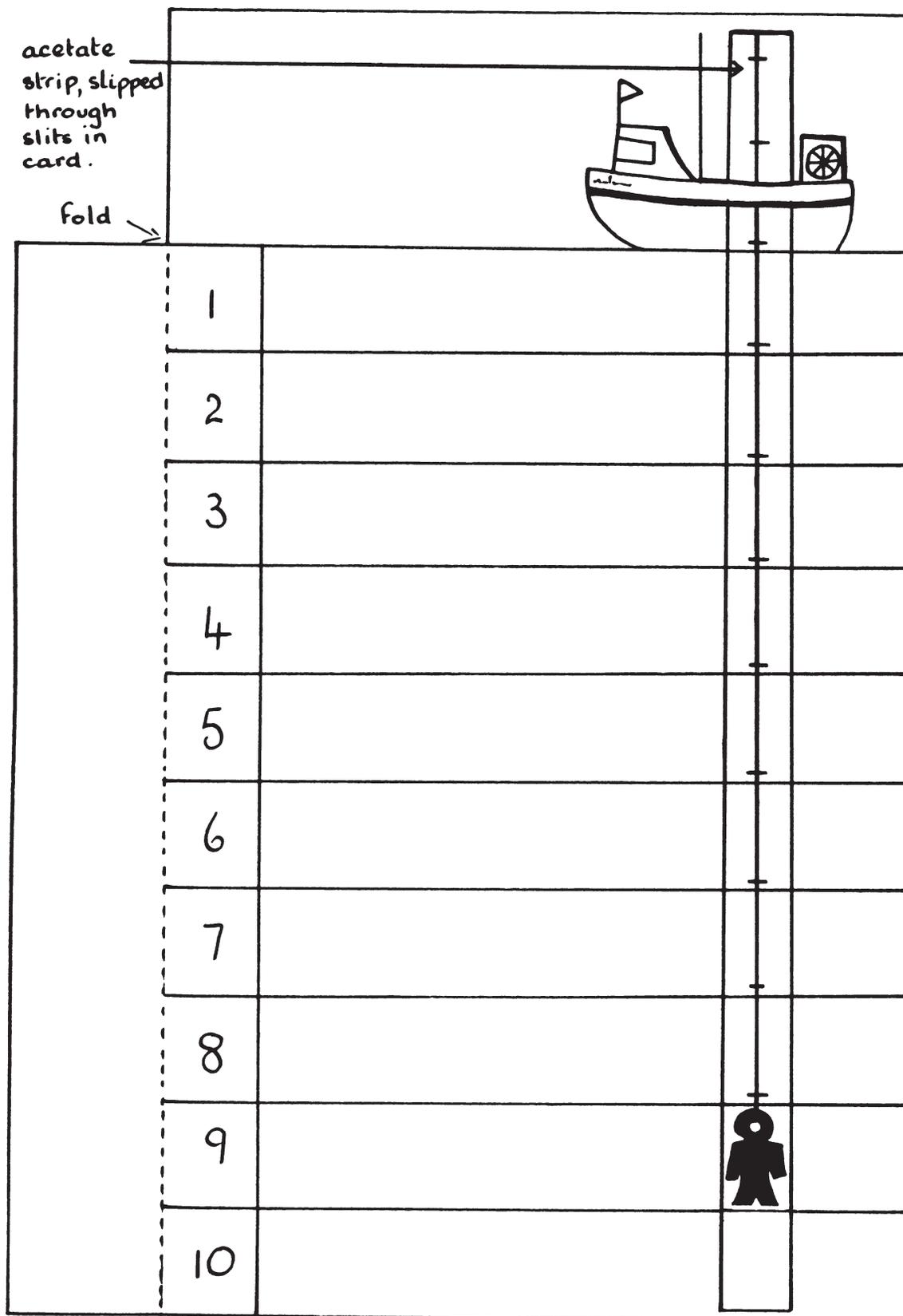


Figure 8 Diver and wincher

- Rules of play*
1. One child is the diver; the other is the wincher on the boat, who winches her up and down as requested.
 2. The diver throws the die and starts at the level indicated: say, level 3. The depth scale is then covered.
 3. The diver throws again, to decide her next level: say, level 7. To get there she calls to the wincher "I'm at level 3 and I want to go to 7. Please give out 4 fathoms [double-metres]."
 4. The wincher does this, counting aloud "1, 2, 3, 4 fathoms; there you are."
 5. The diver then checks by uncovering the depth scale. (You can explain that she has a depth gauge on her wrist, operated by water pressure.) The depth gauge is then covered again.
 6. The diver throws again, say 2. She calls "I'm at level 7 and I want to get to 2. Please haul in 5 fathoms."
 7. The wincher does this, counting as before "Hauling in, 1, 2, 3, 4, 5 fathoms; there you are."
 8. The diver checks her depth.
 9. They continue thus for a while, and then change occupations.

Note A fathom is roughly two metres, and is a measure still used for measuring depth by seamen. For this game a metre is too small a unit.

Activity 4 Number comparison sentences [Num 4.5/4]

An activity for 2 children. Its purpose is to teach two ways of writing a number sentence about the difference between two numbers, and to help them to realize that these are equivalent.

- Materials*
- Instruction board (see Figure 9).*
 - 3 packs of number cards 1-10:
 - 2 single headed, of the same colour
 - 1 of a different colour which is double headed (see Figure 9) *
 - 20 objects for counting, e.g., shells, cubes, bottle tops . . .
 - Pencil and paper for each child.
- * Provided in the photomasters

- What they do*
1. The children sit opposite each other with the instruction board between them. At this stage, the flap at each end is turned under, so that it is not yet visible. Each has a single-headed number pack.
 2. The double-headed number pack is put between them, face down. Each child turns over a card from this pack, and they agree which spaces on the instruction card to put them in.
 3. Each then follows the instructions facing him. (First time through, the teacher may read these with the children following.) The number for the difference is taken from their separate (single-headed) packs. They arrive at this answer by whatever method they like: e.g., by using the counting objects, by counting on or counting back with or without the help of finger counting.

larger number — smaller number = difference

like this :

Write a subtraction sentence which means the same,

the other number sentence underneath yours.

Now carefully turn the board around and copy

Copy this number sentence on your paper.

number? Put your answer here.

How many less are there in the smaller

larger number

smaller number

is less than

is greater than

diff.

diff.

difference

difference

How many more are there in the larger number? Put your answer here

Copy this number sentence on your paper.

Now carefully turn the board around, and copy the other number sentence underneath yours.

Write a subtraction sentence which means the same, like this :

larger number — smaller number = difference

Figure 9 Number comparison sentences

4. To prevent one from simply copying the other, the cards for the difference may be put face down initially and turned over when both are down.
5. They should finish having written something like this on their papers.

$$\begin{array}{ccc} 7 > 2 \text{ diff } 5 & \text{or} & 7 \text{ is greater than } 2 \text{ diff } 5 \\ 2 < 7 \text{ diff } 5 & & 2 \text{ is less than } 7 \text{ diff } 5 \end{array}$$

6. If they disagree, they discuss and if necessary use physical objects.
7. The board is cleared, turned around, and steps 1-4 are repeated. The double headed number cards should be replaced randomly in the pack.

Activity 5 Subtraction sentences for comparisons [Num 4.5/5]

An activity for two children. This is an extension of Activity 4, to be included when you judge that the concepts of Activity 4 are well established. Its purpose is to establish the comparison of numbers as one kind of subtraction, using the general notation for subtraction.

- Materials*
- As for Activity 4, except that on the instruction board the hinged flap at each end, previously turned under, is now brought into view. It reads:

Write a subtraction sentence which means the same, like this:

$$\text{larger number} - \text{smaller number} = \text{difference}$$

- What they do*
- 1-5 the same as in Activity 4.
 6. They then follow the new instructions, finishing with something like this:

$$\begin{array}{ccc} 7 > 2 \text{ diff } 5 & \text{or} & 7 \text{ is greater than } 2 \text{ diff } 5 \\ 2 < 7 \text{ diff } 5 & & 2 \text{ is less than } 7 \text{ diff } 5 \\ & & 7 - 2 = 5 \end{array}$$

7. They should then read all three sentences aloud:

“7 is greater than 2, and the difference is 5.”

“2 is less than 7, and the difference is 5.”

“7 subtract 2, equals 5.”

Discussion of activities Since this topic introduces the comparison aspect of subtraction, which is quite different from the taking-away aspect, there are 5 activities to introduce and consolidate this concept.

The first is a number track activity, using length as a clearly visible difference between the sets. Moreover, the sets to be compared are both sets of the same objects, in this case cubes. Here we have mode 1 schema building.

Where the sets are of different objects, number is not the most obvious way to compare. For example, if we compare a set of 6 children and a set of 4 knives, the difference between children and knives is much more obvious (and more important!) than that between 6 and 4. Once again we note how abstract mathematics is compared with other everyday thinking. In this activity we use a common daily occurrence in which the difference between children and tableware is taken for granted, and the goal of one-to-one correspondence (each child has one of each object) a fairly obvious one. Unlike Activity 1, the comparison is made mentally and the outcome is used for prediction – mode 1 schema testing.

Activity 3 is a straightforward embodiment of the number track concept, numerically comparing present position and desired position. Prediction is again involved.

Activity 4 has two purposes. The first is to introduce recording. The second is thereby to make fully conscious, and crystallize, an idea which may already be present intuitively. This is, that if set A is numerically greater than set B, then set B is less than set A. And the number by which A is greater than B is the same as the number by which B is less than A.

Stated verbally, this seems long-winded. The mathematical notation says it in just 5 symbols. This is the new line introduced by Activity 5. Looked at the other way, we see again how condensed the mathematical statement is, and how necessary to build up its interiority by a variety of activities. This final line relates the comparison aspect of subtraction to the notation already in use for the take-away aspect. It thus begins the process of combining these into a single concept.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 4.6 GIVING CHANGE

Concept Paying a required amount by giving more and getting change.

- Abilities*
- (i) To give the correct change.
 - (ii) To check that one received the correct change.

Discussion of concepts This is another contributor to the comparison aspect of subtraction. In this case, the larger number is the amount tendered, the smaller number is the cost of the purchase, and the difference is the change.

Activity 1 Change by exchange [Num 4.6/1]

An activity for 3 or 4 children (not more). Its purpose is to ‘spell out’ with the coins themselves what is happening when we give or receive change.

- Materials*
- Play money: 30¢ for each customer made up as in step 2, following; 1¢ and 5¢ coins for shopkeeper.
 - A ‘till’ (tray with partitions).
 - Pictures on cards representing objects for sale, with prices marked, all less than 10¢.

- What they do*
1. One child acts as shopkeeper, the rest as customers.
 2. The customers start with 30¢, made up of two 10¢ coins, one 5¢ coin, and five 1¢ coins. The shopkeeper has a plentiful supply of 1¢ and 5¢ coins.
 3. The shopkeeper sets out her wares. If there is not enough table space for all the goods, some may be kept ‘in the stock room’ and put out later.
 4. The customers in turn make their purchases one at a time.
 5. To start with, they pay with exact money. When they no longer have the exact money for their purchases, they pay by giving more and getting change.
 6. Suppose that a customer asks for a 6¢ apple, and hands a 10¢ coin to the shopkeeper.
 7. The shopkeeper says “I have to take 6¢ out of this, so I need to exchange it.” She puts the 10¢ coin into her till and takes out 10 pennies. (With experience, a combination of 5¢ and 1¢ coins will be used.)
 8. Spreading these smaller coins out, she then says “I’m taking this 6¢ for your apple” and does so. “The rest is your change: 4 cents.”
 9. The shopkeeper gives the apple and the 4¢ change to the customer, who checks that she has received the right change.

Activity 2 Change by counting on [Num 4.6/2]

A continuation from Activity 1, for 3 to 6 children. (May be included or by-passed, at your discretion.) Its purpose is to relate the method of giving change which children will often have encountered in shops to its mathematical meaning, by putting it between Activities 1 and 3.

Materials • The same as for Activity 1.

What they do 1-4 are the same as in Activity 1.

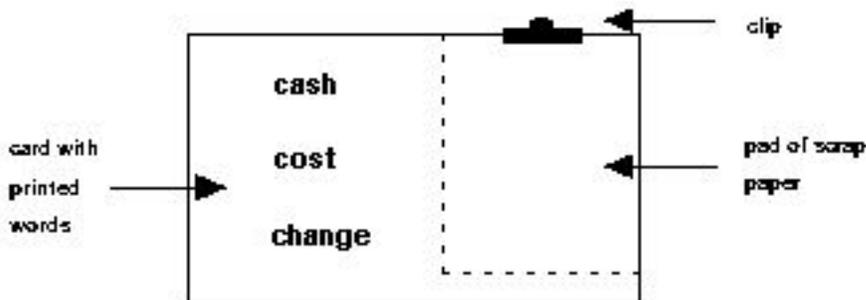
5. The method of giving change is now different. Assume as before that the customer has handed a 10¢ coin to the shopkeeper for a 6¢ apple. The shopkeeper goes to the till and picks up coins to make the total up to 10¢, saying to herself “7, 8, 9, 10.”
6. She says to the customer “6¢ for your apple,” and then while counting the change into the customer’s hand “7, 8, 9, 10.”

Note that with this method, the amount of the change is not explicitly stated or written.

Activity 3 Till receipts [Num 4.6/3]

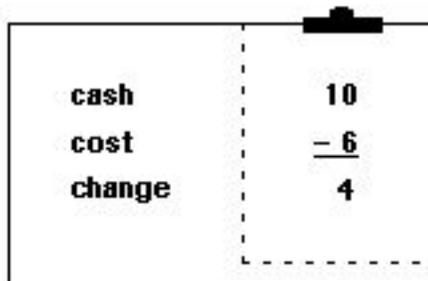
A continuation from Activity 2, for 3 to 6 children. Its purpose is to relate the kind of subtraction involved in giving change (comparison) to the conventional notation for subtraction.

Materials • The same as for Activities 1 and 2, and also
 • A pad of till receipts. (See illustration below.)



What they do 1-4 are as in Activity 1.

5. Having arrived at the right change by any means she likes, the shopkeeper then writes a till receipt for the customer.
6. She shows it to the customer like this, before removing it from the pad and handing it to the customer.



7. This is what the customer receives, together with her purchase and change.

$$\begin{array}{r} 10 \\ - 6 \\ \hline 4 \end{array}$$

Discussion of activities Once again, there is more here than meets the eye. The counting on method for giving change, as usually practised in shops, produces the correct change and allows the customer to check. But it does not say in advance what amount this will be, nor does it lead to subtraction on paper.

So in Activities 1, 2, 3, we have a sequence. In Activity 1, the emphasis is on the concept itself of giving change, using the simplest possible way of arriving at the amount. Note that the customers begin with assorted coins, so that the activity does not begin with giving change, but with paying in the direct way. Giving more and receiving change is then seen as another way of paying the correct amount. We found that when this approach was not used, some children continued to give change even when the customer, having collected the right coins by receiving change, had then paid the exact amount! This shows how easily habit learning can creep in instead of understanding, and also how important it is to get the details right in these activities. Activity 2 uses counting on as a method for first producing the correct change, and then allowing the customer to check. Finally, Activity 3 transfers this to paper and makes explicit the amount of change which the customer should receive. It also relates this new aspect of subtraction to the notation with which children are already familiar. This helps to relate it to the overall concept of subtraction.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 4.7 SUBTRACTION WITH ALL ITS MEANINGS

Concept Subtraction as a single mathematical operation with 4 different aspects.

Ability To relate the overall concept of subtraction to any of its embodiments.

Discussion of concepts In this topic we are concerned with recapitulating the 4 earlier aspects of subtraction: taking away, comparison, complement, and change. Finally, in Activity 5, these are fused together into a concept of subtraction from which can be extracted all of these particular varieties.

Activity 1 Using set diagrams for taking away [Num 4.7/1]

A teacher-led discussion for a small group. Its purpose is to relate the take-away aspect of subtraction to set diagrams.

Materials • Pencil and paper for all.

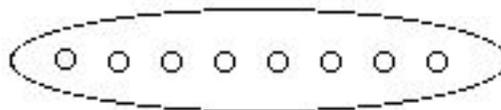
Suggested sequence for the discussion

1. Write on the left of the paper:

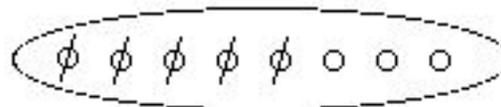
$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$

2. Say and draw (pointing first to the 8, then the 5):

“This says we start with 8.”

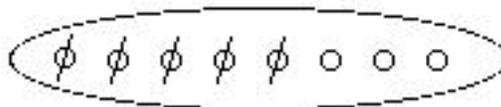


“This says we take away 5.”
Cross out the 5 to be taken away.



3. Write the result, 3.
4. Review the correspondences between the number sentence and the starting set, the action (crossing out), and the result.

$$\begin{array}{r} 8 \\ - 5 \\ \hline 3 \end{array}$$



5. Let the children repeat steps 1 to 4 with another example. Use vertical notation, as above.
6. Give further practice if needed.

Activity 2 Using set diagrams for comparison [Num 4.7/2]

A teacher-led discussion for a small group. Its purpose is to relate the comparison aspect of subtraction to set diagrams, and thereby to what they have just done.

Materials • Pencil and paper for all.

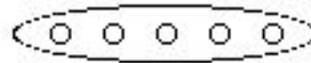
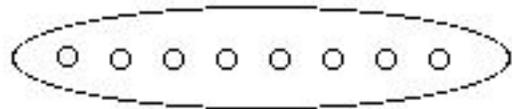
Suggested sequence for the discussion

1. Write as before.

$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$

2. Say, "This subtraction can have another meaning, besides taking away." Here we have two numbers, the larger one above.

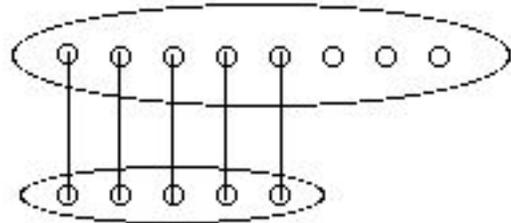
3. Draw these on the right of the subtraction sentence.



4. Ask (pointing): "How many more are there in this set, than this?"

5. If they answer correctly, say "Let's check."

Draw lines like this and say (pointing) "These 5 lines show where the sets are alike, so these 3 without lines show where they are different."

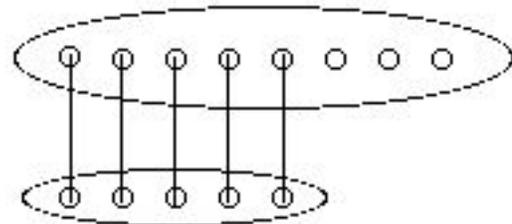


6. If they do not answer correctly, use step 5 to show how they can find the result.

7. Either way, write the result in the subtraction sentence.

8. Review the correspondences between the number sentence, the two sets, the action (comparison), and the result.

$$\begin{array}{r} 8 \\ - 5 \\ \hline 3 \end{array}$$



9. Explain: "We haven't been taking away, so we shouldn't read the number sentence as 'take away.' We say "8, subtract 5, result 3."

10. Give further practice, as required. Use vertical notation only.

11. Say, "Now we have 2 meanings for this subtraction sentence." Review these.

Num 4.7 Subtraction with all its meanings (cont.)

Activity 3 Using set diagrams for finding complements [Num 4.7/3]

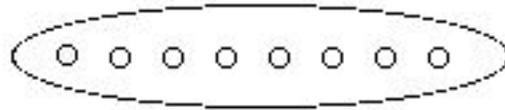
A teacher-led discussion for a small group. Its purpose is to relate the complement aspect of subtraction to set diagrams, and thereby to what they did in Activities 1 and 2. You may wish to review Num 3.5, ‘Complementary numbers,’ Activities 1 and 2, in the Addition network.

- Materials*
- Pencil and paper for all.
 - Red and blue felt tips.

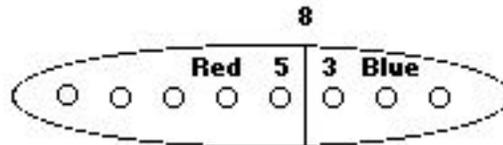
Suggested sequence for the discussion

1. Write, and draw in pencil.

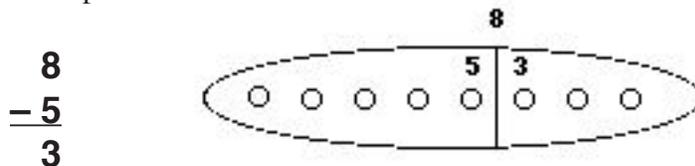
$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$



2. Say, “Here is another meaning. We’re told that these are all to be coloured red or blue. If 5 are coloured red, how many blue?”
3. If (as we hope) they say “Three,” check by colouring. If not, demonstrate.
4. Say, “If we didn’t have red and blue felt tips, what could we do instead?”
5. Accept any sensible answers, and contribute the suggestion below. Tell them that they may continue to use their own way if they like.



6. Invite other meanings, e.g., 8 children, 5 girls, how many boys?
7. Review the correspondences between the number sentence, the whole set, and the two parts of the set.



8. Remind them that the larger number has to be above. This time the larger number is the whole set, the next number is one part, and the last number is the other part.
9. Give further practice, as required. Use vertical notation only.
10. They now have 3 meanings for the subtraction sentence. Review these.

Activity 4 Using set diagrams for giving change [Num 4.7/4]

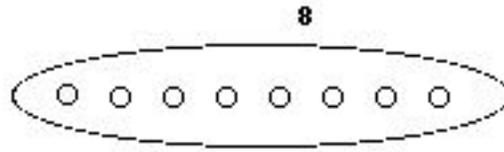
A teacher-led discussion for a small group. Its purpose is to relate the ‘cash, cost, change’ aspect of subtraction to set diagrams, and thereby to what they did in Activities 1, 2, and 3.

- Materials*
- Pencil and paper for all.

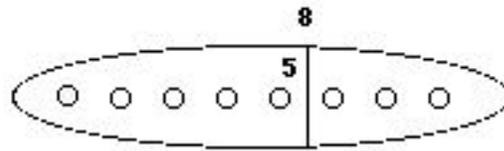
Suggested sequence for the discussion

1. Write, on the left of the paper:

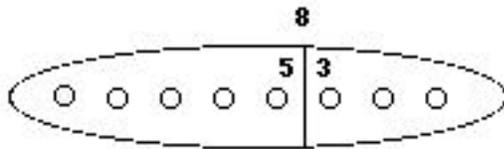
$$\begin{array}{r} 8 \\ - 5 \\ \hline \end{array}$$
2. Say, "There's just one more meaning we can give this. Suppose you are a shop-keeper, and a customer gives you 8¢ for an apple. But the apple only costs 5¢. What money will you give him back?"
3. Assuming that they answer correctly, say, "Yes. Now let's check."
4. Draw.



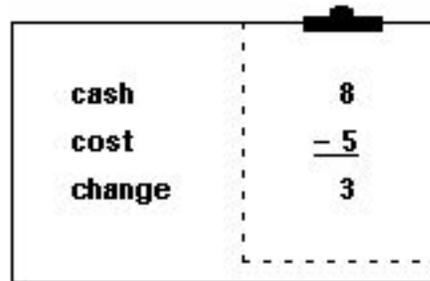
5. Say, "These are the 5 pennies for the apple," and draw the partition line. Write the 5 inside.



6. Continue: "And so these are the pennies you give back to the customer." Point to the right-hand sub-set and write the 3 inside.



7. Relate the foregoing to 'Cash, cost, change.' (Num 4.7/3).



8. Give further practice, as required.
9. Review all 4 of the meanings they now have for subtraction.

Activity 5 Unpacking the parcel (subtraction) [Num 4.7/5]

A game for up to 6 children. Its purpose is to consolidate children's understanding of the 4 different aspects of subtraction.

- Materials*
- 'Parcel' cards, of two kinds, as illustrated in Stage (a) and Stage (b).*
 - A bowl of counters.

* Provided in the photomasters

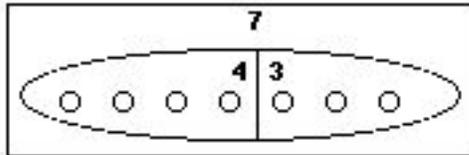
Rules of the game Stage (a)

$$\begin{array}{r} 7 \\ - 3 \\ \hline 4 \end{array}$$

1. The first set of parcel cards is put face down, and the top one turned over. (Reminder: this is read as “7, subtract 3, result 4,” NOT as “7, take away 3 . . .”)
2. Explain that this has a number of different meanings which can be ‘taken out,’ one at a time, like unpacking a parcel.
3. The children take turns to give one meaning. If the others agree, the child whose turn it is takes a counter.
4. There are four different mathematical meanings, as in Activities 1, 2, 3, and 4.
5. An unlimited number of situational meanings can also be found, and these can become repetitive. E.g., if someone says, “7 boxes, 3 empty, so 4 have something in them,” and someone else then says, “7 cups, 3 empty, so 4 have something in them,” this is so little different as to be hardly worth saying. If the rest of the group unanimously think that an example is of this kind, they might reject it even though correct. This might lead to discussion as to what is acceptable as a genuinely different meaning.

Stage (b)

1. This is played in the same way as Stage (a), except that the second set of parcel cards is used, like this one:



These have even more possible meanings, and the children should write these down before expanding them further. E.g.:

$$\begin{array}{r} 4 \\ + 3 \\ \hline 7 \end{array} \qquad \begin{array}{r} 7 \\ - 4 \\ \hline 3 \end{array} \qquad \begin{array}{r} 7 \\ - 3 \\ \hline 4 \end{array}$$

2. We thus have ‘parcels within parcels.’ This activity involves much concentration of mathematical meaning, and children should return to it at intervals until all 4 aspects are mastered. Children find the part-whole relationships harder than the take-away and comparison aspects of subtraction.

Discussion of activities In this topic we have a good example of the highly abstract and concentrated nature of mathematical ideas. Hence its power, but hence also the need for very careful teaching.

In the topics which lead up to this one, the 4 different aspects of subtraction are introduced separately, with the use of materials to provide a less abstract approach. In the present topic these are brought together by using set diagrams, which again provide a less abstract symbolism than the purely numerical symbols whose use, with full understanding, is the final learning goal.

In Activity 5, the children are learning explicitly something about the nature of mathematics, namely its concentration of information. We ourselves have been taking notice of this from the beginning.

Num 4.8 SUBTRACTION OF NUMBERS UP TO 20, INCLUDING CROSSING THE TEN BOUNDARY

Concept Expansion of the subtraction concept to include larger numbers.

Ability To subtract numbers up to 20, including examples which involve crossing the 10 boundary.

Discussion of concepts ‘Crossing the ten boundary’ means calculations like $12 - 3$, $14 - 6$, $17 - 9$. All subsequent examples which involve regrouping, such as $52 - 4$, $82 - 36$, $318 - 189$, depend on this.

Over the years there has been much discussion whether children should be taught to subtract by decomposition or complementary addition. The argument for decomposition has been that it can be demonstrated with physical materials, and so is better for teaching with understanding. For complementary addition, it has been claimed that it is easier to do, and makes for faster and more accurate calculations. The latter can also be justified sensibly, though it seldom is: ‘borrowing’ and ‘paying back’ is a nonsensical explanation.

The present approach is based on the physical regroupings which the children have learned in Org 1, and the concept of canonical form which comes in both Org 1 and Num 2. It has several advantages:

(i) It is mathematically sound.

(ii) It allows children to use whichever technique they find easier: counting back, corresponding to the take-away aspect of subtraction, or counting on, corresponding to complementation. Both of these they have already experienced with physical materials. Taking away across the ten boundary involves decomposing one larger group into 10 smaller groups; complementing involves putting together 10 smaller groups into one larger group. Either way, regrouping and canonical form are the key concepts.

(iii) The same technique (changing out of or into canonical form) is good for adding, multiplying, and dividing.

The present topic is preparatory to the full technique, which follows in Num 4.9. It teaches what we do after changing out of canonical form.

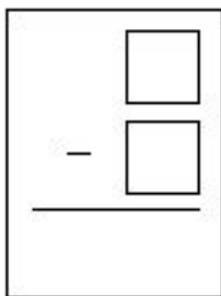
Activity 1 Subtracting from teens: choose your method [Num 4.8/1]

A teacher-led activity for up to 6 children. Its purpose is to show them two ways of subtracting across the tens boundary, help them to see that these are equivalent, and choose which method they prefer.

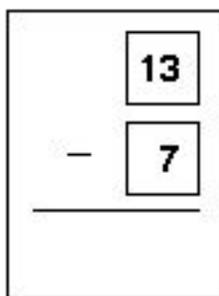
- Materials*
- Two sets of number cards, in different colours. One set is from 10-19, the other is from 0 to 9.*
 - Subtraction board, as illustrated.*

* Provided in the photomasters

Num 4.8 Subtraction of numbers up to 20 (cont.)



- What they do*
1. Before starting, they should review finger counting, including ‘Ten in my head.’ (See Num 1.4/1, Num 1.5/1 and Num 1.7/1.)
 2. The subtraction board is put where all the children can see it the same way up. Both sets of cards are shuffled and put face down, near the board, with the teens set on the left.
 3. The top card from each pack is turned over, and put one in each space on the board to give (e.g.)



4. Demonstrate the two ways of doing this using finger counting.
 - (a) Counting back. “We start at 13 and count back to 7, putting down one finger each time. 12 (one finger down), 11, 10, 9, 8, 7 (six fingers down). The difference is 6.” They all do this.
 - (b) Counting on. “We start at 7 and count on to 13, putting down one finger each time. 8 (one finger down), 9, 10, 11, 12, 13 (six fingers down). The difference is 6.” They all do this.
5. Note that
 - (i) We use the word ‘difference’ in both cases to link with this aspect of subtraction
 - (ii) We do *not* put down a finger for the starting number.
 - (iii) This method gives (as intended) a mixture of examples in which some do and some do not involve crossing the tens boundary.
6. With another pair of numbers, half the children arrive at the difference by counting back, and half by counting on. They should all have down the same number of fingers.
7. Step 5 is repeated until the children are proficient.
8. Tell them that having tried both, they may use whichever method they prefer from now on. They may like to discuss the reasons for their preference.

Activity 2 Subtracting from teens: “Check!” [Num 4.8/2]

A game for 4 or 6 children, playing in teams of 2. Its purpose is to give them fluency in subtracting across the tens boundary.

- Materials*
- Number cards 10-19 and 0-9.*
 - Subtraction boards.*
 - A bowl of counters.
- *The same as for Activity 1.

- What they do*
1. A subtraction board is put where all can see it the same way up. Both sets of cards are shuffled and put near the board, with the ‘teens’ set on the left.
 2. In the first team, each player turns over the top card from one of the packs and puts it on the board, the teens card being on the left.
 3. The two players then do the subtraction independently by any method they like.
 4. Another player says, “Ready? Check!”
 5. On the word “Check,” both players immediately put fingers on the table to show their results. No alteration is allowed.
 6. In some cases the result will be over 10. Example: $15 - 2$. Both players should now put down 3 fingers, saying “10 in my head.” (See Num 1.8/1.)
 7. The other players check, and if both have the same number of fingers on the table (and the others agree that this is the correct answer), this team takes a counter.
 8. Steps 2 to 6 are repeated by the next team.
 9. The game continues as long as desired, the number cards being shuffled and replaced when necessary. All teams should have the same number of tries.
 10. The winners are those with the most counters.

Activity 3 Till receipts up to 20¢ [Num 4.8/3]

A continuation from Num 4.6/3, for 3 to 6 children. Its purpose is to consolidate their new skill in a familiar activity.

- Materials*
- Play money. The customers each have four dimes as well as one nickel, and three pennies. The shopkeeper has a good assortment of all coins.
 - A tray with partitions, used as a till.
 - Pictures on cards representing objects for sale with prices marked, ranging from (say) 3¢ to 19¢.
 - Base 10 material, units and ten-rods.

- What they do*
- 1-7. The same as in the earlier version of ‘Till receipts’ (Num 4.6/3), except that the prices range all the way up to 19¢.
 8. Customers may now purchase several objects at a time, provided only that the total is below 20¢.
 9. At any time when children have difficulty in writing the till receipts or checking them, they should help themselves by using base 10 material as in Activity 1. They may also use the counting on method of Num 4.6/2.

Num 4.8 Subtraction of numbers up to 20 (cont.)

Activity 4 Gift shop [Num 4.8/4]

A game, continuing on from the previous activity, for 3 to 6 children.

Materials The same as for Activity 3, together with a ‘notice’* as illustrated in step 9. Its purpose is further to consolidate their new skills, and extend these to subtraction other than from multiples of 10.

* Provided in the photomasters

Rules of play 1-8. The same as in Activity 3.

9. However, the shopkeeper also displays a notice:

**YOUR PURCHASE FREE
if I give the wrong change.**

10. If a customer thinks he has received incorrect change, the other customers also check. If it is agreed that the change was incorrect, the shopkeeper must give the customer his purchase free. The cash is returned to the customer, and the change to the shopkeeper.
11. If this happens 5 times the shopkeeper goes broke, and someone else takes over the shop. (The number of mistakes allowed to the shopkeeper should be adjusted to the children’s ability.)
12. To make things more difficult for the shopkeeper, customers may pay with whatever amounts they like. E.g., they could hand over 17¢ for an object costing 9¢. (This rule should not be introduced until the children have learned the rest of the game.)

Discussion of activities

It will be noticed that the activities of this topic do not begin with the use of physical materials, in spite of the importance which in general we attach to these. Base ten material is good for teaching the exchange of 1 ten for 10 ones, but they already have plenty of experience of this. It is also good for teaching conversion into and out of canonical form, and this too is done in earlier contributors to the present topic. It lends itself well to teaching subtraction in its ‘take-away’ form, but not nearly so easily to the comparison and complementation forms. The latter are easier to do mentally, since counting forward is easier than counting back. So the approach in this topic relies on the foundations laid by Mode 1 schema-building in earlier topics, and uses finger counting as a transitional technique which applies equally well to either aspect of subtraction. This will fall into disuse as children gradually learn, and use for subtraction, their addition facts.

These are followed by the application of these new techniques in familiar activities. The last activity, ‘Gift shop,’ introduces a penalty for the shopkeeper if he makes too many mistakes, and a reward for the customer who detects a mistake. It is not only in this game that a shopkeeper who cannot do his arithmetic finds himself in difficulties!

Num 4.9 SUBTRACTION UP TO 99

Concept Their existing concept of subtraction, expanded to include subtraction of two-digit numbers.

Abilities (i) To subtract two-digit numbers.
(ii) To apply this to a variety of situations.

Discussion of concept

So far as the concept itself is concerned, all that is new is the size of the numbers to which the operation is applied. However this requires the introduction of new techniques, and it is important that the manipulations of symbols which children learn at this stage should be meaningful in terms of the underlying mathematics.

The method we recommend has been reached by much thought, discussion, and field trials with children. It begins in Num 4.8, and continues here. However, rather than split the discussion, the whole of it was given at the beginning of Num 4.8. It would therefore be useful to re-read this.

Before embarking on this, we need to guard against the common error of subtracting the wrong way round, e.g.,

$$\begin{array}{r} 32 \\ - 17 \\ \hline 25 \end{array}$$

This gives the wrong result because subtraction is non-commutative. This contrasts with addition, which is commutative. The result of these two additions is the same if we interchange the numbers:

$$\begin{array}{r} 7 \\ + 2 \\ \hline 9 \end{array} \qquad \begin{array}{r} 2 \\ + 7 \\ \hline 9 \end{array}$$

Not so for these two subtractions:

$$\begin{array}{r} 7 \\ - 2 \\ \hline 5 \end{array} \qquad \begin{array}{r} 2 \\ - 7 \\ \hline \end{array}$$

cannot be done, with the numbers they know about so far

I leave it to your own judgment relative to the children you teach, whether or not to introduce the term 'non-commutative' at this stage. The important practical result is what Activity 1 is about.

Activity 1 “Can we subtract?” [Num 4.9/1]

A teacher-led discussion for a small group, or for the class as a whole. Its purpose is to emphasize that one can only subtract when the first number of the pair is greater than or equal to the second. In verbal notation, the upper number must be greater than or equal to the lower. (At this stage we are not concerned with negative numbers, but see the note at the end of this activity.)

- Materials*
- Pencil and paper, or
 - Chalk and chalkboard.

Suggested sequence for the discussion

1. Write a subtraction in vertical notation, e.g.,

$$\begin{array}{r} 6 \\ - 2 \\ \hline \end{array}$$

2. Ask for the result.
 3. Remind them of the first of the meanings for subtraction, taking away. (See Num 4.7, ‘Subtraction with all its meanings’)

$$\begin{array}{r} 6 \\ - 2 \\ \hline 4 \end{array}$$



4. Reverse the numbers, and ask “Can we subtract this way round?”

$$\begin{array}{r} 2 \\ - 6 \\ \hline \end{array}$$

5. Depending on their responses, let them see that this would mean making a set with number 2 and crossing out 6.



6. If there is still any doubt, let them try to do it with physical objects.
 7. Repeat steps 1 to 6 with other examples. Include equal numbers, and also cases where one number is zero. Start sometimes with the not-possible case.
 8. Continue, “How about one of the other meanings of subtraction? Let’s try it with cash, cost, change.”
 9. Clearly in this

case you get change.

cash	cents
cost	9
change	<u>- 5</u>
	4

10. But in this case, the shopkeeper would say, in effect, “Can’t be done.”

cash	cents
cost	5
change	<u>- 9</u>
	—

11. Repeat steps 8 to 10 with one or more further examples.
12. Ask “What have we learned?” and obtain agreement on a suitable formulation, in their own words. It should mean the same as the learning goal described in the heading for this activity.

Note Some children may say that they can subtract a larger number from a smaller (e.g., referring to digging a hole, or temperatures below zero, . . .). In this case you could tell them that they are quite right, but they are talking about a different kind of number system, called integers, which they haven’t come to yet. When they do, they will find that it still uses a method for subtraction like the one they are now learning.

Activity 2 Subtracting two-digit numbers [Num 4.9/2]

A teacher-led activity for a small group. Its purpose is to teach subtraction of two-digit numbers, including cases which involve crossing the tens boundary. This now extends its meaning to include that between teens and twenties, twenties and thirties, etc.

- Materials*
- Subtraction board.*
 - Two sets of number cards, in different colours. One set contains twenty cards bearing assorted numbers between 50 and 99, the other contains twenty cards with assorted numbers all less than 50 (provided in the photomasters).**

*Like that for Num 4.8/1

**Suitable assortments are:

- 50 and over – 50, 53, 56, 59, 61, 67, 68, 72, 74, 75,
78, 79, 80, 82, 84, 87, 91, 93, 96, 99.
- Under 50 – 8, 9, 12, 14, 17, 18, 21, 23, 26, 29, 30,
33, 36, 39, 41, 42, 44, 45, 47, 48.

Suggested sequence for the discussion

1. Put out the subtraction board and number cards as in Num 4.8/1. Explain that this is like the earlier activity, but they are going to learn how to do subtraction of larger numbers.
2. Begin with an example such as this, in which both the upper digits are larger than the digits below them.

$$\begin{array}{r} 68 \\ - 23 \\ \hline \end{array}$$
3. Explain: “We subtract a column at a time, ones from ones, tens from tens.”

$$\begin{array}{r} 68 \\ - 23 \\ \hline \end{array}$$
4. Let them practice a few examples of this kind.
5. Next, introduce an example for which this is not so.

$$\begin{array}{r} 74 \\ - 26 \\ \hline \end{array}$$
6. “Can we do ‘4 subtract 6’?” (No.) “So this is what we do. First, we write it in headed columns.”

Tens	Ones
7	4
-2	6

Num 4.9 Subtraction up to 99 (cont.)

7. “Next, we write the upper number differently.”
(This is changing it into a non-canonical form.)

Tens	Ones
6 7	4 14
- 2	6

8. “Now we can subtract a column at a time, as before.”

Tens	Ones
6	14
- 2	6
4	8

9. This is the answer in place-value notation.

$$\begin{array}{r} 74 \\ - 26 \\ \hline 48 \end{array}$$

10. The whole process may be set out as below. (The sign \Leftrightarrow means ‘is equivalent to.’ Its use is optional.)

$$\begin{array}{r} 74 \\ - 26 \\ \hline \end{array} \Leftrightarrow \begin{array}{c|c} \text{Tens} & \text{Ones} \\ \hline 7 & 4 \\ - 2 & 6 \\ \hline \end{array} \Leftrightarrow \begin{array}{c|c} \text{Tens} & \text{Ones} \\ \hline 6\cancel{7} & \cancel{4}14 \\ - 2 & 6 \\ \hline 4 & 8 \end{array} \Leftrightarrow \begin{array}{r} 74 \\ - 26 \\ \hline 48 \end{array}$$

This way they can see all the steps. At present the goal is understanding before speed.

11. The two middle steps may soon be combined mentally.

$$\begin{array}{r} 74 \\ - 26 \\ \hline \end{array} \Leftrightarrow \begin{array}{c|c} \text{Tens} & \text{Ones} \\ \hline 6 & 14 \\ - 2 & 6 \\ \hline 4 & 8 \end{array} \Leftrightarrow \begin{array}{r} 74 \\ - 26 \\ \hline 48 \end{array}$$

12. Now let the children use the subtraction board as in Num 4.8/1, to give assorted examples. That is, the two piles of number cards are shuffled and put face down, and the top card from each pile is turned over and put on the board. The children copy what is there onto their papers, calculate the results, and check.

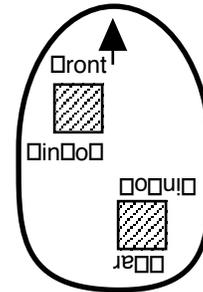
- Notes*
- (i) If step 7 is not understood, use base 10 material to demonstrate the exchange of 1 ten-rod for 10 cubes. The rest of the calculation is more easily dealt with symbolically, with the help of finger counting if necessary. This allows either the use of counting forward from the smaller number (the complement aspect of subtraction), or counting back from the larger number (the take-away aspect of subtraction): see Num 4.8/1 – ‘Subtracting from teens: choose your method.’ Many children find complementation the easier method.
 - (ii) Eventually, the right hand step may be done mentally, but this should not be done until children have had a lot of practice in the written form.

Activity 3 Front window, rear window [Num 4.9/3]

A game for two. Its purpose is to practise subtraction of two-digit numbers.

- Materials.*
- ‘Front window, rear window’ game board.*
 - ‘Car,’ as illustrated.*
 - Pencil and paper for each player.
- * Provided in the photomasters

The ‘car,’ in coloured cardboard, has windows (shaded areas) which are cut out as holes. The arrow shows the direction of travel.



- What they do*
1. This game is based on the fact that the road signs we see looking backward tell us distances of places we have left behind.
 2. The players sit opposite each other with the board between them.
 3. The car is put at the starting town (bottom left), with the arrows pointing in the direction of movement. The players find out which window they look through from the writing they see right way up.
 4. The car moves to the first road sign and each writes the number he sees (say, 87 [km] through the front window and 27 [km] through the rear window).
 5. The car moves on to the next road sign and each again writes the number he sees (say, 68 [km] through the front window and 46 [km] through the rear window).
 6. Each has now recorded two distances. By subtraction, each finds the distance they have travelled between the road signs. Though the numbers are different, the distances are of course the same, so each passenger should get the same result. For the first two signs in this example, the subtractions are:

$$\begin{array}{r} 87 \\ - 68 \\ \hline 19 \end{array} \quad \text{and} \quad \begin{array}{r} 46 \\ - 27 \\ \hline 19 \end{array}$$

7. The car moves on to the next road sign and steps 5 and 6 are repeated with the two latest numbers.
8. This continues to the end of the journey. At the intermediate towns the signs now relate to the next town ahead and the town just left. The car now begins to travel in the opposite direction, so the board needs to be turned around so that the passengers continue to face the same way relative to the car.
9. By changing seats on their next trip, the passengers could get a different set of calculations.
10. Further practice may be given by using other figures, either by making other boards or putting stickers on the existing board.

Num 4.9 Subtraction up to 99 (cont.)



