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THE PSYCHOLOGY  
OF  
LEARNING AND TEACHING MATHEMATICS

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Study No. 1: this is one of several studies, on the various aspects of the teaching of mathematics in secondary schools, which are intended to provide an analysis of how this subject is being taught in different parts of the world.

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I. The aim of this paper is to discuss some of the psychological aspects of the learning and teaching of mathematics in a form which will be usable by teachers and educational psychologists.

The relation between this and teaching itself is that we shall here be concerned mainly with the mental processes involved, rather than with teaching method in detail. The opinion of an expert mathematician may carry much weight; but such a one is all the more likely to think in terms of the end results, while failing to consider the often lengthy and difficult paths whereby these have to be reached by those less expert than himself. Successful teaching of mathematics requires first, a knowledge of the relevant processes of learning; second, and based on this, appropriate methods of teaching; and finally, the application of these in the particular field of mathematics. This paper concentrates on the first of these.

The next distinction to be made is between mathematics and its precursors, such as arithmetic and mensuration. The importance of this distinction lies in the fact that certain mental processes are required for algebra which are not needed for arithmetic; and for Euclidean and projective geometry, which are not needed for drawing and measurement, and so on. The distinction is in the level of abstraction involved; and the term 'mathematics' will be used to include those ideas which cannot be given direct concrete representation. For example, the number seven can be represented by seven physical objects; which cannot be done for the variable  $x$ . One can draw a triangle having sides 3, 4, 5 centimetres but one cannot draw "ABC, any right-angled triangle". (What one does draw is a symbolic representation of a particular class of triangles; quite a different matter.) Similarly the operation "adding 3" can be represented concretely by a physical movement of three objects; but not "differentiate with respect to  $x$ ", nor "project AB to infinity". Anticipating a little, this means that the subject matter of mathematics consists of purely mental objects; and it is from this that many of its particular difficulties arise.

Since the aim is to offer the ideas which follow in a form useful to teachers and educational psychologists, they will be given so far as possible in the form of basic principles having great generality, from which applications to particular teaching situations can be deduced. To supplement this, I have listed in the bibliography several recent papers of the review type which together cover comprehensively the field of current research.

II. In view of the emphasis of the previous section on the need for a better understanding of the mental processes involved, it is logical next to ask what psychological research has been done which bears on the problem. Unfortunately, it must now be admitted that there is as yet remarkably little. This scarcity seems to be generally agreed; e.g. Carr and Peel say, in a summary of research on arithmetic and mathematics, "There is however very little on the learning of arithmetic and mathematics that can be called research". (Reference 20). Likewise Wall and Biggs, of the (British) National Foundation for Educational Research say: "...research into the learning and teaching of mathematics has on the whole been short-term, scattered and piecemeal". (17). The reason for this shortage is, in view of the obvious need, a problem in itself.

An important part of the answer is, I think, the lack of a theory of learning which is applicable to mathematics. Particularly in the USA, and also to a considerable extent elsewhere, behaviourist learning theories have for many years held the field. These reject the study of mental processes on grounds that only behaviour is observable, and that therefore only behaviour is a valid subject for scientific study. The weakness of this argument lies in the failure to distinguish between scientific evidence itself, and theories based thereon. The former should indeed be generally observable (by any one with the necessary knowledge and equipment); but in every science, a theoretical system is developed which unifies the observation in a system of abstract ideas. No physicist would refuse to study magnetic fields on the grounds that those were unobservable, and confine himself to the study of the arrangements of iron filings. The criterion for the scientific respectability of an observation is that others can make it too; but for a scientific concept, the criteria are its powers to unify, to explain and to predict these observables. In psychology, concepts relating to mental processes are as capable as any others of satisfying the above criteria.

I confess to a little hesitation before voicing this criticism of such a great volume of research into learning. This view is however implied by Wall and Biggs, who give (op cit) as one of our chief needs "studies... directed at the development of a comprehensive general theory of learning and mathematical learning in particular"; and in America by Buswell who in his presidential address to the Educational Psychology Division of the American Psychological Association, said: "Without the slightest criticism of experiments in general psychology, we cannot continue to be satisfied with implications for education from results of experiments with simple mental processes, with animals, and at the sub-language level." Though this was said in 1955, it remains true. I am further emboldened by the possibility that this dissatisfaction with currently used theories is more widespread than appears in what gets published, and that perhaps others are only waiting for someone to say "The Emperor has no clothes"!