Front window, rear window [Num 4.9/3]

Another suitable set of figures for the 'Front window, rear window' board:

Road section	Road length					
1	113	Front window	100	79	52	33
		Rear window	13	34	61	80
2	125	Front window	99	76	51	36
		Rear window	26	49	74	89
3	110	Front window	81	55	25	16
		Rear window	29	55	85	94
4	100	Front window	78	57	38	19
		Rear window	22	43	62	81

Activity 4 Front window, rear window – make your own [Num 4.9/4]

An activity and game for two. It is an extension of Activity 3, 'Front window, rear window,' in which the children work out their own road signs. This is appreciably harder than Activity 3.

- Materials*
- A board similar to that for Activity 3, without the numbers written in, but with the lines showing locations of the road signs. This board must be covered in transparent film or laminated.
 - 'Car,' as for Activity 2.
 - 2 dice, one marked 0-2 for tens, the other 1-6 for ones.
 - A washable non-permanent marker, and a damp rag.

- What they do*
1. They first decide on the distance between the first two towns. This must be between 105 and 130 kilometres. They both write this number down.
 2. They throw both dice, to obtain the distance travelled from the start.
 3. One player marks in this distance for the 'rear window,' the other subtracts from the total distance to get the 'front window' number and marks this on the board.
 4. They throw the dice again for the next distance travelled.
 5. They each work out their own number, and mark it on the board.
 6. Steps 4 and 5 are then repeated twice more to complete the first stage of the journey.
 7. They then repeat steps 1 to 6 for the remaining stages of the journey, choosing a different distance between each pair of towns.
 8. They should then play as in Activity 3 to check their calculations, or swap boards with another pair to do this. At first they might prefer to check at the end of each stage of the journey.

Discussion of activities Activity 1 is intended to prevent the error of subtracting the wrong way around, discussed first in the 'Discussion of concepts' for this topic.

Activity 2 then shows children how to subtract two-digit numbers. Since children should already be familiar with moving in and out of canonical form from the addition network, the suggestion is that they work at this level rather than go back to physical embodiments in base ten material. This avoids going right back to the take-away aspect of subtraction, whereas the concept now includes other components. Help from the latter may however be used to demonstrate the change from canonical form, also (correctly) called decomposition and (incorrectly) called borrowing, without detriment to the foregoing.

Activities 3 and 4 are more sophisticated applications of the concept of subtraction than they have encountered so far, using the new technique which they have learned.

OBSERVE AND LISTEN**REFLECT****DISCUSS**

[Num 5] MULTIPLICATION

Combining two operations

Num 5.1 ACTIONS ON SETS: COMBINING ACTIONS

- Concepts*
- (i) The action of making a set.
 - (ii) The action of making a set of sets.
 - (iii) Starting and resulting numbers.

- Abilities*
- (i) To make a given number of matching sets.
 - (ii) To state the number of a single set.
 - (iii) To state the number of matching sets.
 - (iv) To state the total number of elements.

Discussion of concepts Multiplication is sometimes introduced as repeated addition. This works well for the counting numbers, but it does not apply to multiplication of the other kinds of number which children will subsequently encounter; so to teach it this way is making difficulties for the future. This is one of the reasons why so many children have problems with multiplying fractions, and with multiplying negative numbers. The concept of multiplication which is introduced in the present topic is that of combining two operations, and this continues to apply throughout secondary school and university mathematics. And as a bonus, the correct concept is no harder to learn when properly taught.

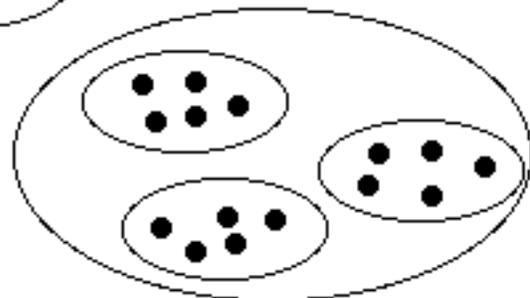
In the present case, we are going to multiply natural numbers. A natural number is the number of objects in a set, and we start with the concept as embodied in physical actions.

First action: make a set of number 5.
Second action: make a set of number 3.

To combine these,
we do the first
action



and then apply the second
action to the result
(make a set of 3 sets of 5).



This is equivalent to making a set of number 15.

At this stage, there is not a lot of difference between this and adding together 5 threes, just as near their starting points two diverging paths are only a little way apart. But in the present case one of these paths leads towards future understanding, while the other is a dead end.

So instead of the sequence 'Start, Action, Result' used in the addition and subtraction networks, we shall be using the sequence 'First action, Second action, Combined result.' Later in this network we shall discuss notations for this.

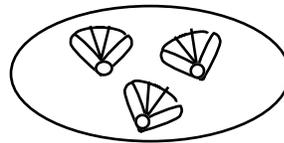
Activity 1 Make a set. Make others which match [Num 5.1/1]

An activity for up to 6 children. Its purpose is to introduce the concept of multiplication in a physical embodiment.

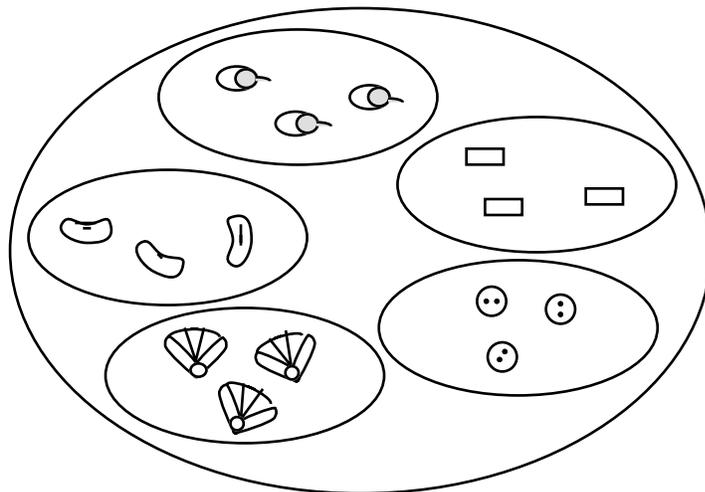
- Materials*
- Five small objects for each child. These should be different for each child, e.g., shells, acorns, bottle tops . . .
 - 6 small set ovals. *
 - Large set loop.

* Oval cards, about 6 cm by 7.5 cm, provided in the photomasters.
See illustrations for steps 1 and 3.

- What they do*
1. The first child makes a set, using some or all of his objects. A small set oval is used for this. It is best to start with a set of fairly small number, say 3.



2. Everyone makes a set which matches this, i.e., has the same number. They too use set ovals and then check with each other.
3. All the sets are put in the set loop to make one combined set, which is counted.



Num 5.1 Actions on sets: combining actions (cont.)

4. With your help, they say (in their own words) what they have done. E.g., “Vicky made a set of 3 shells. We all made matching sets, so we made 5 sets of 3. When we put these together, there were 15 things altogether.” or “We made 5 sets of 3, making 15 altogether.” Or “5 sets, 3 in each, makes 15.”
5. The children take back their objects and steps 1 to 4 are repeated.
6. To give variety of numbers, sometimes only some of the children should make matching sets. E.g., everyone on this side of the table, or all the boys, or all the girls.

Activity 2 Multiplying on a number track [Num 5.1/2]

An activity for up to 6 children. Its purpose is to expand the concept of multiplication to include larger numbers. The use of a number track saves the time and trouble of counting the resulting sets.

- Materials*
- A number track 1 to 60 (2 cm spaces suggested).*
 - 10 cubes of one colour for each child (2 cm cubes suggested).
 - Actions board, see Figure 10.*
 - First action cards 2 to 10, e.g., “Each make a rod with 7 cubes.”*
 - Second action cards 1 to 6, e.g., “Join 4 rods.”*
 - Slips of paper to fit ‘combined result’ space on Actions Board.
 - Pencil.

* Provided in the photomasters

- What they do*
1. The pack of first action cards and second action cards are shuffled and put face down.
 2. The top ‘first action’ card is turned over, and put face up on the lower space.
 3. Each child makes a rod as instructed. The rods are then pooled for general use.
 4. The top ‘second action’ card is turned over, and put face up on top of the pack.
 5. The number of rods indicated is taken, and joined together on the number track.
 6. The result is recorded on a slip of paper, which is put in the ‘combined result’ space.
 7. The ‘second action’ card used is put face up at the bottom on the pile. Steps 3 to 6 are then repeated using the same first action card. This saves re-making the rods every time.
 8. When a face-up ‘second action’ card is reached, this means that all of this pack have been used once. The pack is then shuffled and put face down again.
 9. Steps 2 to 7 are then repeated with the next first action card.

Activity 3 Giant strides on a number track [Num 5.1/3]

An activity for up to 6 children. Its purpose is to begin the process of freeing children’s concept of multiplication from dependence on physical objects.

- Materials*
- A card number track 1 to 50 (1 cm squares suggested).*

ACTIONS BOARD

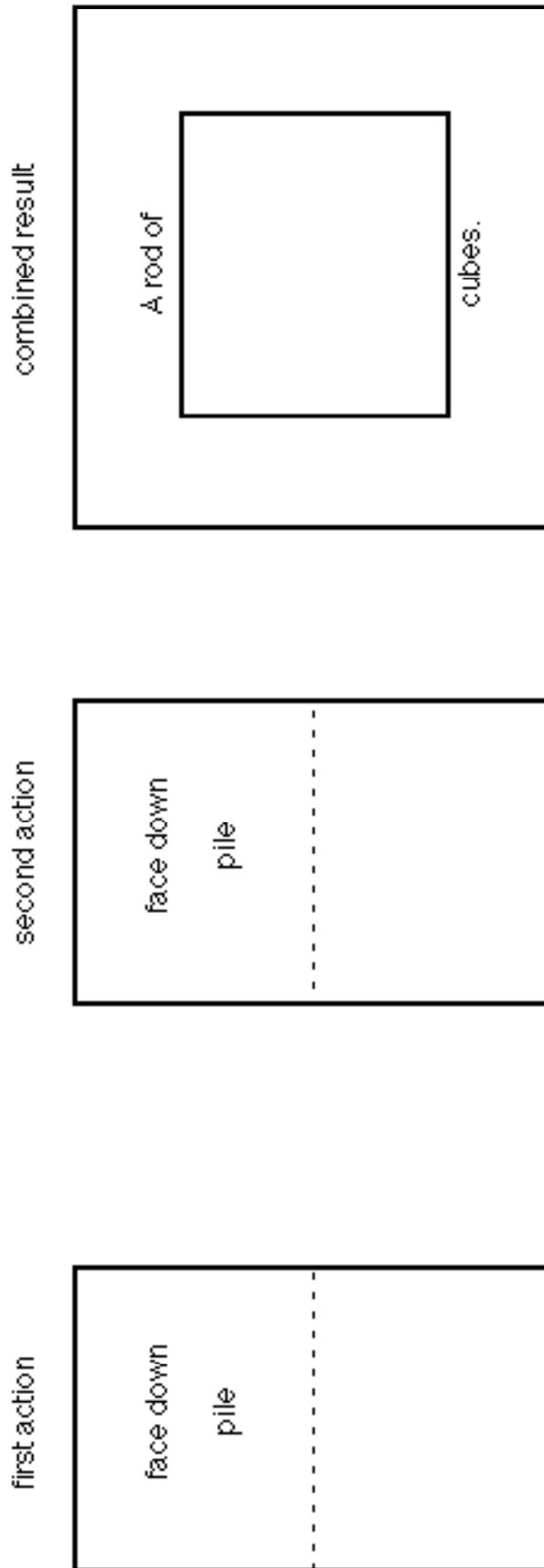


Figure 10 Actions board for multiplying on a number track.

Num 5.1 Actions on sets: combining actions (cont.)

- Activity board, see Figure 11.*
- Length-of-stride cards 2 to 5.**
- Number-of-stride cards 2 to 10.**
- Plasticine.
- Pencil and paper.

* Provided in the photomasters

** Single-headed and of a size to fit the dotted spaces on the activity board. Each pack should be of a different colour, provided in the photomasters.

- What they do*
1. The two packs are shuffled and put face down in their respective dotted spaces on the activity board.
 2. The top card in each pile is turned over.
 3. Suppose that each stride is 3 spaces, and they take 7 strides.
 4. One child puts down 'footprints' on the number track (small plasticine markers) at spaces 3, 6, 9 and so on, representing strides each of 3 spaces. The others help by making the plasticine markers for him, and also checking that they are put in the right spaces.
 5. This continues until (in this case) 7 strides have been taken. The last footprint will be in space 21.
 6. Another child records 21 on a strip of paper, which is put in the last space on the board.
 7. The cards are then replaced face down at the bottom of the pile, and steps 2 to 6 are repeated.

Discussion of activities The first two activities embody, as physical actions, the concept of multiplication as described at the beginning of this topic. In Activity 1, the first action is making a set of (say) shells, and the second action is making a set of these matching sets. In Activity 2, the first action is making a rod out of unit cubes, and the second action is making a rod using a given number of these rods. In Activity 3, this is repeated at a slightly more abstract level. The last activity also makes a start with relating multiplication to number stories. The first action is making a stride of a given number of spaces, and the second action is making a given number of strides.

All of the activities in the present topic use Mode 1 schema building. No mental calculations, and no predictions, are yet involved. Very simple recording is introduced in Activities 2 and 3.

OBSERVE AND LISTEN

REFLECT

DISCUSS

We are GIANTS.

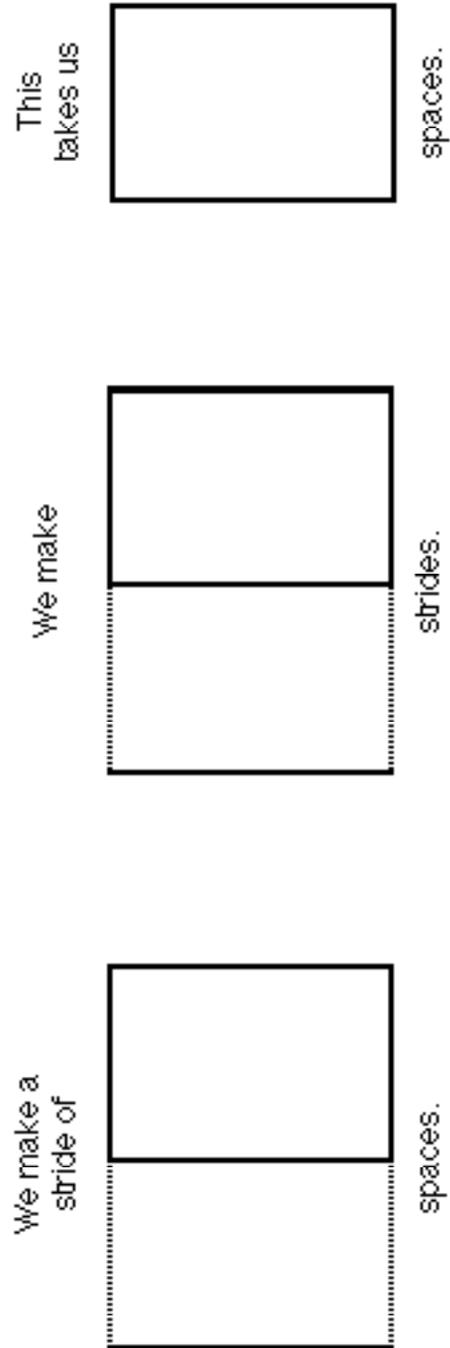


Figure 11 Giant strides on a number track.

Num 5.2 MULTIPLICATION AS A MATHEMATICAL OPERATION

Concept Multiplication as a mathematical operation.

Ability To do this mentally, independently of its physical embodiments.

Discussion of concept

Multiplication becomes a mathematical operation when it can be done mentally with numbers, independently of actions on sets or other physical embodiments. At this stage we concentrate on forming the concept, using the easiest possible numbers as operands. These are first 2, then 5, since the children have already learned to count in twos and fives. Afterwards they will expand the concept to include multiplication of 4 and of 3.

Note that this corresponds to subitizing 2-sets and 5-sets, etc. It does not start us down the path of repeated addition. Note also that in this topic the children are already multiplying these numbers by numbers up to ten.

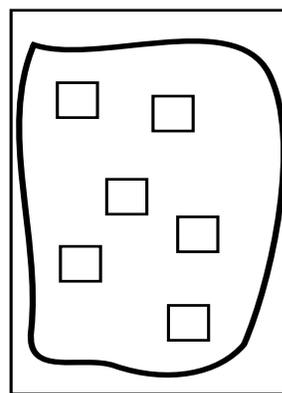
Activity 1 “I predict – here” using rods [Num 5.2/1]

An activity for up to 6 children. Its purpose is to introduce multiplication used predictively, as a mental operation followed by testing. It is a development of “I predict - here” (NuSp 1.1/1).

- Materials*
- Set cards 1 to 10, as further described below.*
 - Number track 1 to 50 (2 cm suggested).
 - 50 cubes (2 cm suggested).
- * Provided in the photomasters

Set cards

On each is drawn a set loop, and within the loop are drawn squares the size of a cube, in number from 1 to 10. These squares should be randomly placed.



Set card

- What they do*
1. The set cards are shuffled and put in a pile face down. (Use only cards 2 to 6 to begin with.)
 2. Together the children make 10 2-rods which are pooled for communal use. Each rod must be of a single colour.
 3. The top set card is turned face up, and a 2-rod is placed standing on end on each square.
 4. One of the children then predicts where these will come to when put end to end on the number track. This may be done by counting in twos. He makes his prediction, e.g., with a piece of plasticine. Adjacent rods should be of different colours.

5. His prediction is tested physically.
6. The track is cleared, the long rod broken up into 2-rods, and steps 3, 4, 5 are repeated with another child making the prediction.
7. When the children can do this well, the activity is repeated using 5-rods. The predictions are now made by counting in 5's.
8. After that, the activity is repeated using 3-rods and 4-rods. For these, the predictions may be made by pointing to each rod in turn and counting (for 3-rods): "1, 2, 3; 4, 5, 6; 7, 8, 9;" etc.

Activity 2 Sets under our hands [Num 5.2/2]

An activity for up to 6 children. Its purpose is to give further practice in the operation of multiplication.

- Materials*
- Five small objects for each child.*
 - Number cards 2 to 6.**
 - 6 small set ovals.*
 - Large set loop.*
 - Pencil and paper for each child

*As for Num 5.1/1

** Provided in the photomasters

- What they do*
1. The first child makes a set, using some or all of her objects. As before, a small set oval is used.
 2. A number card is put out to remind them what the number of this set is.
 3. All the children then make matching sets, using set ovals.
 4. They cover the sets with their hands.
 5. They try to predict how many objects there will be when they combine all these sets into a big set. This can be done by pointing to each hand in turn and mentally counting on. E.g., if there are 4 in each set: (pointing to first hand) "1, 2, 3, 4"; (pointing to second hand) "5, 6, 7, 8"; etc.
 6. They speak or write their predictions individually.
 7. The sets are combined and the predictions tested.
 8. Steps 1 to 7 are repeated, with a different child beginning.
 9. As in Num 5.1/1, the number of sets made should be varied, by involving only some of the children. All however, should make and test their predictions.

Discussion of activities In topic 1, the physical activities were used for schema building. The activities came first, and the thoughts arose from the activities. In the present topic it is the other way about: thinking first, and then the actions to test the correctness of the thinking. First Mode 1 building, then Mode 1 testing. By this process we help children first to form concepts, and then to develop them into independent objects of thought.

Activity 1 gives visual support for the mental activity of counting on, by which the children are predicting. They can see the number of cubes in each rod, as well as the number of rods. In Activity 2, this visual support is partly withdrawn. They can see how many hands there are, but they have to imagine how many objects there are under each hand. In this way we take them gently along the path towards purely mental operations.

Num 5.3 NOTATION FOR MULTIPLICATION: NUMBER SENTENCES

Concept The use of number sentences for representing the operation of multiplication and its result.

- Abilities*
- (i) To write number sentences recording multiplication as embodied in physical materials.
 - (ii) To use number sentences for making predictions.

Discussion of concept Several notations for multiplication are currently in use. Each can be read aloud in ways which fit the meaning of multiplication which we are using. For example,

$$5 \times 3 = 15$$

can be read: "Make a set of 5. Make it 3 times. Combined result, a set of 15."

This may be shortened to: "5, 3 times, equals (or makes) 15."

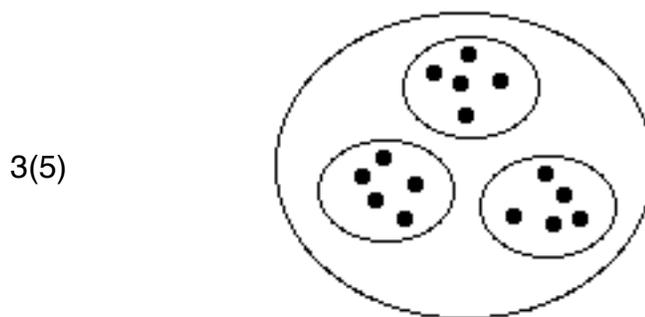
This is easier to say than "5 multiplied by 3 equals 15. (I do not recommend 'timesed by', as one sometimes hears. It is not grammatically correct, and no easier for the children.) The above readings have the advantage that the order 'First action, Second action' is preserved.

Other notations for the same operation use parentheses, with an equals sign or an arrow.

$$3(5) = 15 \quad \text{or} \quad 3(5) \rightarrow 15$$

"3 sets of 5 equal (or make) 15," or "3 fives are 15."

This reverses the order of the operations, but corresponds well to the diagram below which shows the combined result.



I suggest that you use whichever notation you and the children are happiest with, until topic 5.5. Here I introduce a notation for binary multiplication which combines both, and when properly understood makes everything much simpler.

Until then, the activities can be used with either notation. You might think it useful for the children to understand both notations, since they will certainly meet both in their future work.

Activity 1 Number sentences for multiplication [Num 5.3/1]

An activity for up to 6 children. Its purpose is to introduce number sentences for recording multiplication as embodied in physical materials.

- Materials*
- Actions board.*
 - First action cards 2 to 5.*
 - Second action cards 2 to 6.*
 - Five small objects for each child.**
 - 6 small set ovals.**
 - 1 set loop.**
 - Pencil and paper for each child.

* See Figure 12. This shows the actions board for () notation with two cards in position (see step 2). The actions board, 'first action' cards 2 to 5, and cards for both () and X notation are provided in the photomasters.

** As for Num 5.1/1 and Num 5.2/2.

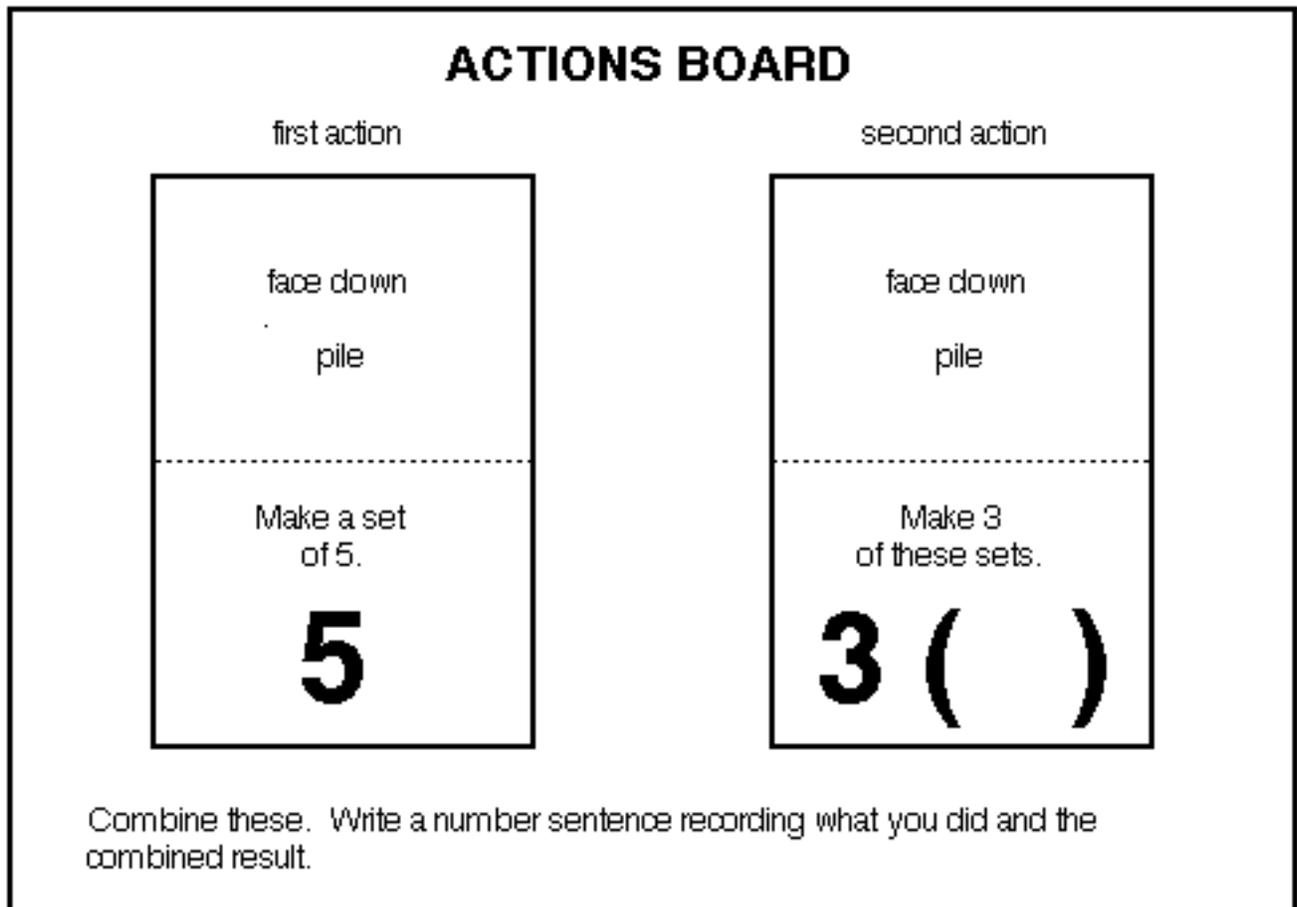


Figure 12 Actions board.

- What they do*
1. The action cards are shuffled and put in 2 piles face down on the Actions board.
 2. The first child turns over the 'first action' card. He does what it says, using some or all of his objects and a small oval set card.

Num 5.3 Notation for multiplication: number sentences (cont.)

3. The next child turns over the top 'second action' card, and puts it beside the first. The board will now look like Figure 12 (or the equivalent when the alternative cards in the photomasters are used).
4. The children make as many of these sets as it says on the 'second action' card.
5. They combine the sets by putting them in the large set loop, and count the result.
6. They write number sentences in whichever notation you show them. The meaning of these should be carefully explained. In the example shown, either this:

$$3(5) = 15$$

"Three sets of five make a set of fifteen," or

"Three fives make (or equal) fifteen."

or this:

$$5 \times 3 = 15$$

"A set of five, made three times, makes a set of fifteen," or

"Five, three times, makes (or equals) fifteen."

7. They should learn how to read their number sentence aloud, as in these examples. It is good to be able to say these in several ways. In the process of step 6 they are comparing their results. Any discrepancies offer opportunity for discussion.
8. The objects are taken back, and steps 1 to 6 are repeated, beginning with a different child.

Activity 2 Predicting from number sentences [Num 5.3/2]

An activity for up to 6 children. Its purpose is to teach children to write number sentences which predict the result of multiplying.

- Materials*
- Number cards 2 to 5.**
 - 6 small set ovals.*
 - Large set loop.*
 - Die (1 to 6, then 1 to 9).
 - 5 small objects for each child.*
 - Pencil and paper for each child.

* As used in Num 5.1/1.

** Provided in the photomasters

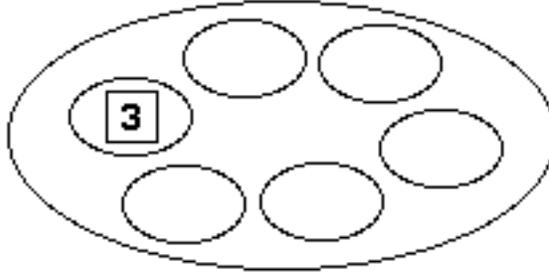
- What they do*
1. The number cards are shuffled and put in a pile face down.
 2. The top card is turned over and put face up on one of the oval small-set cards. This represents the 'first action.' E.g.,



Make a set
of 3

Num 5.3 Notation for multiplication: number sentences (cont.)

3. The die is thrown, and that total number of small-set ovals are put out (counting the first). The set loop is put round them all, to make a big set. This represents the 'second action.' E.g.,



4. Each child writes the beginning of a number sentence for the above. In this case,

either $6(3)$
or 3×6

5. They then complete their number sentences to predict the combined result when they actually make the sets represented. To do this, they may use the method for predicting learned in topic 2. In this case the completed sentences would be

either $6(3) = 18$
or $3 \times 6 = 18$

6. Finally they test their predictions by putting 3 objects on each small-set card, and counting the combined set.
7. The objects are taken back, and steps 2 to 6 are repeated.

Discussion of activities The children have already learned to use multiplication predictively, in topic 2. The new factor here is the use of notation.

In this topic they progress from 'Do and say', to 'Do and record', and then to 'Predict and test.' In Activity 1 they use number sentences to record past events; in Activity 2 they use number sentences to predict future events. This parallels the progression in topic 2, in which thought begins to become independent of action. Here, we begin to link thinking with notation.

Initially, writing number sentences is an extra task rather than a help, so it needs to be learned in a situation where the rest is familiar. For more difficult calculations, written notation is no longer an extra chore but a valuable support. It means that we do not have to 'keep everything in our head' at the same time. Pencil and paper give us an external, easily accessible, extra memory store.

Another step has been taken here. Until now, the number symbols have stood for sets of single objects. Now, some of them stand for sets of sets. So we are now handling more information at a time – one might say, in set-sized packages. This is another of the sources of the power of mathematics.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 5.4 NUMBER STORIES: ABSTRACTING NUMBER SENTENCES

Concept Numbers and numerical operations as models for actual happenings, or for verbal descriptions of these.

- Abilities*
- (i) To produce numerical models in physical materials corresponding to given number stories, to manipulate these appropriately, and to interpret the result in the context of the number story: first verbally, then recording in the form of a number sentence.
 - (ii) To use number sentences predictively to solve verbally given problems.

Discussion of concepts The concept of abstracting number sentences is that already discussed in Num 3.4, and it will be worth reading this again. In Num 4.4 it was expanded to include subtraction, and here we expand it further to include multiplication.

In some applications of multiplication, we need to place less emphasis on the first action and the second action, and more on their results: namely a small set (of single objects) and a large set (a set of these sets). The use of small set ovals and a large set loop from the beginning provides continuity here.

Activity 1 Number stories (multiplication) [Num 5.4/1]

An activity for 2 to 6 children. Its purpose is to connect simple verbal problems with physical events, linked with the idea that we can use objects to represent other objects.

- Materials*
- Number stories on cards, of the kind in the example over. Some of these should be personalized, as in Num 3.4/1, but now there should also be some which do not relate to the children, more like the kind they will meet in textbooks. Also, about half of these should have the number corresponding to the small set coming first, and about half the other way about.**
 - Name cards for use with the personalized number stories.
 - Number cards 2 to 6.**
 - 30 small objects to manipulate: e.g., bottle tops, shells, counters, . . .
 - 6 small set ovals. *
 - A large set loop.
 - Slips of blank paper, pencils.

*As for Num 5.1/1.

** Provided in the photomasters

What they do (apportioned according to how many children there are)

1. A number story is chosen. The name, cards and number cards are shuffled and put face down.
2. If it is a personalized number story; the top name card is turned over and put in the number story. Otherwise, explain "Some of these stories are about you, and some are about imaginary people."

Num 5.4 Number stories: abstracting number sentences

3. The top number card is turned over and put in the first blank space on the story card.
4. The next number card is turned over and put in the second blank space.
5. The number story now looks like this.

There are children in each rowing boat
and rowing boats on the lake. So
altogether children are boating on the lake.

6. Using their small objects (e.g., shells) to represent children, the oval cards for boats, and the set loop for the lake, the children together make a physical representation of the number story. If it is a personalized number story, this should be done by the named child.
7. The total number of shells is counted, and the result written on a slip of paper. This is put in the space on the card to complete the story.
8. While this is being done, one of the children then says aloud what they are doing. E.g., “We haven’t any boats, so we’ll pretend these cards are boats, and put shells on them for children. We need 3 children in each boat, and 4 boats inside this loop which we’re using for the lake. Counting the shells, we have 12 children boating on the lake.”
9. The materials are restored to their starting positions, and steps 1 to 8 are repeated.

Activity 2 Abstracting number sentences [Num 5.4/2]

An extension to Activity 1 which may be included fairly soon. Its purpose is to teach children to abstract a number sentence from a verbal description.

- Materials*
- As for Activity 1, and also:
 - Pencil and paper for each child.

What they do As for Activity 1, up to step 8.

9. Each child then writes a number sentence, as in step 6 of Num 5.3/1. They read their number sentences aloud.
10. The materials are restored to their starting positions, and steps 1 to 8 are repeated.

Activity 3 Number stories, and predicting from number sentences [Num 5.4/3]

An activity for 2 to 6 children. It combines Activities 1 and 2, but in this case completing the number sentence is used to make a prediction as in Num 5.3/2.

- Materials.*
- Number stories asking for predictions.**
 - Name cards for use with the personalized number stories.*
 - Two sets of number cards 2 to 6.*
 - 30 objects to manipulate.*
 - 6 oval small-set cards.*
 - A set loop.*
 - Pencil and paper for each child.*

*As for Activity 1.

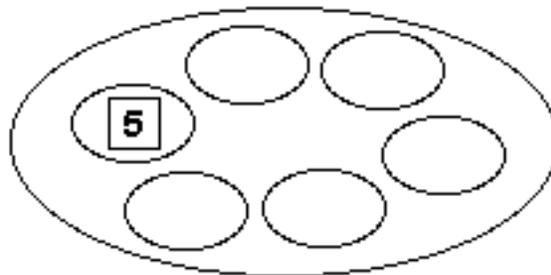
** Provided in the photomasters

What they do (apportioned according to how many children there are).

1. A number story is chosen. The name cards and number cards are shuffled and put face down.
2. If it is a personalized number story, the top name card is turned over and put in the number story. Otherwise, explain “Some of these stories are about you, and some are about imaginary people.”
3. Two more children turn over the top two number cards, and put these in the first two spaces on the story card.
4. The number story now looks something like this:

Giles is collecting fir cones in a wood.
He gets **5** cones into each pocket and he has **6** pockets.
When he gets home and empties his pockets, how many fir cones will he have?

5. The named child (if there is one) then puts out an appropriate number of small-set cards, and a number card from the second pack, to represent the situation. (This corresponds to step 6 of Activity 1.) For the present example, this is what he should put.



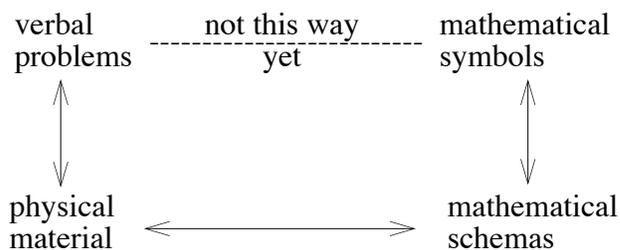
Num 5.4 Number stories: abstracting number sentences

6. He explains, using his own words and pointing: “These 6 ovals represent my 6 pockets, and this 5 is the number of fir cones I have in each.” (This verbalization is an important part of the activity.)
7. All the children then write and complete number sentences, as in Activity 2. For the present example, these would be
 either $6(5) = 30$
 “six fives make 30”
 or $5 \times 6 = 30$
 “five, six times, equals 30”
 They read these aloud.
8. Another child then tests their predictions by putting the appropriate number of objects in each small-set oval. The number sentences are corrected if necessary.
9. They then write their answers to the question as complete sentences.
 E. g. (in this case), “Giles will have 30 fir cones.”
10. The materials are returned to their starting positions, and steps 1 to 9 are repeated using a different number story and numbers.

Discussion of activities

The activities in this topic parallel those in Num 3.4 (addition) and Num 4.4 (subtraction), and in view of their importance it will be worth re-reading the discussion of these.

Solving verbally stated problems is one of the things which children find most difficult, as usually taught. This is because they try to go directly from the words to the mathematical symbols. In the early stages, it is of great importance that the connection is made via the use of physical materials, since these correspond well both to the imaginary events in the verbally-stated problem, and to the mathematical schemas required to solve the problem. The route shown below may look longer, but the connections are much easier to make at the present stage of learning.



Since these physical materials have already been used in the earlier stages for schema building, they lead naturally to the appropriate mathematical operations. Side by side with this, they learn the mathematical symbolism which will in due course take the place of the physical materials.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Num 6] **DIVISION**

Sharing equally, grouping, factoring

Num 6.1 GROUPING

Concepts (i) Grouping.
(ii) Remainder.

Abilities (i) Given a set of given number, to re-arrange this into equal groups of a required number.
(ii) To state how many groups, and the remainder.

Discussion of concepts

Like subtraction, the mathematical operation of division is derived from several different kinds of physical actions on sets of objects. Though the actions themselves are quite different, they have something in common mathematically; and when the children have realized what this is, they have the higher order concept of division.

The two chief kinds of action are grouping and sharing. We also include organizing into rectangles, which is closely related to factoring, and shows particularly clearly the relation between multiplication and division.

We need to emphasize that ‘grouping’ is short for ‘arranging in groups of equal number.’

Activity 1 Start, Action, Result: grouping [Num 6.1/1]

An activity for up to 6 children. Its purpose is to introduce the grouping aspect of division.

Materials

- SAR board (grouping), see Figure 13.*
- Start cards 10-25.*
- Action cards 2-5.*
- 25 (or more) small objects.
- Pencil and paper for each child.

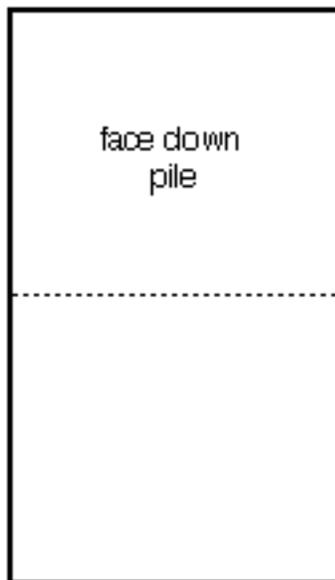
* Provided in the photomasters

What they do

1. The start cards and the action cards are shuffled and put face down in the upper part of their spaces on the activity board.
2. The top card of each pile is turned over.
3. One child then puts out a set of the given number.
4. Another child then rearranges this set into equal groups of the number stated on the action card.
5. Another child counts how many groups there are, how many objects are left over, and puts this into words.
E.g., “There are 5 groups, and 2 objects remaining.”

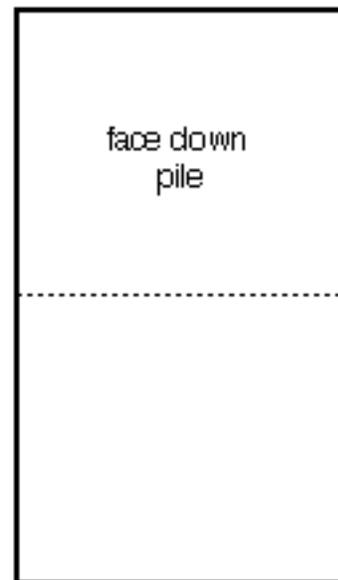
START

Put out a set
of this number



ACTION

Arrange it into
equal groups
of this number



Write a number sentence recording what was done, and the result.

Figure 13 Start, Action, Result: Grouping

Num 6.1 Grouping (cont.)

- All the children then write a number sentence recording what was done, and the result. Example:

$$17 \xrightarrow{\text{make groups of 3}} 5 \text{ groups rem } 2$$

This is read as

“Start with 17 and arrange into groups of 3, result 5 groups, remainder 2.” If there is no remainder, they should say either “. . . remainder zero” or “. . . exactly.”

- Steps 2 to 6 are repeated.

Activity 2 Predictive number sentences (grouping) [Num 6.1/2]

A game for up to 6 children. Its purpose is to consolidate the concept of grouping by using it in a predictive game.

- Materials*
- Game board, see Figure 14.**
 - Start cards 10-25.*
 - Action cards 2-5.*
 - 25 (or more) small objects.*
 - Pencil and paper for each child.*

* The same as for Activity 1.

** Provided in the photomasters

- Rules of the game*
- The start cards and action cards are shuffled and put face down in the upper part of their spaces on the game board. For most children, you will need to read through the instructions on the board with them.
 - The top card of each pile is turned over.
 - A set of the given number is put out.
 - All the children write their predictions. These are in the form of number sentences, as in Activity 1.
 - They check their predictions in two ways: by comparison with each other's, and physically, by putting groups of the required size into the boxes.
 - To preserve the connection between the number sentences and their physical meanings, the sentences should be expanded when read aloud, as in the example on the game board for Activity 1.
 - Each correct prediction scores 1 point.
 - The board is cleared, and steps 2 to 6 are repeated.

Activity 3 Word problems (grouping) [Num 6.1/3]

An activity for up to 6 children. Its purpose is to use the skills acquired in Activities 1 and 2 for the solution of problems in the form of number stories, and for checking their solutions. We are now using the term ‘word problem’ for a number story with a request for a prediction.

START

Put out this number of objects

face down
pile

ACTION

If you put a group of this number into each box

face down
pile

How many boxes will be filled?
How many objects will remain?

Each write you prediction in a number sentence

Test your predictions in two ways

1	2	3	4
5	6	7	8
9	10	11	12

Figure 14 Board for predictive number sentences (grouping).

Num 6.1 Grouping (cont.)

- Materials*
- Problem cards (as illustrated in step 2, following).**
 - Start cards 10-25.*
 - Action cards 2-5.*
 - 25 (or more) small objects.
 - Pencil and paper for each child.

* As for Activities 1 and 2.

** Provided in the photomasters

- What they do*
1. The problem cards are shuffled and put face down. Likewise for the start and action cards.
 2. The top card on the problem pile is turned over. The top cards in the start and action piles are turned over and put in the first and second spaces on the problem card. The result should look something like this.

26 children are going boating.
Each boat takes **4** children.
How many boats will be filled?
Will there be any children remaining
when these boats are full?
Write your prediction and test it.

3. Each child first writes a number sentence, abstracting the numerical part of the problem (see Num 3.4/2) and completing the sentence. E.g., (in this example):

$$26 \xrightarrow{\text{make groups of 4}} 6 \text{ groups rem } 2$$

4. They then relate this result back into the number story, and write this also. In this example,
5. They test their predictions in two ways:
 - (i) by comparison with each others' results;
 - (ii) physically, using the small objects to represent (in this case) children.
6. If all get the same result, they may agree that physical testing is not necessary. Otherwise, it should be used.
7. Each correct prediction scores 1 point.
8. Steps 1 to 7 are repeated with a different problem and different numbers.

Note About one-third of the problems should have the start and action numbers in reverse order, so that children do not develop a mechanical routine. An example is given on the next page.

A raft is big enough to support **A** children. If there are **S** children who want to cross a river on it, how many fully loaded trips will there be? How many on the last, partly loaded trip?
Write your prediction and then test it.

Discussion of activities Activity 1 has a familiar look. It is the ‘Start, Action, Result’ sequence with physical materials for building a new concept. The concept is grouping, one of the contributors to the higher order concept of division.

Activity 2 tests their grasp of the concept, and consolidates it, by using it in a predictive game. At this stage we leave it to the children to devise their own methods. Some may use counting on in twos, threes, fours, or fives, with or without the help of finger counting. Others may discover for themselves the relation between this and multiplication. For larger numbers it is necessary to use known multiplication facts, and in topics 5 and 6 this will be taught. For the present, I think it is good to allow room for children to exercise their own ingenuity.

Activity 3 applies the new skill to solving problems given as number stories. This uses and further consolidates two abilities already developed for the operations of addition, subtraction, and multiplication: namely abstraction of a number sentence from a number story, and using objects to symbolize other objects.

In Activity 2, we ask children to test their results by both Mode 2 (comparison) and Mode 1 (physical prediction). This is how results are tested in engineering, navigation, electronics, and other areas of applied mathematics. In Activity 3, we allow that if everyone agrees on a result, it may be accepted as correct. This is how the pure mathematicians do it.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 6.2 SHARING EQUALLY

- Concepts*
- (i) Equal shares.
 - (ii) Remainders.

- Abilities*
- (i) Starting with a set of given number, to separate this into a required number of equal shares.
 - (ii) To state the number in each share, and the remainder.

Discussion of concepts Sharing is the next contributor to the mathematical operation of division. Physically it is quite different, as may be seen from the diagrams at the beginning of the next topic (Num 6.3).

Here we have yet another example of how the same mathematical model can represent quite different physical situations. It is because such models are multi-purpose that they are so useful; but this quality can also cause confusion if we do not take care in the building up of these multi-purpose, higher-order, concepts.

Activity 1 Sharing equally [Num 6.2/1]

An activity for up to 6 children. Its purpose is to introduce the sharing aspect of division. Note that in the present context, sharing always means *equal* sharing.

- Materials*
- SAR board (sharing), see Figure 15.**
 - Start cards 10-25.*
 - Action cards 2-5.*
 - 20 (or more) small objects.*
 - Pencil and paper for each child.

*As for Num 6.1/1 and 2.

** Provided in the photomasters

- What they do*
1. The start cards and the action cards are shuffled and put face down in the upper parts of their spaces on the activity board.
 2. The first child turns over the top start card, and puts out a set of this number.
 3. The second child turns over the top action card, and distributes this set equally between as many children as appears on the action card. If there are 5 or 6 children, there will always be enough children to do this. If (e.g.) there are only 3 children, and the action card indicates that the set is to be shared by 5 children, then 2 children must ask for (and receive) “A share for my friend, who isn’t here.”
 4. All the children then write a number sentence recording what was done, and the result. Example:

$$23 \xrightarrow{\text{share among } 5} 4 \text{ each rem } 3$$

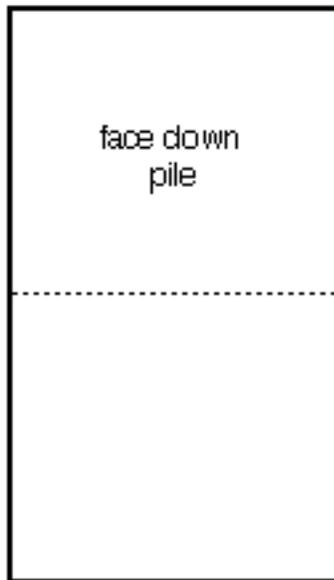
This is read as

“Start with 23, share among 5, result 4 in each share, remainder 3.” If there is no remainder, they should say either “. . . remainder zero” or “. . . exactly.”

5. They compare their results.
6. Steps 2,3,4 are then repeated.

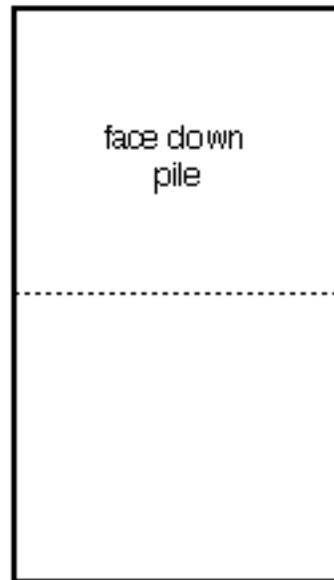
START

Put out a set
of this number



ACTION

Share it equally
among this
number of children



Write a number sentence recording what was done, and the result.

Figure 15 Start, Action, Result: Sharing

Activity 2 “My share is . . .” [Num 6.2/2]

A game for up to 6 children. Its purpose is to consolidate the concept of sharing by using it in a predictive game.

- Materials*
- Game board, see Figure 16.**
 - Start cards 10-25.*
 - Action cards 2-5.*
 - 25 (or more) small objects.*
 - Pencil and paper for each child, and for scoring.
- * As for Activity 1.
** Provided in the photomasters

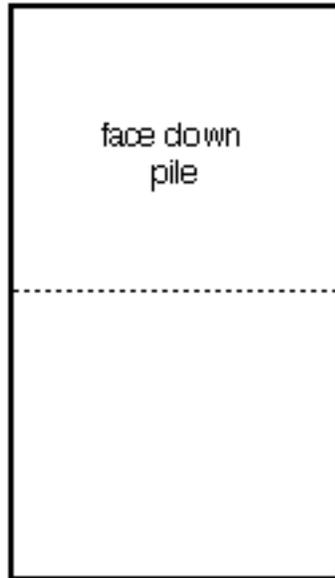
- Rules of the game*
1. The start cards and the action cards are shuffled and put face down in the upper parts of their spaces on the game board.
 2. The top card of each pile is turned over.
 3. A set of the specified number is put out.
 4. Players then take turns, as follows.
 5. The player whose turn it is looks at the action card, and decides what her (equal) share will be (using pencil and paper if she likes). She then says “My share is . . . ,” and takes this number of objects.
 6. She may need some help, initially. Suppose that 19 objects are to be shared among 5. “If you gave everyone 1 each, how many would that use? If you gave everyone 2 each, how many would that use?” (And so on).
 7. The correct number of other players take the same number of shares as the player in step 5. This will show whether she has decided correctly or not.
 8. One point is scored for a correct prediction.
 9. It is now another player’s turn, and steps 2 to 7 are repeated.
 10. When players are proficient, they may agree to play without pencil and paper, except for scoring.

Activity 3 “My share is . . . and I also know the remainder, which is . . .” [Num 6.2/3]

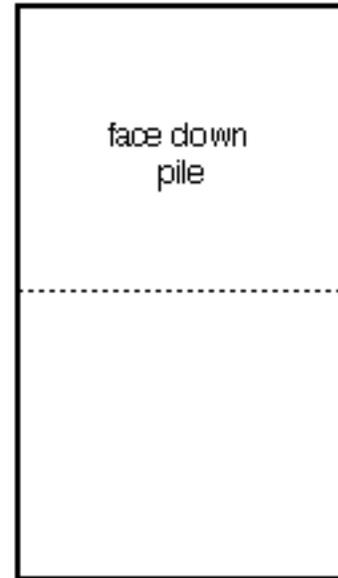
This is a more advanced version of Activity 2. The rules are the same, except that in step 5 the player also predicts the remainder. If correct, she scores a second point. E.g., if the start number was 23 to be shared by 4, she might say, “My share is 5, and I also know the remainder, which is 3.” Since this is correct, she would score a second point.

START

Put out a set
of this number



It is to be
shared by this
number of players



ACTION (by the player whose turn it is).

Take your fair share.

ACTION (by the other players).

Take the same shares.

Was the first player correct?

Figure 16 Board for “My share is . . .”

Activity 4 Word problems (sharing) [Num 6.2/4]

An activity for up to 6 children. It parallels ‘Word problems (grouping)’ (Num 6.1/3), and its purpose is similar: to use the skills acquired in Activities 1 and 2 for the solution of number problems.

- Materials*
- Problem cards, as illustrated in step 2 below.**
 - Start cards 10-25.*
 - Action cards 2-5.*
 - 25 or more small objects.*
 - Pencil and paper for each child.

*As for Num 6.1/3.

** Provided in the photomasters

- What they do*
1. The problem cards are shuffled and put face down. Likewise for the start and action cards.
 2. The top card in the problem pile is turned over. The top cards in the start and action piles are turned over, and put in the start and action spaces on the problem card. The result should look something like this.

A packet containing **17** candies is to be shared by **5** children.
How many candies will they each have?
Write your prediction and then test it.

3. Each child first writes a number sentence, abstracting the numerical part of the problem and completing the sentence. E.g., (in this example):

$$17 \xrightarrow{\text{shared by } 5} 3 \text{ each rem } 2$$

4. They then relate this back to the problem, and write this also. In this example,

They will have 3 candies each. (2 left over)

5. They test their predictions in two ways:
 - (i) by comparison with each other’s results;
 - (ii) physically, using the small objects to represent (in this case) candies.
6. If all get the same result, they may agree that physical testing is not necessary. Otherwise it should be used.
7. Each correct prediction scores 1 point.
8. Steps 1 to 7 are repeated with a different problem and different numbers.
Note About one-third of the problems should have the start and action numbers in reverse order, so that children do not develop a mechanical routine. An example is given on the next page.

A gardeners are asked to plant **S** trees altogether. If they each plant the same number, how many each will that be?
How many more trees will there still be to plant?
Write your prediction and then test it.



Word problems (sharing) [Num 6.2/4]

Discussion of activities These activities closely parallel those in topic 1. However, we have found that many children find sharing more difficult than grouping, so plenty of practice is necessary, especially at Activities 2 and 3.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Num 6.3 DIVISION AS A MATHEMATICAL OPERATION

Concept The connection between grouping and sharing.

Ability To explain this connection. (This is most easily done with the help of physical materials.)

Discussion of concept Physically, grouping and sharing look quite different.

Start with 15.



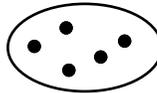
Make groups of 3.



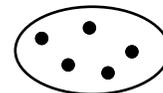
Resulting number of groups is 5.



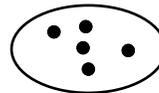
Start with 15.



Share among 3.



Number in each share is 5.



It is only at the level of thought that we can see that these are, in a certain way, alike. When children have grasped the connection between grouping and sharing, they have the higher order concept of division.

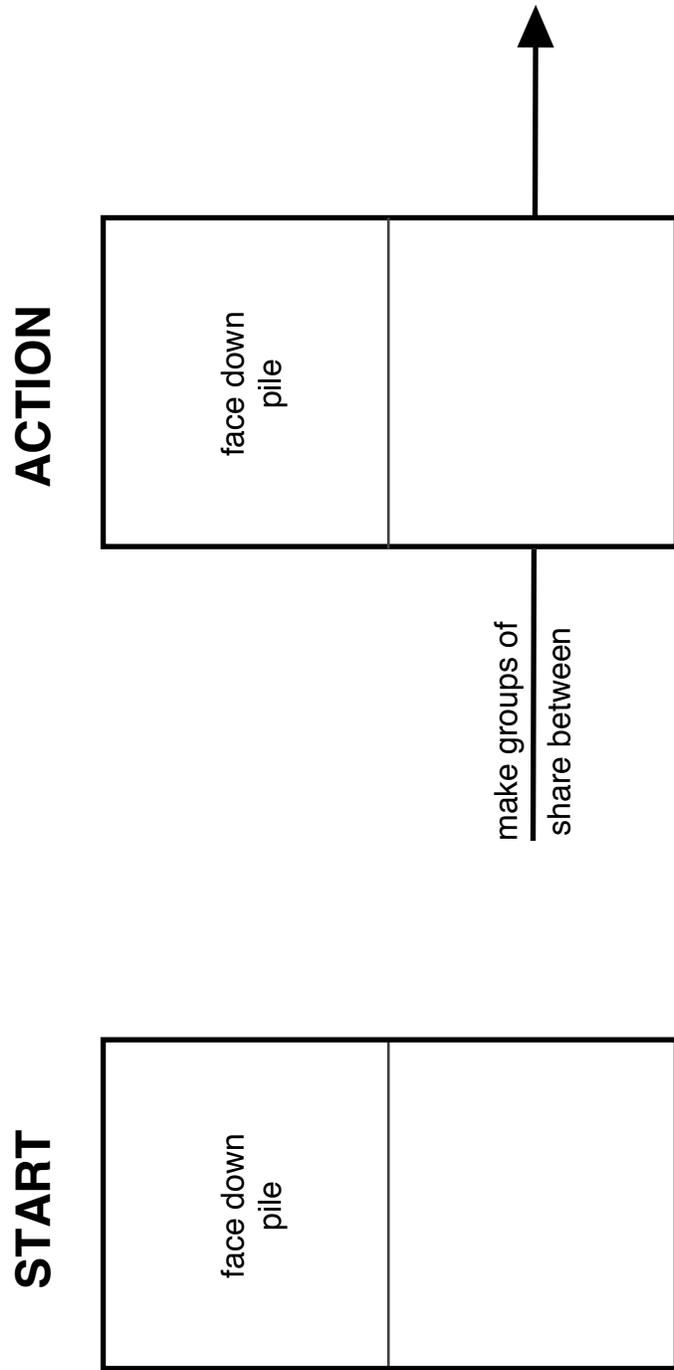
Activity 1 Different questions, same answer. Why? [Num 6.3/1]

A problem for children to work at in small groups. (I suggest twos or threes.) The purpose is for them to discover for themselves the connection between grouping and sharing.

- Materials*
- 'Different questions, same answer . . .' [two-question] board, see Figure 17.**
 - Start cards 10 to 25.*
 - Action cards 2 to 5.*
 - 50 (or more) small objects.
 - Pencil and paper for each child.

* As for Num 6.1/1, 2

** Provided in the photomasters



Copy and complete the above number sentences.

Show what you have done by diagrams.

Repeat several times.

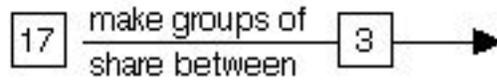
What do you notice? Can you explain why?

Figure 17 Different questions, same answer. Why?

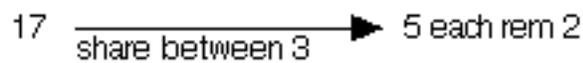
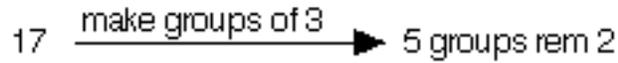
Num 6.3 Division as a mathematical operation (cont.)

Introducing the problem

1. The start and action cards are shuffled and put face down in the upper spaces on the 'Different questions, same answer . . .' [two-question] board.
2. The top card of each pile is turned and put face up in the lower space.
3. By reading above and below the line, there are now two unfinished number sentences. E.g.,



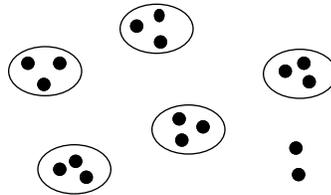
4. One (or more) in each group copies down the upper sentence, and one (or more) the lower sentence. They then complete whichever sentence they have written, using physical objects if they like. E.g.,



N.B. They should write these neatly, and keep them for later use in Activity 2.

5. They draw diagrams to show what they have done. These diagrams could be used in step 4, instead of physical objects. E.g.

Grouping



Sharing



6. They compare the two results.
7. Steps 2, 3, 4 are repeated. Step 5 need not be repeated every time: its purpose is to emphasize that these are two different questions.
8. Ask, "Will the two results always be the same? If so, why?"
9. Leave them to discuss this, and to arrive at a clear explanation. (One suggestion will be found in the 'Discussion of activities.')
10. Return and hear their explanation, discussing it if necessary.

Activity 2 Combining the number sentences [Num 6.3/2]

An activity for a small group. Its purpose is to teach the notation for the mathematical operation of division.

- Materials*
- The number sentences which they have written in Activity 1.
 - Pencil and paper for each child.

What they do 1. Have them compare the first number sentence of each kind. E.g.,

17 $\xrightarrow{\text{make groups of 3}}$ 5 groups rem 2

17 $\xrightarrow{\text{share between 3}}$ 5 each rem 2

2. Tell them that these two meanings may be combined in one number sentence. In this case, it would be

$$17 \div 3 = 5 \text{ rem } 2$$

This is read as

“17, divide by 3, result 5 remainder 2.”

or as

“17 divided by 3 equals 5 remainder 2.”

3. They then repeat steps 1 and 2 for the other number sentences.

Activity 3 Unpacking the parcel (division) [Num 6.3/3]

A continuation of Activity 2, for a small group. Its purpose is to remind children of the two possible meanings of a number sentence for division.

- Materials*
- Pencil and paper for each child.

What they do 1. They all write a division number sentence on their own paper. The first number should not be greater than 20.

$$11 \div 4 = 2 \text{ rem } 3$$

2. The first child shows his sentence to the others.
3. The next two on his left give the grouping and sharing meanings. In this case, “Start with 11, make groups of 4, result 2 groups remainder 3” followed by “Start with 11, share among 4, 2 in each share (or, each gets 2), remainder 3.” Note that “. . . groups,” or “. . . in each share,” are important parts of the expanded meaning.
4. Steps 2 and 3 are repeated until all have had their sentences ‘unpacked.’

Num 6.3 Division as a mathematical operation (cont.)

Activity 4 Mr. Taylor's Game [Num 6.3/4]

This game for 2 players was invented by Mr. Stephen Taylor, now Headmaster of Dorridge Junior School, and I am grateful to him for permission to include it here. Its purpose is to bring together addition, subtraction, multiplication, and division, in a simple game.

- Materials*
- Number cards: 1 set 0 to 25, 3 sets 0 to 9.*
 - Game board, see Figure 18.*
 - Counters of a different colour for each player.
- * Provided in the photomasters

- What they do*
1. The object is to get 3 counters together in a line. They must be in the same row, column, or diagonal.
 2. The number cards are shuffled and put face down, the 0 to 25 cards in one pile and the 3 sets of 0 to 9 cards in another.
 3. The first player turns over the top card of each pile. He may choose to add, subtract, multiply, or divide the numbers shown. Division must, however, be exact.
 4. He puts one of his counters on the square corresponding to the result of the operation chosen.
 5. The other player turns over the next cards on the two piles and carries out steps 3 and 4.
 6. Play continues until one player has 3 in a row.
 7. Another round may now be played. The loser begins.

Discussion of activities Activity 1 poses a problem, for children to solve by the activity of their own intelligence. When they have seen the connection between grouping and sharing, they have the higher order mathematical concept of division.

The easiest path to seeing the correspondence is, I think, a physical one. If we have 15 objects to share among 3 persons, a natural way to do this is to begin by giving one object to each person. This takes 3 objects, a single 'round,' which we may think of as a group of 3. The next round may be thought of as another group of 3, and so on. Each round gives one object to each share, so the number of rounds is the number in each share.

This verbal description by itself is harder to follow than a physical demonstration accompanied by explanation. This I see as yet another demonstration of the advantage of combining Modes 1 and 2.

Activity 2 provides a notation for these two aspects of division. Note that the first number is the operand, that on which the operation is done. The operation is the division sign together with the second number, e.g., $\div 3$.

Activity 3 is another example of 'Unpacking the parcel.' There is much less in this one than in the subtraction parcel, but is still a useful reminder of the two physical meanings combined in a single mathematical notation.

After all this, they deserve a game. Mr. Taylor's game fits in nicely at this stage.

OBSERVE AND LISTEN

REFLECT

DISCUSS

5	20	19	25	1
21	7	14	6	16
3	23	11	18	24
9	17	8	13	2
12	4	10	22	15

Figure 18 Mr. Taylor's Game.

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[Num 7] FRACTIONS

Double operations Numbers which represent these Fractions as quotients

Num 7.1 Making equal parts

Concepts (i) The whole of an object.
(ii) Part of an object.
(iii) Equal parts and their names.

Abilities To make, name, and recognize wholes, halves, third-parts, fourth-parts, fifth-parts, etc., of a variety of objects.

Discussion of concepts Many children have difficulty with fractions, and I think there are several causes which continue to bring this about.

(i) Fractions *are* difficult. Work with fractions is begun too early, and taken too far for children of elementary school age.

(ii) The same notation is used with three distinct meanings. For example, $\frac{2}{3}$ can mean a fraction, or fractional number, or a quotient.

(iii) The authors of many text books for children appear to confuse these three meanings, and they pass on their confusion to the children. It is rather like confusing the different physical embodiments of $7 - 3$: take-away, comparison, complement. But in this case the confusion is at a more abstract level. The distinction is not an easy one, and a full discussion of this I would consider as at least high school mathematics.

In the present network I have tried to present only the truth, but for the reasons above not the whole truth. We begin as usual by introducing the concept in several different physical embodiments. The terms 'third-part,' 'fourth-part,' etc., are used to distinguish fractions from the ordinal numbers third, fourth, etc.; and also as reminders that we are talking about parts *of something*. This is implicit in the word 'part,' but to start with we need to say so explicitly. Note that in this topic we are not yet talking about fractions, but about wholes and parts. This is just the first stage of the concept.

Activity 1 Making equal parts [Num 7.1/1]

An activity for up to 6 children. Its purpose is to make a start with the concept of a fraction, using two different physical embodiments.

Materials

- S-A-R boards, Fractions 1, Fractions 2, and Fractions 3. *
- Plasticine.
- Blunt knives.
- Cutting boards.

* See Figure 19. A full-size version, together with boards 2 and 3, will be found in the photomasters. For board 2, it is helpful also to have a cardboard template the size and shape of a granola bar, which children can cut around.

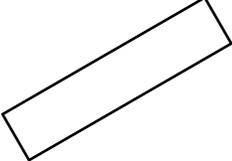
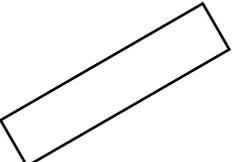
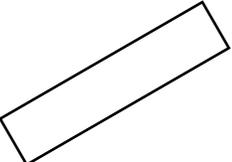
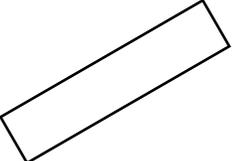
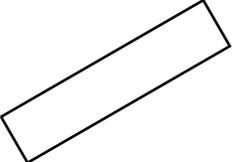
START	ACTION	RESULT	NAME
<p>A plasticine sausage.</p> 	<p>Leave this one as it is.</p>	<p>(Put it here.)</p>	<p>The <u>whole</u> of a sausage.</p>
<p>A plasticine sausage.</p> 	<p>Make 2 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>halves</u> of a sausage.</p>
<p>A plasticine sausage.</p> 	<p>Make 3 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>third-parts</u> of a sausage.</p>
<p>A plasticine sausage.</p> 	<p>Make 4 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>fourth-parts</u> of a sausage. Also called quarters.</p>
<p>A plasticine sausage.</p> 	<p>Make 5 equal parts.</p>	<p>(Put them here.)</p>	<p>These are <u>fifth-parts</u> of a sausage.</p>

Figure 19 SAR board 1. Making equal parts

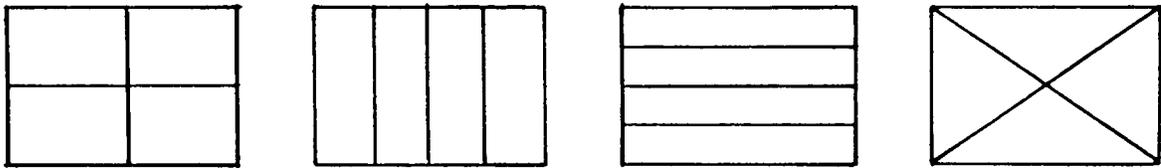
Num 7.1 Making equal parts (cont.)

What they do (Apportioned between the children.)

1. Six equal-sized round plasticine 'sausages' are made, by rolling 6 equal amounts of plasticine and trimming the ends to fit the sausage shapes on the left of board 1. One sausage is put into each outline.
2. The children do the actions described on the board. The lines of division may be marked lightly before cutting. In this way, trials can be made and corrected by smoothing out the marks.

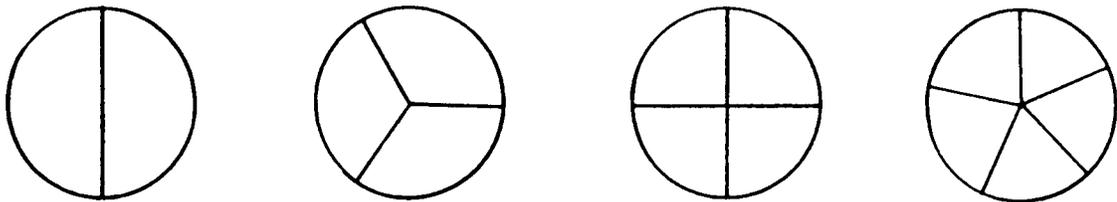


3. After cutting, the separated parts are put in the RESULT column next to their descriptions.
4. Steps 1 to 3 are repeated using Fractions Board 2 and fresh plasticine. Board 1 should if possible remain on view. With board 2, a variety of division lines are easily found. E.g., fourth-parts:



At this stage accept any correct results. These possibilities will be explored further in the next activity.

5. Steps 1 to 3 are now repeated using Fractions Board 3. If possible, fresh plasticine should be used, the other two boards remaining on view. The lines of division should be radial, as shown below.

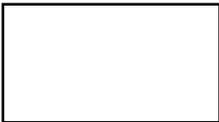


Activity 2 Same kind, different shapes [Num 7.1/2]

An activity for up to 6 children. Its purpose is to develop the idea that parts of the same kind may not look alike. In Activity 1, this arose from the use of different objects. Here we see that this can be so even with the same object.

- Materials*
- SAR boards: halves, fourth-parts, third-parts, see Figure 20.*
 - Plasticine.
 - Blunt knives.
 - Cutting boards.

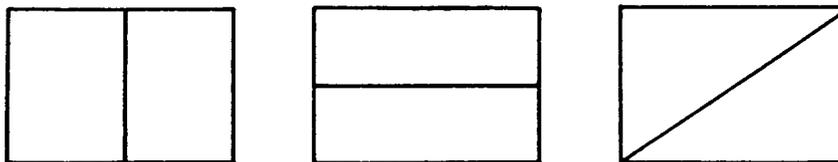
* Figure 20 illustrates the first of the SAR boards for this activity; the others are in the photomasters.

START	ACTION	RESULT	NAME
A granola bar. 	Make 2 equal parts.	(Put them here.)	These are <u>halves</u> of a granola bar.
	Make 2 equal parts in a different way.	(Put them here.)	These are also <u>halves</u> of a granola bar.
	Make 2 equal parts. in a different way.	(Put them here.)	These are also <u>halves</u> of a granola bar.

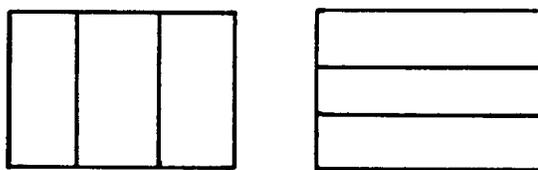
There are three simple ways. Later you may be able to find others.

Figure 20 SAR board 1. Same kind, different shapes. Halves

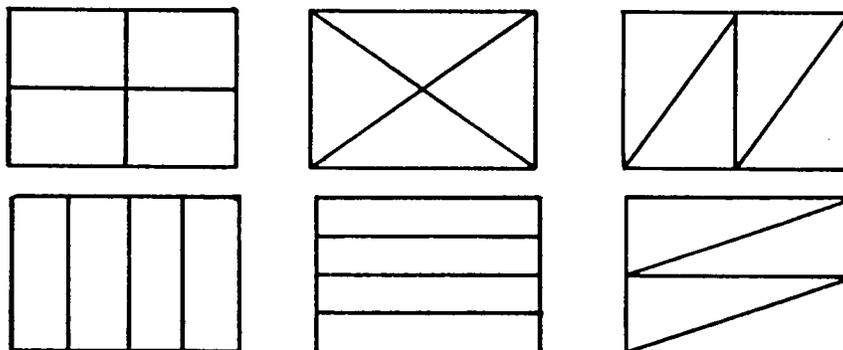
What they do 1. Begin with the halves board. This is used in the same way as the SAR board for Activity 1. The three straightforward ways are:



2. Next, they use the third-parts board. This offers only two straightforward ways.

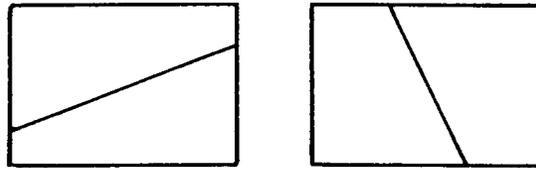


3. Next, they use the fourth-parts board. There are six ways of doing this which are fairly easy to find.



Num 7.1 Making equal parts (cont.)

4. Finally, they may like to return to the halves board, and try to find more ways. Here are 2 more, which can be varied indefinitely.



Activity 3 Parts and bits [Num 7.1/3]

A teacher-led discussion for up to 6 children. Its purpose is to emphasize that when we talk about third-parts, fourth-parts, etc., we mean equal parts. To make this clear, we use the word ‘bits’ when these are unequal.

- Materials*
- Plasticine.
 - Blunt knives.
 - Cutting boards.

- Suggested sequence for discussion*
1. Make 2 plasticine ‘sausages’ of equal size. Cut one into 3 equal parts, the other into three bits. Ask whether both of these have been cut into third-parts.
 2. If necessary, explain the difference, as described above.
 3. Have the children make some more sausages, all of the same size, and cut some into halves, fourth-parts, fifth-parts, and some into two, four, five bits.
 4. These are put centrally together with the parts and bits made in step 1.
 5. The children then ask each other for parts or bits. E.g., “Sally, please give me a fourth-part of a sausage”; or “Mark, please give me a bit of a sausage.”
 6. The others say whether they agree.
 7. Steps 5 and 6 are repeated as necessary.

Activity 4 Sorting parts [Num 7.1/4]

An activity for up to 5 children. Its purpose is to consolidate the concepts formed in Activities 1 and 2, moving on to a pictorial representation.

- Materials*
- Parts pack of cards.*
 - 5 name cards.**
 - 5 set loops.

* Some of these are illustrated in Figure 21. The complete pack is in the photomasters.

** These are marked WHOLES, HALVES, THIRD-PARTS, FOURTH-PARTS, FIFTH-PARTS.

- What they do*
1. Begin by looking at some of the cards together. Explain that these represent the objects which they made from plasticine, in the last activity – sausages, granola bars, cookies; and some new ones. They also represent the parts into which the objects have been cut, e.g., third-parts, fourth-parts, halves, fifth-parts. Some have not been cut: these are wholes.

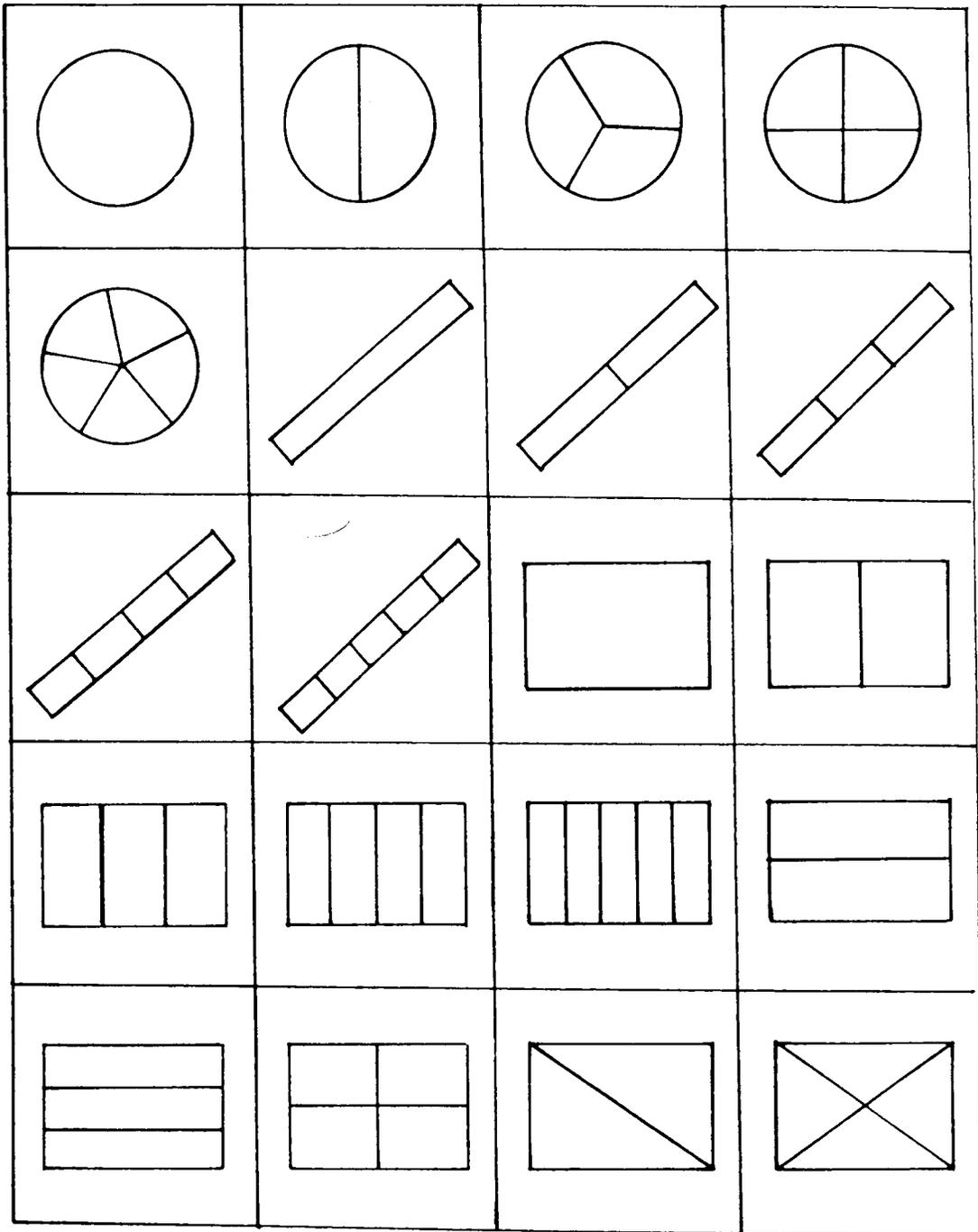


Figure 21 Cards for Sorting parts and Match and mix: parts

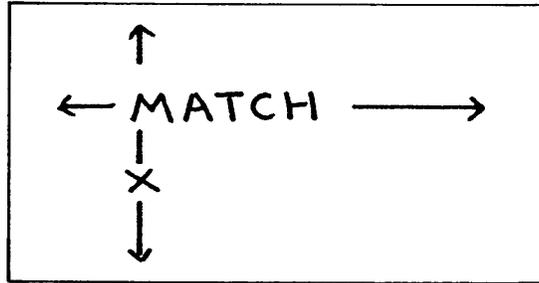
Num 7.1 Making equal parts (cont.)

2. The parts pack is then shuffled and spread out face upwards on the table.
3. The name cards are put face down and each child takes one.
4. Each child puts in front of him a set loop with the name card he has taken inside, face up.
5. They all then collect cards of one kind, according to the name card they have taken.
6. They check each other's sets, and discuss, if necessary.
7. If they are fewer than 6 children, they may then work together to sort the remaining cards.
8. Steps 2 to 7 may be repeated, children collecting a different set from before.

Activity 5 Match and mix: parts [Num 7.1/5]

A game for 2 to 5 players. Its purpose is further to consolidate the concepts formed in Activity 1.

- Materials*
- Parts pack of cards.*
 - A card as illustrated below (provided in the photomasters).
- *The same as for Activity 4.



- Rules of the game*
1. The cards are spread out face downwards in the middle of the table.
 2. The MATCH and MIX card is put wherever convenient.
 3. Each player takes 5 cards (if 2 players only, they take 7 cards each). Alternatively the cards may be dealt in the usual way.
 4. They collect their cards in a pile face downwards.
 5. Players in turn look at the top card in their piles, and put cards down next to cards already there (after the first) according to the following three rules.
 - (i) Cards must match or be different, according as they are put next to each other in the 'match' or 'mix' directions.
 - (ii) Not more than 3 cards may be put together in either direction.
 - (iii) There may not be two 3's next to each other.(‘Match’ or ‘mix’ refers to the kind of part.)

Examples (using A, B, C . . . for different kinds of part).

A typical arrangement.

C
AAA
BBB
DDD
B
A

None of these is allowed.

A	BBBB	CCC	AA	A
B		DDD	BB	B
C			CC	A
D				

5. A card which cannot be played is replaced at the bottom of the pile.
6. Scoring is as follows.
 - 1 point for completing a row or column of three.
 - 2 points for putting down a card which simultaneously matches one way and mixes the other way.
 - 2 points for being the first ‘out.’
 - 1 point for being the second ‘out.’
 - (So it is possible to score up to 6 points in a single turn.)
7. Play continues until all players have put down all their cards.
8. Another round is then played.

Discussion of activities

Activities 1, 2 and 3 make use of Mode 1 schema building (physical experience). This is every bit as important when introducing older children to fractions as physical sorting is when introducing young children to the concept of a set. In the present case, it is the physical action of cutting up into equal parts from which we want the children to abstract the mathematical operation of division. In this case, the operand is a whole object, so it is important to have something which can easily be cut up. Plasticine is ideal for this, since children can first make the right shapes and then cut them up.

Activities 4 and 5 move on to pictorial representation of these physical actions. This is where most text books begin. What is not understood is that the diagrams condense no less than six ideas, as will be seen in Num 7.3 in *Volume 2*. These diagrams are likely to be helpful if and only if the children already have the right foundation schema to which these can be assimilated. Otherwise, here is the first place where children can get confused.

So in this topic, we begin with physical actions on objects: making equal parts. We then introduce diagrams representing these and no more. We are not yet into fractions; just objects, and parts of objects. The key ideas are that the parts must all be equal in size (or we call them ‘bits’); and that the kind of parts we are talking about depends only on how many the object is cut up into, not on their shape, nor on what the object is. These ideas are encountered first with physical objects, then with diagrams.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Space 1] **SHAPE**

Shapes in the environment and in mathematics

Space 1.1 SORTING THREE DIMENSIONAL OBJECTS

Concept Simple three-dimensional shapes, applied to everyday objects.

Ability To sort suitable selections of everyday objects.

Discussion of concept The concept of shape is part of our everyday life and vocabulary. Here we are singling out some particular shapes which have mathematical counterparts.

Activity 1 Sorting by shape [Space 1.1/1]

A teacher-led discussion for a small group. Its purpose is to concentrate children's attention on shape, among the various other attributes which objects may have. The word will no doubt already be part of their everyday vocabulary.

- Materials*
- For Stage (a): a collection of everyday objects (convenient sizes) which approximate the shapes of spheres, cuboids, and cylinders; and others which are none of these.
 - for Stage (b): as for Stage (a) plus cones.

- Note*
- (i) A cuboid is a solid of which every face is a rectangle. If all the faces are squares, the solid is called a cube.
 - (ii) While the 'everyday' cone-shaped object that is most likely to come to mind is a traditional ice cream cone, there are others! (E.g., some party hats, some 'funnels,' . . .)

- Suggested outline for the discussion*
- Stage (a)*
1. The children should already be familiar with the activity of sorting, which is the first topic in the network Org 1, 'Set-based organization.'
 2. Put out the objects and ask for suggestions for sorting these.
 3. There are of course many ways which do not include shape, such as eatable/not-eatable (since some of the cans are likely to be food cans). Accept any of these which are valid, and allow them to sort in one or more of these ways.
 4. If the objects are well chosen, shape will be a salient attribute and there is a good possibility that someone will suggest this. If not, call attention to it.
 5. Ask them to sort the objects by shape.
 6. Invite suggestions for what we will call the different kinds of shape into which the objects have been sorted. Suitable everyday words are balls (spheres), boxes (cuboids, including cubes) and cans (cylinders). I have given the mathematical names in brackets, and in the event that any of these are suggested, this is of course acceptable.

Stage (b) As for Stage (a), but including the cone-shaped objects.

Activity 2 Do they roll? Will they stack? [Space 1.1/2]

A teacher-led activity for a small group. Its purpose is to introduce further attributes of objects which are related to and dependent on shape.

Materials • As for Activity 1.

- Suggested outline for the discussion*
1. This follows the same lines as Activity 1.
 2. If none of the children thinks of these attributes, you could demonstrate rolling/not rolling and ask what they notice.
 3. When they have sorted by this attribute, do likewise for stacking/not stacking.
 4. Now choose an object which has not been used before, and ask the children if they can tell by looking at it whether it will roll. Allow them to test their prediction.
 5. Ask them to sort all the objects into two groups, will roll/won't roll, just by looking at them.
 6. Let them test their predictions for both groups, one object at a time.
 7. Repeat steps 5 and 6 for will stack/won't stack.



Do they roll? Will they stack? [Space 1.1/2]

Note A can (or cylinder) is an interesting example, since it rolls on its side, and stacks on its end. So which set should we put it in? For my own suggestion, please see Org 1.2/4, 'Which *two* sets am I making?', steps 4 and 5.

Discussion of activities Starting with selected objects from everyday life, the children are laying foundations for the idea of shape in its more abstract and mathematical meaning. They are also using shape as a basis for testable predictions. Since this is a group activity, discussion is also involved.

Space 1.2 SHAPES FROM OBJECTS

Concept Outlines taken from objects, in various positions and aspects.

Ability To match an object to one or more of its outlines.

Discussion of concept Though most of the objects in our environment are three-dimensional, our view of them at any given moment is two-dimensional. What we usually see is an outline, with other detail within this outline from which we derive an awareness of solidity. This we can do without knowing how we do it. Our perceptual processes take care of it for us, usually without conscious thought.

When we describe a geometrical solid, such as a cube or a pyramid, we do so by saying how many faces it has and what are their shapes, how many edges, how many vertices. This is a more conscious activity, involving seeing, thinking, and naming. In this topic, we begin to make conscious the relationships between objects and the outlines of their faces.

Note that in Space 1.5, Activity 1 (p. 295) we use the terms 'square' and oblong.' These are two kinds of rectangle, in the same way that boys and girls are two kinds of children.

Activity 1 Matching objects to outlines [Space 1.2/1]

A game for a small group. Its purpose is to teach children to abstract the outline of an object.

- Materials*
- A tray of objects, each of which can be put flat and drawn around.
 - An outlines card for each child. Each card has on it five outlines (more if you like) obtained by putting one of the objects flat on the card, and drawing around it.

Note on materials Suitable objects might be a nickel, an eraser, a paper clip, bottle tops of various sizes and shapes, little stones which have at least one face flat or nearly so. If some of the objects can be drawn round in more ways than one, so much the better. Altogether, there must be at least one object for every space on every card. E.g., if there are 6 cards each with 5 outlines, the tray will have on it at least 30 objects. This will become clearer after you have read the description of the game. The objects need not all be different.

- Rules of play*
1. The tray of objects is put in the middle. Each child has an outlines card.
 2. The purpose of each player is to fill his card with objects matching the outlines.
 3. Each in turn takes an object from the tray and puts in on his card to match an outline.
 4. If it does not match, she has to return it to the tray and it is the next player's turn. (This rule may be relaxed at the learning stage.)
 5. The winner is the one who first fills her card. However, the others should continue until all have filled their cards.

Discussion of activity This activity involves quite a lot of abstracting. An outline on card looks very different from a solid object. The differences are greater than the resemblances. Activity 1 allows physical comparison of objects and outlines – Mode 1 testing.

Space 1.3 LINES, STRAIGHT AND CURVED

Concept Straight lines and curved lines.

Abilities (i) To recognize examples of each.
(ii) To produce examples of each.

Discussion of concepts These concepts are straightforward. However, it is worth noting that there is a distinction between the everyday usage of the word 'line' and the mathematical usage. In everyday usage, we talk (e.g.) about a thick line and a thin line. To a mathematician, a line has no thickness and no width. Just as the numeral 5 (which is a mark on paper) is a symbol for the number 5 (which is a concept, a mental object), so also the (everyday) lines we draw on paper are ways of representing mathematical lines.

At this stage we will of course not be explaining this distinction to children, but it is as well to know it ourselves.

Activity 1 Drawing pictures with straight and curved lines [Space 1.3/1]

An activity for up to 6 children.

- Materials*
- A picture for each child.*
 - Coloured non-permanent pens.
 - A spinner, with spaces marked 'straight line' or 'curved line,' each with an example of a straight or curved line, or a die with 'straight line' written on 3 faces and 'curved line' on the others (each with an example).

**Note* Specimen pictures are provided in the photomasters. Each is made up of straight and curved lines. It should be clear where each line starts and finishes. The lines should be somewhat faint.

- What they do*
1. Each child in turn spins the spinner.
 2. He is then allowed to colour one line in his picture, straight or curved according to where the spinner points.
 3. If (e.g.) there are only straight lines left uncoloured, and the spinner points to 'curved line,' he cannot use his turn.
 4. As they complete their pictures, players withdraw from steps 1 and 2.
 5. Since this activity is for emphasizing straight and curved lines, which are used as outlines, it is probably better not to encourage them to colour in their pictures. If they particularly want to, they could be allowed to do so.