Another weakness of contemporary learning theory is that it only deals with the learning of isolated facts and responses, rather than integrated systems of knowledge and skills. A characteristic of all higher

subjects of learning is the way in which earlier stages lead to and make possible the later ones; and this hierarchic dependency is particularly characteristic of mathematics. So any learning theory which is to be applicable to mathematics must deal not only with thinking processes, but with a highly organized and interconnected system of thinking processes. No such theory has yet been published.

The foregoing criticisms of available learning theories have not yet shown any reason why the crisis which has arisen should be in the particular subject of mathematics. This leads to the question, what is the nature of mathematical thinking? Is there some difference between this and other subjects which makes it harder to learn? Or perhaps it involves a different learning process, so that the poorer results are due to the unappropriate use of methods which may be quite suitable for other subjects?

There have been various studies of mathematical thinking, of which some are listed in the bibliography (12, 13, 14). These however tend to be less descriptions of mathematical thinking than themselves examples of it; that is, they are high order generalizations about mathematical thought, these generalizations being themselves of a nature which is more like higher mathematics than anything else. Their other limitation is that they describe only the end result; and knowledge of a goal is not a sufficient condition of knowing how to get there. However, the former information is also necessary; and for our present purpose, mathematics may be described as one particular structure of abstract ways of thinking which is applicable to a wide variety of situations, including many which are very different from those in which the former were first developed. This definition makes explicit three points of particular importance for the present study; the conceptual and highly structured nature of mathematics, and the detachability of its concepts from their original context. These points will now be developed further.

Firstly, concepts are different from facts, and the process of concept-formation is different from the learning of facts. Secondly, the learning of a structure is something more than the learning of a collection of isolated details. Thirdly, whereas the responses which form part of stimulus-response learning theory are automatic, stimulus-bound, and capable of being rote-learned, true mathematics requires deliberate and generalized operations, and their acquisition depends on understanding. Further, the detachability of these concepts and operations from their origins, and their deliberate extrapolation to new developments and applications, is closely related to their becoming themselves objects of consciousness. These objects of mathematical thinking have, unlike those of other sciences, only mental existence. Mathematics is thus essentially a reflective process, and is qualitatively different from activities based on sensori-motor interaction with the outside world.

It is the failure to realize these differences which has caused much of the lack of success by teachers and researchers. The former often try to teach mathematics as if it were a collection of facts and responses, instead of a structure of concepts and operations. The latter, with a few noteworthy exceptions, mainly still try to use the stimulus-response kind of learning theory. In Europe, which has never been wholly behaviourist in its psychology, an awakening of interest in the development of conceptual thinking is indicated by the rapid recent growth of interest in and research deriving from the work of Piaget. This has as yet, however, received little attention in the United States.

III. For a proper understanding of the psychological bases of learning mathematics, we need to know about

- a) the formation of concepts in general, and mathematical concepts in particular.
- b) the learning of organized structures; and in relation to this, about the internal organization of mathematical knowledge.
- c) the processes of mathematical generalization and exploration whereby new concepts and operations are derived at the abstract level from existing ones.
- d) problem solving and the correction of errors.
- e) the function of mathematical symbolism.
- f) motivations for learning mathematics.

a) Concept formation: A concept is not easy to define, partly because there are many different kinds of concept, and partly because of certain intrinsic characteristics which will appear later. To begin with one of the simpler sort, a class-concept is a set of properties which characterizes a particular collection of objects. For example, "tree" is the name of a class concept, and so is "five". The former concept is the set of properties common to oaks, ashes, elms, birches, pines; the latter is the set of properties common to five pennies, five buttons, five men, five ships. "Oak" is itself the name of<sup>x</sup> a class-concept, for it represents not one particular oak tree but the set of properties held in common by all oak-trees. And if we group the class-concept "tree" in a collection with other class concepts such as "plant", "fern", "grass", "fungus", we can derive another class-concept "vegetable". These concepts form a hierarchy in which "tree" is subordinate to "vegetable", superordinate to "oak", and of the same order as "fungus".

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<sup>x</sup> Hereafter, quotation marks will be taken as sufficient reminder that a word used to refer to a concept is not the concept itself. This reminder is important, because it is possible to have the word without having the concept.