Activity 2 “I have a straight/curved line, like . . .” [Space 1.3/2]

A game for up to 6 children. Its purpose is to relate the concepts straight and curved to objects in the environment.

Materials • A pack of cards, each of them having a single line, either straight or curved: about 10 of each.*

*Provided in the photomasters

- Rules of the game*
1. The cards are shuffled, and the pack is put on the table face down.
 2. The first player picks up the top card, looks at it, and says (e.g.) “I have a curved line, like the edge of that flower pot.”
 3. The others respond by saying “Agree,” or perhaps “Disagree.”
 4. If the latter, there is discussion until agreement is reached.
N.B. Curved lines need not be exactly similar to the shapes in the environment.
 5. The next player in turn picks up a card, and steps 2, 3, and 4 are repeated until all the cards have been used.

Activity 3 “Please may I have . . . ?” (Straight and curved lines) [Space 1.3/3]

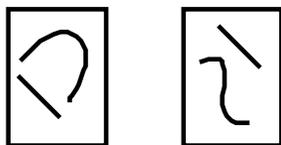
A game for 4 or 5 children.

Materials • A pack of cards like those illustrated in step 2 below.*

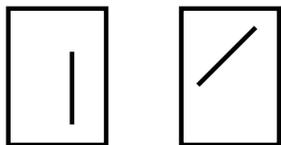
* Provided in the photomasters

- Rules of the game*
1. The cards are shuffled, and all are dealt to the players.
 2. The object is to form pairs of cards which are alike in their kinds of line,

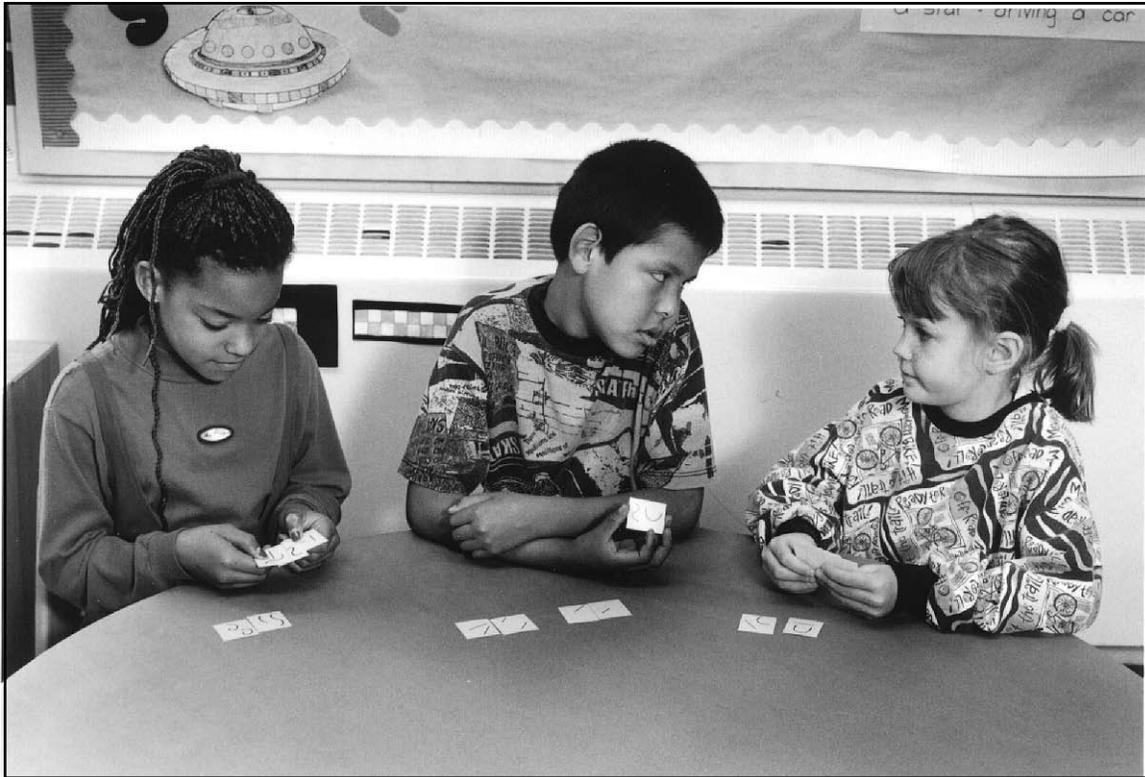
such as these



or these



3. Players begin by putting down any pairs which they have.
4. Then, starting with the player to the left of the dealer, children ask in turn for cards which they need to make pairs with cards they already have.
5. They may ask whoever they like. E.g., “Please, Denise, may I have one straight (line) and one curved (line)?” (The words in brackets may be omitted if you find it necessary to shorten the sentence.)
6. If the player asked has the card, he gives it. Otherwise he replies “Sorry.”
7. The first player to put down all his cards is the winner, but the others continue play until all have put down all their cards.



“Please may I have . . . ?” (Straight and curved lines) [Space 1.3/3]

Discussion of activities The first activity, drawing pictures with straight and curved lines, requires children to distinguish between straight and curved lines in their pictures. These lines they then copy, in the simplest possible way by tracing over them. Here we have Mode 1 concept building by two kinds of experience, seeing and doing. Activity 2 involves putting these concepts to use by seeking out examples in the environment. Differences such as length of line, degree of curvature if curved, have to be ignored, centring only on the property straight or curved. Activity 3 is particularly for linking these concepts with their mathematical vocabulary.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.4 LINE FIGURES, OPEN AND CLOSED

- Concepts*
- (i) Open and closed figures.
 - (ii) Boundaries.
 - (iii) Inside and outside.

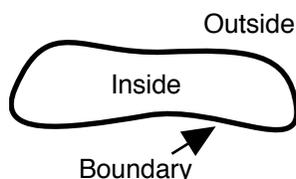
- Abilities*
- (i) To distinguish between open and closed figures.
 - (ii) To associate these with the property of a boundary as described below.

Discussion of concepts Here, for a change, the everyday and the mathematical meanings of the words are not very different. If the door is open, we can enter and leave the room. While it is closed, we cannot.

In mathematics, a figure is *closed* if it makes a boundary between two regions, an inside and an outside, such that we cannot get from one to the other without crossing the boundary. Otherwise the figure is *open*.

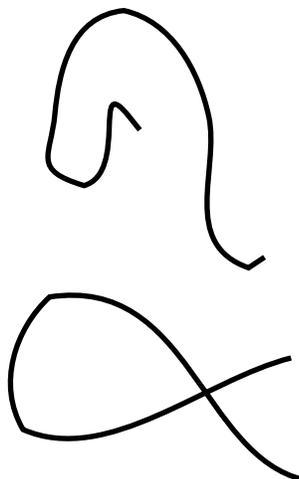
An open figure does not have an inside and an outside.

Closed figure



This figure also has an inside and an outside, so we may agree to call it closed. Not all of the figure is part of the boundary.

Open figure

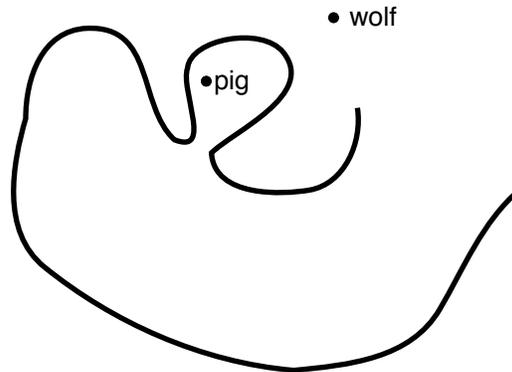


Activity 1 “Can they meet?” [Space 1.4/1]

A game for 2 to 6 children. Its purpose is to help children to form the concept of a boundary.

- Materials*
- Four small animal models. You might choose a pair which are friends, e.g., a sheep and a cow, and a pair which are enemies, e.g., a pig and a wolf.
 - A length of cord.

- Rules of play*
1. Explain that the string represents a fence, which the animals cannot get past.
 2. Explain that some animals are friends, and some are enemies.
 3. One child now arranges the length of string on the table to represent the fence.
 4. He also places two of the animals on the table, and asks “Can they meet?” (Initially this should be made easy, but as the game progresses children will enjoy making it more difficult to decide: see illustration.)



5. When all the children have decided on their answers, the animals are ‘walked’ towards each other by one of the other children to find the correct answer.
6. Steps 1 to 5 are then repeated, by other children.

- Notes*
1. There is no fixed rule that friendly animals must always be allowed to meet, or that enemies must always be kept apart. This makes the prediction more interesting!
 2. It may be a good idea for the other children to close their eyes during steps 3 and 4.

Activity 2 Escaping pig [Space 1.4/2]

A game for 2 of more players. Its purpose is to consolidate the concept of a boundary.

- Materials*
- Game board (see Figure 22), covered with transparent film.*
 - Pig.
 - Non-permanent marker.
 - Damp rag to clean board.
 - Die marked ‘open’ on 3 faces, ‘closed’ on 3 faces.
- * Provided in the photomasters

- Rules of play*
1. Player A puts his pig outside the sty.
 2. The other player(s) throw the die for each gate in turn. With the erasable marker, they draw in the gate open or closed according to the fall of the die.
 3. When all the gates are drawn, player A predicts whether or not the pig can reach the wood. If she thinks that it can, she tests by ‘walking’ the pig there.
 4. The board is cleaned, and steps 1 to 3 are repeated with another child as player A.

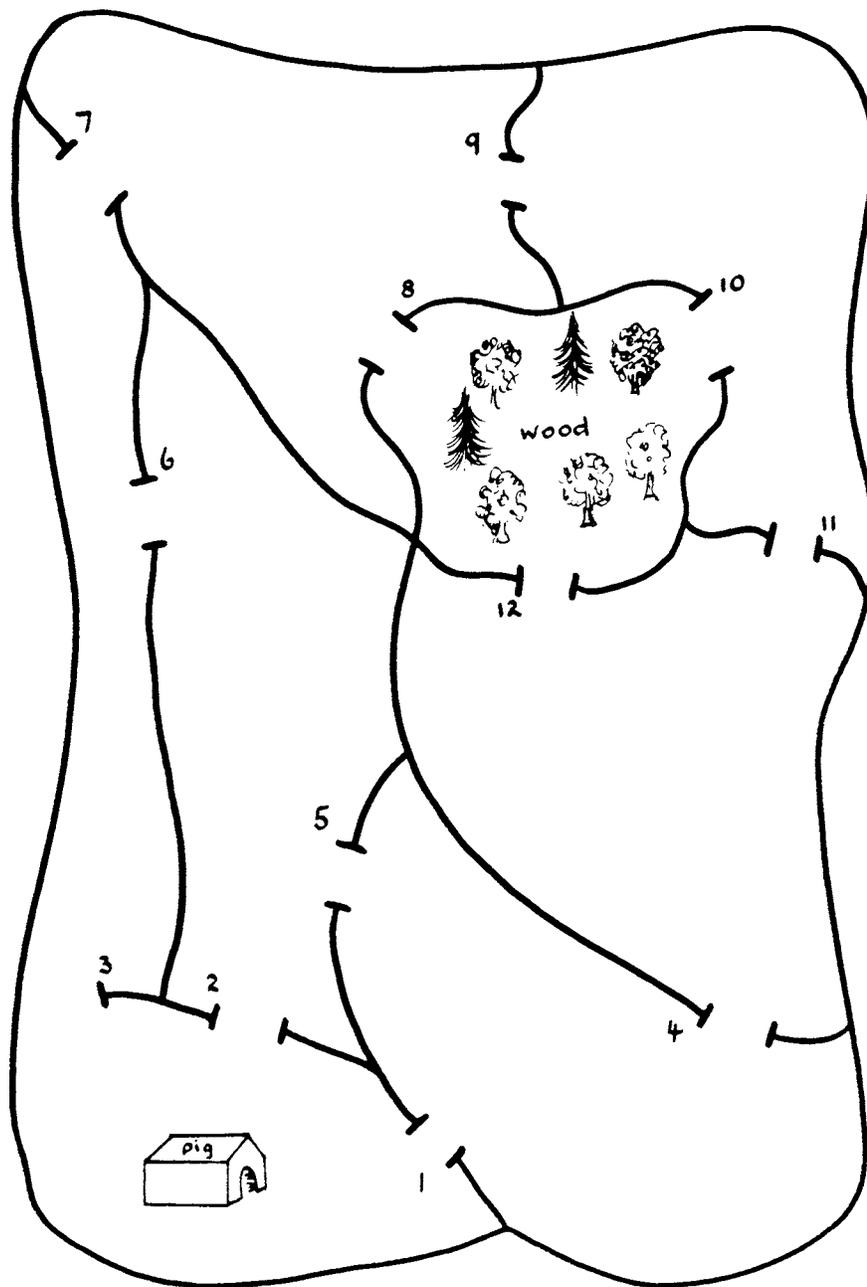


Figure 22 Escaping pig

Activity 3 Pig puzzle [Space 1.4/3]

A game for 2 players. It follows on from Activity 2, and its purpose is the same.

Materials As for Activity 2, except that the die is replaced by a pack of 6 cards, 3 marked 'Yes' and 3 marked 'No.'

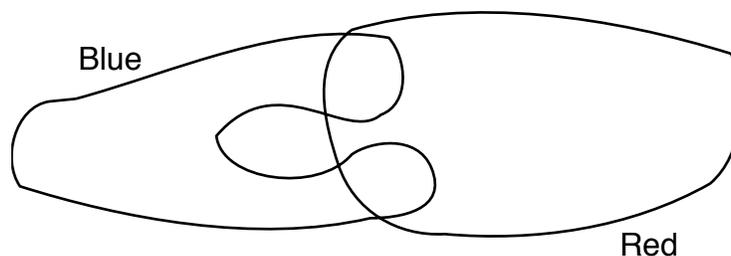
- Rules of play*
1. The 'Yes' and 'No' cards are shuffled and put face down.
 2. Player A takes a card and looks at it herself, but does not show it to the other player.
 3. Player A now marks the gates so that the pig can or cannot reach the wood, according to whether her card is 'Yes' or 'No.' She may make this a complicated route (or non-route), to 'fool' the other player.
 4. Player B then predicts whether the pig can or cannot reach the woods.
 5. B's prediction is compared with A's card. If these disagree, one of them tests physically by 'walking' the pig to the woods. (The testing could be done by the player who thinks that the pig can reach the woods, since a negative result is harder to prove.)
 6. After clearing the board, the children change roles.

Activity 4 Inside and outside [Space 1.4/4]

An activity for 2 or more children. Its purpose is to exercise the new concepts in a more difficult situation.

- Materials*
- 2 loops of cord, in different colours: say red and blue.
 - 2 packs of about 8 cards. Each is of a colour matching one of the cords. In each pack half the cards are marked 'inside' and half are marked 'outside'.
 - A marker.

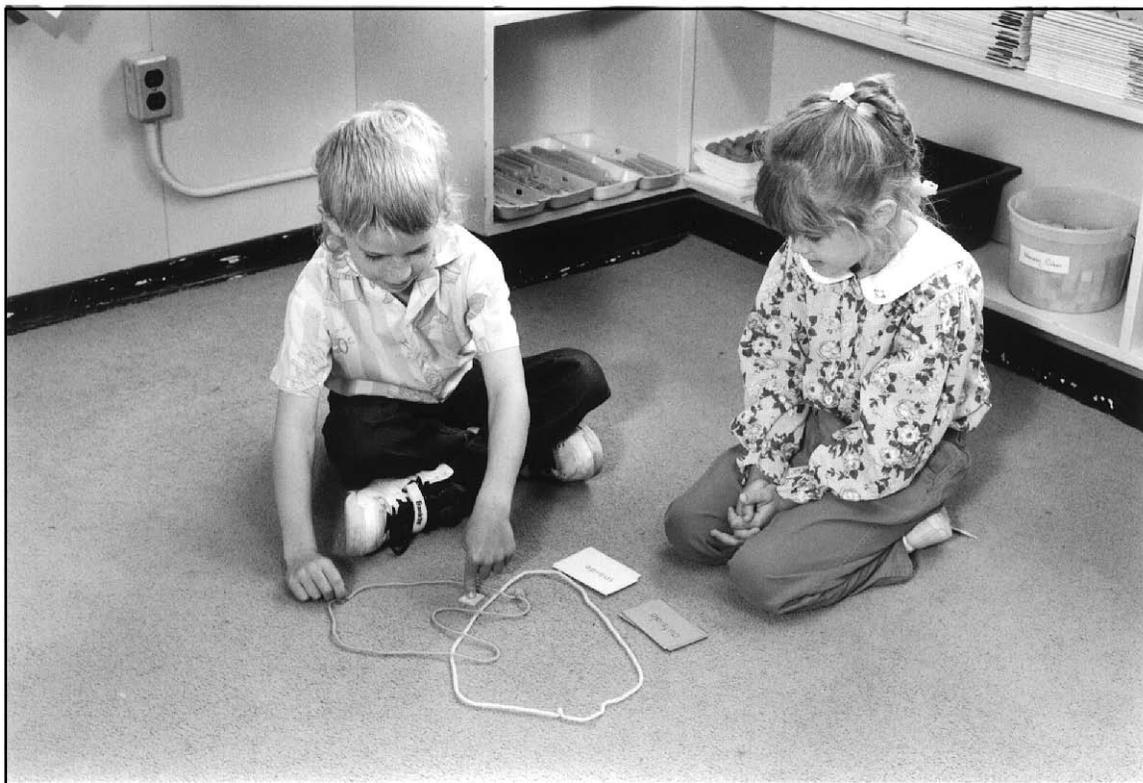
- Rules of play*
1. The loops are laid out in any shapes so that they overlap. E.g.:



2. The cards are shuffled and put face down on the table.
3. Player A takes one card from each pile. He looks at these but does not show them. Suppose that the red card reads outside and the blue card reads inside. The marker is put on the table accordingly.
4. Player B must now describe the position of the marker. In this example he would say, 'Outside the red loop and inside the blue loop.'

Space 1.4 Line figures, open and closed (cont.)

5. B's description is compared with A's cards.
6. They then check by holding a finger on the table where the marker is, and pulling gently on the loops. If the marker is outside, the loop will come away; if inside, the finger will hold it.



Inside and outside [Space 1.4/4]

7. Steps 1 to 6 are repeated with other children as A and B.

Note This game may be played with a single loop and one pack of cards. This provides a simple introduction to the version described.

Discussion of activities Activities 1, 2, and 3 all use physical objects for learning and applying the concepts of inside/outside, and boundary. These are properties which a closed figure has, and an open figure does not have. They lead to physical predictions which can be tested. Activity 4 is more sophisticated, and not only in the way which is easily apparent, that of using two overlapping figures. It also uses another property of a closed loop, that it may have its shape changed (as long as we do not cut it) without affecting its inside/outside property.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Space 1.5 SORTING AND NAMING TWO DIMENSIONAL SHAPES

- Concepts*
- (i) Those of a circle, square, oblong, rectangle, triangle.
 - (ii) That of a geometric shape, of which the above are some examples.

- Abilities*
- (i) To recognize examples of the above shapes.
 - (ii) To distinguish between these and non-examples.

Discussion of concepts These are easily identifiable geometric shapes, which children may already have met when using attribute cards (Org 1.1/7) and attribute blocks (Org 1.2/3,4). There is only one point which needs discussion, which is the use of the terms 'square' and 'rectangle.' I prefer the usage explained in step 4 of Activity 1, since the alternative usage seems to me like referring to 'boys' and 'children' instead of 'boys' and 'girls.' However, I would not go so far as saying that one was correct and the other incorrect. (There are other cases where I would.)

Activity 1 Sorting and naming geometric shapes [Space 1.5/1]

A teacher-led activity for a small group of children. Its purpose is to introduce children to the concepts described above, or consolidate these if they have met them already.

- Materials*
- Cut-out cardboard shapes of all the above shapes, in a variety of sizes. In the case of triangles and rectangles, these should also vary in shape. The number of each kind should vary from six to eight or nine. All should be of the same colour.

- What they do*
1. Spread these out on the table, and ask the children if they can sort them. There should be no difficulty about this, since they will have done a variety of sorting activities already.
 2. In the event that the squares and oblongs are put together in the same set, ask if they can sort them any further. (Usually, children distinguish between these.)
 3. Ask if they know the names of these different shapes. If they come up with meaningful everyday names, such as ring for circle, accept these, and explain that the mathematical name is (in this case) 'circle.'
 4. They are quite likely to use the name 'rectangle' for any rectangle which is not a square. You could explain that many people use the word in this way, but since the name means that is right angled, it is more accurate to use this for all shapes which have right angles at each corner, and use the words 'squares' and 'oblongs' to distinguish the two different kinds of rectangle.
 5. They should now be ready to do Activity 2 by themselves.

Activity 2 Sorting and naming two dimensional figures [Space 1.5/2]

This is a direct follow-on from Activity 1, for a small group of children. Its purpose is to consolidate the above concepts in a slightly more abstract form.

- Materials*
- A set of cards, all the same size, on which are geometrical figures of the same description as the cut-out shapes used for Activity 1. They should vary in both size and orientation on the card.

- What they do*
- (i) Ask them to sort these cards in the same way as they sorted the cut-out shapes.
 - (ii) When they have finished, they should carefully check all the sets and discuss any disagreements.

Activity 3 I spy (shapes) [Space 1.5/3]

An activity for a small group of children. Its purpose is to give children practice in relating abstract geometrical figures to objects in the everyday environment.

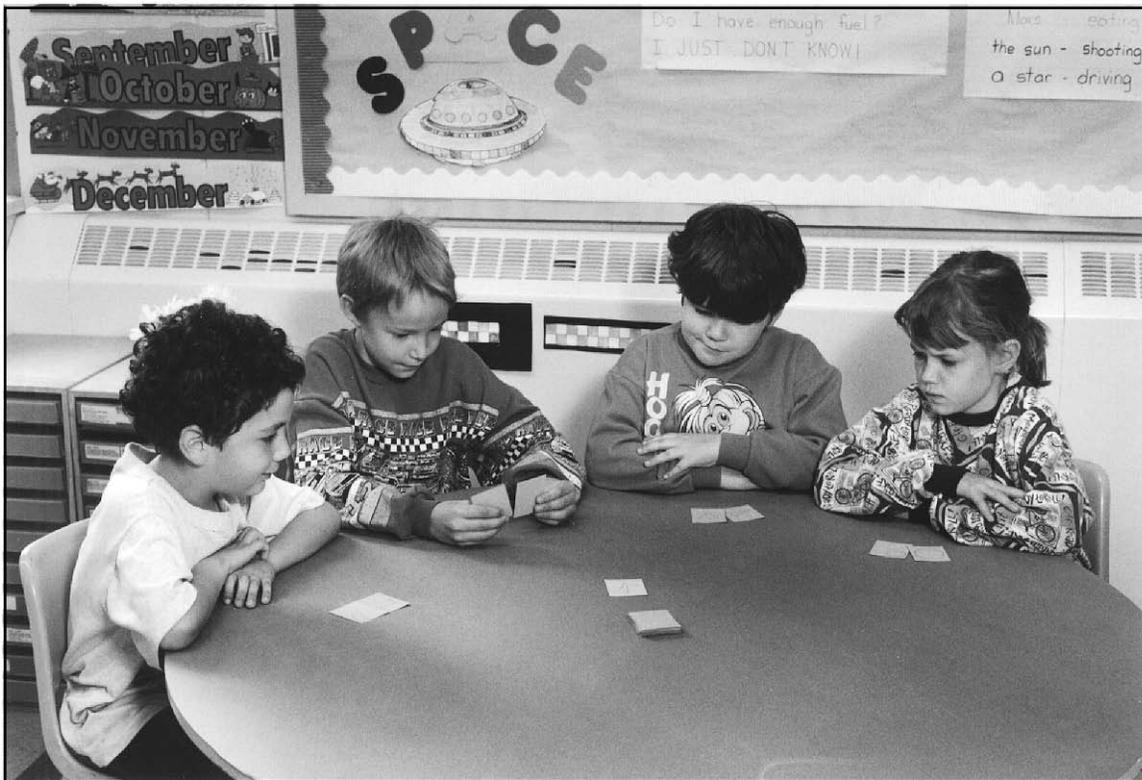
- Materials*
- One card of each of the shapes they have been using, namely circle, square, two oblongs, triangle. One oblong should be about 6 cm by 4 cm, the other about 10 cm by 3 cm. The triangle should be scalene, i.e., not of any particular kind, such as right-angled or equilateral.
 - A blank sheet of paper.

- What they do*
1. One of the players chooses a shape which she can match with something in the environment. This might be, say, a door and an oblong. She puts this shape on top of the paper, the others being out of sight underneath.
 2. She then says, “I spy, with my little eye, something shaped like this – an oblong.”
 3. The others then try to guess what this is. (You may decide that they take turns, or let players speak when they are able to make a guess.)
 4. The first player to guess right is the next to do the “I spy”
 5. If no one can guess right, they may give up and the spier tells what the object is.

Activity 4 Claim and name (shapes) [Space 1.5/4]

An activity for a small group of children. Its purpose is to consolidate the concepts they have been using, and introduce them to the idea of non-examples.

- Materials*
- A set of cards similar to those used in Activity 2, but with the addition of about a dozen non-examples. These should be figures which are not any recognized geometric shape, such as a figures with two curved and two straight sides. It is better not to include figures which they will learn later, such as rhombi and semi-circles, since we do not want to give the impression that these are not recognized geometric kinds of shape, even though we are not teaching them yet.



Claim and name (shapes) [Space 1.5/4]

- What they do*
1. The cards are shuffled and put in a pile face down on the table. The top card is then turned face up and put separately.
 2. The players then take turns to turn over the top card from the face-down pile, and put it face up separately from the others.
 3. If the player whose turn it is sees two cards of the same shape, he claims them and says what shape they both are. E.g., "I claim these two, both oblongs." He then collects that pair.
 4. The turn then passes to the next player.
 5. As the number of face-up cards increases, clearly the chance of there being two alike increases. Some of them will however be non-examples, which makes it a little more difficult.
 6. If the player whose turn it is overlooks a pair, the next player may claim it before taking his turn in the normal way. And if this player overlooks it, the same applies to the next player and so on.
 7. The winner is the player with most pairs.
 8. At the end, they look at the cards remaining, and discuss why no one has claimed any of them. These are the shapes which do not fall into any recognized category.

Discussion of activities In these activities, children are repeating in two dimensions what they have already done in three dimensions.

Space 1.6 SHAPES FROM OBJECTS AND OBJECTS FROM SHAPES

- Concepts*
- (i) Those of a sphere, cylinder, cuboid, cone.
 - (ii) That of a mathematical (geometric) shape.

- Abilities*
- (i) To name correctly physical models of these concepts.
 - (ii) To relate them to everyday objects.

Discussion of concepts In the earlier topic Space 1.1, ‘Sorting three-dimensional objects,’ the children have formed the above concepts, under everyday names, from everyday objects. In this topic, they meet physical embodiments of these concepts, and form connections in the reverse direction, from mathematical object to everyday object.

Activity 1 “This reminds me of . . .” [Space 1.6/1]

An activity for a small group of children. Its purpose is to help them begin relating geometric models to everyday objects. At this stage, these are physical models, rather than mental models.

Materials

- Geometric models of spheres, cuboids, cylinders and cones.

- What they do*
1. For this activity, the children take turns around the table.
 2. The models are put out on the table, and the first child chooses one of them. You tell him the name of the one he has chosen.
 3. He holds it so that all can see it, and says what it reminds him of. E. g., if he is holding a cuboid he might say “This cuboid reminds me of a box of chocolates.” Note that he should use the name of the object, for practice in remembering this name. The others either say “I agree,” or if they don’t agree they explain why not.
 4. The child whose turn it is next chooses from the remaining models, and steps 1 and 2 are repeated.
 5. This continues until all the models have been used. If each child retains her chosen model for the time being, this ensures that all the models are used.
 6. When all the models have been used, steps 1 to 4 may be repeated with a different child starting. Since the choice gets less for each successive child, it might be fairest for the child who was left with the last object to have the first choice next time.

Discussion of activity Though the objects they are here working with are physical objects, they are nevertheless abstracting in that they omit all the qualities we find in everyday objects except shape. Moreover, there are few everyday objects which have these shapes ‘without extras.’ So the children are now working from mathematical and general ideas to everyday particular objects, and checking their own thinking against that of others in their group.

Space 1.7 NAMING OF PARTS

Concepts (i) The principal parts, with names, of the following geometrical solids and figures: spheres, cylinders, cuboids, cubes, prisms, pyramids, circles, oblongs, squares, triangles.

(ii) The relation between the faces of these solids and the two-dimensional figures.

Abilities (i) To name any part indicated.

(ii) To indicate any part named.

(iii) To demonstrate the relationships between two-dimensional figures and faces of the solids.

Discussion of concepts It is sometimes argued that since we live in a three-dimensional world, the study of geometry should start with three dimensions. In fact, the mathematics of three dimensions is much more difficult than that of two, and is nearly always approached by reducing it to two at a time. (Try to draw a diagram for a path going obliquely up a hillside, then consider the problem of calculating its slope, and you will see what I mean.) Moreover, if we want to describe a solid exactly, we do so in mainly terms of two- and one-dimensional attributes. (A cuboid is a solid of which all the faces are rectangles.) So this topic is almost as far as we shall take the study of three-dimensional shapes, and this has been almost entirely descriptive. However, in the last activity of this topic, we do take a short step beyond by looking at the relation between three-dimensional and two-dimensional shapes.

Activity 1 “I am touching . . .” (three dimensions) [Space 1.7/1]

A teacher-led activity for a small group of children. Its purpose is to give practice in identifying the principal parts of geometric solids and figures, and remembering their names. The titles of this and the next three also introduce the terminology of dimensions.

Materials • A set including all the geometrical models they have been using so far, namely spheres, cylinders, cones, cuboids, with the addition of pyramids and prisms. There needs to be at least as many models as children, and preferably more to give a variety of examples. Also, there should be at least two kinds of prism, say triangular and hexagonal. (A square prism would normally be called a cuboid. A circular prism is a cylinder.)

What they do

1. The solids are put on the table where all can see them.
2. Since they have not yet worked with pyramids or prisms, I suggest you begin by asking whether there are any they haven't seen before, and telling them their names. Explain also why the two kinds of prism are both called by the same name, and invite suggestions for what other kinds of prism there could be .

Space 1.7 Naming of parts (cont.)

3. Each child then takes one of the models, and you take one yourself. (I suggest that you don't begin by taking a cone.)
4. Suppose that you have taken a cuboid. Touch one of its vertices, and say "I am touching a *vertex* of my cuboid."
5. In turn, the children now do the same with their own objects. E.g., "I am touching a vertex of my pyramid."
6. A child with a sphere would need to say something like "I can't touch a vertex of my sphere because it doesn't have one." This makes things more interesting.
7. When all have had their turn, everyone takes different objects and steps 3 to 6 are repeated. In step 4 you might now say, e.g., "I'm touching a *face* of my pyramid."
8. Repeat as before for *edge* and *curved surface*. The children will discover that not every object has every kind of part. In fact there is only one kind of object which has a part of every kind, and one of them has only one kind of part.
9. When they have once done this with you, they should then be able to do activity 2 on their own.

Activity 2 "Everyone touch . . ." (three dimensions) [Space 1.7/2]

An activity for a small group of children. Its purpose is to consolidate the new concepts and vocabulary introduced in Activity 1.

Materials • All the geometric models used in Activity 1.

- What they do*
1. Each child takes one of the objects.
 2. The child whose turn it is first says (e.g.) "Everyone touch an edge."
 3. Each child then touches an edge of her own solid, and looks around to check that the others are doing it correctly.
 4. Any child whose solid has not the part in question says so. In the present example someone with a sphere would say "I can't because a sphere doesn't have an edge." The others decide whether this is so.
 5. They all change their objects for different ones, and steps 1 to 4 are repeated.

Activity 3 "I am pointing to . . ." (two dimensions) [Space 1.7/3]

Activity 4 "Everyone point to . . ." (two dimensions) [Space 1.7/4]

These are repeats of Activities 1 and 2, except that instead of solids, flat shapes are used (circles, squares, oblongs, triangles). The parts to be identified and named are now: side, vertex, and (for circles only) circumference.

Materials • Two or three of each of the cut-out shapes first introduced in 'Sorting and naming geometric shapes' [Space 1.5/1].
• It may be useful to have something to help more exact pointing, such as pencils.

What they do These activities are the same, step by step, as Activities 1 and 2, except for the differences in materials and vocabulary.

Activity 5 “My pyramid has one square face . . .” [Space 1.7/5]

An activity for a small group of children. Its purpose is to call their attention to the relation between the shapes of the solids, and the number and shape of their faces. This will be useful preparation for the more advanced study of the nets of solids.

Materials • The geometric solids used in Activities 1 and 2.

- What they do*
1. Each child takes one of the solids and studies the shapes of the faces, and the number of each shape.
 2. In turn, they share with the others what they have found. The first child might say, for example: “My pyramid has one square face and four triangular faces.” The next child might then say, “My cuboid has six oblong faces.” and so on. A child who took a sphere might say something like: “My sphere doesn’t have a face at all – only a curved surface and only one of those.” This would emphasize that by ‘face’ we mean a flat face.
 3. When all have had a turn, they return their models to the middle, and take a different one. Steps 1 and 2 are then repeated.
 4. It is worth continuing for several rounds, since something one has checked out for oneself is likely to be better remembered than something one has only heard.

Activity 6 Does its face fit? [Space 1.7/6]

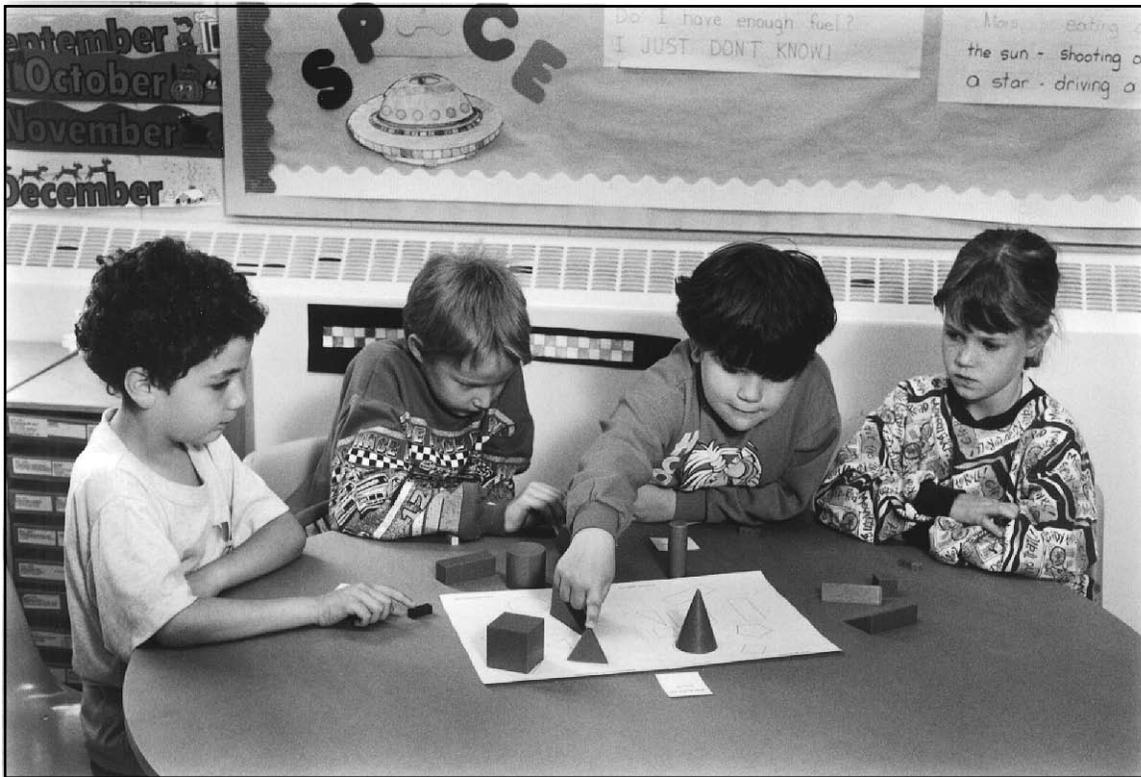
An activity for a small group of children. Its purpose is further to consolidate the relation between a geometric solid and its faces.

- Materials* • A set of geometric solids, similar to those used in Activity 5, leaving out the sphere(s). If there can be some new examples, so much the better; but the set should be fully representative of those listed under *Concepts* at the head of this topic.
- A large sheet of paper on which are drawn outlines of all the different faces of these solids. So a cuboid will be represented by three rectangles; a square pyramid by one square and one triangle; and so on. The faces for a given solid should be scattered, not all together.
 - A set of cards, one for each of the solids, with the name of that solid on it. (So if there are two cuboids, there should be two cards bearing the word ‘cuboid.’)
 - Some small objects for markers, such as counters or 1 cm cubes.

- What they do*
1. The paper of outlines is put in the middle, and the solids where children can reach them. The cards are shuffled, and put in a pile face down.
 2. The first child then takes the top card from the pile, and takes an object whose name is on the card.
 3. She then points to what she thinks is the outline of one of its faces.

Space 1.7 Naming of parts (cont.)

4. She tests this by putting the solid on to its outline. If it fits, it is left there and she retains the card and scores a point. If it does not, the object is removed and the card is replaced randomly in the pack.
5. In turn, the children repeat steps 2 to 4 until all the cards are used and all the objects placed.
6. The child whose turn it is next then tries to find another outline to which one of the solids now on the paper can be moved. For example, if the square pyramid has been put on the square which fits its base, somewhere there should be a vacant triangle which fits one of the sloping sides. She predicts as in step 3, and tests as in step 4. If correct she scores a point as before, and one of the markers is put in the outline vacated so that it cannot be used again.
7. This continues until all the outlines are occupied either by a geometrical solid or a marker.
8. The winner is the one who scores most points.



Does its face fit? [Space 1.7/6]

Discussion of activities Having formed and consolidated the concepts of the basic geometric shapes in three and in two dimensions, the children are learning about these in a little more detail. They are also making a start with relating three- and two-dimensional attributes.

Space 1.8 PARALLEL LINES, PERPENDICULAR LINES

- Concepts*
- (i) 'Is parallel to'
 - (ii) 'Is perpendicular to'
- as relationships between two lines.

- Abilities*
- (i) To recognize examples of parallel or perpendicular lines.
 - (ii) To construct examples of parallel or perpendicular lines.

Discussion of concepts Just as we have relationships between two numbers (e.g., 'is greater than,' 'is equal to'), so also we have relationships between two lines such as those in the present topic. If line a is parallel to line b, then line b is parallel to line a, so we may also say that these lines are parallel (meaning, parallel to each other). This is true also of the relationship 'is perpendicular to.' This reversibility does not hold for all relationships. E.g., it is true for 'is equal to,' but not for 'is greater than.' Here, however, our main concern is with the particular relationships named in this topic, not with the ways in which relationships themselves may be classified.

Activity 1 "My rods are parallel/perpendicular" [Space 1.8/1]

An activity for up to 6 children. Its purpose is to help children learn these two relationships.

- Materials*
- A pack of 20* cards. On 10 of these is written 'parallel' with an example, and on the other 10 is written 'perpendicular' with an example.**
 - For each child, 2 rods of different lengths. A square section is useful to prevent rolling.

* Any even number of cards will do, provided that there are enough to give a good variety of examples. In the examples, it is important that the pairs of lines should be of different lengths, and oblique relative to the edges of the paper. (See illustrations in 'Discussion of activities.')

Parallel and perpendicular are relationships between lines, independently of how these lines are positioned on the paper.

** Provided in the photomasters

What they do Stage (a), with cards

1. The cards are shuffled, and put face down.
2. Each child in turn takes a card and puts it in front of him face up.
3. The children then put their rods on top of the lines in the illustration.
4. In turn, they show their cards to the others and say, "My rods are parallel," or "My rods are perpendicular," as the case may be.
5. Each child then takes another card, which he puts face up on top of the card he has already. Steps 3 and 4 are then repeated.

Stage (b), without cards

1. Each child in turn puts his rods either parallel or perpendicular, and says "My rods are parallel/perpendicular" (as the case may be).
2. The others say "Agree" or "Disagree."
3. A child may deliberately give a false description if he chooses. The others should then all disagree.

Activity 2 “All put your rods parallel/perpendicular to the big rod” [Space 1.8/2]

An activity for up to 6 children. Its purpose is to consolidate the concepts parallel and perpendicular.

- Materials*
- Ruler or big rod.
 - Small rod for each child (It is good if the small rods are of assorted length.)

- What they do*
1. The teacher puts down the big rod and says “All put your rods parallel to the big rod.”
 2. The children do so.
 3. Steps 1 and 2 are repeated several times. Encourage variety in the placing of the children’s rods.
 4. Then, after step 2, the teacher removes the big rod and asks the children what they notice about their own rods. It should be brought out in discussion that the children’s rods are all parallel to each other.
 5. Steps 1,2,3,4 are repeated with the instruction “. . . perpendicular to the big rod.”

Activity 3 Colouring pictures [Space 1.8/3]

An activity for 2 to 6 children.

- Materials*
- A picture for each child, made up of lines which are all in parallel or perpendicular pairs. The pictures should be in lines that are faint [i.e., feint lines]. (See examples in the illustration below.) *
 - A pack of parallel/perpendicular cards, as used in Activities 1, 2, 3.
- * Provided in the photomasters



- What they do*
1. The pack is shuffled and put face down.
 2. The first child turns over the top card.
 3. He is then allowed to colour two lines in his picture, which must be either parallel or perpendicular according to the card.
 4. The other children in turn do steps 2 and 3.
 5. Putting pencils on top of the lines helps to show up which lines are parallel or perpendicular.
 6. Continue until all the pictures have been coloured.

Discussion of activities Activity 1 is for building the concepts parallel and perpendicular from a variety of examples. Again I emphasize the importance of choosing examples which do not link these concepts either with length of line, or position on paper. If this mistake is not avoided, children will be able to recognize examples like these:



but not like these:



Activity 1 is also for linking the concepts with the appropriate vocabulary. ‘Perpendicular’ is quite a hard word, so you may decide to give children practice in saying it. Initially they are not expected to read these words from the cards. They learn them orally, linked with the visual examples of parallel and perpendicular lines. In this way they will gradually learn to recognize the written words.

Activity 2 uses the newly formed concepts to generate examples, with peer-group checking. Activity 3 is similar, but develops a situation in which more than two rods are involved. Of these, a given pair may be either parallel or perpendicular.

Activity 4 is the payoff. We now have a picture with many lines. Correct recognition of relationships between these lines allows children to colour their lines. You will recognize the development of the earlier activity, Space 1.3/1, to make use of these more advanced mathematical ideas. Why more advanced? Because being straight or curved is a property of a single line; being parallel or perpendicular is a property of a pair of lines – a relational concept.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[NuSp 1] THE NUMBER TRACK and THE NUMBER LINE

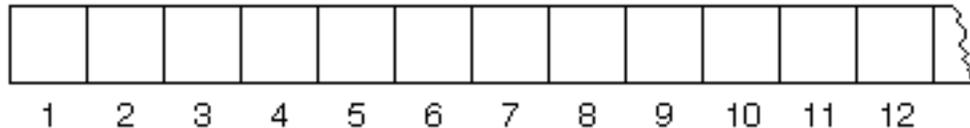
Powerful support to our thinking about numbers

NuSp 1.1 CORRESPONDENCE BETWEEN SIZE OF NUMBER AND POSITION ON TRACK

Concept Correspondence between these two orders:
size of number,
position on track.

Ability To relate the number of a set to length on a number track.

Discussion of concept



The number track is a spatial representation of number. It leads on to the number line, and these together are of importance throughout mathematics, from the reception or kindergarten class in the elementary school to mathematics at university level. They provide a valuable support for our thinking about numbers in the form of a pictorial representation.

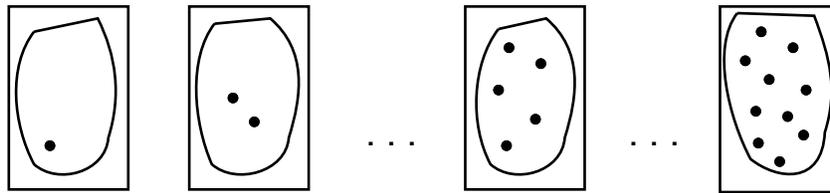
Some teachers use the term ‘number strip’ for a number track represented on paper, as in the diagram above. Conceptually, however, it is the same as the physical number track made (usually) of plastic, so I shall use the term ‘number track’ for both.

Activity 1 “I predict - here” on the number track [NuSp 1.1/1]

A game for children to play in pairs. Its purpose is to establish the relation between size of number and position on the track. A copy of this activity also appears as Num 1.5/3 in the ‘Numbers and their properties’ network.

- Materials*
- Number track up to ten.*
 - 10 small objects such as bottle tops, pebbles, small shells.
 - 10 cubes for number track.
 - Set cards as illustrated (1 to 10).*
- * Provided in photomasters.

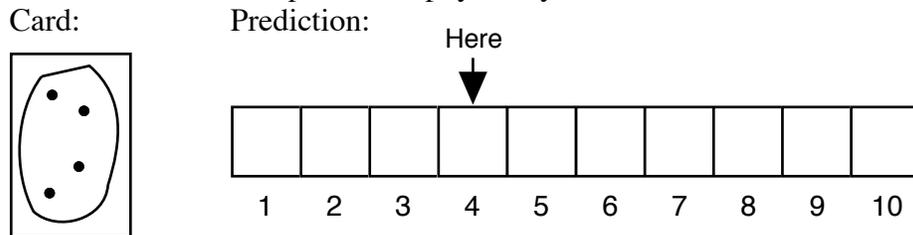
NuSp 1.1 Correspondence between size of number and position on track (cont.)



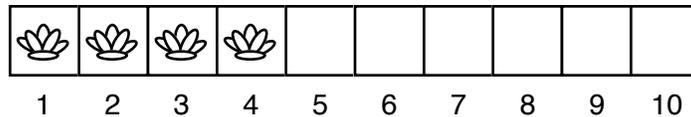
Note The set cards are not numbered, to avoid short-circuiting the conceptual activity by simply matching the numerals on card and track. The only numerals are on the track, and these are written outside the squares.

What they do Form (a)

1. The set cards are shuffled and put face down.
2. Child A turns over the top card and puts it face up on the table.
3. Together the children put one of the small objects on each dot in the set loop, to make a physical set matching the set of dots on the card.
4. Child A then predicts how far these objects will come on the number track when one of them is put in each space.
5. He says “I predict - here” and marks his prediction in some way. (If the number track is covered in plastic film, a blob of plasticine has the advantage that it will stay put.)
6. Child A then tests his prediction physically, as illustrated below.



Test of prediction (using shells):

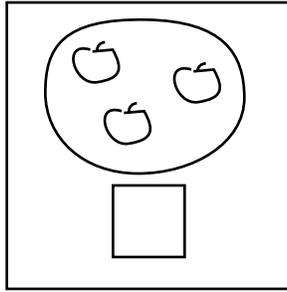


“I predict - here” on the number track [NuSp 1.1/1, Form (a)]

NuSp 1.1 Correspondence between size of number and position on track (cont.)

Form (b) This is played exactly as in form (a), except that Multilink or Unifix cubes are used and a number track. Please use both forms. We do not want the concept linked only with cubes.

Variation.*
This uses pictures instead of dots, with a box under the set loop.



1. The child counts the pictures and writes their number in the box.
2. He puts one cube on each picture.
3. These are then transferred to the number track. The last square reached should have the same numeral as has been written in the box.

* Suggested by Mrs. Marion Jones, of Lady Katherine Leveson's School, Solihull.

Discussion of activities

The number track and the number line have much in common. In the number line, however, numbers are associated with points on a line, not spaces on a track. The number line begins at zero, the number track begins at one. The number line schema, moreover, is developed further in a variety of ways. It is extrapolated backwards to represent negative numbers; points are interpolated to represent fractional numbers, and later on irrational numbers. The number track is much less abstract, and lends itself more readily to activities with physical embodiments of the concepts we want children to acquire. So we shall stay with the number track throughout the earlier part of this network.

Even here, however, abstraction has begun. When in Activity 1, Form (a) we let the size of the space used up on the track be independent of the size of the object, we are already moving towards the idea of a unit object. The introduction of unit cubes takes this idea a step further.

OBSERVE AND LISTEN

REFLECT

DISCUSS

NuSp 1.2 CORRESPONDENCE BETWEEN ORDER OF NUMBERS AND POSITION ON TRACK

- Concepts*
- (i) Successor, predecessor (from Num 1.3).
 - (ii) Spatial order on a number track.
 - (iii) The correspondence between these two orders.

Ability To match these two orders correctly, in either direction.

Discussion of concepts These concepts are closely related to those of the topic just before, but are developed here in greater detail. In the topic before, number and position on the number track were related. 'One larger' corresponds to 'next on the right.' In this topic, the verbal sequence of counting numbers corresponds to the spatial sequence of squares.

Activity 1 Sequences on the number track [NuSp 1.2/1]

A game for children to play in pairs. Its purpose is to teach the concepts and abilities described above.

* Based on an idea from Mrs. Yvonne Selah, advisory teacher with the Inner London Education Authority. A copy of this activity also appears as Num 1.5/4 in the 'Numbers and their properties' network.

- Materials*
- Number track 1 to 10 (later, 1 to 20).*
 - 4 cubes for each player (for 1 to 10), different colours for different players. [Later, 7 cubes each for 1 to 20]
 - Number cards 1 to 10 (later, 1 to 20).*
- * Provided in the photomasters

- What they do*
1. The pack of number cards is shuffled and put face down. The top card is then turned face up, starting a separate pile.
 2. Each player in turn may
 - (i) either use the card showing if the other player did not, or turn over another card;
 - (ii) put down one of her cubes in the corresponding position on the number track, or not. Both players use the same track, and only one cube is allowed in each space.
 3. The aim is to get as many cubes next to each other as possible.
 4. The game finishes when one player gets 3 in a row [for 1 to 10] (5 in a row for 1 to 20), or when both players have put down all their cubes.
 5. If before this the pack has been finished, it is shuffled and used again as in step 1.
 6. Scoring is as follows. 1 by itself scores zero; 2, 3, 4, 5 cubes in a row score respectively 2, 3, 4, 5.



Sequences on the number track [NuSp 1.2/1]

Discussion of activity

This game combines both the first and the second kind of use of mathematics described in the Introduction. Co-operation in playing the game depends on a shared mathematical schema; but choosing the best alternative involves prediction, not in this case at a level of certainty, but based on a variety of possibilities.

Though the concepts of predecessor and successor are strongly involved in these activities, and are not difficult ideas, these terms may well be thought unsuitable for children of this age. Though we ourselves need names for these concepts, they are not particularly required by the children.

What are being matched in these activities are not objects, but orders of two kinds. A relation (matching) is involved between two different order relationships, size of number and spatial order. Once again we see how even at so elementary a level as this, mathematics is a really abstract subject. Yet young children master it without difficulty if it is presented right. If we can help them to use their intelligence to the full, they show themselves much more clever than they are usually given credit for.

OBSERVE AND LISTEN

REFLECT

DISCUSS

NuSp 1.3 ADDING ON THE NUMBER TRACK

Concept Correspondence between addition and actions on the number track.

Abilities (i) To link mathematical ideas relating to addition with number track activities.
(ii) To use the number track as a mental support for adding.

Discussion of concept Addition as a mental operation, corresponding to the physical action of putting more, is discussed at length in Num 3. On the number track, putting more objects has the result of taking up more length on the track. Since the track is numbered, we have a built-in technique for showing the number of the result without counting. Alternatively, the result can be predicted by counting on spaces, and tested physically.

This synthesis of numerical and spatial ideas continues to increase in usefulness as we apply it to a variety of mathematical operations. Addition is the first of these.

Activity 1 Putting more on the number track (verbal) [NuSp 1.3/1]

An activity for two to four children. Its purpose is to introduce the use of the number track for adding. This should be used as Activity 2 in Num 3.1. Another copy of this activity is provided for that purpose in the 'Addition' network.

- Materials*
- An SAR board, see Figure 23.*
 - A number track.*
 - Cubes to fit the track, in two colours.
 - Start cards 1-5 (later 0-5), which say (e.g.) "Start with a set of 3."*
 - Action cards 1-5 (later 0-5), which say (e.g.) "Put 2 more," "Increase it by 5," "Make it 4 larger."*
 - Result cards numbered from 0-10.*
 - A reversible card.*

On side one is written:

"Find the card to show your result. Say what you did, and the result."

On side two is written:

"Predict the result."

* Provided in the photomasters

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces.
 2. The reversible card is put in its place with side one showing.
 3. One child turns over the top start card into the space below, and puts a set of the required number either singly into the number track or joined into rods.
 4. Next he turns over the action card, and puts more cubes into the track as instructed, using a different colour. Note that at this stage we do not talk about adding.
 5. Next he finds the appropriate result card to show the number of the resulting set.
 6. Finally he must describe to the others what he did, and the result.
 7. Steps 3 to 5 are then repeated by the next child.

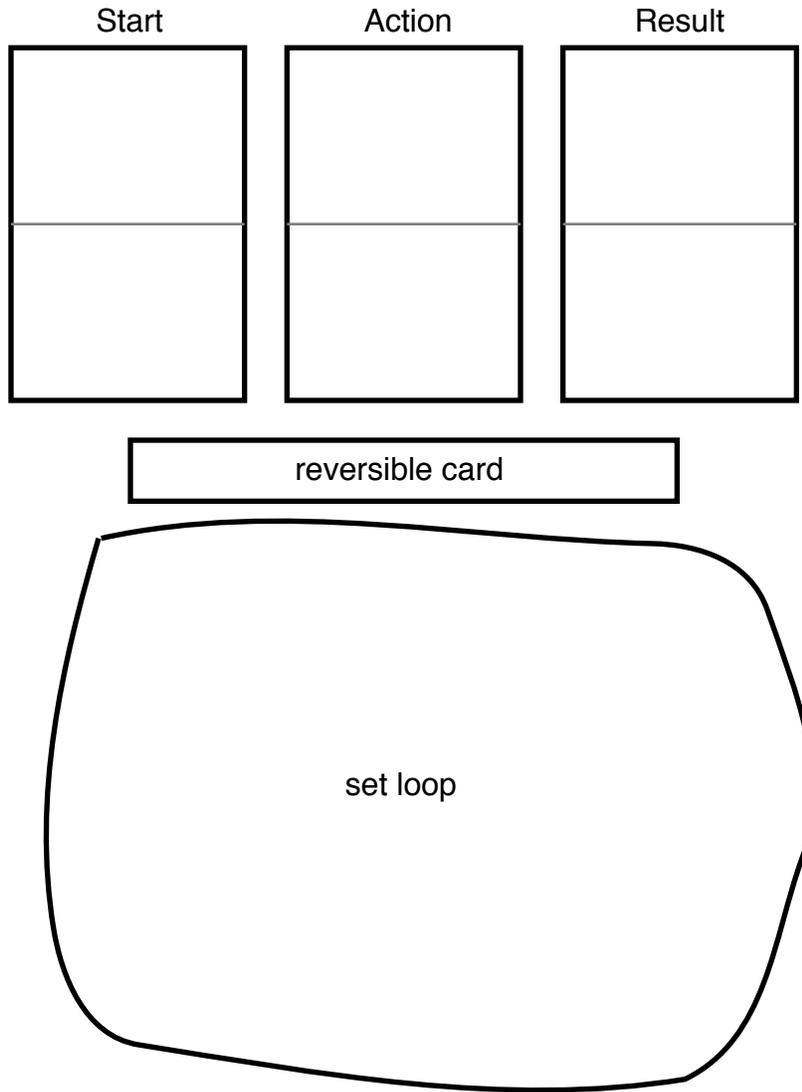


Figure 23 SAR Board

Activity 2 “Where will it come?” [NuSp 1.3/2]

An activity for two. This is a predictive form of Activity 1, and should be used as Num 3.2, Activity 2. A copy of this activity also appears as Num 3.2/2 in the ‘Addition’ network.

- Materials*
- The same as for Activity 1.
 - Also, some plasticine.

- What they do*
1. The cards are shuffled and put face down in the upper part of their spaces on the SAR board. The reversible card now shows side two.
 2. The children make up a 1-5 staircase each, in different colours.
 3. Player A turns over the top start card, selects a rod of this number and puts it into the number track.
 4. Player B turns over the top action card, but does not yet take out a rod. First, he says “I predict that it’ll come to *here*” pointing, and marking his prediction with a piece of plasticine. This will involve some form of counting on, which on the number track corresponds to movement to the right.
 5. Then he tests his prediction physically by joining a rod of the given number to the first on the number track.
 6. Steps 3, 4, and 5 are repeated until all the cards are turned.
 7. Then the cards are shuffled, and the children begin again with their roles interchanged.

Activity 3 Crossing* [NuSp 1.3/3]

A board game for 2 or 3 children. Its purpose is to consolidate the abilities described above in a situation which requires several predictions to be made in order to choose the best action.

*This attractive improvement to the original version of the ‘Crossing’ game was suggested by Mrs. Mary Hamby of Leegomery County Infant School, Telford. A copy also appears as Num 3.2/4 in the ‘Addition’ network.

- Materials*
- Game board, see Figure 24.*
 - 3 markers for each player, a different kind or colour for each player.
 - Die 1-6 and shaker.
- * Provided in the photomasters

- What they do*
1. The blank squares on the board represent paving stones. Some of these have been removed to allow flowers to grow. The object is to get across the board from START to FINISH, treading only on the paving stones and not on the flowers.
 2. Each player starts with all 3 markers off the board, at the START.
 3. Players throw the die in turn. The number thrown shows how many steps they may take. This means that from START, they may put a marker on the board at the square with that number; and from a square on the board, they may move one of their markers forward that number of squares.

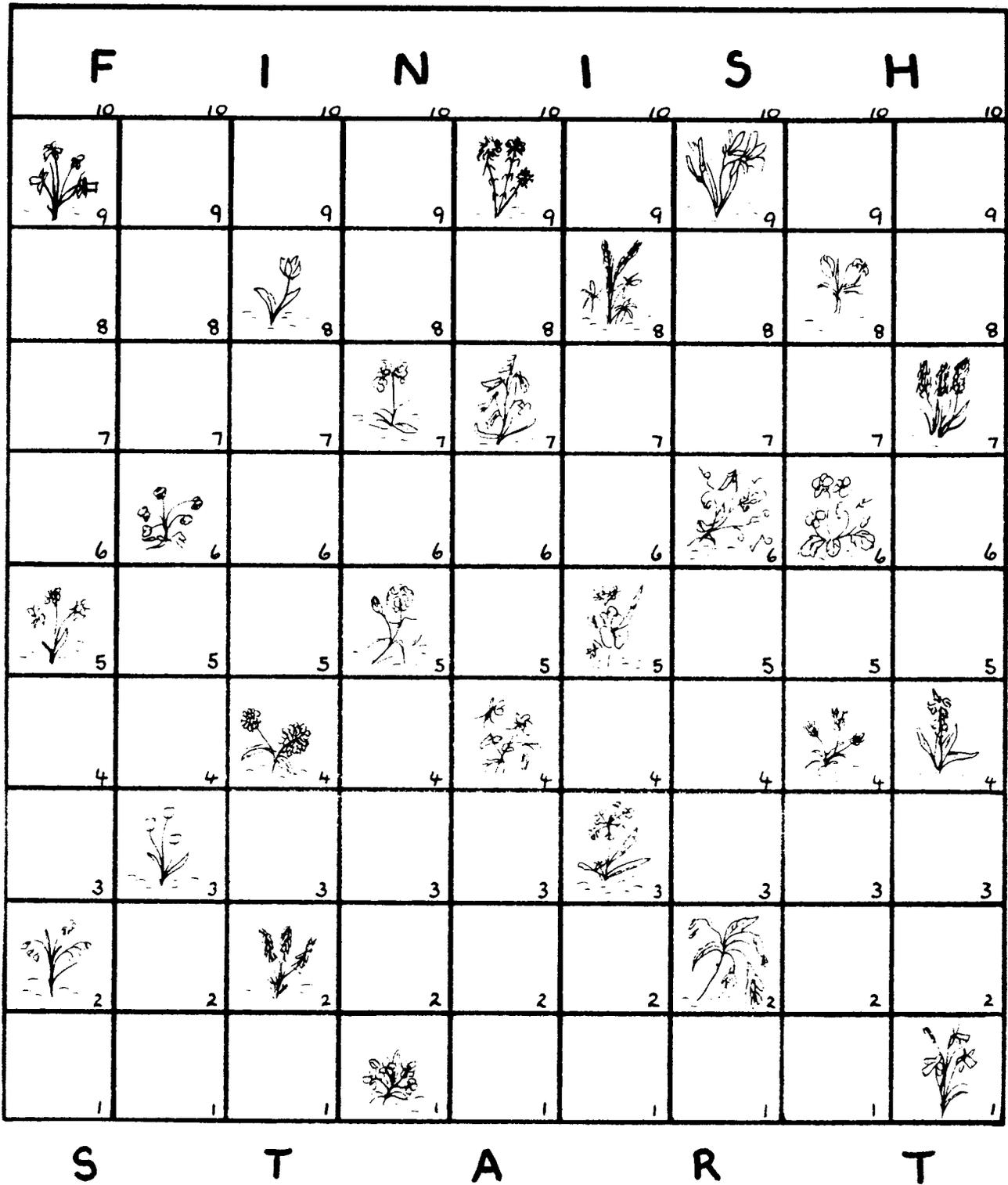


Figure 24 Crossing.

4. They may move whichever of their markers they like. When starting, they may choose any vacant track. After that, they must keep moving straight forward along the same track.
5. They may not land on a square marked with a flower. Players may move their markers over them normally, but if they make a move which stops on a square with a flower, that marker must go back to the start.
6. If they touch a marker they must move it if they can, or go back to the START if they cannot. This rule may be relaxed when learning.
7. The exact number must be thrown to finish. The first player to get all his markers to FINISH is the winner, but the others may continue playing until all are across.

Variation Players learning the game may start with just 2 markers.

Activity 4 “Where will it come?” (Through 10) [NuSp 1.3/4]

This is the same game as Activity 2, but with results greater than 10. It should be used as Num 3.7, Activity 2 in the ‘Addition’ network.

- Materials*
- As in Activity NuSp 1.3/2 except:
 - Both players use staircases 1-9, the first 5 cubes being of a different colour from the remainder. E.g., an eight rod would be 5 blue, 3 white. The five rod would be all blue, and the one to four rods would all be white. These give the same grouping as finger counting.
 - Use ‘Start’ and ‘Action’ cards 4-9.* These will give a mixture of results which do/do not cross the ten boundary.
 - Result cards 1-20.*
- * Provided in the photomasters

Discussion of activities We have now moved on from lengths on the number track, corresponding to numbers, to actions (putting more) on the number track, corresponding to mathematical operations. Activity 1 is for schema building by physical experience (Mode 1); Activities 2, 3, 4 consolidate this schema by using it for making and testing predictions. The number track encourages the transition from counting all to counting on.

OBSERVE AND LISTEN

REFLECT

DISCUSS

NuSp 1.4 SUBTRACTING ON THE NUMBER TRACK

Concept Correspondence between subtraction and actions on the number track.

Abilities (i) To link mathematical ideas relating to subtraction with number track activities.
(ii) To use the number track as a mental support for subtracting.

Discussion of concepts Though subtraction might seem to be no more than the inverse of addition, it is in fact a more complex concept, derived from as many as four simpler concepts. These are discussed fully in Num 4.

The simplest of these contributory concepts is ‘taking away,’ opposite of ‘putting more.’ It is this aspect of subtraction which is used in Activities 1, 2, and 3. Activity 4, ‘Capture,’ uses the comparison aspect of subtraction.

Activity 1 Taking away on the number track (verbal) [NuSp 1.4/1]

An activity for a small group. Its purpose is to introduce the use of the number track for subtracting. This should be used as Activity 2 in Num 4.1.

Materials

- An S.A.R. board.*
- A number track.*
- Cubes to fit the track.*
- Start cards 5-10, which say (e.g.) “Start with a set of 5.” *
- Action cards 1-5 (later 0-5), which say (e.g.) “Make it 2 less,” “Take 3 away,” “Make it 4 smaller.” *
- Result cards numbered from 0-10.*
- A reversible card.*
 - On side one is written.
“Find the card to show your result. Say what you did, and the result.”
 - On side two is written
“Predict the result.”

* The same as for NuSp 1.3/1.

What they do This is done in the same way as ‘Putting more’ (NuSp 1.3/1), with appropriate changes to step 4.

Reminders

1. The children should verbalize their results each time.
2. It is useful to have colour codes for different kinds of card.

Activity 2 What will be left? [NuSp 1.4/2]

This is an activity for two. It is a predictive form of Activity 1, and should be used as Num 4.2, Activity 2.

Materials

- As Activity 1, except that:
- Player A has start cards 5-10.
- Player B has action cards 1-5.
- Plasticine.

- What they do*
1. Player A makes a staircase 1-10.
 2. The cards are shuffled and put face down in the upper part of their spaces.
 3. Player A turns over the 'Start' card, selects the rod of this number and puts it onto the number track.
 4. Player B turns over the top 'Action' card. She now has to predict what will be left on the track when a rod of that number is taken away. She marks her prediction with a piece of plasticine.
 5. She then tests it by physically removing the appropriate number of cubes.
 6. The length left in the track is checked against the marker, and the number taken away against the 'Action' card.
 7. Steps 2 to 5 are then repeated with roles interchanged.

Activity 3 Crossing back [NuSp 1.4/3]

A board game for 2 or 3 children.

What they do This is played on the same board as 'Crossing' (Activity 3 in the previous topic), and in a similar way. The difference is that they all begin in the finishing space, corresponding to 10, and end where START is printed.

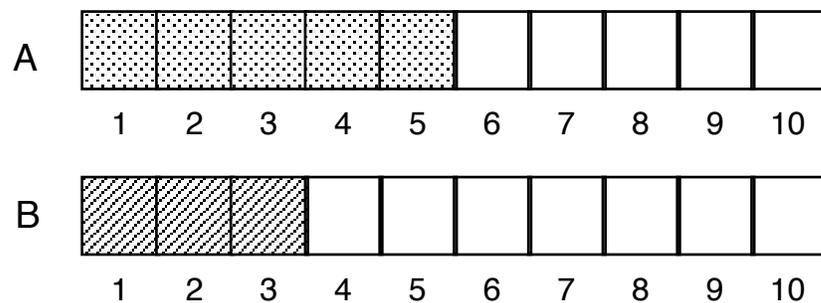
- Notes*
- (i) Since this game is a little harder than its predecessor 'Crossing,' it may be wiser not to use the 'Move if touched' rule initially. But it should come in when the children have sufficient experience in order to make the game a predictive one.
 - (ii) If children find it confusing to begin where 'Finish' is written, you could make another board. But using the same board helps to show the inverse relation between adding and subtracting.

Activity 4 Capture [NuSp 1.4/4]

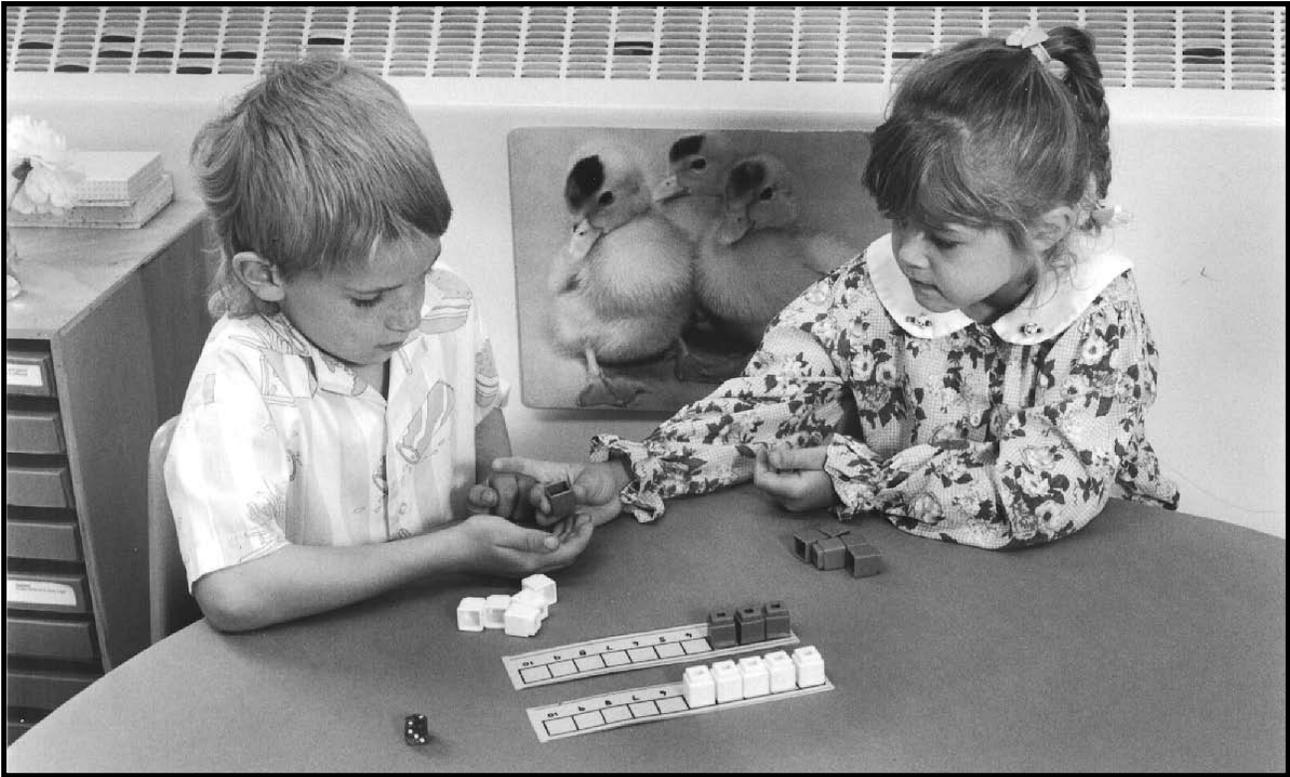
This is a game for two. Its purpose is to introduce the comparison aspect of subtraction.

- Materials*
- 2 number tracks 1-10.*
 - 1 die 1-6.
 - 10 cubes for each player, a different colour for each.
- * Provided in the photomasters

- What they do*
1. The two players sit side by side and their number tracks are placed in front of them, parallel and with the 1s lined up to the left.
 2. Each player throws the die, and puts the number of cubes indicated on the track. The result might look like this:



Since A has filled two more spaces than B, B must give A two cubes.



Capture [NuSp 1.4/4]

3. The cubes are then taken off the track.
4. Both players throw again, and the process is repeated. Captured cubes may not be used to put on the track, but may be used if cubes have to be given to the other player.
5. The game finishes when either player has had all his cubes captured, or cannot put down what is required by the throw of the die.
6. The other player is then the winner.

Discussion of activities The first three activities are direct analogues of those in the previous section, using subtraction instead of addition. This gives continuity and helps to give confidence: there is not too much novelty. Activity 4 is different, and provides variety. It uses a different aspect of subtraction, namely comparison of two sets. This game could be played without number tracks. Their use has however the advantage of showing numerals alongside the rods. This links the activity with the symbols which will later be used in number sentences.

OBSERVE AND LISTEN

REFLECT

DISCUSS

NuSp 1.5 RELATION BETWEEN ADDING AND SUBTRACTING

Concept That addition and subtraction are inverse operations.

Ability To translate into action the opposite effects of these two operations.

Discussion of concepts If we add 4 and then subtract 4, we are back to the number we started with. These two operations cancel each other out, so each is called the inverse of the other.

In more advanced mathematics (e.g., group theory) this concept plays an important part. At this stage, children need only to be introduced to the concept. Also, since equal operations of addition and subtraction, which exactly cancel, get one literally nowhere, activities embodying the concept in this form are not likely to be very interesting. So those which follow emphasize simply the opposite effects of these two operations, which is what is important at this stage.

Activity 1 Slow bicycle race [NuSp 1.5/1]

A game for a small group. Its purpose is to introduce the inverse relationship between adding and subtraction.

Materials For each player:

- A 1-10 number track.*
- A cube.

For the group:

- Two 1-6 dice.
- A shaker.

* Provided in the photomasters

- What they do*
1. The number tracks are put alongside each other.
 2. To decide who starts, they each throw a single die, and the highest starts.
 3. They then throw both dice together, taking turns clockwise round the group.
 4. Each player moves his cube forward according to the larger number shown on the two dice, backward according to the smaller number. (The start, corresponding to zero, is just before 1 on the track.) If for example a player throws 6 and 2, he must actually move forward 6 and then back 2, and not just go forward 4.
 5. If his forward move takes him past the finishing line, i.e., past 10, he is out of the race.
 6. The winner is the last player to be left in the race.

Variation Players combine the forward and backward moves mentally, and move forward the resulting amount. This will keep them longer on the track. Players must agree beforehand which form of the game they are going to play.

Activity 2 Ups and downs [NuSp 1.5/2]

This is a board game of the 'snakes and ladders' kind for up to four players. It relates the mathematical symbols for adding and subtracting to movements on the number track.

- Materials*
- Ups and downs board,* see Figure 25 (provided in the photomasters).
 - 1 small marker for each player.
 - 1 die 1-6.
 - Shaker.

*Interesting situations arise at one or two places. You can devise your own variations.



Ups and downs [NuSp 1.5/2]

- What they do*
1. Each player throws the die in turn and moves forward according to the number thrown.
 2. If they land on an action square they go forward or backward according to whether the operation is addition or subtraction.
 3. There is no rule against two players occupying the same space.
 4. The exact number must be thrown to finish, and the winner is the player who does so first.

Discussion of activities Both of these activities are of a schema building kind. Mode 1 schema building is used – from physical embodiments of these two operations, children's awareness of their opposite nature is strengthened. This relationship is a higher order concept derived from ones which they already have, so Mode 3 schema building, creativity, is also involved.

OBSERVE AND LISTEN

REFLECT

DISCUSS

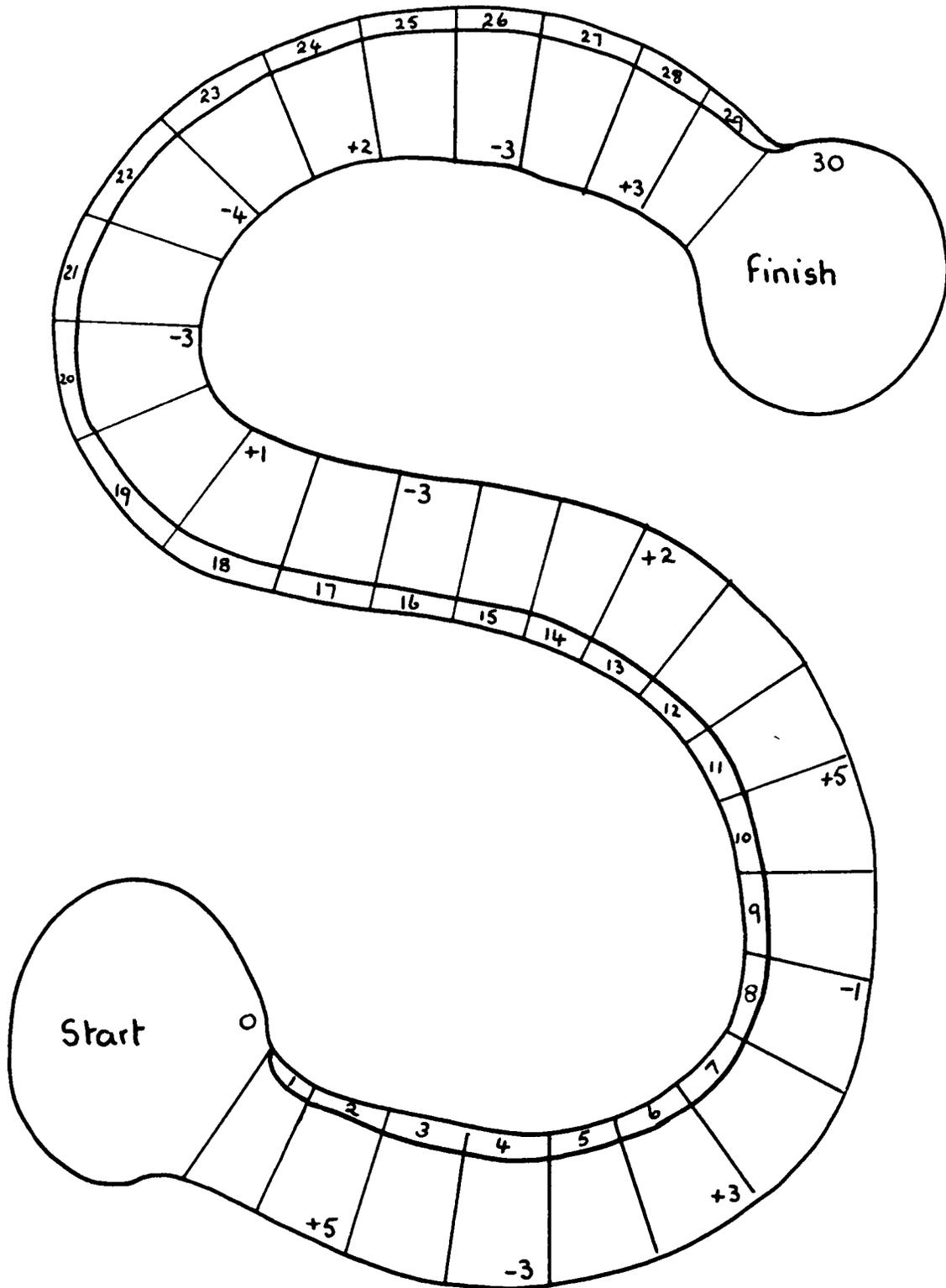


Figure 25 Ups and downs.

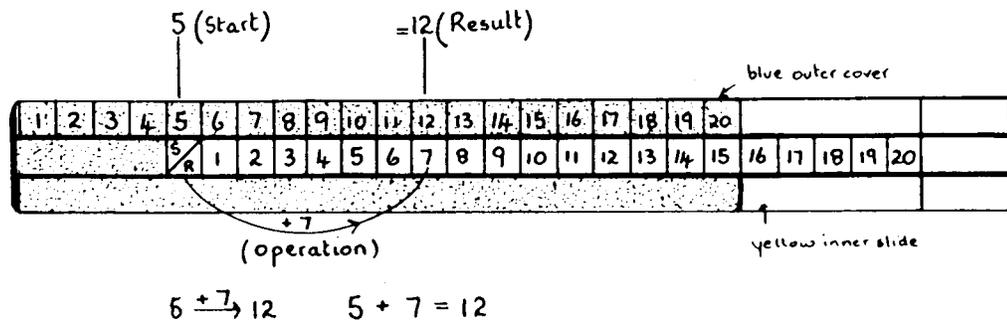
NuSp 1.6 LINEAR SLIDE RULE

Concept The linear slide rule as a method of adding and subtracting.

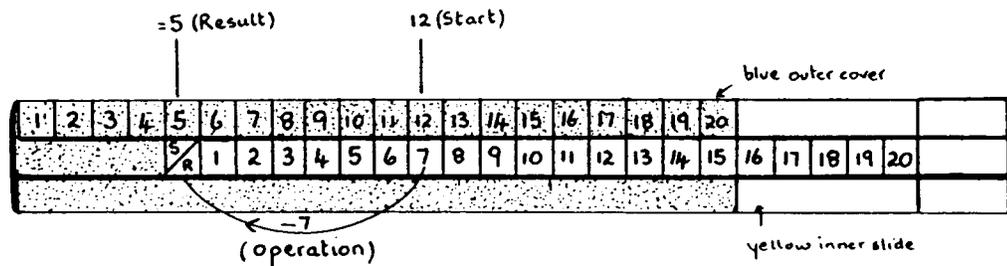
Abilities Simple whole-number addition and subtraction by the linear slide rule.

Discussion of concept This is, in essence, two number tracks side by side, one of which can be slid relative to the other. The slider needs an extra space at the beginning, marked S/R corresponding to 'Start' for addition, and 'Result' for subtraction. The slide rule may be of any length you choose to make: 1-20 is a practical size to begin with. Its working is best conveyed by illustrations.

Adding



Subtracting



You may wonder why this is called a *linear* slide rule. This is to distinguish it from a logarithmic slide rule, a valuable aid for many more advanced mathematical operations until replaced by electronic calculators.

Activity 1 Add and check [NuSp 1.6/1]

An activity for two players. This provides an interesting way of practising addition facts.

- Materials*
- Linear slide rule 1-20 (provided in the photomasters).
 - 2 dice 1-9, or spinners.
 - Shaker for die.

- What they do*
1. Child A has the pair of dice.
 2. B has the linear slide rule.
 3. A throws the dice, and speaks the numbers and their sum.
 4. B then checks A's result using the linear slide rule.
 5. Examples:
A: "Six add two, result eight." or "Six add two equals eight."
B: "Correct."
A throws again. "Five add seven, result thirteen."
B: "No, twelve."
 6. They continue until A has (say) ten correct, and then change around.

Activity 2 Adding past 20 [NuSp 1.6/2]

A problem suitable for some of the brightest children.

Materials As in Activity 1.

- What they do*
1. Ask, "If your linear slide rule only goes up to 20, can you find a way of using it to add when the result comes to more than 20? Say, $17 + 8$, or $13 + 15$?"
 2. One way – there are others – is to use 20 as the start mark, and add 20 to the result. E.g., $17 + 8$. Set 20 (instead of letter S) on the slider against 17 on the upper rule. Opposite 8 on the slider, read 5 on the upper rule in the usual way, and add 20: result 25. (When you do it physically, this is much easier than it sounds.)
 3. If the problem in this form is too hard, you could show them the above method, and ask them why it works.
 4. When this method has been mastered, it may be used to repeat Activity 1 with more difficult numbers.

Discussion of activities The linear slide rule is, in effect, a useful physical tool for counting on and counting back. It provides further connections between concepts which children already have; and its use in the foregoing activities serves to strengthen these concepts.

OBSERVE AND LISTEN

REFLECT

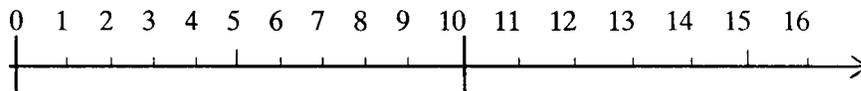
DISCUSS

NuSp 1.7 UNIT INTERVALS: THE NUMBER LINE

- Concepts* (i) Unit intervals on a line.
(ii) The number line.

Abilities To use the number line in the same ways as the number track, in preparation for other uses of the number line.

Discussion of concepts



The differences between a number track and a number line are appreciable, and not immediately obvious.

The number track is physical, though we may represent it by a diagram. The number line is conceptual – it is a mental object, though we often use diagrams to help us think about it. The number track is finite, whereas the number line is infinite. However far we extend a physical track, it has to end somewhere. But in our thoughts, we can think of a number line as going on and on to infinity.

On the number line, numbers are represented by points, not spaces; and operations are represented by movements over intervals on the line, to the right for addition and to the left for subtraction. The concept of a unit interval thus replaces that of a unit object. Also, the number line starts at 0, not at 1. For the counting numbers, and all positive numbers, we use only the right-hand half of the number line, starting at zero and extending indefinitely to the right. For positive and negative numbers we still use 0 for the origin, but now the number line extends indefinitely to the right (positive numbers) and left (negative numbers).

Activity 1 Drawing the number line [NuSp 1.7/1]

This is a simple activity for introducing the concept.

Materials • Pencil and paper for each child.

- What they do*
1. Ask the children to draw a line, as long as will conveniently go on the paper.
 2. They mark off on it equal intervals.
 3. They number these 0, 1, 2 ... as in the diagram above.
 4. At this stage, the two differences between the number track and the number line which need to be pointed out are: (i) With the number track, numbers are represented by spaces; with the number line, numbers are represented by points on the line. Though it is helpful to use different marks for tens, fives, and ones, it is the points on the line which represent the numbers. (ii) The number track starts at 1, the number line starts at 0.

Activity 2 Sequences on the number line [NuSp 1.7/2]

NuSp 1.2/1, 'Sequences on the number track' may usefully be repeated here. Smaller markers like the one suggested in the following activity will be needed.

Activity 3 Where must the frog land? [NuSp 1.7/3]

A game for two players. Its purpose is to introduce the use of the number line for adding.

- Materials*
- As long a number line as you like (some number lines are provided in the photomasters).
 - A marker representing a frog for each player, occupying as small a base as possible.*
 - 1 die 1-6, or 1-10 for athletic frogs.
- * A short length (about 2 cm) of coloured drinking straw, with a bit of plasticine on the end, makes a good marker for this and many other activities.



- What they do*
1. The frogs start at zero.
 2. Player A throws the die and tells the frog what number it must hop to. This is done mentally, using aids such as finger counting if he likes. (To start with, children may use counting on along the number line, but should replace this by mentally adding as soon as they have learned the game.)
 3. Player B checks, and if he says "Agree" the frog is allowed to hop.
 4. If B does not agree, he says so, and they check.
 5. If A has made a mistake, his frog may not hop.
 6. Two frogs may be at the same number.
 7. They then exchange roles for B's throw of the die.
 8. The winner is the frog which first hops past the end of the line. (The exact number is not required).

Activity 4 Hopping backwards [NuSp 1.7/4]

The subtraction form of Activity 3, starting at the largest number on the number line and hopping backwards past zero.

Activity 5 Taking [NuSp 1.7/5]

Another capture game for two, but quite different from NuSp 1.4/4, 'Capture.' Its purpose is to give further practice in relating numbers to positions and movements on the number line.

- Materials*
- 1 number line 0-20.*
 - 3 markers for each player.
 - 1 die 1-6.
- * Provided in the photomasters.

NuSp 1.7 Unit intervals: the number line (cont.)

What they do Form (a)

1. The markers begin at zero.
2. The die is thrown alternately, and according to the number thrown a player may jump his marker forward that interval on the line.
3. A piece which is jumped *over* is taken, and removed from the board for the rest of the game.
4. An occupied point may not be jumped *onto*.
5. A player does not have to move at all if he doesn't want to. (We introduced this rule when we found that starting throws of low numbers were likely to result in the piece being taken next throw, with no room for manoeuvre.)
6. The winner is the player who gets the largest number of pieces past 20. (It is not necessary to throw the exact number.)

Form (b)

This may also be played as a subtraction game, backwards from 20.

Activity 6 A race through a maze [NuSp 1.7/6]

This is a board game for up to 3 players. Its purpose is to bring out the correspondence between the relationships smaller/larger number and left/right hand on the number line.

- Materials*
- Board, see Figure 26*.
 - Number line 1-20 *
 - Number cards 1-20 *
- * Provided in the photomasters

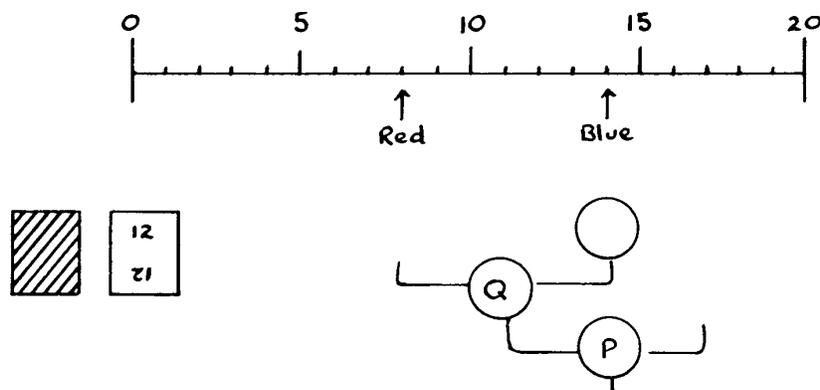
For each player:

- 1 coloured pointer for the number line.
- 1 marker to take through the maze.