Suppose now that a child asks us "What is a tree?". Should we answer by attempting a botanical definition, or by saying "Oaks, ashes, pines, apple trees, and so on -- these are all trees"? Undoubtedly the latter: for even if we knew the former, he would not understand it. Nor would it suffice to say "An oak is a tree", for this would probably not convey to him that birches and firs are also trees, but not gooseberry bushes. In other words, one of the easiest ways to communicate a class-concept is to give a number of different examples which belong to the class. The more the number and the greater the variety, the greater the likelihood that the person will abstract just those properties which we intend. If we gave as example "Oaks, ashes, elms", the child might confine his class-concept to deciduous trees only; if "Apple-trees, pear-trees, plum-trees", he might regard gooseberry bushes as trees, but not larches.

An important characteristic of this way of communicating a concept is that it makes use only of concepts of lower order than itself. If the answer given had been "A tree is a member of the vegetable kingdom which is spormophytic, perennial, woody, and has one or few main stems", the child would not have understood, because this definition uses concepts which are of equal or higher order than "tree". If the child does not possess the concept "tree", he is unlikely to have the others either.

The reader may well be wondering what all this has to do with mathematics. A great deal; for in teaching mathematics we are concerned with the communication of very high order concepts, and the burden of the foregoing is that to anyone who is not already in possession of a sufficient number of concepts of a given order, new concepts of that order cannot be communicated directly. All that can be done is to give a number of carefully chosen examples from which the person will, it is hoped, form his own concept. One cannot teach someone a concept in the same way as one can teach facts: one can only try to arrange for him to learn it. It is for this same reason that it has been necessary to approach the concept "concept" by such a roundabout method. So far, the concept "class-concept" has (it is hoped) been communicated by several examples of class-concepts, including a mathematical example. To complete the process, it would be necessary first to communicate a number of other particular kinds of concept in the same way, and to hope that from these, readers would form the superordinate concept "concept".

Some of the kinds of concept used in mathematics are; relational concept, operation, abstracting, generalizing. Examples of relational concepts are "equal", "greater than", "tends to", "function of", "isomorphic with". Examples of operations are "differentiation", "addition", "projection", "substitution". Examples of abstracting are "concept formation", getting the equation", and every case in which arithmetical or mathematical data are derived from a practical problem. Examples of generalizing are the processes of developing ideas about fractional and negative indices starting from positive whole number indices, and about trigonometrical ratios of angles of any magnitude from the definitions for an acute angle.

If the reader has succeeded in forming, from so few instances, these particular concepts, he may be able to derive from them the superordinate concept "concept". The reason for the difficulty of giving a direct definition of "concept" should also now be apparent: namely, an insufficiency of existing concepts of as high an order as itself.

This indirect method will probably seem very unsatisfactory to most teachers. How can one ensure that pupils form the right mathematical concepts, or that they form them at all, except by clear and unambiguous definitions? Unfortunately, a teacher's desire that his pupils shall learn accurately is likely to defeat its own ends. Definitions like "a parallelogram is a plane four-sided figure having its opposite sides parallel" use concepts of the same or higher order than parallelogram, and cannot therefore communicate the concept of parallelogram to a beginner in geometry. The first stage has to be the formation of a set of concepts, by the building-up process already described. Definitions are (among other things) a tidying-up process, in which one decides exactly what set of characteristics the concept comprises. That is to say, having formed the concept, it may be possible to formulate it -- to make the concept the object of consideration in itself, apart from the members of its class from which it was abstracted.

This formulating and defining process also has the effect of relating it to other concepts of the same or higher orders. When the concept "parallelogram" is formulated by the definition given earlier, since it is shown thereby to be included in the class "plane four-sided figures", everything known about the latter becomes applicable to parallelograms. As soon as a square is known to have its opposite sides parallel, it follows that it possesses all the properties of a parallelogram. This activity of reflecting on concepts, investigating their mutual relationships, developing systematic techniques of inference from a small number of basic concepts, and deducing statements about particular instances (riders) from a combination of more general ones (theorems), is very characteristic of formal geometry. But no one can reflect on concepts which are not there; and this, I suspect, is what many pupils at the secondary school stage are trying to do. Formal geometry only becomes possible if the basic concepts involved have first been formed in the mind of the pupil by the indirect process of helping him to arrive at them for himself.