What they do

1. The pack of number cards is shuffled and put face down.
2. In turn, each player turns over the top card and puts it face up, starting a new pile. He may then use this number to position his pointer on the number line, or decide not to (since an extreme left or right position is not helpful). In that case he repeats this step at his next move.
3. When he does position his pointer, he also puts his marker at the start of the maze.
4. After taking steps 2 and 3, players at subsequent turns move forward through the maze if they can. The number they turn over determines whether they can or not.

An example will make it clear.



Since 12 is to the left of the blue pointer's position, player 'Blue' can move forward if he is at P on the maze, but not if he is at Q. However, the reverse is the case for player 'Red', since 12 is to the right of his pointer.

5. There is no limit to the number of players at a given position on the maze.
6. When all the cards are turned over the number pack is shuffled and used again.
7. The winner is the first player to reach the finish.

Note If the children have difficulty in remembering their left and right, you could help with labelled arrows.

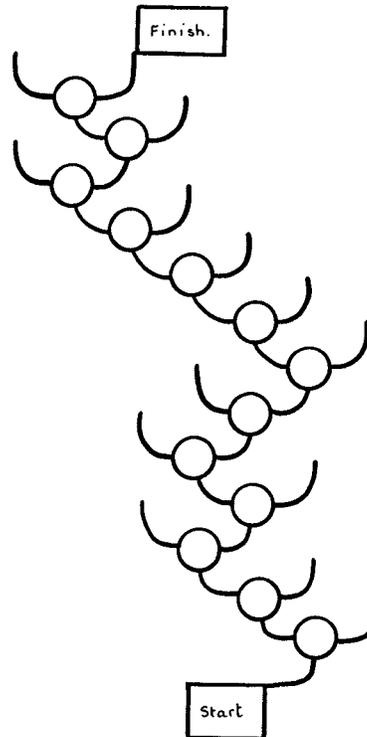


Figure 26 A race through a maze

Discussion of activities The first four activities are concerned first with introducing the number line, and then with linking it to concepts which are already familiar. Activity 5, 'Taking,' is more difficult, since it involves mentally comparing a number of possible moves before deciding which one to make.

Activity 6 involves correspondence not between objects, but between two different relations. One is the relation of size, between two numbers, and the other is the relation of position, between two points on the number line. Here is another good example of the conceptual complexity of even elementary mathematics. Yet young children manage this without difficulty if we make it possible for them to use their intelligence to the full.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Patt 1] PATTERNS

Patt 1.1 PATTERNS WITH PHYSICAL OBJECTS

Concept That of a simple linear pattern.

Abilities To copy and invent simple patterns in a variety of materials.

Discussion of concept In everyday life, the word ‘pattern’ has a number of related meanings. One of these is a decorative design, and this is what we mean in the present context. The essential feature of this kind of pattern is regular repetition, so that once we have seen the pattern we know what will come next. Its importance in the context of early mathematical learning is that children are learning to rely on their own perception of pattern to decide whether or not something is correct, rather than on someone else’s say-so, for which they may not understand the reason. (Do you know why the same spelling ‘...ough’, is pronounced differently in ‘bough’, ‘trough’, and ‘enough’? I don’t, and there is no pattern which I can recognize.)

Activity 1 Copying patterns [Patt 1.1/1]

An activity for children to do in pairs. After you have introduced it, they should be able to continue on their own. Its purpose is to heighten their innate awareness of pattern, and link it with vocabulary.

Materials • Interlocking cubes. Each pair should have these in just two colours for stage (a). For stage (b), three or more colours should be available.

What they do Stage (a)

1. Sitting side-by-side in pairs, one child makes a rod with two colours in a simple pattern, such as:
red white red white red white . . .
2. The other copies it.
3. They check that the patterns are the same by putting their rods together.
4. The rods are broken up and the second child then makes a different pattern for the first child to copy. This might be, for example:
white red red white red red white red red . . .
5. This is checked as before, and steps 1 to 4 are then repeated.
6. If one child cannot see what the other’s pattern is, he asks for an explanation. What has to be decided is whether there is a pattern but the other did not see it, or whether there is no recognizable pattern.

Note At this stage it is reasonable to regard (say)

white white red white white red white white red . . . and
red red white red red white red red white . . .

as two different patterns. With this assumption, up to six patterns can be formed by alternating one, two, or three whites and one, two, or three reds. If someone argues that these are the same pattern with the colours reversed, congratulations – they have anticipated a future activity without help.

Stage (b)

This is similar to the above, except that each pair now has three colours of cubes. This allows a much greater variety of patterns. At this stage, some children may invent more sophisticated patterns such as

red white red white white red white red white white . . .

Activity 2 Patterns with a variety of objects [Patt 1.1/2]

A continuation of Activity 1, for children working on their own or in pairs. Its purpose is to show that patterns can be made with all sorts of objects.

- Materials*
- Containers with a variety of objects convenient for making patterns. For example, one container might have counters in three colours; another might have sunflower seeds, macaroni shells, and dried beans; another might have three different kinds of buttons. There needs to be enough of each variety so that children do not run out while making their patterns: I suggest not less than ten of each. This would allow, e.g., three repetitions of a pattern like
A B B B A B B B . . . with one to spare.

- What they do*
1. Each child chooses a container, and makes a pattern as in Activity 1. If there are not enough containers for one each, they may work in pairs.
 2. Every pattern should be recognizable by another child or another pair.
 3. When they have finished, they may change around the containers and repeat steps 1 and 2.

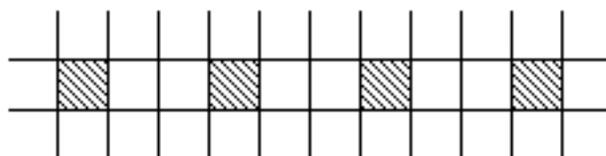
Activity 3 Making patterns on paper [Patt 1.1/3]

A continuation of Activities 1 and 2, in which the patterns are now drawn on paper. This makes possible much greater variety. This activity may be done in pairs or in small groups.

- Materials*
- For each child, squared and plain paper (2 cm squares recommended).
 - For each child, pencil and eraser.
 - A central pool of coloured pencils may be provided if desired.

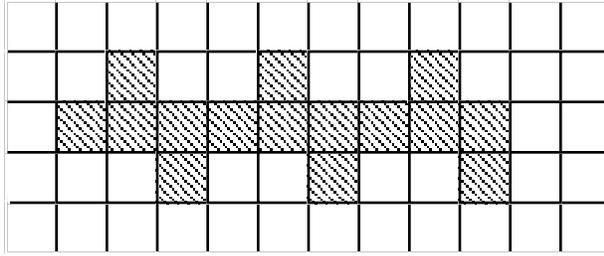
- What they do*
- (i) Each child makes a pattern on paper. Here are some (reduced scale) examples which you might show to get them started.

Pattern 1



Patt 1.1 Patterns with physical objects (cont.)

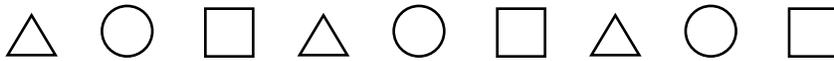
Pattern 2



Pattern 3



Pattern 4



Pattern 5



2. They look at each other's patterns, and describe them. Discussion may arise out of this.
3. **Important.** These patterns (omitting those on which there was no agreement) should be kept for future activities.



Making patterns on paper [Patt 1.1/3]

Patt 1.2 SYMMETRICAL PATTERNS MADE BY FOLDING AND CUTTING

Concept Patterns having symmetry about one, two, or three lines, such as can be made by folding paper and cutting.

Abilities (i) To make patterns of this kind.
(ii) To distinguish shapes which might have been made in this way from those which could not.

Discussion of concept This topic uses the activity of making patterns by folding paper, and cutting at the folds, as an introduction to the mathematical concept of symmetry about a line. Everything on one side of the fold line matches everything on the other side, and these two sides will coincide when the paper is folded again. In the present case they must, since this is how they were made.

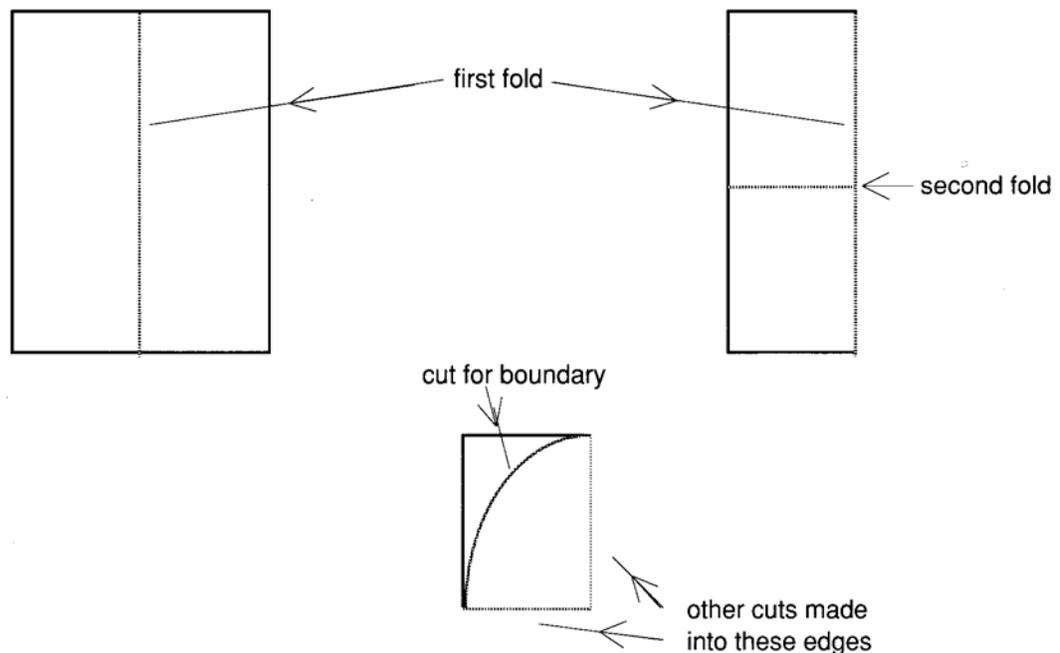
This pleasant activity will, I hope, already be familiar to many children.

Activity 1 Making paper mats [Patt 1.2/1]

An activity for children to do individually. Its purpose is to introduce children to these if they have not made them before.

Materials. For each child, and also for yourself as demonstrator:

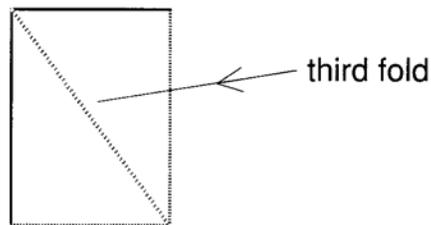
- One or more sheets of paper.
- Scissors.



Patt 1.2 Symmetrical patterns made by folding and cutting (cont.)

What they do Stage (a)

1. The paper is folded edge-to-edge in one direction, and again in the perpendicular direction.
2. A cut is made to remove all of the unfolded edges, which is to say all of the original four sides of the paper. This is to produce a symmetrical boundary when the paper is opened flat.
3. A number of small shapes are cut away at the folds, of various shapes and sizes.
4. If desired, the same may be done at the unfolded edges.
5. The paper is then opened flat. The more-or-less random cuts, of no particular shape, are now transformed into attractive symmetrical patterns.
6. The children compare and admire each other's patterns, and agree how clever they were to make them!



Stage (b)

As for stage (a), except that after step 3 or 4 the paper is folded once again to bring the two folded edges together. More cuts are then made at the new fold. In this case it is better not to make too many cuts in step 3, to leave space for those to come.

Activity 2 Bowls, vases, and other objects [Patt 1.2/2]

A variation of the previous activity. Its purpose is to call attention to the connection between folding and cutting, and symmetry.

Materials. As for Activity 1. The sheets of paper can be smaller, say one quarter of those for Activity 1.

What they do

1. Begin by making just a simple fold.
2. Cut a simple curve from the side opposite the fold, and, optionally, from the other two open edges, but in this case not from the folded edge.
3. Open flat. The result may look like the profile of a bowl or a vase, according to the original cut.
4. Children may like to experiment in producing shapes of other everyday objects. These may be profiles of flat objects, or outlines of flat objects.

Activity 3 Symmetrical or not symmetrical? [Patt 1.2/3]

An activity for a small group of children. Its purpose is to make explicit the concept of symmetry embodied in both the previous activities, and attach vocabulary.

- Materials*
- A selection of the cut-out shapes from Activity 2.
 - About the same number of non-symmetrical cut-outs. These could resemble everyday objects, e.g., bananas, or they could be abstract shapes.

- What they do*
1. Begin by calling attention to the difference between the symmetrical and non-symmetrical shapes. Because of the way they were made, each half of the symmetrical shapes fits exactly onto its opposite half when turned over at the fold line. There is no fold line for which this is true for the non-symmetrical shapes.
 2. The shapes are then mixed up and put in the middle of the table.
 3. One of the children shuts her eyes and takes a shape from the middle.
 4. She looks at it and says “symmetrical” or “non-symmetrical” as the case may be.
 5. If the others agree, they in turn do likewise.
 6. If there is any disagreement, the shape is tested by folding.

Note The presence or absence of a fold line is of course a strong clue. For this reason, it may be desirable to make some folds in the non-symmetrical shapes also. These will, of course, not be axes of symmetry.

Discussion of activities Folding and cutting provides an excellent physical embodiment with which to begin forming the concept of symmetry about a line. Using this to make attractive paper shapes is an activity with which I hope many children will already be familiar, and this is why I use it as an introduction. In so doing, I am departing briefly from the important general principle of introducing a new concept for the first time in a low-noise situation, since Activity 1 starts with two axes of symmetry, then three. At this stage, it is offered as another example of pattern. We then use a single line of symmetry in Activity 2, in which symmetry is salient, in preparation for Activity 3 where the concept is made explicit and named. Contrasting examples with non-examples is another important part of concept formation, and this is also used in Activity 3.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Patt 1.3 PREDICTING FROM PATTERNS

Concept That of a pattern as something which enables us to know what is coming next.

Ability To predict what will come next when a pattern is followed.

Discussion of concept “Next” can mean next in space, or next in time, or sometimes both as in the case of a bus stop. To perceive a particular pattern is to know not only that there is a regular repetition, but also what it is that repeats, and thereby what comes next in either or both of these meanings.

Activity 1 What comes next? [Patt 1.3/1]

A predictive activity for children to play in pairs, based on ‘Copying patterns’ [Patt 1.1/1]. Its purpose is to allow children to make and test predictions based on their perception of pattern. It may be played at two levels, corresponding to stages (a) and (b) in Activity 1.

- Materials*
- As for the corresponding stages in ‘Copying patterns’ [Patt 1.1/1].
 - A sheet of paper or an empty paper towel tube.
 - A rubber band.

What they do Stage (a)

1. Each child makes a rod in some pattern out of sight of the other, and conceals it by rolling the paper around it and keeping it there with the rubber band.
2. Child A then pushes her rod into view, a little at a time.
3. When child B thinks she knows what will come next, she says so, and her prediction is tested by pushing the rod further into view.
4. When child B can predict reliably, they exchange roles and steps 2. and 3. are repeated.

Activity 2 Predicting from patterns on paper [Patt 1.3/2]

An activity for children working in pairs. It is similar to Activity 1, ‘What comes next,’ but uses patterns on paper.

- Materials*
- Patterns produced in ‘Making patterns on paper’ [Patt 1.1/3].
 - A sheet of paper.

- What they do*
1. The patterns are put in a central pool, face down.
 2. In each pair, child A takes a pattern from the pool. She looks at it, without letting the other child see it, and checks that it is not one which either of them has seen already.
 3. Still without letting child B see it, child A puts the pattern face up on the table, and covers it with the sheet of paper.

4. Child A then uncovers the pattern from the beginning, a little at a time.
5. When child B thinks she knows what will be seen next, she says so, and her prediction is tested by uncovering the next object in the pattern.
6. When child B can predict reliably, she further demonstrates her understanding of the pattern by extending A's pattern on the paper. Child A checks whether or not this is correct.
7. If so, the pattern in use is returned to the pool. The children then exchange roles, and steps 2 to 5 are repeated.

Discussion of activities These two activities help children to realize consciously that making and testing predictions is a way in which they can check for themselves whether their ideas are correct or incorrect. In a simple way, they are forming and testing hypotheses, which brings them into the company of all scientists, from beginners to the most eminent.



Predicting from patterns on paper [Patt 1.3/2]

Patt 1.4 TRANSLATING PATTERNS INTO OTHER EMBODIMENTS

- Concepts* (i) That the same pattern can be shown by different objects.
(ii) That a pattern is therefore independent of any particular embodiment.

Ability To show a given pattern with different objects.

Discussion of concept One of the reasons that mathematics is so useful and adaptable a mental tool is the great variety of particular situations to which the same mathematical knowledge can be applied. This means that we must help children's mathematical concepts to become context-free: that is, independent of any particular embodiment, and, in particular, independent of the manipulatives which are such a useful help in forming them initially. The present topic is a good example.

Activity 1 Different objects, same pattern [Patt 1.4/1]

A teacher-led discussion for a small group. Its purpose is to help children to realize that the same pattern can be made with a variety of different sets of objects.

Materials • The containers used in 'Patterns with a variety of objects' [Patt 1.1/2].

- Suggested sequence for the discussion*
1. Make a simple pattern using just two kinds of object from one of the containers. For example,
shell bean bean shell bean bean shell bean bean . . .
 2. Ask if any one thinks they can copy this pattern using objects from one of the other containers.
 3. If no one offers, it may be that they need more time with 'Patterns with a variety of objects' [Patt 1.1/2]. Alternatively, you could make a copy yourself, and ask whether they think that this is the same pattern, although the objects are different. If they appear doubtful, then they do need more time with Patt 1.1/2.
 4. Assuming that in step (ii) someone does offer, let him try, and ask whether the others agree with what he has made. If they do not agree, discuss what changes are needed so that the second pattern is the same as the first, albeit with different objects.
 5. Working in pairs, the children may now choose other containers and make further embodiments of the given pattern. They check each other's results.
 6. The objects are replaced in the containers, and steps 1 and 5 are repeated.

Activity 2 Patterns which match [Patt 1.4/2]

An activity for a small group of children. Its purpose is to consolidate the concept formed in Activity 1, by applying it to different materials.

- Materials*
- The example patterns which were used in ‘Making patterns on paper’ [Patt 1.1/3].
 - The patterns on paper which they made themselves in that activity.

- What they do*
1. Begin by showing the example patterns. Ask if any of these show the same pattern with different objects.
 2. In the set of examples given in Patt 1.1/3, Patterns 2 and 4 do show the same pattern with different objects. For short, we say that they match. In this set there are no other matching patterns.
 3. The match will show even more clearly if we make a pattern matching Pattern 2 using the letters A B C If we now do the same with Pattern 4, we find that we get A B C A B C A B C for both.
 4. This is also a good way to make sure that we have not overlooked any other patterns which are alike.
 5. Their own patterns are now spread out on the table. Each child takes a pattern, and collects all the patterns which match it. It is advisable for them to begin by checking that no two of them have matching patterns.
 6. Finally they check each other’s collections.

Activity 3 Patterns in sound [Patt 1.4/3]

A continuation of the two previous activities, for a small group. Its purpose is to show how the concept of pattern can be embodied in a completely different modality.

- Materials*
- Three or more objects which make different sounds when struck. Instruments from a percussion band offer one possibility.
 - One pattern from each of the sets of matching patterns collected in Activity 2.

- What they do*
1. The patterns are spread out where all can see them. It would be advisable to begin with, say, three or four, in which not more than three different objects are used.
 2. One child then chooses a pattern without telling the others which it is. He translates this into a sound pattern, in which each object in the visual pattern corresponds to a different sound.
 3. The other children try to identify which of the visual patterns is being played.
 4. The first child who thinks he knows says so (without indicating which pattern it is), and demonstrates by continuing the sound pattern.
 5. If the second child is right, that child continues until another child thinks he knows the pattern. If he is wrong, the first child says so and takes back the role of striker.
 6. Steps 3, 4, 5 are repeated until all have identified the pattern.
 7. Steps 2 to 6 may then be repeated with a different child starting.

Activity 4 Similarities and differences between patterns [Patt 1.4/4]

A teacher-led discussion for a small group of children. Its purpose is to have children reflect on and organize what they now know about patterns, within the framework of similarities and differences.

Materials • The example patterns from ‘Making patterns on paper’ [Patt 1.1/3].

- Suggested outline for the discussion*
1. Begin with the example patterns. Ask in what ways they can be alike, and in what ways different.
 2. We have already seen that they can be alike in that the pattern is the same though the objects themselves are different. So we can have two patterns which use the same objects but are different patterns, or which use different objects but are the same pattern.
 3. In the second case, we need a way of saying what the pattern is. I suggest that the translation into letters introduced in Patt 1.4/2 is convenient and universal.
 4. Another way in which patterns can be alike or different is the number of different objects used, and the number of objects in each repetition. These are independent.

For example,

A B A A B A A A B A B A A B A A A B A B A A B A A A B

uses only two objects, but there are nine objects in each repetition (which we may call a cycle). But

A B C D A B C D A B C D

uses four objects, and the number in a cycle is also four.

5. Given a number of patterns on paper or in sound (though not limited to these) we can identify the following attributes.
 - The pattern itself.
 - The objects used to show the pattern.
 - The number of different objects used.
 - The number of objects in a cycle.
6. These are also ways in which they can be alike or different, which prepares the way nicely for the next activity. I suggest that you make a start with this right away, after which they can do it on their own when it is convenient.

Activity 5 Alike because . . . and different because . . . [Patt 1.4/5]

An activity for a small group of children. Its purpose is to consolidate the concepts which were discussed in Activity 4.

Materials • The children’s own patterns.

- What they do*
1. The patterns are put in a pile face up. The top pattern is then put separately to start a second pile.
 2. The children then take turns describing the likenesses and differences between the two patterns showing. E. g., if these were the example Patterns 4 and 5, he might say:

Patt 1.4 Translating patterns into other embodiments (cont.)

“These are alike because they use the same objects: triangles, circles, and squares. They are different because one has a cycle of three and the other has a cycle of five, so they can’t be the same pattern.”

Or if they were Patterns 2 and 4, he might say:

“These patterns use different objects, but they both have three different objects and they both have a cycle of three. They are both the same pattern, which in letters would be A B C A B C A B C . . .”

The other children say whether they agree or not, and may contribute further suggestions.

3. When the first child has done this, the next child moves the top pattern from pile 1 across to the top of pile 2, so that there is a different pair showing (though one was in the previous pair). He then describes the likenesses and differences between these, as described in step 2.
4. They continue as in step 3 until each child has had at least one turn.



Alike because . . . and different because . . . [Patt 1.4/5]

Discussion of activities In this topic children are expanding their concept of a pattern using all three modes of schema construction. They are learning from physical experience, using a variety of manipulatives; they are extrapolating pattern concepts to new embodiments; and they are checking for consistency with their existing schemas by frequent discussion.



Tricky Micky [Meas 1.1/2]

Setting up shop

[Meas 1] **LENGTH**

The measurement of distances

Meas 1.1 MEASURING DISTANCE

- Concepts*
- (i) Invented units of distance.
 - (ii) Combining units to fill a distance.
 - (iii) Counting units to measure a distance.

Abilities To measure a straight-line distance by combining unit lengths to fill the distance, and counting these units.

Discussion of concepts Once again I find myself saying “There is more here than meets the eye.” The present topic introduces the idea of measurement, which opens up a whole new field in the application of mathematics.

The key concept is the transition from what may be called natural units to invented units. If we have a set of shells, or chairs, or books, we can say how large in number this set is by regarding each separate object as a unit, and counting these unit objects. But if we have a strip of wood, a piece of string, or a garden hose, we cannot use this method. The answer to “How many?” is in each case “One.” But now we also want to know, for each object, “How big?” Or, in this case, more specifically, “How long?”

To answer the second question we choose a particular distance which we agree to call “One.” We then count how many of these, combined together, fill the distance in question.

This does far more than allow us to answer the question just put. We have already developed much knowledge about the use of numbers derived from counting separate objects. The introduction of units of measure extends the application of this knowledge into a great variety of new applications: beginning here with distance, and continuing with area, volume, weight, time. Other applications include the measurement of speed, of temperature, of electrical phenomena, of value (money) – the list is a long one.

Before a unit can be used for measurement, however, we also need a way of combining these so that the result is equivalent to whatever we want to measure. For length this is straightforward: we put them end to end in line, without gaps or overlap. Likewise for weight – we put the units of weight in the same scale pan. For area it is not so straightforward: some shapes which might be chosen as units will not combine to fill a surface – they leave gaps. Straightforward or otherwise, the ways in which units are combined are an essential part of any system of measurement. So in the activities which follow, both in this network and those which follow, I call attention to this aspect rather than let it be passed over without being made explicit. This is what happens if measurement starts with rulers.

Meas 1.1 Measuring distance (cont.)

Activity 1 From counting to measuring [Meas 1.1/1]

A teacher-led discussion for up to 6 children. Its purpose is to introduce the concept of a unit of distance.

- Materials*
- Set loop.
 - Small objects for set.
 - Popsicle sticks.
 - Medium sized paper clips (not joined).
 - Any other objects of uniform length.

- Suggested sequence for the discussion*
1. Put out the loop, with some objects in it.
 2. Ask “How do we say how large (or how big) this set is?” (Answer, we count how many objects there are.)
 3. “Here’s a more difficult question. How do we say how big this distance is, from here to here?” (The distance can be anything you find suitable, e.g., width of a table, length of a pencil box,)
 4. If some one suggests using a ruler, agree that this is a good way. But from a teaching point of view, this gets there too quickly. It is rather like a condensed notation, which is quick and convenient but does not easily reveal all the contributory concepts which it combines. So in this case continue “If we didn’t have a ruler, but we had some of these, is there a way we could use them?” (“These” being suitable objects from the materials listed.)
 5. Discuss any other suggestions, and try them out if practicable. At this stage we are not looking for a fully developed system, but for something which involves (a) putting objects of the same length end to end, to fill the distance, and (b) counting these.
 6. Help the children to formulate what has been done. Since to begin with there was nothing to count in order to say how big the distance was, we filled the distance with things we could count.

Note The term ‘unit’ will be introduced in Activity 2.

Activity 2 Tricky Micky [Meas 1.1/2]

An activity for up to 6 children. Its purpose is to make explicit the concept of a unit of distance, and its use for measurement.

- Materials*
- ‘Stick candy’ of assorted lengths.*
 - Three boxes of paper clips: small, medium, large.
 - Notice[†]:

LOVELY STICK CANDY ONLY A PENNY A PAPER-CLIP-LENGTH. LENGTHS SHORTER THAN A PAPER CLIP FREE.
--

- Role cards[†] for children taking parts of Tricky Micky and his assistants.
- Pennies, real or play money.

* I suggest the extra thick gaily coloured milk straws often provided with milk shakes.

[†] Provided in the photomasters.

- What they do*
1. One child takes the role of Tricky Micky, who sells ‘stick candy.’ He sets up his stall with the ‘stick candy,’ the notice, and a box of medium paper clips on view. The other children take the roles of customers except when it is their turn to act as an assistant. The children should only see their own role cards.
 2. The first customer is served by Fair Clare. When the customer has chosen his stick of candy, Fair Clare measures it using medium paper clips from the box on view, and charges accordingly.
 3. The second customer is served by Tricky Micky, who measures using a box of smaller paper clips which he has at the back of the counter. This way, the customer is overcharged. He tries to do this without it being noticed.
 4. This continues until the trick is noticed, and its effect discussed.
 5. The next customer is served by Lazy Daisy. She uses large paper clips from a box of her own, to save herself trouble.
 6. This continues until Tricky Micky realizes that customers are now paying too little. The reason for this is discussed.
 7. Another assistant now takes over, Mixy Trixy. She uses paper clips of mixed lengths.
 8. This continues until someone realizes that now we don’t know whether customers are being charged too much, or too little, or what.
 9. Silly Sam now takes over, with strict instructions only to use medium paper clips. The other boxes are put away to make sure.
 10. However, Sam still gets it wrong. Sometimes he leaves gaps, sometimes he overlaps the paper clips, sometimes he puts them crooked. When one mistake is pointed out he starts making a different mistake. These are pointed out and the effect discussed.
 11. Finally, the customers decided that the only way to be sure of fair trading is to get together with Tricky Micky and his assistants, and make a list of rules for correct measurement. Tricky Micky also sees the need for this, since his assistants have sometimes under charged.
 12. The children work on this together. The points to be included are:
 - (a) All the objects used must be of the same size.
 - (b) Everyone must know what objects are being used.
 - (c) The objects must be used to fill the distance in a straight line, end to end (which means no gaps, no overlaps).
 - (d) If the distance cannot be filled exactly, we give the answer to the nearest number of units below “and a bit over.”
 13. Tell them that whatever the distance we agree to count as ‘one’ is called a unit of distance. So in this case, the unit of distance agreed on is the length of a medium paper clip.
 14. One of these may then be displayed on the classroom wall, labelled

A STANDARD PAPER CLIP

Note You may prefer to omit some of the roles. The essential ones are Fair Clare and Tricky Micky (other roles: Lazy Daisy, Mixy Trixy, Silly Sam).

Activity 3 Different names for different kinds of distance [Meas 1.1/3]

Introductory note for teachers Here, as elsewhere, we encounter words with more than one meaning. For example we speak of the length of a table, as against its width. But we also say that a centimetre is a measure of length, as against area; and use centimetres to measure not only length, but also width, height, depth, thickness,

This distinction between some physical quality, and the number of units by which we measure that quality, is an important one which is often overlooked. Fortunately, the particular meaning is usually clear from the context; so the important thing is to know that there are these two meanings, and to be aware which one we are currently using.

Height, depth, width, thickness, are straightforward, and the first meaning of length belongs with these.

Length is distance along the longer dimension of something, from one end to the other.

Width is distance across, between one side of something and the other.

Breadth means the same as width.

Height is vertical distance above, e.g., from the bottom to the top of something.

Depth is vertical distance below, from the top to the bottom of something.

Thickness is distance through, between one surface of something and the other.

So we see that distance is the more general term. But (like length), we use it to mean not only the space between one point and another, as above; but also the number of units by which we measure that space.

Since it helps to keep our thinking clear if we use different words for different ideas, I shall hereafter use ‘distance’ for the space itself, and ‘length’ for the number of units which tells us the size of that space. Note also that in this network we are concerned with one-dimensional space: that is, space on a line. In area and volume, we shall deal with two-dimensional and three-dimensional space, respectively.

Though we need to be clear about the above in our own minds, so far as the children are concerned I think that this may be treated fairly lightly.

The activity itself This may take the form of a teacher-led discussion, as suggested below.

1. A table is a good example to begin with. Explain that we have different names for different kinds of distance. Ask “What do we call this distance, from here” (touching one end of the table) “to here?” (touching the other end). The answer is of course “Its length.”
2. Width (= breadth) and thickness may be dealt with in the same way.
3. For height, the classroom itself is a good example to start with. Its height is the distance from floor to ceiling.
4. For depth, you could use a swimming pool, a lake or a river. We would also talk about the depth of a hole.
5. The children may then be invited to give examples of each kind of distance, and also the other way about, asking the others “What do we call *this* distance?”

Discussion of activities Activity 1 introduces the key concept of a unit of distance, using physical objects. Sometimes for this purpose a single object such as a pencil or a hand-span is used repeatedly. This is of course perfectly correct, but for introducing the measurement of length I think that it is harder for children in two ways. It is more difficult to combine these unit distances correctly, end to end in a straight line without gaps or overlaps. And it is harder to count them. (Neither of the foregoing objections applies to pacing a distance, but this exemplifies rather well the ideas in Activity 2. The measured length of the school hall would vary quite a lot if it were paced first by the smallest child, then the largest, and then their teacher!) The method used in Activity 1 shows visually the relation between unit objects in a set, and units of distance; and it allows all the units of distance to be seen and counted.

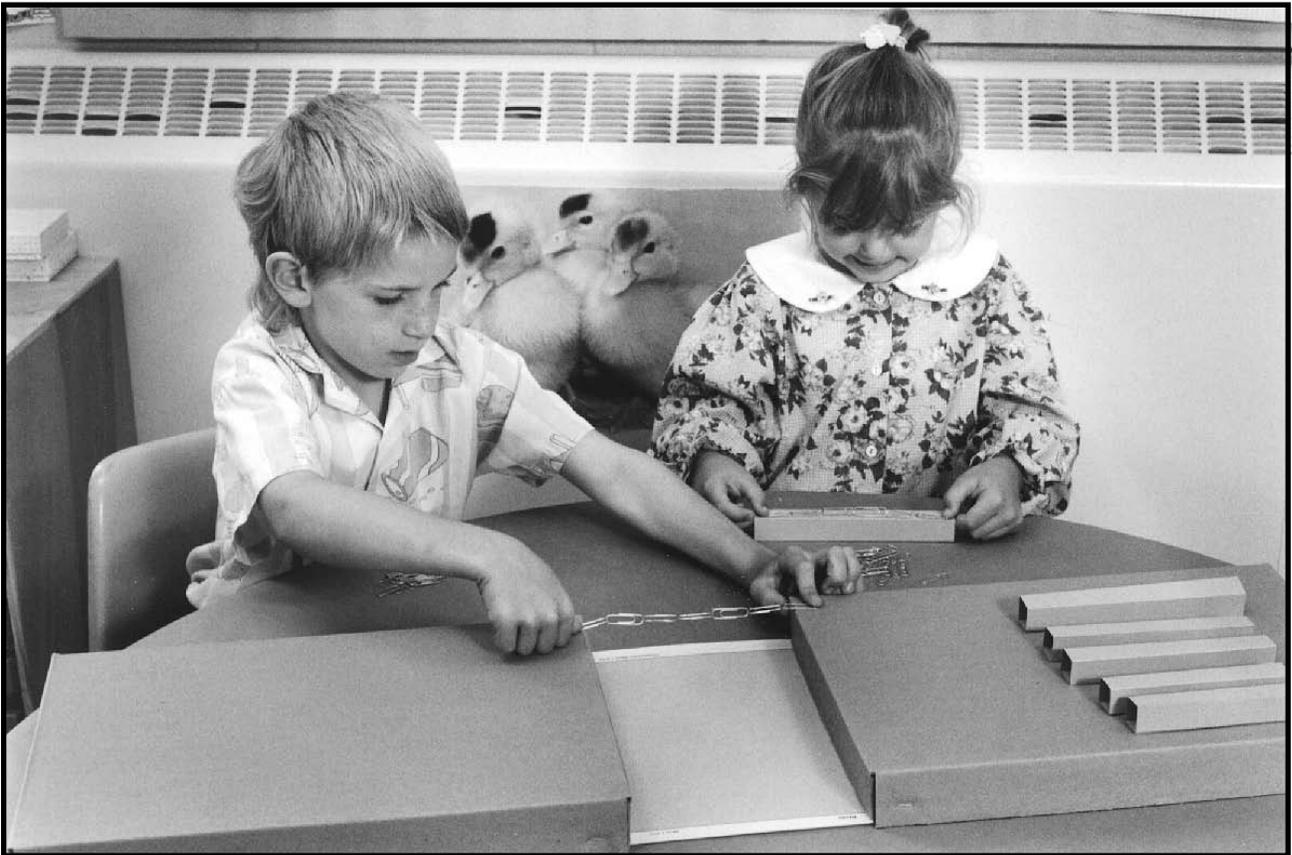
Activity 2 develops in detail the requirements for successful co-operation (in this case, fair trading) by a combination of physical materials and discussion: Modes 1 and 2. In the process, the concept of a unit is itself given more detail and precision, since what we mean by a unit is closely related to what it is for and how it is used.

Activity 3 is about vocabulary and the use of words for accurate communication.

OBSERVE AND LISTEN

REFLECT

DISCUSS



Building a bridge [Meas 1.2/1]
(see following page)

Meas 1.2 THE TRANSITIVE PROPERTY; LINKED UNITS

- Concepts*
- (i) The transitive property of measurement.
 - (ii) Units linked end to end, ready for using to fill a distance.

- Abilities*
- (i) To use measurement to compare distances in different locations.
 - (ii) To use linked units to measure a distance.

Discussion of concepts A major practical use of measurement depends on the transitive property. This enables us to compare sizes and distances without putting the two objects concerned alongside one another, which is not always convenient and may be impossible. We use it to buy curtains to fit our windows without having to take them home to try them for size first. We can ask for tires to fit the wheels of our car without first trying them on.

In mathematical terms,

if the length of A = the length of B

and the length of B = the length of C

then the length of A = the length of C

So the middle length B may be used as a go-between. For this purpose we need something which is easily transportable, and also easy to use.

Instead of a number of separate objects, which have to be carefully put in place to satisfy the requirements arrived at in Activity 2 of the last topic, it would be much better to have some kind of device which makes this easy. By linking paper clips together, we have a measuring chain which does just this. It only needs to be pulled tight to put the paper clips in a straight line, end to end. We can then count those which together fill the distance, ignoring the others. This makes a direct connection between the use of separate objects, and the use of measuring instruments such as rulers and measuring tapes.

Activity 1 Building a bridge [Meas 1.2/1]

A predictive activity for two teams. Its purpose is to teach the concepts and ability discussed above, and also to introduce children to another practical use of measurement.

- Materials*
- River.*
 - Banks.*
 - Box girders. *
 - Box of medium paper clips.

*The river is made of paper or card, painted to look like water, about letter-size (21.5 by 28 cm). The banks can be books, boxes, or the like. The box girders are made from cardboard, square in cross-section. You need 2 each of the following lengths, cut as accurately as possible: 14 cm, 16 cm, 18 cm, 20 cm, 22 cm, 24 cm. These are also used in Meas 1.5/1 in *Volume 2*.

- What they do*
1. One team consists of engineers starting work on a bridge, the other is in charge of materials and transport.
 2. The river, with its banks, is at one end of the table.



- The engineers are here. The box girders, with the other team is at the other end of the table. We imagine a distance of several kilometres in between.
3. Having arrived at the river, the engineers want to measure its width so that they can call for a box girder of suitable length. Each team has half of the box of paper clips, for measuring.
 4. The engineers, however, have a problem: how to fill the distance across the river with paper clips end to end. They discuss this. Eventually someone has the idea of linking the paper clips to make a measuring chain.
 5. With this, they measure the width and decide on a length for two box girders to span it safely but without waste.
 6. Using their walkie-talkie radio, the engineers call the other team for two box girders of the required length. They also explain the use of the measuring chain.
 7. When they arrive, the girders are put across the river with space between, to form the support for a roadway. This tests the engineers' prediction.
 8. Steps 1 - 7 may be repeated with the teams interchanging roles, and with another river of a different width.
 9. In Meas 1.5/1 (see *Volume 2*) we shall repeat this activity and add the roadway.

Discussion of activity

This simple activity nevertheless embodies two further contributors to the theory and practice of measurement. Theory, since transitivity is a mathematical property of great generality. Practice, because the activity introduces ready-for-use combinations of units. Both, because the transitive property is being put to practical use whenever we use measurement to make things fit; from simple everyday cases such as measuring a room for a carpet, to sophisticated technological uses in which electrical properties of components must be 'fitted together' to give a working circuit. Forgive me if I point out once again, that there is nothing so practical as a good theory.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 1.3 CONSERVATION OF LENGTH

Concept That the length of an object does not change with its position.

- Abilities*
- (i) To match and compare lengths independently of position.
 - (ii) To match and compare lengths independently of perceptual distractors.

Discussion of concepts The term ‘length’ is here used to mean both the distance from one end of an object to the other, and also the measure of this distance.

This and transitivity of measurement are interdependent. We assume that whatever we use for measuring does not change its length as we move it from one place to another. This is easier to accept for something made up of unit lengths, and in topic 2 it was assumed intuitively. Here we make it explicit, and expand the concept to include more difficult embodiments.

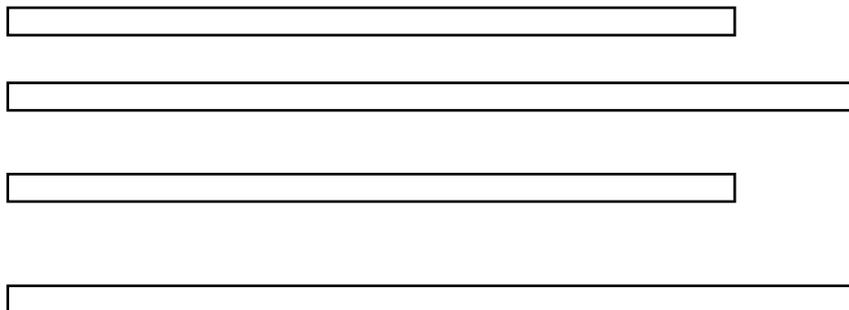
Activity 1 “Can I fool you?” (length) [Meas 1.3/1]

A game for two teams of equal number. Its purpose is to help children’s perception of length to become increasingly independent of position. It is not an activity for the use of measurement.

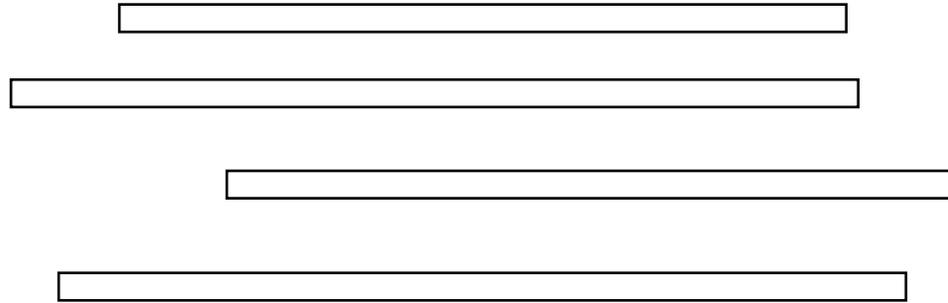
- Materials*
- 6 wooden rods, each 20 cm long.
 - 6 wooden rods, each 24 cm long.
- * Other lengths may also be used for variety of experience.

- What they do*
1. Team A has the rods to begin with. They sort these into sets of equal length, and put two rods of each size on the table. See step 4 for further details of how they are positioned.
 2. Each child of team B, in turn, tries to pick up a pair of rods of equal length. He checks by physically matching, but must not do this until after he has taken them, one with each hand.
 3. After each attempt by a member of team B, team A replaces the rods taken by another pair from their stock. They may also re-arrange the rods if they like.
 4. Team A tries to ‘fool’ the other team by the ways in which they put down the rods. Initially, these are restricted, as follows.

Stage (a) Rods parallel, all even at one end. E.g.,



Stage (b) Rods parallel, ends need not be level. E.g.,



Stage (c) Unrestricted.

5. When each child in team B has had his turn, the teams exchange roles.
6. The difficulty may also be increased by using more rods. My own inclination is however to keep the number down, so that position rather than number is the difficulty to be overcome.

Activity 2 Grazing goat [Meas 1.3/2]

An activity for up to 6 children. Its purpose is to extend the conservation of length to curved lines, with the help of measurement.

- Materials*
- ‘Grazing goat’ game board (see Figure 27).**
 - Model goat.
 - Cord, representing rope
 - Measuring chain made from paper clips.*
 - Plasticine to mark predictions.

* As used in Meas 1.2/1

** Provided in the photomasters

- What they do*
1. The goat is put somewhere in the middle of the grassy part of the field, with one end of the rope tied round its neck. The other end is threaded under the paper clip. The rope is laid in a curve.
 2. One of the players acts as goat boy. The farmer has explained to him that goats eat not only grass, but nearly everything else. He has told the goat boy to allow enough rope for the goat to eat as much grass as possible, but nothing else.
 3. However, the goat boy has been butted by the goat and doesn’t want to go in the same field as it again. So he adjusts the rope from behind the fence to the length which he thinks will best do what the farmer has ordered.
 4. Each child estimates where he thinks the goat is able to reach.
 5. The measuring chain is then laid along the rope, and used to make a more accurate prediction.
 6. Finally, the goat walks to the full extent of its rope, and everyone is able to check their predictions.
 7. Steps 1 to 6 are repeated with a different goat boy. The most successful may be named goat boy of the year.

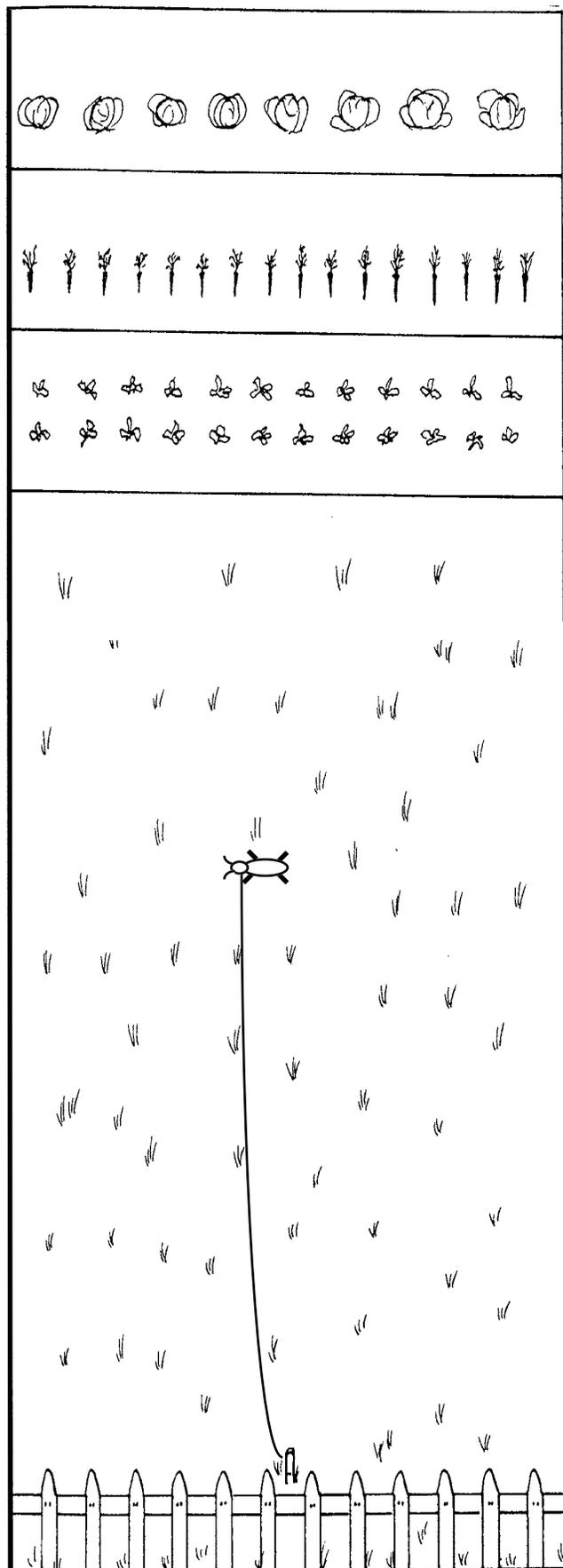


Figure 27 Grazing goat

Discussion of activities Activity 1, Stage 1, is quite easy and serves mainly to introduce children to the rules of the game. It only involves comparing the rods at one end.

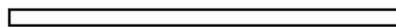
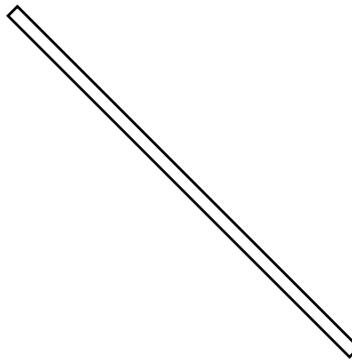
Stage 2 will be recognized as based on a well-known experiment of Piaget. It involves comparing both ends, and compensating.

This rod comes further
at this end



This rod comes further
at this end.

Stage 3 is much harder, and it may not be until children are older that they will be able to make correct judgements in cases like the one below. It involves the combination of more developed perception, and also the use of inference. The use of this game will help to develop these abilities.



OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 1.4 INTERNATIONAL UNITS: METRE, CENTIMETRE

- Concepts*
- (i) The need for international units.
 - (ii) Metre, centimetre.

- Abilities*
- (i) To measure in metres and centimetres.
 - (ii) To put this to practical use.

Discussion of concepts We now introduce the metric system of units, based on the metre. The scientific name for these is S.I. units, where S.I. stands for 'Système Internationale.' S.I. units could also be thought of as referring to Standard International units, which would incorporate explicitly the idea of a standard unit.

The S.I. system has at least three important advantages. First, it is used internationally by scientists, and by an increasing number of countries for technology, commerce, and everyday life. Second, the system uses base 10 for relating units of the same kind. E.g., one kilometre is 1000 metres, one metre is 100 centimetres; whereas a mile was 1760 yards, a yard was 3 feet, a foot was 12 inches. So S.I. units are much more easily convertible, which is a great advantage for calculations using place-value notation. Third, S.I. units of different kinds often have simple relationships. E.g., one gram is the weight of one cubic centimetre of water (at 4 °C), one kilogram is the weight of one litre of water (1000 cubic centimetres). We shall gradually introduce children to all of these advantages.

Anyone who wants to use standard units needs to have something which is either a copy of the international standard, or a copy of a copy of a copy of . . . this. This is what we get when we buy a metre rule, or a smaller measure. So the use of standard units implies the use of standardized measuring instruments. The accuracy of these also depends on the transitive property of measurement.

Activity 1 The need for standard units [Meas 1.4/1]

A teacher-led discussion, for any number of children. Its purpose is to show the need for a standard international system of units.

Materials For each table:

- a ruler marked and numbered in centimetres.*
- a metre rule.
- source book for historical material.

* It does not matter if it is marked in centimetres and millimetres also, but any *numbering* must be in centimetres.

- Suggested outline for the discussion*
1. Ask: “Who can name one of the countries that we trade with? And another? And another? . . . (Countries of origin of our clothes, watches, playthings, calculators, . . .)
 2. If we use things like paper clips to measure with, how do we know that they all use the same size of paper clip as we do? Why does it matter? (Recall Tricky Micky, Meas 1.1/2.)
 3. What about units for longer distances, such as the length and width of a room, the distance from school to home? Paces would be useful here, but whose? Different people have paces of different lengths.
 4. And we need to know how many small units in a larger unit. The number of paper clips in a pace may not be a convenient number. What would be a convenient number?
 5. So we need at least two units, agreed internationally, one fairly small and one larger. And the large one should be equivalent to a convenient number of the smaller. Convenient for what? Especially, for conversions and calculations.
 6. Some will no doubt have heard of centimetres and metres. (Clothes these days are often marked with sizes in both centimetres and inches.) Let them examine a metre rule and a centimetre ruler. Explain that these units are now used in many countries, all over the world.
 7. There are 100 centimetres in a metre, just as there are 100 cents in a dollar. In both cases we use the larger unit (metres, dollars) for large amounts, and the smaller units (centimetres, cents) for smaller amounts. And we sometimes use them together, e.g., 3 metres, 45 centimetres; 3 dollars, 45 cents.
 8. Canada adopted the S.I. system relatively recently, but the United States has been unable to move as quickly in this direction. Consequently, many of the older units are still encountered (e.g., inches, feet, yards, and miles, as units of distance). Can anyone say how these relate to each other?
 9. Within living memory, even more units were in use which from a present viewpoint are quite hard to believe. The simplicity and convenience of the S.I. system is brought out more strongly by comparison with these. So I think that it is worthwhile investigation for children to find out about these and write a short description, with comments.

Activity 2 Counting centimetres with a ruler [Meas 1.4/2]

An activity for up to 6 children, to be played in pairs. Its purpose is to establish the equivalence of counting a number of units, and reading a number on a ruler. This activity should also help children to place the starting point on the ruler correctly.

- Materials*
- Plenty of Centicubes.
 - A 30 cm ruler for each pair, marked in centimetres.

- What they do*
1. (Preliminary). All the children make pairs of rods of the same colour and length. No two pairs should be of the same colour. They should vary between 10 cm and 30 cm. These are mixed together and put in the middle for everyone to use.
 2. From now on, they work in pairs. Child A has the ruler.

Meas 1.4 International units: metre, centimetre (cont.)

3. Child A names a colour, say blue.
4. Each then takes one rod of that colour. A finds the number of cubes by counting, B by using the ruler.
5. Whoever finishes first says the number. The other finishes his way, and says "Agree" or "Disagree."
6. If "Agree," they check by putting the rods alongside to see if they match.
7. If "Disagree," they change roles and measure/count again.
8. If counting cubes and measuring with the ruler do not agree, this might be because the starting point of the ruler has not been positioned correctly.
9. The rods are then replaced, and steps 2 to 7 are repeated.
10. This activity should be continued until children can always get a correct result more quickly with the ruler.

Activity 3 Mountain road [Meas 1.4/3]

A game for two, four, or six children working in pairs. Its purpose is to use measurements to make predictions of practical interest.

- Materials*
- 3 model trucks, on which loads of different sizes are secured. Some loads are high and narrow, some are wide and low. Furniture vans, vans with cranes, and the like, could be used for the high narrow ones.
 - Table sized enlargement from the photo-master, with models of bridges replacing the symbols. *
 - Centicubes.

For each child:

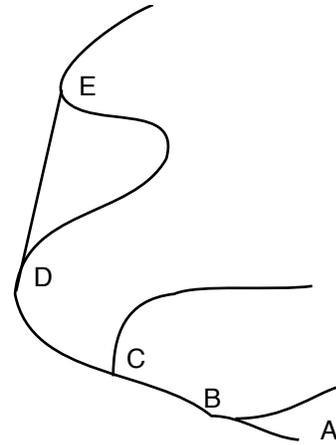
- Road map.†
- Ruler marked in centimetres.
- Non-permanent marker.

* The bridges are made from centicubes to the heights or widths shown in the illustration. These measurements should be adjusted to the sizes of the model vehicles which you have. The high loads should always be able to cross the bridges of limited width (marked W), and the wide loads should be able to cross bridges with restricted heights (marked H).

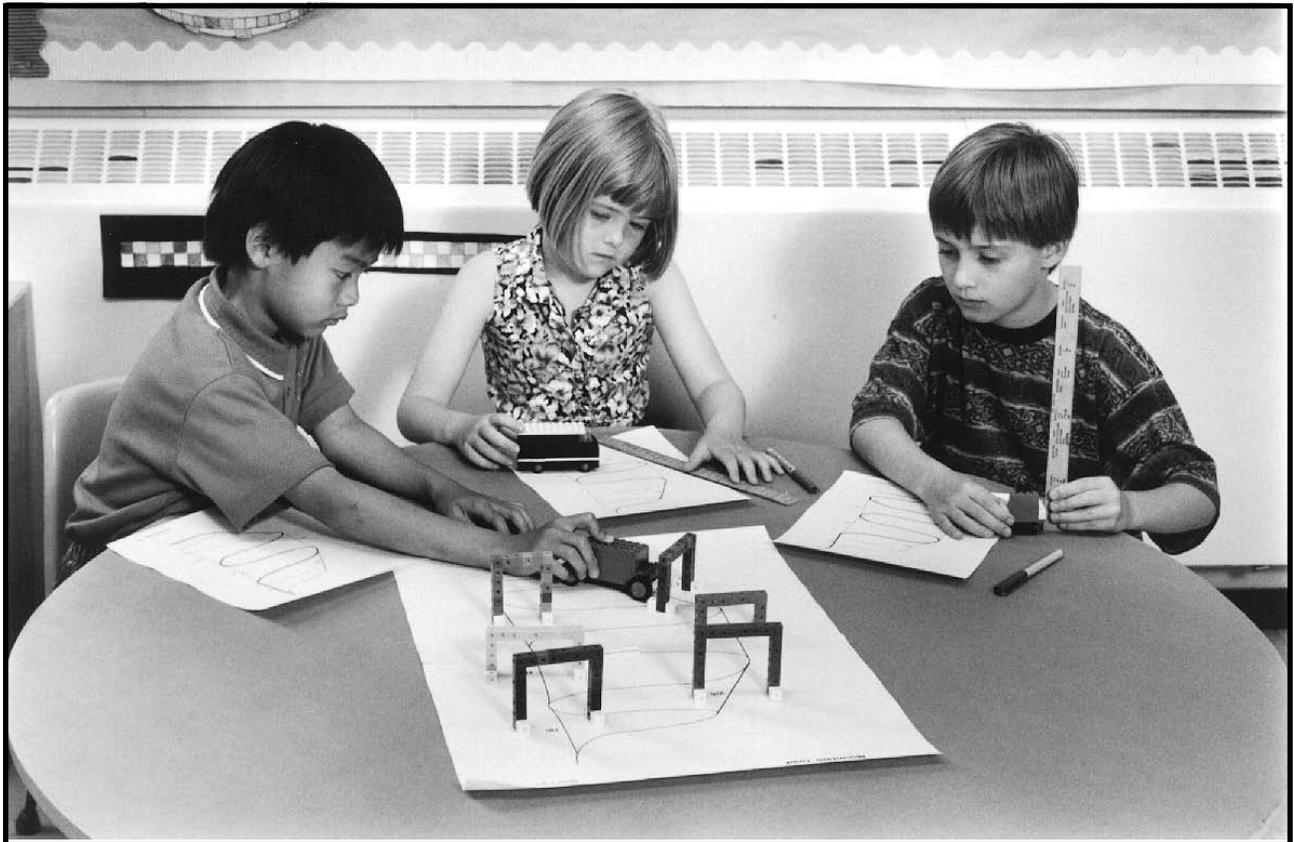
† A copy from the photo-master, covered with transparent film to allow use and re-use with non-permanent markers.

- Rules of the game*
1. The children are shown the table-sized model of a mountain road.
 2. Each is given a road map, and the correspondence between this and the road and bridges explained. The high vehicles can always use the bridges marked W (restricted width), but have to be careful with the bridges of restricted height (marked H). Likewise, the other way about, for trucks with wide loads.
 3. Another factor to be taken into account is that the trucks cannot get around hair-pin bends.

So a truck coming from A could go in either direction at junctions C and D, but at B and E it could not take the hairpin turn to the right.



3. Back at their own table, each pair is given a vehicle. One acts as driver, the co-driver acts as navigator.
4. The drivers measure their vehicles, and together they plan their route. They mark this route on their road maps with a non-permanent marker.
5. Watched by the others, each driver in turn takes his vehicle along the chosen route. The co-driver navigates with the help of his road map.
6. Every pair which gets there without mishap, preferably by the shortest way possible for that vehicle, is a winner.
7. The vehicles may be re-allotted, and steps 4 to 6 repeated.



Mountain road [Meas 1.4/3]

Activity 4 Decorating the classroom [Meas 1.4/4]

An activity for as many children as you like. Its purpose is to introduce the use of the metre as a unit for larger measurement.

- Materials*
- Metre measures. *
 - Party paper streamers.
 - Plasticine.

* These do not need to be graduated in cm, so additional measures can be made from bamboo, dowel, and the like.

- What they do*
1. A suitable reason is found for decorating the classroom.
 2. In small groups, children are allotted distances in which the streamers are to be hung. E.g., a wall, the space between two windows.
 3. They use the metre rules to measure these distances, and then to measure the right lengths of streamer.
 4. The streamers may be fixed in position with plasticine, or otherwise.
 5. We are not here concerned with great accuracy, but with the use of a larger unit for larger distances. So measurements like “3 and a bit over” or “3 and a half” are suitable for this job.

Discussion of activities Activity 1 is a general discussion, calling attention to the need for international standard units, and some of the requirements to be satisfied. It relates the knowledge they have acquired while working on this network to their general knowledge.

Activity 2 introduces the use of a ruler, which like a number track greatly speeds up the counting of units. It calls attention to the correspondence between counting a number of units and reading this number directly from a ruler, using both Mode 1 (physical matching) and Mode 2 (“Agree”).

In Activity 3, I have tried to provide a game which would make use of measurement to arrive at a sound plan of action in the same kind of way as in the adult world. This was not easy, since full-sized objects are not easy or convenient for children to move around. Models are much more manageable, and the amount of mathematics used in a given time is greater. But please note that we are not here using scale models. The sizes are actual sizes, in centimetres. If children spontaneously suggest that 1 centimetre could represent (say) 1 metre, good for them. But conceptually, this is much more advanced, since in their general form scale models use ratio and proportion. At this stage, a centimetre is a centimetre, and the models are objects in their own right.

In Activity 4, measurement is put to another of its major uses in the adult world: making things fit. Here we are concerned with introducing the concept, rather than with accuracy. Party streamers provide an convenient and inexpensive material.

OBSERVE AND LISTEN

REFLECT

DISCUSS

[Meas 2] **AREA** See *SAIL, Volume 2*.

[Meas 3] **VOLUME and CAPACITY**

Meas 3.1 **CONTAINERS: FULL, EMPTY**

Concepts (i) Full and empty, in the context of
(ii) A container.

Ability To recognize whether a container is full or empty.

Discussion of concept A container is a hollow object that we can put other objects inside. If there is nothing inside, we say it is empty. By ‘full,’ we mean that there is so much in it that there is no space left. The simplest case of this is when we fill the container with a liquid, since this takes the shape of the container and leaves no gaps. A bowl full of sugar lumps would not be full with the above meaning. In everyday life, by ‘full’ we usually mean ‘nearly full’ as against ‘brim-full.’

The capacity of a container is the volume it will hold when full. So the concept of volume is prerequisite for that of capacity (see concept map).

Activity 1 Full or empty? [Meas 3.1/1]

This activity does not need to be set up separately, but may be included as part of normal conversation during other occupations such as painting, sand and water play, meal or snack times.

Materials

- Containers such as glasses, cups, mugs, jugs.
- Water, milk, fruit juice, or whatever is in use.

What they do You might, for example, ask them to put the full glasses on a tray to take them to the table, and bring the empty glasses to the sink for filling. This is less a matter of specific planning than of being open to possibilities of the moment.

Discussion of activity This activity simply involves giving a little more emphasis to an everyday experience, and linking it with vocabulary.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 3.2 VOLUME: MORE, LESS

Activity 1 Which is more? [Meas 3.2/1]

A teacher-led activity for a small group of children. Its purpose is to help children to form the concept described above.

- Materials*
- Identical drinking glasses, one for each child and one for yourself.
 - A jug of water.

- Suggested sequence for the activity*
1. Each child pours herself a glass of water. You pour one for yourself, to a height which is clearly different from any of the others.
 2. Put your glass alongside one of the children's, and ask "Which glass has more water in it? And which has less?"
 3. Repeat, first with your own glass and a different child's; and then, with two of the children's glasses, which are likely to be closer together in height. Sometimes the questions should be asked the other way round, beginning with "Which has less?" For some pairs, the answer may be "They both have the same amount."
 4. Repeat step three, but this time ask one of the children to pour more into the one with less water until both glasses have the same amount of water in them.

- Notes*
- (i) In the course of this activity, the word 'volume' may be gradually introduced as meaning the same as 'amount' in the present context. (In other contexts, amount might mean something different. For example, the same amount of string would probably mean the same length.)
 - (ii) If convenient, this activity can also be done at meal or snack times.

Discussion of activity	This activity begins with a fairly typical everyday situation, but takes it a little further in that it concentrates attention on the comparison aspect, and again links it with vocabulary.
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OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 3.3 CONSERVATION OF VOLUME

Concept That the volume of a liquid remains the same when it is poured without spilling into another container.

Ability To recognize this fact.

Discussion of concept One of Piaget's classical experiments was concerned with just this concept. It has been replicated many times since, and each time you do Activity 1 you will be doing another such replication. Until children have this concept, measurement of volume using non-standard or standard units has no real meaning. If we pour a glassful of water into a measuring cylinder of a different shape, we are assuming that the reading tells us not only the volume of the water while it is in the measuring cylinder, but its volume beforehand while it was in the glass. If we measure a certain amount of (say) milk for a recipe, we are assuming that this volume will remain the same after we have poured it into the mixing bowl. The same argument applies for non-standard measures, such as cups full.

Activity 1 Is there the same amount? [Meas 3.3/1]

A teacher-led activity for a small group of children. Its purpose is to help children to realize that the volume of a given amount of water remains the same, independently of the shape of the container it is in. If they already realize this intuitively, then the discussions in this activity will raise their awareness from intuitive to reflective, and relate it to the present schema.

- Materials*
- Two identical drinking glasses.
 - A tall thin glass.
 - A short wide glass.
 - A jug of water (or other drinkable liquid).

- Suggested sequence for the activity*
1. Fill the two identical glasses to the same height.
 2. Ask "Is there the same amount to drink in these two glasses?" If some do not think that there is, pour a little more into whichever they think is less (or out of that which they think has more) until all agree that there is the same.
 3. Now pour all the water from one of the glasses into the tall thin glass, and ask the same question for the two differently shaped glasses in which the liquid now is. The answer here will vary, according to the age and conceptual development of the children.
 4. If some of them think that the amount is more (because they see that the height is greater), and others think that it is less (because the thickness is less), this will give rise to an interesting discussion.
 5. Likewise, if some of them think that the amount is the same and others think that it is different.

Meas 3.3 Conservation of volume (cont.)

6. In either of the last two cases, pour the water back from the tall thin glass into the glass it started in, so that we now have two identical glasses filled to the same level, and ask the same question once again.
7. Now repeat steps 2 to 6 as before, but this time using the short wide glass.

Note The ages at which children attain constancy of volume varies substantially, and has been the subject of considerable debate in the psychological literature. As a very rough guide, the transitional age is around four or five, and they seem able to do so without specific instruction. Children who have formed the concept may show surprise that you ask such an obvious question! In this case it is instructive for oneself (it certainly was for me) to do the experiment with children young enough not to have formed the concept.

Activity 2 “Can I fool you ?” [Meas 3.3/2]

A continuation of Activity 1, for a small group of children working in two teams. Its purpose is to consolidate their concept of the conservation of volume.

- Materials*
- An assortment of glasses, jugs, jars, varying as widely as possible in height and width. Among these are two identical containers, of capacity equal to or greater than any of the above. (These are used repeatedly in subsequent activities, which will repay the time taken to collect them.)
 - Water.

- What they do*
1. Team A chooses two of the containers, and pours some water into each.
 2. Team B then has to decide which has more water in it. The first team tries to make this difficult, by using two containers of different shapes and (e.g.) by filling the wider of the two containers to a slightly lower height.
 3. They then have to find a way of finding whether team A was right. The easiest way is, of course, by pouring the water from the two containers in use into the two identical containers. If one of them happens to belong to the identical pair, this will make it easier.
 4. The teams then change about, and steps 1 to 3 are repeated.

<p>Discussion of activities At this stage we need to make sure that children have achieved conservation of volume, since the measurement of volume by transferring the contents to a measuring cylinder would be invalid were this not so. The first activity introduces the concept specifically, and the second exercises and consolidates it further.</p>

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 3.4 CAPACITY OF A CONTAINER: COMPARING CAPACITIES

Concept The capacity of a container.

- Abilities*
- (i) To compare capacities of two given containers using
 - (a) two identical containers,
 - (b) another container larger than either,
 - (c) a non-standard unit of measure.
 - (ii) To compare the capacities of more than two containers as in (c) above.

Discussion of concept The capacity of a container is the amount of a liquid which it will hold when full. This is therefore a simple extension of the concept of volume.

Activity 1 Which of these can hold more? [Meas 3.4/1]

An activity for a small group. Its purpose is to extend their concept of volume as described above.

- Materials*
- The same as for the preceding activity (Meas 3.3/2). For stage (b) it is helpful for them to be labelled in some way.
 - Water

What they do Stage (a)

1. Taking turns, the first child chooses two containers.
2. All then say which they think will hold more water when full.
3. The first child then fills the two containers in question, and tests her prediction by the method described in the activity 'Can I fool you' (Meas 3.3/2) .
4. It is then the next child's turn, and steps 1 to 3 are repeated.

Stage (b)

One of the two identical containers is then removed, and steps 1 to 4 are repeated. It is now somewhat harder, since it is no longer possible to compare the water in two identical containers side by side. A good method is to start by pouring all the water from one container into what appears to be the largest available, and measuring how high it comes by putting a ruler vertically alongside it with one end on the table. This water is then poured away, and the process repeated for the other container.

With this method, the children are close to the concept of a measuring cylinder.

Discussion of concepts In this and all the preceding activities, the children have been learning by a combination of Modes 1 and 2; that is, by shared experiences which they discuss together, and about which they make and test predictions.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 3.5 MEASURING VOLUME AND CAPACITY USING NON-STANDARD UNITS

Concept The capacity of a container as distinct from the volume held at a particular time.

Ability To order containers according to capacity or volume held, using non-standard units.

Activity 1 Putting containers in order of capacity [Meas 3.5/1]

Discussion of concept An activity for a small group of children working together. Its purpose is to extend their abilities to measure, using a non-standard unit of a different kind, and to introduce the term ‘capacity.’

- Materials*
- The same set of containers as for stage (b) of the preceding activity: that is, an assortment of containers with no two alike.
 - For each child, an identical container smaller than any of these, such as an egg cup.
 - Water.

- What they do*
1. Explain that their task is to arrange the containers in order of how much they can hold when full. This is called their capacity.
 2. To begin, each child takes one of the containers.
 3. After that, it is for them to work out the best method. They could, of course, use the same method as in the preceding activity, but this would be slow since there is only one largest container available. The fact that they each have an egg cup provides a strong clue.
 4. The method is, of course, to use this as a non-standard unit, and record the number of these which it takes to fill their respective containers. This needs to be recorded.
 5. If there are more containers, steps 2 to 5 are repeated. Those not thus engaged can use the results already obtained to begin putting in order those already measured.

Activity 2 “Hard to tell without measuring” [Meas 3.5/2]

A continuation of Activity 1 for two teams. Its purpose is to underline the need for measurement, and give an experience of applying the concept of volume more generally.

- Materials*
- As for Activity 1, except that the containers are now lettered.

- What they do*
1. Using their egg cups, team A pours a different volume of water into each of the containers. The amount in each container is recorded. Team B does not watch while this is done.
 2. Team B now tries to arrange the containers in order of how much water is in them.
 3. While team B is doing this, team A writes the letters in order of how many egg cups full were poured into the containers.
 4. These two orders are now compared.
 5. Steps 1 to 5 may now be repeated with the teams changing about.

Discussion of activities These two activities, like the earlier ones only more so, extend the children’s concepts and abilities by giving them problems to solve, embodied in physical materials. As can be seen from the concept map, these activities bring together all the concepts and methods formed in the previous topics. The first one concentrates on capacity, and the second on volume in general.

[Meas 4] MASS AND WEIGHT

Meas 4.1 INTRODUCTORY DISCUSSION

Mass and weight are closely related, and often confused for reasons which will emerge in this discussion. Though we do not want to cause difficulties for children in the early stages by making too much of this distinction, it is easier to understand something which is accurate than something which is confused. So the object of the present discussion is to get the relations between mass, weight, and inertia as clear as possible in our own minds, and to outline the ideas which underlie the approach I have taken in this topic.

At an everyday level, we can say that a mass is anything which has weight. So a stone, a book, a house, a person are masses, whereas a day of the week, a poem, a joke, are not. This is much the same at a scientific level, since weight is the gravitational attraction of the earth on all other masses. Every mass attracts every other mass with a gravitational force; but this is very small, unless at least one of the masses is very large, which is the case with the earth.

Weight is one of the characteristics of mass, the other being inertia. On the moon, the same mass would weigh less, and at some location in between, where the gravitational pull of the moon was equal and opposite to that of the earth, it would be weightless. But in all these places its inertia would be the same. When we push a car in neutral on smooth level ground, it is its inertia which requires a hard push to get it started, and an equally hard push backward to stop it. The forces required would be much less for a bicycle, and much more for a railway car, though the frictional resistance is small even for the last of these. This would still be the case at each of the three locations described, though the weights would now be different.

Both weight and inertia are common everyday experiences. It is the weight which makes our arms tired when we carry a child; it is inertia which causes hurt when a child falls over. However, weight is much easier to measure (e.g., by bathroom scales). Given that the weights of two bodies are proportional to their masses if both are the same distance from the centre of the earth, which for everyday situations they are, then the easiest way to measure a body's mass is to compare its weight with that of a standard mass. So for both these reasons, we shall approach the concept of mass through the experience of weight, and the measurement of mass by the measurement of its weight.

In this topic, the hardest concepts come at the beginning and I would not expect children to grasp them in the form outlined above. My aim in the activities which follow is not to sweep anything important under the carpet, but to keep it visible though not necessarily fully discussed. The children will then, I hope, not have anything to unlearn later.

Activity. No specific activity is suggested for the children at this level. The words 'heavy,' 'light,' 'heavier,' 'lighter,' should already be part of their everyday vocabulary, so in preparation for the next topic it will be useful to ensure that this is the case, and that their meanings are understood correctly.

Meas 4.2 COMPARING MASSES: HEAVIER, LIGHTER (ESTIMATION)

Concepts Heavier, lighter.

- Abilities*
- (i) To use these words correctly in context.
 - (ii) To estimate which of two masses is the heavier and which the lighter.

Discussion of concepts Heavier and lighter are really more basic concepts than heavy and light. When we use the terms ‘heavy’ and ‘light,’ we mean in relation to some comparison or expectation. Often we mean ‘relative to what is comfortable to lift,’ as in “This suitcase is heavy, and I’m glad I bought one with wheels.” Sometimes we mean relative to what is usual or average for that kind of object, such as “A lightweight raincoat.” Heavy for an acrobat would be light for a sumo wrestler. From this topic onwards, the emphasis is on comparison, giving greater precision to the foregoing concepts and vocabulary.

Activity 1 “Which one is heavier?” [Meas 4.2/1]

An activity for children to do in pairs. Its purpose is to consolidate and make explicit children’s everyday knowledge of the above-named concepts, and to focus attention on the comparison of weights and relate this to language. It is not to develop expertise in estimation, since a balance does this both more easily and more accurately.

- Materials*
- (Set A) An assortment of eight or more objects, most of them clearly distinguishable in weight, but with (say) three of the same weight. In this set, all should look different, and there should be as little connection as possible between appearance and weight. So there should be small heavy objects and large heavy objects, small light objects and large light objects. (Note that we are using ‘light’ and ‘heavy’ here with the meaning ‘. . . relative to the other objects.’) All should be light enough for children to lift easily.
 - (Set B) As for set A, except that now all the objects look alike. In other words, there is a set of similar opaque containers containing different amounts of something fairly heavy, e.g., sand, so that they are clearly distinguishable in weight. To tell them apart, they should be marked in some way, such as patches of colour. I suggest that letters or numerals are not used, since these have an order of their own.

What they do Stage (a)

1. Starting with set A, the objects are all put in the middle of the table. If there are enough objects, several pairs of children can share the same set of objects.
2. Child A takes any two of the objects, compares their weights while holding them, and says “This one is heavier.”
3. She passes them to child B who likewise compares them, but this time uses the other word, in this case ‘lighter.’
4. In the case of two objects of the same weight, they would say something like “These are about the same weight,” or “I can’t tell any difference in weight.”

Stage (b)

As for Stage (a) but using set B.

Discussion of activities These activities provide for concept building by Mode 1, physical experience, and testing by comparison with the experience of the other child in the pair, Mode 2.

Meas 4.3 COMPARING MASSES: THE BALANCE

- Concepts*
- (i) Weight as the earth's attraction on all physical objects.
 - (ii) The greater the mass of an object, the greater is this attraction; that is, the greater its weight.
 - (iii) The balance as a way of comparing the weights of two objects, and thereby comparing their masses.

Ability To compare masses by comparing their weights.

Discussion of concepts Some physical qualities, such as length, we can measure directly. Others, such as temperature, we have to measure indirectly, e.g., by measuring the length of a thread of mercury or alcohol in glass. Mass is measured indirectly. Though two methods are available, measuring a body's weight or its inertia, the former is easier in most cases. (For a railway car, inertia might be easier than weighing.) Physicists tell us that both of these qualities are directly proportional to mass. Strictly, 'kilogram' refers to the mass itself rather than its weight, and in physics and applied mathematics, a force equal to the weight (at sea level) of a kilogram is called a kilogram-weight. In the present network, however, we shall normally use the word 'kilogram' for both mass and weight. I would say that weight and inertia are both primary concepts, formed from direct experience of the physical world; and that mass is a secondary concept, describing the quality of an object which gives rise to both of these. It is therefore arguable that we should continue to use the word 'weight' until the children also have the concept of inertia. However, since this view is not shared by everyone, I introduce the word 'mass' in this topic so that children will be familiar with the word even if they have as yet an incomplete grasp of the concept.

In this as in other cases, we use the same word for a quality and its measure. Thus we talk about an area of land, and also say that the area of a field is 6 hectares. Usually the meaning is clear from the context.

Activity 1 "Which side will go down? Why?" [Meas 4.3/1]

An activity for a small group of children, introduced by a teacher-led discussion. Its purpose is to help them to form and relate the above concepts, and to use them predictively.

- Materials*
- A balance.
 - One of the sets of objects used in Meas 4.2/1, Stage (a).

- Suggested outline for the discussion*
1. Put two objects of different weights in the two pans of the balance.
 2. Ask which side went down, the one with the lighter object, or the heavier.
 3. Why? "Because it is heavier" is not an answer: it is simply re-stating what has been observed.
 4. Explain that the earth attracts everything towards itself, as though it was pulling with an invisible rope. This attraction is what we feel as weight when we lift something. The greater the mass, the stronger the attraction, so the earth's pull was stronger on the side with the greater mass.

Meas 4.3 Comparing masses: the balance (cont.)

5. Now we know that in the earlier activities when we were comparing weights we were also comparing masses. Mass and weight are not the same, but they are very closely connected, and the easiest way to compare masses is by comparing their weights. The balance does this more accurately: it can show smaller differences than we can feel.
6. Ask them to predict what will happen if the two objects have equal masses. Since you know which these are, this prediction can be tested.

- What they do*
1. Each child then takes two of the objects, and tries to predict which of them will go down when one is put in each pan of the balance. If a child thinks he has a pair of equal mass, he predicts accordingly.
 2. In turn, they state and test their predictions. To make things more interesting, alternate children should predict which object will go up.

Activity 2 Find a pair of equal mass [Meas 4.3/2]

An activity for a small group of children. It gives them lots of practice in using the balance.

- Materials*
- A balance.
 - One of the sets of objects used in Meas 4.2/1, Stage (b) (These are the ones which all look alike except for some distinguishing mark.)

- What they do*
1. Remind them that in this set there are at least two of the same weight.
 2. Their task is to find which there are, and how many.

Note This could take a long time if they do not work out a system.

Activity 3 Trading up [Meas 4.3/3]

An activity for a small group of children. It gives further practice in estimating, and in the use of the balance.

- Materials*
- The same as for Activity 2.

- What they do.*
1. The aim is to obtain the object of greatest mass by 'trading up,' as described below.
 2. Each child in turn takes one of the objects from the pool.
 3. The next time round, each child takes another object, and manually compares its mass with that of the one he already has. He may then if he wishes 'trade up,' i.e., keep the new one and return the other to the pool.
 4. After an agreed number of rounds, they use the balance to find who is the winner.

<p>Discussion of activities Like those in topic 2, these activities provide Mode 1 learning experiences which expand their concept of weight to that of a force which they feel with their muscles. This is then used predictively, as an introduction to experiencing the behaviour of a balance.</p>

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 4.4 MEASURING MASS BY WEIGHING, NON-STANDARD UNITS

Concepts (i) Weighing as a way of measuring mass.

Ability (i) To measure the mass of a given object in terms of non-standard units.
(ii) To compare the masses of two or more objects, using non-standard units.

Discussion of concepts (Step 5 in the introductory discussion in Meas 4.3/1, “Which side will go down? Why?”, is directly relevant here also.)

If you look at the concept map for this network, you will see that weighing with non-standard units is not shown as conceptually prerequisite for weighing with standard units. This is because the need for standard units has already been shown in Meas 1, LENGTH. I have therefore used the present topic to underline the utility of weighing, by introducing it in a problem-solving situation (Activity 1). Activity 2 again calls attention to the importance of standard units.

When measuring length, we combine units by putting them end to end. For area, we need units which tessellate, to cover a surface with no gaps and no overlap. Combining unit weights turns out to be straightforward, since if we put them in the same pan of a balance the gravitational forces on all of them are combined into a single downward force.

Activity 1 Problem: to put these objects in order of mass [Meas 4.4/1]

A group problem-solving activity. Its purpose is to lead children to the concept of weighing as a means of solving a problem which has been presented to them.

- Materials*
- All the objects in set A from Meas 4.2/1, ‘Which one is heavier?’
 - A balance
 - A suitable set of unit objects (see discussion in the Note below) in a transparent plastic bag. Collectively, these must be heavier than the heaviest of the objects in set A.
 - Pencil and paper
 - Two envelopes containing hints (see steps 4 and 6 below).

- Notes*
- (i) It is not easy to think of unit objects for mass which can readily be found in schools and which fulfil the requirements of being all of the same mass, fairly heavy for their size (the plastic cubes so readily available are much too light) and about 50 to 100 grams in mass so that we can weigh up to 1 kilogram with up to 20 of them. After much thought, my suggestion is that you buy a bag of bolts (the metal kind which screw on to nuts) from a hardware store. It should be possible to find some of suitable size for our present purpose. If you think of something better, I hope that you will share it with me and other teachers.
 - (ii) You will need another bag of bolts (or whatever you choose) of slightly smaller mass for Activity 2, so I suggest that you read this activity also before you go shopping.

Meas 4.4 Measuring mass by weighing, non-standard units (cont.)

- What they do*
1. Give them all the materials listed, with the bag of bolts included among the others.
 2. The problem is to put all of these objects in order of mass.
 3. To do this by comparing them in pairs would be possible, but laborious. If they start out this way, it might be worth letting them find out how laborious this is, before asking if they can think of a better way.
 4. If they find themselves stuck, they may ask you for an envelope containing a hint.
 5. The first hint reads: “You may open the bag of bolts.” This suggests that they may have some special significance, and implies the possibility of using one or more of these separately.
 6. If after a time they are still stuck, they may ask for another hint. This reads: “Choose one of the objects, and see how many bolts it takes to balance this.”
 7. This, of course, as good as tells them the solution. After this, it is straightforward to order the numbers of bolts equal (or approximately equal) in mass to each object, and put these numbers in order.
 8. (Optional) Ask them to write a short account of the ways they tried, and how measurement was found to be the key to success.

Activity 2 Honest Hetty and Friendly Fred [Meas 4.4/2]

An activity for a small group of children. Its purpose is to call attention to the need for standard units, and thus prepare for the next topic.

- Materials*
- Twelve or more large potatoes (six or more for each stall).
 - Two balances.
 - The bag of bolts used in Activity 1, for Honest Hetty.
 - A similar bag, in which the bolts are like the others but lighter, for Friendly Fred.
 - Notices as illustrated below.*
- * Provided in the photomasters.

**COME TO HONEST HETTY
FOR YOUR
DELICIOUS BAKED POTATOES
CHOOSE YOUR OWN
AND PAY
ONLY ONE CENT
FOR EACH BOLT MASS**

**COME TO FRIENDLY FRED
FOR YOUR
FAVOURITE BAKED POTATOES
CHOOSE YOUR OWN
AND PAY
ONLY ONE CENT
FOR EACH BOLT MASS**

Meas 4.4 Measuring mass by weighing, non-standard units (cont.)

- What they do*
1. Honest Hetty and Friendly Fred are baked-potato sellers in the same mall. They lay out their stalls, with potatoes, balance, and the bolts which will be used for weighing, with their notices prominently displayed. The other children act as customers.
 2. Some customers are served by Honest Hetty and some by Friendly Fred.
 3. After several purchases from the two stalls, several of the more enterprising customers get together to find out which of the two stalls gives the most for the money, or whether they are both the same. They need to work out a method for doing this.
 4. Finally a report of the findings is given.

Discussion of activities Both of these activities involve plenty of social interaction and discussion. The practical usefulness of weighing is brought out in Activity 1, and the social importance of reliable measures is shown in Activity 2. Both of these, especially the latter, are small scale counterparts of applications of weighing which are important in everyday life.

OBSERVE AND LISTEN

REFLECT

DISCUSS



*'Crossing' [Num 3.2/4 & NuSp 1.3/3] **

* A frame electronically extracted from
NUMERATION AND ADDITION,
a colour video produced by the
Department of Communications Media,
The University of Calgary

[Meas 5] **TIME**

Meas 5.1 PASSAGE OF TIME

- Concepts*
- (i) The passage of time.
 - (ii) Past, present, future.
 - (iii) Remembering, being aware of, forecasting.

Abilities To show that they have the above concepts by their use of language, though not necessarily at a reflective level.

Discussion of concepts The concept of time is a difficult one, the subject of discussion by philosophers and others over many centuries. Nevertheless, in everyday life we use terms relating to time freely, and as though we understand what time is. Indeed, we could hardly do without ideas relating to time, and for practical purposes it does not seem to matter that we do not fully understand its nature. In the present network, I have tried to conceptualize our intuitive everyday usages, and to embody the result in a coherent concept map, without getting into deep water. (Perhaps it would be truer to say: without hereafter calling attention to the depth of the water we are exploring.)

There seems to be general agreement that time is a single dimension, more like length than area. We use metaphors based on length freely in our descriptions of time phenomena. These work well for communication, so our individual experiences of time must have much in common. They are also successful for predicting, both at everyday and at scientific levels, so our model of time based on these may be regarded as a good working model.

In the present topic, we use the concept of time as something which passes us by, or through which we pass. We do not know which of these is the case, but we do know that relative to ourselves, there is only one possible direction of movement. Behind us is the past, which we can partially remember but not change. Ahead is the future, which we can partially forecast and in some ways influence. Separated by these is the present instant, our point of awareness. Our concept of 'now' extends a little way before and after this instant.

I am not, of course, suggesting that we try to communicate these concepts to children of the ages we are presently thinking about. They are offered as a background schema to organize our own side of the discussions which form an important part of this network.

Activity 1 Thinking back, thinking ahead [Meas 5.1/1]

A teacher-directed activity, suitable for a group of almost any size. Its purpose is to begin organizing children's everyday knowledge of the ideas of past, present, and future, with associated vocabulary. Words to be brought in are:

Relating to the past: *Remember, think back, . . .*

Relating to the future: *Think ahead, expect, forecast, plan to . . . , hope to . . . , will be . . . , might happen . . . ,* (The number of these reflects the varying levels of probability with which we can foretell the future, from near-certainty to ignorance.)

Relating to the present: *now*.

Suggested outline for the activity

1. Remembering. Let's all think back to yesterday. Who can remember what day it was? Can you remember something you did yesterday? And something else? . . . Something we all did?
2. Apart from things we ourselves did, can someone remember something which happened yesterday?
3. Now let's all think ahead to tomorrow. Who can say what day it will be? Can you say something you expect to do tomorrow? Something you will do tomorrow? Something you hope to do tomorrow?
4. Still thinking ahead, can any one tell us something which is sure to happen tomorrow? Something that will probably happen tomorrow?
5. Do any of you watch the weather forecasts on television? What is the forecaster trying to do?
6. And what day is it today? Can you tell me something you are doing now? And something which is happening now?

Note It is not suggested that all the above should be included every day. It is offered as an informal activity, which may conveniently form part of the daily routine, varying in detail but centered around these three organizing concepts: past, present, future.

Meas 5.2 ORDER IN TIME

Concept Order in time of events and experience.

Abilities To understand and use correctly terms denoting order in time.

Discussion of concepts

Some order relationships are easy to show in physical embodiments, such as order of length, order in position from left to right or top to bottom, order of weight. In this topic and the two which follow, we shall be developing two much more abstract order relationships: order of events (before and after in time), and order of duration (shorter and longer lengths of time). It is therefore desirable that children have already formed the concept of order in more concrete embodiments such as those provided in Org 1.4 and Org 1.5. The present topic is concerned with the first of these relationships, order of events and experiences in time.

Now is a moving item in the order relationship before and after, always coming after anything which is past and before anything which is future. For this reason, I introduce it last.

Activity 1 Thinking about order of events [Meas 5.2/1]

A teacher-led activity similar to Meas 5.1/1, with a similar purpose. In the present case the words to be brought in are:

yesterday, today, tomorrow; before, after; now, not now.