The hierarchic nature of mathematical concepts is particularly clear for arithmetic and algebra. Let us consider by way of example the simple expansion

$$(x + y)^2 = x^2 + 2xy + y^2$$

The truth of this depends on the truth of the following general statements:

- (i)  $(x + y)^2 = (x + y)(x + y)$
- (ii)  $(x + y)(x + y) = x(x + y) + y(x + y)$
- (iii)  $xy = yx$
- (iv)  $xy + xy = 2xy$

The first appears to represent chiefly an agreement among mathematicians about notation and will not be discussed further at this stage (but see also IV(a)). On the other three, however, depends the validity of most of the manipulations of algebra.

Let us consider briefly what subordinate concepts are involved in the third of these,  $xy = yx$ . This represents symbolically the collection of all statements such as  $2 \times 7 = 7 \times 2$ ,  $5 \times 8 = 8 \times 5$ ,  $4 \times 9 = 9 \times 4$ ; which is to say that it represents a concept. This concept can only exist as such in the mind of a pupil who has abstracted it himself from a sufficient number of separate examples. Each of these examples will involve the concepts of the individual numbers, of the operation of multiplication, and of the relationship of equality.

To discuss the process of formation of the individual number concepts would take too long here (see 1). But it may be worth pointing out that "multiply" is an operational concept, representing that which all the individual operations like  $2 \times 7$ ,  $8 \times 5$ ,  $4 \times 9$ , etc., have in common. Equality, again, will only exist as a true concept if a pupil has abstracted it himself from a number of instances, those instances all being exemplars of the mathematical concept of equality and non-exemplars of concepts rather like it such as "the same as". (Compare with the formation of the concept "tree").

Similar discussions are desirable for statements (ii) and (iv), but must be omitted to leave space for a few words about the variables  $x$  and  $y$ . So far, the examples given have all used the natural numbers only, which are concepts derived from collections of separate objects. The concept of multiplication has also only been developed in this context, where  $4 \times 7$  means that four similar collections each containing seven unspecified objects are to be made into a single collection. The successive transitions by which  $x$  and  $y$  can come to represent positive and negative integers and fractions, and the generalization of the concept of multiplication to make it applicable to these new concepts, cannot be dealt with at this stage because the processes of mathematical generalisation have not yet been discussed. But it may be pointed out that this process of generalization is hardly likely to be possible if the concepts which are to be generalized have not been formed, as true concepts. (This suggests, in parentheses, a reason for the trouble that many children have with fractions etc., even at the arithmetic stage). Even such a simple algebraic process as the expansion of  $(x + y)^2$  turns out to be dependent on a complex hierarchy of class-concepts, relations, and operations; and a pupil who lacks any of these subordinate concepts will not understand what he is doing.

It is particularly difficult for a teacher who himself possesses a concept to realize that a pupil does not, for once a concept is formed it seems obvious to its possessor. The concept, being evoked by and structuring the perception of the data, is perceived as if it were part of the data; and it is then difficult to imagine any other way of perceiving this data.



It is made even harder for a teacher to know whether a pupil really has a concept by the fact that a reasonably intelligent and industrious pupil can rote-learn all the manipulations which are necessary in the early stages. He will find it more difficult, for (to anticipate a later section) conceptual learning implies schematic learning, and the latter is easier and more lasting.

The first of the basic psychological principles on which the learning of mathematics needs to be based is therefore that of the process of concept formation. When this principle has been fully grasped, it can be applied at all levels of mathematics upwards from those of the basic arithmetical and elementary algebraic levels which have here been used as examples. The best way to grasp this principle is contained within itself, namely to experience as many different examples as possible of the process. For this, the reader cannot do better than read as much as he can of what has been written on this topic by Dienes (see bibliography). This work stems from Piaget's studies of concept development; but I think that many of the ideas which he attributes to Piaget are really Dienes' own, which he had as a result of reading Piaget's work and subsequently perceived as part of it. In schools at Leicester, and recently at Harvard in collaboration with Professor Bruner, he has pioneered the formulation and methodical application of the foregoing principles of concept development in the fields of arithmetic and early algebra. Extension of these upwards to the secondary stage is a task which still remains to be done. Before it can be begun, however, there are further differences between mathematics and arithmetic to be discussed.