- Suggested outline for the activity*
1. *Yesterday, today, tomorrow.* "What day is it today? What day was it yesterday? And what day will it be tomorrow?"
 2. "Tell me something which happened yesterday. And something which has happened today. Can you think of something which you think will happen tomorrow?"
 3. *Before, after.* "Where were you before you came into this classroom? And where were you before that? What did you do as soon as you came into the classroom? And what did you do after that?"
 4. "Where will you go after you leave this classroom? And where will you go after that?"
 5. "Tell me some of the things you did after you got out of bed this morning, and before you came to school."
 6. "Tell me some of the things you will do after you get home, and before you go to sleep."
 7. *Now, not now.* "Tell me something you are doing now. Tell me something you have done in the past, but are not doing now. Tell me something you expect to do in the future, but are not doing now."
 8. "Tell me where you are now. Tell me somewhere you have been in the past, but are not now. And somewhere you expect to be in the future, but are not now."

Activity 2 Relating order in time to the ordinal numbers [Meas 5.2/2]

- Suggested outline for the activity*
1. “Tell me any two things you have done today.”
 2. (When these are named). “Which did you do first? And which did you do second?”
 3. “Tell me two more things you have done today, and say which was first and which was second.”
 4. “Tell me two more things you have seen today, and say which was first and which was second.”
 5. “Tell me two more things you have heard today, and say which was first and which was second.”
 - 6, 7, 8. As for 3, 4, 5 but replacing ‘today’ by ‘yesterday.’
 9. The foregoing may be repeated for the future, using questions like those in Activity 1, steps 3 and 5.

Discussion of activities In Activity 1 we are bringing together experiences which children have already had, and using these to expand their concept of an order relationship to include order of events in time.. In Activity 2 we relate this to another well-known order relationship, that of the ordinal numbers.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 5.3 STRETCHES OF TIME AND THEIR ORDER OF LENGTH

Concepts (i) Different stretches of time, and their names.
(ii) Their order of length.

Abilities (i) To say which of two stretches of time is the longer and which is the shorter.
(ii) To say these in order of length, both ascending and descending.

Discussion of concepts I have not been able to find a better word to describe what seconds, minutes, hours, days, weeks, years . . . have in common than to say they all name a particular stretch of time. 'Length of time' was another possibility, but this would be to use the same word as we shall use for the measure of how long a given stretch of time is, and it does not lend itself well to what we shall be doing in this and the next topic, putting named stretches of time in order of length (duration) and in order of occurrence (where they come in the long succession of time through which we all pass).

We shall not be using this word with the children, but we shall be forming (or consolidating) the concept by bringing together examples of particular stretches of time — those already named in the first two lines of this discussion. So we need a general name, for our own use; and 'a stretch of time' fits well into the length model we are using.

Activity 1 "If this is an hour, what would a minute look like?" [Meas 5.3/1]

A teacher-led discussion for a small group of children. Its purpose is to introduce children to the length model for time, at this stage only qualitatively.

Note We shall be using three ways of representing relative lengths. In the first, the open hands are held a little way apart, say 5 cm. I shall call this 'hands close together.' In the next, they are held more than shoulders' width apart, which I shall call 'hands wide apart.' In the third, one hand is held with thumb and forefinger about a centimetre apart, the other fingers being bent into the palm. I shall call this 'thumb and forefinger.' (All much easier to show than to describe.) The purpose of these is to provide a quick and easy qualitative representation of 'considerably shorter/longer than.'

Materials None required.

Suggested outline for the discussion

1. Ask "Which is longer, a minute or an hour?" "So if this was an hour (hands wide apart), what would a minute look like?"
2. The children should hold their hands close together. If some hold their hands only a little closer together, Ask "Is a minute just a little smaller than an hour, or much smaller?"

Meas 5.3 Stretches of time and their order of length (cont.)

3. Now ask “Which is longer, a minute or a second?” “So if this was a minute (hands close together), what would a second look like?”
4. For this, we are likely to get an assortment of responses. Discuss these, and suggest that since it is hard to hold our hands a lot closer than the ‘hands close together’ position, we might use this (showing the thumb and forefinger position). If there are other good suggestions, these may be included as alternatives.
5. If there are any children who are unclear about seconds, you could tap seconds on the table. These do not need to be exact, and I find that I can come close by saying mentally “One second, two seconds, three seconds” (You may prefer to use a watch.) Then return to step 4.
6. The children may now in turn take over your role..
7. If they stay with the three representations described above, it is not necessary to continue beyond this step, since we have made our point.
8. However, variations may now occur spontaneously. E.g., “If this (hands close) is a second, what would a minute look like?” Hands wide would be a correct response to this, and indicate awareness that we are talking about relative lengths.
9. Another variation might be “If this (hands close) is a second, what would an hour look like?” Hands wide would here not be wide enough, since we have gone up two sizes. If this is the response of some children you might say, “But if this (hands close) is a second, what would a minute look like?” Since this would be ‘hands wide,’ an hour would have to look something bigger still. “From this wall to this wall,” pointing to opposite walls of the classroom would be appropriate for this.
10. Another variation might be “If this (thumb and forefinger) is a minute, what would a second look like?” Finger and thumb almost touching, or “Too small to show,” would both be acceptable.

Activity 2 “If this is a month, what would a week look like?” [Meas 5.3/2]

A continuation of Activity 1. Its purpose is to extend the foregoing representations to days, weeks, months, years.

Suggested outline for the discussion This follows the same lines as Activity 1.

Activity 3 Winning time [Meas 5.3/3]

A card game for 2 to 6 children. Its purpose is to consolidate the concepts of relative duration established in the first two activities

- Materials*
- A pack of two-headed cards on which are the words second, minute, hour, day, week, month, year. Six sets of these is a suitable number, making a pack of 42.*
*Provided in the photomasters.

- What they do*
1. The pack is shuffled, and seven cards are dealt to each player. (This number may be varied if desired.)
 2. The players hold their cards all together, face down.
 3. Each player in turn puts down his top card face up, in the middle for all to see.
 4. The player who puts down the card with with the longest time wins the trick (as it is called), and is the starter for step 3.
 5. Steps 3 and 4 are repeated until all the cards are played.
 6. The winner so far is the player who wins most tricks.
 7. Another round may then be played.

Activity 4 Time whist [Meas 5.3/4]

A harder variation of Activity 3, involving consideration of relative lengths of time and weighing of possibilities. Its name is taken from the well-know card game, of which it is a simplified version.

- Materials*
- The same pack of cards as for Activity 3.

- What they do*
1. The pack is shuffled, and five cards are dealt to each player. (This number might be increased, but at this age five is probably enough for them to manage.)
 2. The player on the left of the dealer begins, and the others each put down a card in turn, as in Activity 3.
 3. However, in this game the players look at their cards, and may play whichever they choose.
 4. As before, the player who puts down the card with the longest time wins the trick, and is the starter for the next round.
 5. Steps 3 and 4 are repeated until all the cards are played.
 6. The winner so far is the player who wins most tricks.
 7. Another round may then be played.

Discussion of activities Activities 1 and 2 provide simple physical experiences whereby to relate children's existing knowledge of the various measures of time in daily use to the length model of time. At present this is done at a qualitative level only. Activity 4 is appreciatively harder, since the decision which card would be best to play depends on a purely mental comparison of lengths of time.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 5.4 STRETCHES OF TIME: THEIR ORDER OF OCCURRENCE

- Concepts*
- (i) The names of days of the week.
 - (ii) Their order of occurrence.
 - (iii) The names of the months of the year.
 - (iv) Their order of occurrence.

- Abilities*
- (i) To say the days of the week in order.
 - (ii) To say the months of the year in order.
 - (iii) To relate each of these to a spatial order.

Discussion of concepts We organize our thinking about time first by giving names to equal lengths of time, such as hours, days, months, years; and then by thinking of long stretches of time as a succession of days, of months, and of years, to each of which we give names. In this way, we can say where we are along the road.

Activity 1 Days of the week, in order [Meas 5.4/1]

An activity for a small group of children. Its purpose is to consolidate Concepts (i) and (ii) above, to fill any gaps or uncertainties, and to develop and practise Abilities (i) and (ii) above.

- Materials*
- Days of the week track*
 - Seven blank cards to cover the days on the track*
- * Provided in the photomasters

What they do Stage (a)

1. They say the days of the week, in order, in turn around the table, prompting each other if necessary until they are confident that they know this.

Stage (b)

1. The cards are placed to cover Sunday through Saturday on the 'days of the week' track.
2. The first child points to the first card on the left, says "Sunday" and turns it over to uncover the name of the day, which is left uncovered. The next child does likewise with the second card, and so on, until all of the days are uncovered.
3. The children correct and prompt each other if necessary.

Stage (c)

1. As for Stage (b).
2. The first child acts as pointer, pointing to any of the cards covering the names of the days.
3. The other children in turn have to name the day to which he is pointing and check by removing the card. It is then covered again with the blank card.
4. For the next round a different child acts as pointer.

Discussion of activities Part of the theory of intelligent learning presented in *Mathematics in the Primary School* indicates that it is useful to memorize information which is frequently required, so that it can be easily recalled when needed. This frees our conscious thinking for what is new in a situation. Within an overall context of intelligent learning, memorizing can be a useful servant.

Activities 1 and 3 are concerned with memorizing the names of the days of the week and of the months of the year, respectively, and their respective orders in time. Activities 2 and 4, in contrast, require the use of intelligence, since every new card turned encounters a new pattern of cards on the table. Also, the order relationships learned in Activities 1 and 3 have to be transferred to two spatial order relationships, left to right and downwards. This spatial representation forms the basis for representing dates in a calendar.

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 5.5 STRETCHES OF TIME: THEIR RELATIVE LENGTHS

Concepts The relative lengths of a second, minute, hour, day, week, month, year.

Abilities To say, for any unit of time, how many of these are equivalent to an appropriate larger unit. In some cases there will be more than one of the latter.

Discussion of concepts There is no easily discernible pattern relating the above units of time, whose origins lie far back in the history of more civilizations than one. The only general concept is that of equivalence, between a single larger unit and a particular number of a given smaller unit.

Activity 1 Time sheets [Meas 5.5/1]

A game for a small group of children. Its purpose is to help them memorize the above relative sizes. This goes best with 2 to 4 children.

- Materials.
- A time sheet for each child (see illustration below).*
 - Two reminder cards (time sheets in which the spaces have been filled with the correct numbers).*
 - A pack of 48 number cards which fit the spaces in the time sheet, six for each of the eight spaces.*
- * Provided in the photomasters

TIME SHEET

There are	seconds in a minute
There are	minutes in an hour
There are	hours in a day
There are	days in a week
There are	days in a month
There are	weeks in a year
There are	months in a year
There are	days in a year

Meas 5.5 Stretches of time: their relative lengths (cont.)

What they do Stage (a)

1. A reminder card is put where all can see. A second one is put if necessary, so that all can see it right way up.
2. The pack of cards is shuffled, and put in a pile face down in the middle.
3. The top three cards are then turned over, and put face up.
4. The child whose turn it is first uses one of the face up cards, if he can, to fill one of the spaces on his time card. He then turns over another card so that there are still three cards for the next player.
5. If he cannot use one of the face-up cards, the turn passes to the next player.
6. If there is a complete round in which no player can use any of the face-up cards, these are set aside and three new ones turned over.
7. When a player has filled his card he drops out. Play continues until all have filled their time cards.

Stage (b)

This is the same as Stage (a), except that the reminder card is not used.

Activity 2 Thirty days hath November . . . [Meas 5.5/2]

An activity for any number of children. Its purpose is to help them commit to memory the number of days in every month of the year.

What they do This activity simply consists in their saying together the well-known rhyme:

*Thirty days hath September,
April, June, and November.
All the rest have thirty-one,
except for February alone,
and this hath twenty-eight days clear
and twenty-nine in each leap year.*

Variation As above, but the children recite a line at a time, in turn.

Sequel When they ask, explain what a leap year is.

Discussion of activities Here the only overall pattern is that there is a given number of each smaller unit in any particular larger unit. If the larger unit is a month or a leap year, even this number is not constant. This is therefore another case where memorizing is the only available approach. Such cases are fewer in mathematics than in almost any other subject.

OBSERVE AND LISTEN

REFLECT

DISCUSS

MEAS 5.6 LOCATIONS IN TIME: DATES

Concept 'Where we are' in time.

- Ability*
- (i) To find and name given dates in a calendar.
 - (ii) To find how long a time it is from one date to another.

Discussion of concepts If we think of our passage in time as movement along a road, the date tells us where we are to the nearest day, and the time of day tells us within that day. The present topic deals with the first of these.

A calendar is a table of days in which each day is given a unique name by specifying the year, month, and day of the month. This name is called its date. In addition, it keeps track of the days of the week, within which much of our daily life is planned for work and leisure. And it enables us to work out how long in time it is between two given dates.

Activity 1 What the calendar tells us [Meas 5.6/1]

A teacher-led activity for a group of any size. Its purpose is to familiarize children with the processes of naming and locating dates in the calendar.

- Materials*
- A sufficiency of calendars.*
 - If the print is small, they also need something to point with.
- * If possible one per child, otherwise one between two. For a large group, it is useful to have also a wall calendar, large enough for all to see, and a pointer.

- Suggested outline for the activity*
1. Before the activity begins, have today's date written on the blackboard. (Many teachers make a daily practice of this.)
 2. Ask someone to read this aloud, and ask what it tells us. (It tells us which day of the week we are in, which day of the month, which month of the year, and which year. That is to say, it tells us where we are in time and gives it a name. This kind of name is called a date.
 3. Tomorrow we shall have moved on a day, so what will the date be then?
 4. And what was the date yesterday?
 5. Ask the children to find today's date in their calendars. If they need help, tell them to begin by finding the right month; Then, the number for the day of the month.
 6. Point out how when we have found the right date in the month, the calendar also tells us which day of the week we are in. So we can check one against the other.
 7. Ask them if they can find out from their calendars what will be the date one week on from today. (If this takes them into another month, do step 8 before this one.)

Meas 5.6 Locations in time: dates (cont.)

8. Ask them to find what was the date a week back from today.
9. Repeat for two, three weeks on and back. It becomes harder when the month changes.
10. What day of the week will it be one month on from today? That is to say, on the day with the same number as today, next month. And one month back from today?
11. Likewise for other numbers of months forward and back.
12. Ask if they know the dates (month and day) of their birthdays. They can then use their calendars to find what day of the week it is.
13. Repeat for other important dates.

Activity 2 “How long is it . . . ?” (Same month) [Meas 5.6/2]

An activity for a small group of children, which they should be able to do on their own after the preparation they have had in Activity 1. Its purpose is to give further practice in finding their way around in a calendar. They may need help in getting started.

Materials • As for Activity 1.

- Pencil and paper for each child.

- What they do*
1. One child chooses a date between the present day and the end of the month, and asks (e.g.) “How long is it until the 27th of this month?”
 2. They all count forward in weeks, and then the extra days. E.g., if ‘today’ was the 4th of the month, they would count, starting from 4 on the calendar, “1 week” (to the 11th), “2 weeks” (to the 18th), “3 weeks”(to the 25th), “and two days.”
 3. When they have all written down their answers, they compare results and discuss any disagreements.
 4. After all have had a turn, steps 1, 2, and 3 may be repeated for a date earlier in the month. In this case, the question asked would be “How long is it since . . . ?”

Activity 3 “How long is it . . . ?” (Different month) [Meas 5.6/3]

An extension of Activity 2. The steps are exactly the same, except that in step 1 the starting child chooses a date in the same year but a different month. E.g., she might ask how long it is until her next birthday, if it is not yet past; or since her last birthday, if it is.

Activity 4 “How long is it . . . ?” (Different year) [Meas 5.6/4]

A further extension of the previous two activities. An easy beginning would be to ask “How long is it since the day I was born?” They already know how many years this is, so it is only a matter of counting forward in months and days since their last birthdays. Other possibilities are historical events, preferably ones which have some interest for the children.

Computer programs are available which will print a calendar for any required year, past, present or future, and it adds to the interest if these can be made available for this activity. They can then find on which day of the week the event took place, as well as how long ago it was.

Future as well as past events should of course be included. It may be useful to have some suggestions ready in case they are short of ideas.

<p>Discussion of activities The calendar is an excellent representation of the combination of three separate orderings by which we we organize our experience of the passage of days, and give names to them. These have already been dealt with separately in previous topics, and the present topic brings them together. Giving them activities which involve repeated use of the calendar should help consolidate children’s knowledge of the relationships between days of the week, days of the month, and months of the year, and develop their ability to put this knowledge to use.</p>

OBSERVE AND LISTEN

REFLECT

DISCUSS

Meas 5.7 LOCATIONS IN TIME: TIMES OF DAY

- Concepts* (i) Where we are during a day.
(ii) The clock as an instrument for measuring the time of day.

Abilities To tell the time of day by the clock, in digital and analogue notations.

Discussion of concepts The preceding topic was concerned with location in time to the nearest day. In the present topic, we continue the development of this concept to specify locations within the time stretch of a single day. The clock replaces the calendar as the means whereby we do this, and the symbolism by which we describe and record it.

Activity 1 “How do we know when to . . . ?” [Meas 5.7/1]

A teacher-led discussion for a group of any size. Its purpose is to underline the importance of being able to tell the time, as a way of fitting one’s own activities in with those of others.

Materials None required.

- Suggested outline for the discussion*
1. Ask “How do we know when to . . . ?” This can be anything which must be done at a specific time, such as catch the bus for school, switch on the television for a favourite program, come home from visiting a friend. In some cases, e.g., end the lesson and go out for recess, the answer may well be that someone else says so: e.g., most children will be told by a parent when it is time to get up in the morning. In this case, we want to know how they know. If bells ring at programmed times in your school, you need to be prepared for the answer “Because the bell rings.”
 2. In nearly every case, the answer eventually comes back to because it is a particular time of day. It was in order not to imply this answer in the wording of the question that I did not name this activity “How do we know when it’s time to . . . ?”
 3. “So when do we know when it is . . . (whatever time is given)?”
 4. The answer will nearly always be “By looking at a clock or watch.”
 5. So we need to be able to read the time from a clock or watch, if we cannot already.

Activity 2 “Quelle heure est-il?” [Meas 5.7/2]

A teacher-led discussion for a group of any size. Its purpose is to introduce children to the analogue clock face.

Materials • An analogue clock face, with a movable hour hand only. This needs to be large enough for the whole group to see easily.

Suggested outline for the discussion

1. Explain that long ago, people only needed to tell the time to the nearest hour, so when they wanted to know the time they used to ask “What hour is it?” In French this is how they still ask what the time is.
2. So the early clock faces only showed hours, and this is how we are going to begin.
3. Earlier still, before there were any clocks, there was one time of day which they could tell by the sun (if it was shining). Can anyone say what this might be?
4. If anyone suggests sunrise or sunset, these are good ideas, but why won’t they do? (These change from day to day.)
5. What doesn’t change even though the days get shorter or longer is the middle of the day when the sun is half-way between rising and setting. This is when it is highest in its path across the sky. (Later, you might introduce the word ‘zenith.’)
6. So they divided the day into two equal parts, before mid-day and after mid-day. When writing, these are often written ‘a.m.’ and ‘p.m.’ for short. ‘A.m.’ stands for ‘*ante meridiem*,’ which is Latin for ‘before mid-day.’ ‘P.m.’ stands for ‘*post meridiem*,’ which means . . . ? Which is it now, a.m. or p.m.?
7. There are twelve hours in each half day. The a.m. half starts at . . . ? And ends at . . . ? And the p.m. part starts at . . . ? And ends at . . . ?
8. Now show the clock face, with the hand pointing to the actual time, to the nearest hour. Ask “Quelle heure est-il?”, or “What hour is it?”, whichever you prefer. Since we are imagining ourselves to be in the position of people long ago, they might like to answer as they did, and say (e.g.) “It is eleven of the clock.” Later this will be shortened to “o’clock.”
9. The next question is how we know whether the hour shown on the clock face is a.m. or p.m. This is usually clear from where we are and what we are doing. In the foregoing example, where would they be if it was eleven p.m.? We hope, in bed and asleep! If it is clear which we mean, we do not need to say a.m. or p.m. unless we want to.
10. They are then ready to practise reading the hour from the clock face, with the hour hand always pointing to an exact hour. It should not take long for them to become fluent at this, after which they will be ready for the next activity.

Activity 3 Hours, halves, and quarters [Meas 5.7/3]

A continuation of Activity 1, which uses the minute hand to show halves and quarters of an hour.

- Materials*
- The same clock face as used for Activity 1.
 - A second one, similar but with a minute hand as well as an hour hand.

- Suggested outline*
1. Use the first clock face to explain that the hour hand moves slowly round the dial, taking an hour to get from one figure to the next.
 2. Position the hour hand half way between two hours, and ask what they would call this time.
 3. Accept any sensible answers. Now show the second clock face, and explain that this helps us to tell the time more exactly. The longer hand makes a complete turn in every hour.
 4. Start the minute hand at 0, and ask: "Starting here, what part of a turn has it made when it gets to here?" moving it to the figure 6. (Half a turn.) "So how long after the hour does it show?" (Half an hour.)
 5. Put the hour hand half way between any two figures, say 9 and 10, with the minute hand at 6, and ask what time this is showing. "Half an hour after nine" is a good answer, which may be shortened to "Half past nine."
 6. Move the hour hand to half way between two other figures, and ask "What time does the clock show now?" Repeat until they are fluent, which should not take long.
 7. Repeat steps 4, 5, 6 as before, except that the minute hand now shows a quarter after the hour.
 8. Repeat steps 4, 5, 6 as before, except that the minute hand now shows a quarter before the next hour. Though "A quarter before . . ." is the usual answer, I think that we should also accept "Three quarters after . . ."; first, because it is also correct, and second because it corresponds to the way it would be both spoken and written in hours and minutes, e.g., 9:45.
 9. Finally, repeat the above with the minute hand at zero. Explain that when the time is right on the hour, we read this as (e.g.) "Three o'clock."
 10. On a subsequent occasion, the children may usefully practise reading the clock face as above, with the minute hand in one of the four positions above. They may take it in turns to set the hands while the others read the time.

Activity 4 Time, place, occupation [Num 5.7/4]

An activity for a small group of children. Its purposes are to give further practice in reading the time from a clock face, and to relate times of day to a wider context of where they go and what they do.

- Materials*
- A clock face, with movable hour and minute hands.
 - A two-sided card, with 'a.m.' on one side and 'p.m.' on the other.

- What they do*
1. One child has the clock face, and sets it to a time in which the minute hand is at one of the four quarters and the hour hand is in approximately the right position relative to the minute hand. The two sided card may be either side up, but for the first round the time should be earlier than the present time.
 2. He first asks the others “What time is this?”
 3. When they have answered, he then asks the other children in turn, “Where were you at this time today? And what were you doing?”
 4. When all have answered, the turn passes to the next child who repeats steps 1, 2, and 3 as before, except that the time shown is later than the present time, and the questions in step 3 are changed to “Where will you probably be at this time today? And what will you probably be doing?”
 5. The next child sets the time to any desired time of day, the questions in step 3 being changed to “Where were you at this time yesterday? And what were you doing?”
 6. The next child does likewise, but asks “Where will you probably be at this time tomorrow? And what will you probably be doing?”
 7. Four children will now have acted as questioner. If there are others who have not yet had a turn at this, they may make any sensible choice of day for their questions.

Discussion of activities

The object of all this is not only for them to be able to tell the time by the clock, but to understand this within meaningful contexts. Here I suggest two. First, is the practical need of an agreed system for describing locations in time, for social cooperation. Many of the ways in which we coordinate our own actions with those of others would fall to pieces without clocks and watches. Because of this practical need, the origins of measuring time have a long history, and I have tried to show just a glimpse of this.

Both analogue and digital clock faces are now widespread, and some teachers prefer to teach the latter first on the grounds that it is easier. This may be so, though it begins with hours and minutes rather than hours and quarters. Myself, I prefer to start with the analogue clock face, partly for the historical background which I have already mentioned, and partly because for some purposes I find it preferable. The analogue clock face shows intervals graphically, so that it is easy to read off how long it is (e.g.) from 9:45 to 10:20. At, say, 06:55 (digital) the position of the minute hand at 5 minutes before the hour shows directly that the time is coming up to seven o'clock. The digital reading requires this to be inferred from the difference between 55 and 60. Finally, the digital readings give a greater degree of accuracy than is needed for many purposes. When I look at my own watch, it tells me the time to the nearest minute and second, whether I want these or not. (Yes, it is digital, because I like the other facilities which come with this.) But in our living room we have an analogue clock to an old German design.

Children need to be at home with both symbolisms, the spatial and the numerical. Above I have explained why I have begun with the former, but if you prefer the other way about, the foregoing activities can be taken in a different order.



Sequencing numerals 1 to 10 [Num 2.3/5]

[Meas 6] TEMPERATURE

MEAS 6.1 TEMPERATURE AS ANOTHER DIMENSION

Concept Temperature as another dimension.

Abilities (i) To say which of two bodies is the hotter, and, conversely, which is the colder.
(ii) To sequence two or three objects in order of temperature.

Discussion of concepts

It is important to distinguish between temperature and heat. To boil an electric kettle takes longer if the kettle is full than if it is only partly full, because although the starting and finishing temperatures are the same, the amount of heat required is different. Since heat energy is input at a constant rate, we can quantify this, and find that (as we would expect), twice as much water requires twice as much heat.

We may not wish to explain this distinction to children of the ages we are presently thinking about, but we need to be clear in our own minds so that we use the terms correctly. An object at a higher temperature is hotter. Temperature is the measure of how hot it is, as against how much heat it has. Heat is a form of energy, measured in calories or joules. Temperature is the level of this energy, related (according to the molecular theory of heat) to how fast the molecules are moving to and fro.

We use a height model for temperature, in the same way as we use a length model for time. Heat tends to flow from a body at a higher temperature to one at a lower temperature. One of these 'bodies' is often the body of air surrounding the other body, so that (e.g.) a hot drink cools and a chilled drink gets warmer.

In this and the next topic, we use blocks of varying height to represent this model for the children.

Activity 1 Comparing temperatures [Meas 6.1/1]

An activity for children to do on their own or in pairs. Its purpose is to call their attention to what they already know in everyday life: that some things are hotter than others, to relate it to vocabulary, and to include it in their existing concept of an order relationship.

- Materials*
- For each child or pair, three objects at different temperatures. E.g., bottles, jam jars, filled with water at different temperatures by mixing from hot and cold taps. Other objects may be prepared by keeping some at room temperature and some in a refrigerator, some might be warmed in a microwave if suitable (e.g., potatoes).
 - For each child or group, three blocks of different heights on which these objects can be put. The thicknesses are not critical; e.g., 1 cm, 3 cm, 5 cm.

Note At stage (a) only two objects and two blocks are used.

Meas 6.1 Temperature as another dimension (cont.)

What they do Stage (a)

1. Ask the children to feel the objects, and put the hotter one on the higher block.
2. If they are working in pairs, one does this and the other checks. If working on their own, they check with each other in the same group.
3. Explain that we use the word ‘temperature’ for the level of heat, so we say that the hotter object is at the higher temperature, the colder is at the lower temperature. The different heights are to help us to think of it in this way.

Stage (b)

1. Tell the children that this is like what they did before, but now they are to put the hottest object on the highest block, the coldest object on the lowest block, and the one in between on the block of middle height.
2. What other words could they use to describe what they have done? (Similar, but using the terms higher and lower temperature.)

Activity 2 “Which is the hotter?” [Meas 6.1/2]

An activity for children working in pairs. Its purpose is to relate the concepts used in Activity 1 to their everyday experience.

Materials None required.

- What they do*
1. Child A names two objects and asks child B which is the hotter. E.g., an ice cream and a baked potato, a cup of tea and a chilled drink, a summer day and a winter day.
 2. When child B has answered, child A either says “I agree,” or explains why he disagrees.
 3. It is then the turn of child B to name two objects.
 4. In some cases the reply might be “It could be either.” For example, indoors and out of doors. This would sometimes be different for winter and summer.
 5. The wording of the question should be varied, using sometimes “Which is colder?” and also sometimes “Which is at the lower/higher temperature?”

<p>Discussion of activities In these activities, children are working with physical materials to begin systematizing their everyday knowledge and relating this to a scientific model. As is often the case, this will be of a mathematical kind.</p>
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OBSERVE AND LISTEN

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MEAS 6.2 MEASURING TEMPERATURE BY USING A THERMOMETER

Concept The thermometer as an instrument for measuring temperature.

Abilities (i) To use a thermometer correctly for measuring temperature.
(ii) To sequence two or three objects in order of temperature.

Discussion of concept Children will already be familiar with the idea of measuring instruments, such as rulers for measuring length, kitchen scales, speedometers in cars. They will probably also have encountered a thermometer in some form or other — most children have had their temperatures taken by a parent or doctor. So here again, we are consolidating, organizing, and extending their everyday knowledge.

Activity 1 The need for a way of measuring temperature [Meas 6.2/1]

A teacher-led experiment followed by a discussion. Its purpose is to underline the need for a way of measuring temperature which is more accurate than estimation, and to think of other reasons for this. Below I suggest two forms of the experiment. You may like to try others.

Materials • Four containers for water, such as mugs, jam jars.

Suggested procedure for the experiment

1. The four containers are nearly filled with water at three different temperatures. One should contain cold water, chilled if necessary with a lump of ice which should be removed before the experiment. We may call this one **C**. Another, container **H**, should contain hot water, as hot as would be comfortable to wash one's hands in. The remaining two should contain warm water, both at exactly the same temperature. This should be just a little above room temperature. We will call these **W** (for warm). Note that these terms are for our own convenience: the containers should not be labelled with these letters.
2. First put out containers **C** and **H**.
3. A volunteer is now needed. Ask her to put two fingers of her left hand in **C** and two fingers of her right hand in **H**, and say which she thinks is hotter and which is colder. She should keep her fingers in the containers for not less than 20 seconds before the next step.
4. Now give her the two **W** containers and ask her to compare these. (In this case it is better to say "compare" rather than ask which is hotter/colder.)
5. Although they are in fact at the same temperature, this will feel warmer to the fingers coming from the cold water.
6. Now ask another volunteer to compare the two **W** containers. She is likely to say that they are both the same.

Meas 6.2 Measuring temperature by using a thermometer (cont.)

7. So here we have one reason why we need a way of measuring temperature more reliably than by feeling. Can anyone think of others? (E.g., too hot or cold to be safe to touch; inaccessible; accuracy required is greater than would be possible by estimation.)

Variation In this simpler form of the experiment, only one **W** container is used. After step 3, the volunteer puts the fingers of both hands in **W**, which will feel warmer to one hand than to the other. Since this does not take long, more children can take part. May I suggest that in any case you try this for yourself beforehand? It is a little strange to have ones hands sending messages which are contrary to what one knows to be the case.

Activity 2 Using a thermometer [Meas 6.2/2]

An activity for a small group. After a teacher-led introduction, they should be able to continue on their own. Its purpose is to introduce children to the use of a thermometer.

- Materials*
- Two Celsius thermometers, scaled from 0° (or below) to 100° (or above).
 - At least two blocks of different heights, as used in Meas 6.1/1. The more of these can be made available, the better.
 - A number of containers such were used in Meas 6.2/1.
 - Hot and cold tap-water.
 - Pencil and scrap paper for each child.

- Suggested introduction*
1. Fill two of the containers with water at clearly different temperatures.
 2. Have the children agree which is the hotter and which the colder, and let them show this as before by putting them on blocks of different heights.
 3. Now put a thermometer in each of the containers, and ask the children to watch carefully and report what happens.
 4. We hope that at least the nearer ones will observe that the thread of mercury or alcohol changes length, doing this more slowly until it is steady.
 5. Put the thermometers vertically side by side, in the same relative positions. We hope that they will now notice that the greater height in the thermometer corresponds to the higher temperature.
 6. Looking more closely, they can see that the thermometer is marked with a scale. These are degrees Celsius, which are the international units used to measure temperature.
 7. Pass them around, and let each say what the reading is. Unless the liquid was at room temperature, the reading will gradually change. What do they learn from this?
 8. They may now each take a container and fill it with water mixed from cold and hot taps. Working in pairs, one estimates which is hotter and which is colder, and shows this by putting them on blocks.

Meas 6.2 Measuring temperature by using a thermometer (cont.)

9. They then check with thermometers. Since there will certainly not be two of these per pair of children, they will have to make the comparison by reading the temperatures in degrees Celsius which they record on a small piece of scrap paper.
10. These are wetted so that they adhere temporarily to the containers.
11. (Optional further step.) Pairs may then cooperate to put all their containers in order of temperature.

Note The temporary nature of the labels' attachment corresponds well to the fact that the temperatures will not stay the same for long!

Discussion of activities Note that in Activity 2, the children begin by using their own senses to compare temperatures, so that the higher thermometer reading corresponds to what they already know to be the higher temperature. This relates the new experience to their existing schemas.

I have not included any explanation of how a thermometer works, mainly because these are resources for learning and applying mathematics rather than an introduction to science. It will however be as well to be prepared with an explanation if children ask. The most common kind, and the one which you will need for use in the classroom, is the mercury in glass thermometer, with alcohol thermometers available for sub-zero temperatures. Non-liquid thermometers, which use a bimetallic strip, are also fairly common, so you might want to check out this kind too.

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MEAS 6.3 TEMPERATURE IN RELATION TO EXPERIENCE

Concept The relationship between temperatures as shown by a thermometer and everyday experience.

Abilities (i) To say roughly what temperatures are to be expected in a variety of everyday examples.
(ii) Conversely, to say whether a given temperature is a likely one in any given case.

<p>Discussion of concept The concept itself is the same as the children learned in Meas 6.2. Here they are expanding its field of application.</p>

Activity 1 Everyday temperatures [Meas 6.3/1]

A project-like activity for a small group of children to work at cooperatively, as described below.

Materials

- Not less than two thermometers for the group; more if possible.
- Pencil and paper for each child.

What they do

1. Explain that now they know how to measure temperature accurately with a thermometer, you would like them to make a list of everyday temperatures which they have met.
2. They should start by 'brainstorming' to produce a list of what they want to measure. Here are some suggestions to start with.
Inside temperatures, in the classroom and elsewhere in the school
Outside temperatures, at different times of year.
Temperature in the same room at floor level and as near the ceiling as practical.
Temperatures of liquids left in shade and sun.
Temperatures of ice water, and (under supervision) boiling water.
Body temperature using ordinary and (if available) a clinical thermometer. Why is body temperature important?
3. They work in pairs to read and record the temperatures they have listed, having first apportioned this among themselves.
4. Finally they combine their information in a list for the classroom wall.

Activity 2 “What temperature would you expect?” [Meas 6.3/2]

An activity for any sized group of children, working in pairs. Its purpose is to give practice in using and extrapolating their knowledge of likely temperatures, based on the experience gained in Activity 1.

Materials For each pair:

- The query list below* (photomaster provided).
- Pencil and paper.
- Their rough drafts of the lists made in Activity 1 would be useful, as an aid to memory.

* *Note* This is provided for your convenience, here and in the photomasters, but you may wish to extend this to include material of local and topical interest.

- What they do*
1. They discuss with their partners what would be reasonable answers to the queries on the list, and fill these in on their sheet. In most cases a range of temperatures, or “Around ...”, would be sensible replies.
 2. All the pairs at the same table may then share their conclusions, and discuss any points of divergence. If these cannot be resolved, they will need to consult you.
 3. As in Activity 1, the final results might make a suitable display for the classroom wall.

Query List

What temperature would you expect to find in the following?

A classroom or living room.

A refrigerator.

A freezer.

An oven ready for cooking.

A shopping mall.

Water from the cold tap.

Water from the hot tap.

A swimming pool.

A hot bath or shower.

A mountain stream.

Meas 6.3 Temperature in relation to experience (cont.)

Activity 3 Temperature in our experience [Meas 6.3/3]

This is a long-term project, for the whole class, to be organized in whatever way you think best. It is based on the idea that temperature is an interesting source of data for environmental studies.

- Materials*
- Suitable display material for the classroom wall, to incorporate the data below as it is collected.
 - Celsius thermometers.
 - An outdoor thermometer, which can be read through the window, is an interesting source of data. A maximum and minimum thermometer would be even more interesting, and could be used to make a graph of year-long changes.

What they do (These are suggestions offered as a starting point.)

Temperatures are recorded and collated, in some cases graphically, for the following temperatures:

Outdoors (locally) in summer, winter, spring, fall, at the same times of day.

Highest temperature recorded, and lowest.

Hourly readings inside and outside for the whole of a school day.

Discussion of activities In Meas 6.1 we based the scientific concept of temperature on children's everyday experience of this. In Meas 6.2, we introduced the instrument (thermometer) and units (degrees Celsius) by which this is measured. In the present topic, children are developing connections in the reverse direction, from temperature as measured with a thermometer back to everyday experience. This continues the overall process by which we accept the importance and validity of the knowledge which children already have, lead them to consolidate and organize it, and then help them to expand it further.

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GLOSSARY

These are words which may be unfamiliar, or which are used with specialized meanings. The definitions and short explanations given here are intended mainly as reminders for words already encountered in the text, where they are discussed more fully. This is not the best place to meet a word for the first time.

<i>abstract</i>	(verb) To perceive something in common among a diversity of experiences. (adj.) Resulting from this process, and thus more general, but also more remote from direct experience.
<i>add</i>	This can mean either a physical action, or a mathematical operation. Here we use it with only the second meaning, in order to keep these two ideas distinct.
<i>addend</i>	That which is to be added.
<i>base</i>	The number used for grouping objects, and then making groups of these groups, and so on. This is a way of organizing large collections of objects to make them easier to count, and is also important for place-value notation.
<i>binary</i>	Describes an operation with two operands.
<i>canonical form</i>	When there are several ways of writing the same mathematical idea, one of these is often accepted as the one which is most generally useful. This is called the canonical form. A well known example is a fraction in its lowest terms.
<i>characteristic property</i>	Property which is the basis for classification, and for membership of a given set.
<i>commutative</i>	This describes a physical action or a mathematical operation for which the result is still the same if we do it the other way about. E.g., addition is commutative, since $7 + 3$ gives the same result as $3 + 7$; but subtraction is not.
<i>concept</i>	An idea which represents what a variety of different experiences have in common. It is the result of abstracting.
<i>congruent</i>	Two figures are congruent if one of them, put on top of the other, would coincide with it exactly. The term still applies if one figure would first have to be turned over.
<i>contributor</i>	One of the experiences from which a concept is abstracted.
<i>counting numbers</i>	The number of objects in a set. The cardinal numbers, 1, 2, 3 . . . (continuing indefinitely). Zero is usually included among these, but not negative or fractional numbers.
<i>digit</i>	A single figure. E.g., 0, 1, 2, 3 . . . 9.
<i>equivalent</i>	Of the same value.
<i>extrapolate</i>	To expand a schema by perceiving a pattern and extending it to new applications.
<i>higher order concept</i>	A concept which is itself abstracted from other concepts. E.g., the concept of an even number is abstracted from numbers like 2, 4, 6 . . . so <i>even number</i> is a higher order concept than 2, or 4, or 6 . . .
<i>interiority</i>	The detail within a concept.
<i>interpolate</i>	To increase what is within a schema by perceiving a pattern and extending it inwards.

<i>Levels</i>	Specific learner expectations which describe the developmental path of mathematics learning for elementary school children. This continuum is set out in eight levels. Levels 1 & 2 are appropriate for the majority of four and five year olds; Level 3 for the majority of six year olds, etc.
<i>low-noise example</i>	An example of a concept which has a minimum of irrelevant qualities.
<i>lower order concept</i>	The opposite of <i>higher order concept</i> , q.v.
<i>match</i>	To be alike in some way.
<i>mathematical operation</i>	See <i>operation</i> .
<i>Mode 1</i>	Schema building by physical experience, and testing by seeing whether predictions are confirmed.
<i>Mode 2</i>	Schema building by receiving communication, and testing by discussion.
<i>Mode 3</i>	Schema construction by mental creativity, and testing whether the new ideas thus obtained are consistent with what is already known.
<i>model</i>	A simplified representation of something. A model may be physical or mental, but here we are concerned mainly with mathematical models, which are an important kind of mental model.
<i>natural numbers</i>	The same as counting number.
<i>notation</i>	A way of writing something.
<i>numeral</i>	A symbol for a number. Not to be confused with the number itself.
<i>oblong</i>	An oblong and a square are two kinds of rectangle, in the same way as boys and girls are two kinds of children.
<i>operand</i>	Whatever is acted on, physically or mentally.
<i>operation</i>	Used here to mean mental action, in contrast to a physical action.
<i>pair</i>	A set of two. Often used for a set made by taking one object from each of two existing sets: e.g., a knife and a fork.
<i>place-value notation</i>	A way of writing numbers in which the meaning of each digit depends both on the digit itself, and also on which place it is in, reading from right to left.
<i>predict</i>	To say what we think will happen, by inference from a suitable mental model. Not the same as guessing. Prediction is based on knowledge, guessing on ignorance.
<i>schema</i>	A conceptual structure. A connected group of ideas.
<i>set</i>	A collection of objects (these may be mental objects) which belong together in some way.
<i>subitize</i>	To perceive the number of objects in a set without counting.
<i>sum</i>	The result of an addition. Often used, incorrectly, to mean any kind of calculation.
<i>symmetry</i>	A relation of a figure with itself. If a line can be found which divides a figure into two congruent halves, the figure has symmetry about this line. This is the only kind of symmetry considered in this book. Rotational symmetry is another kind.
<i>transitive</i>	A property of a relationship. E.g., if Alan is taller than Brenda, and Brenda is taller than Charles, then we know also that Alan is taller than Charles. So the relationship 'is taller than' is transitive.
<i>unary</i>	Describes an operation with a single operand.

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A Sequencing Guide for Pre-grade Level 1

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/1 Making successive sets	101	Num 2.3 <i>Single-digit numerals recognized and read</i>	129
/2 Putting one more	101	/5 Sequencing numerals 1 to 10	131
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A Sequencing Guide for Grade Level 1

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/2 Predicting from patterns on paper	334	/4 Number comparison sentences	219
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/1 Sorting and naming geometric shapes	295	Num 4.6 <i>Giving change</i>	223
/2 Sorting and naming two dimensional figures	296	/1 Change by exchange	223
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/3 Patterns in sound	337	/1 Building a bridge	346
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A Sequencing Guide for Grade Level 2

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S A I L T H R O U G H M A T H E M A T I C S

V O L U M E 1 F O R T H E E A R L Y Y E A R S

S T R U C T U R E D A C T I V I T I E S

F O R I N T E L L I G E N T L E A R N I N G

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Volume 2: for the later years

Volume 1a: photomasters

Volume 2a: photomasters

Convinced that practical work is essential throughout the elementary school years, and not just for younger children, Richard Skemp provides in these volumes a fully structured collection of more than 300 classroom-tested activities. The collection covers a core curriculum for children aged four to eleven years, using practical work extensively at all stages and providing a carefully graded transition from practical work to abstract thinking, and from oral to written work. The activities range from teacher-led discussions to games which children can play together without direct supervision, in which success depends largely on mathematical thinking. These promote discussion and co-operative learning, consolidate children's knowledge and lead to fluency in mathematical processes.

Volumes 1 and 2 also provide:

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- clear statements of what is to be learned from each group of activities
- for each activity, a list of materials and step-by-step instructions.

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Sail Through Mathematics will be invaluable to classroom teachers, special resource teachers and administrators, as well as in the pre-service and inservice education of teachers.

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Calgary, Alberta

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Structured Activities for Intelligent Learning

Progress Record

Name _____

	Pre-grade 1	Grade Level 1	Grade Level 2
Org 1	○○○○○○○○ 1 2 3 4 5 6 7 8 9	○○○○○○○○ 10 11 12	○○○○○○○○ 13
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Meas 4	○○○○○○○○ 1 2 3 4	○○○○○○○○ 5 6 7 8 9	○○○○○○○○ 10 11 12
Meas 5	○○○○○○○○ 1 2 3 4 5 6	○○○○○○○○ 7 8 9	○○○○○○○○ 10 11 12
Meas 6	○○○○○○○○ 1 2 3 4 5 6	○○○○○○○○ 7 8 9	○○○○○○○○ 10 11 12