b) Schematic learning: One of the most serious defects of contemporary learning theories for our present purpose is their failure to take account of the way in which existing knowledge makes possible, and also influences, subsequent learning. The more structured the subject itself, the more serious is this deficiency in any theory which attempts to deal with the learning of it. In arithmetic and mathematics for example, knowledge of addition and of the multiplication tables make possible the learning of long multiplication; and the latter is necessary (together with several other processes) for simple interest. Knowledge of all the elementary algebraic processes is necessary before solving equations can be learnt, and the former depend in turn on arithmetical knowledge. These facts are so obvious that they are likely to be taken for granted; so it is usually overlooked that later learning is possible only because earlier learning has taken a particular form. Long division, for example, would be very difficult for a child who had learnt his multiplication tables correctly, but had learnt to write the results only in Roman numerals. Differentiation of  $x^n$  is only possible because the expansion of the binomial  $(x + x)^n$  is known; and, where  $n$  is other than a positive integer, the latter expansion depends on a theoretical structure of considerable complexity. This hierarchical arrangement appears in other subjects to varying degrees, but to a less extent. When learning a language, ignorance of (say) one page of irregular verbs does not preclude learning the next page, nor does it interfere with the translation of the whole of a passage in which those verbs do not appear. Ignorance of the history of the tenth century matters little to a pupil learning that of the fifteenth. In the sciences, interdependence is much greater, but still less than for

mathematics. Taking physics as an example, most of the theory of electricity and magnetism can be acquired without knowing anything of mechanics, or hydrostatics. Mathematics is probably the most interdependent and hierarchical of any structure of knowledge currently taught.

This effect of previous learning on a particular task has been called by psychologists "transfer", and has been the subject of a limited amount of rather scattered research. Probably the best contribution has been that of Katona, whose ingenious series of experiments the reader is strongly recommended to read in the original work (9). One general result of his researches has been to show that there was positive transfer, i.e. that the learning of one task or solution of one problem was a help towards learning another task or solving a further problem, if and only if some basic principle of structure or method was perceived in the former, and could be applied to the latter.

Katona's work has received much less attention than it deserves. Part of the reason for this is to be found in the prevailing climate of psychological theory at the time of its publication, and after. Ever since the introduction by Ebbinghaus, in 1885, of nonsense syllables such as KED, WUL, NAD, etc., as material for learning experiments, these and similar tasks have been used extensively as a means of eliminating the effect of previous learning; which has been regarded as an irrelevant and uncontrolled variable, to be got rid of as far as possible. Transfer effects, where studied, have been regarded as something additional to and superimposed upon a basic learning process, rather than one of its major determinants.

Katona's experiments were conceived, and the results formulated, in terms of Gestalt theory. This school of psychological thought developed as a reaction against associationist approaches to learning, and its chief merit lies in its emphasis that a 'whole' when learnt is more than the sum of its individual parts. The discovery of this 'whole' (or Gestalt, as it is termed) results in an altered perception of all the individual parts, with easier learning and better retention. So great is the effect of this change on perception that the Gestalt school has tended to locate this 'whole' in the perceived material, and regard this aspect of learning (which they rightly stress) as an insight into the underlying structure of the material itself, or, when the task is solving problems, into the basic principle of the problem. The true site of the Gestalt is, of course, in the mind of the subject; and the process of insight is the formation therein of a concept -- class-concept, relational concept, operational concept, or several of these together. The difference between this abstracting process and the perception of something pre-existing in the material is crucial, for the following reason. In the former case, but not in the latter, the abstraction remains as part of the mental equipment of the subject, and is thereby capable of being used with new material and for new tasks and problems. Further, each set of abstractions is available in combination with abstractions derived from

other material encountered at other times, and as a basis for further abstractions. It is this cumulative process especially which Gestalt theory fails adequately to subsume; and which is so central to learning activities which (like mathematics) may extend over periods of the order of ten, twenty, or forty years.

The full importance of the existing body of knowledge seems to be realized by very few contemporary psychologists, and there is virtually no research into its formation and effects. Piaget, in his 'Psychology of Intelligence', is almost alone in his insistence thereon: and even by those who are now using Piaget's ideas extensively, this particular one has not yet been taken up. This may be because the book in which it appears is one of his most difficult, and the implications for learning theory and educational psychology are not made explicit.

Piaget, following Bartlett and Head, calls these structures of existing knowledge schemata. Closely related to this are two other important ideas, those of assimilation and accommodation. By the former, Piaget means the fitting of new experiences into an existing schema; and by the latter, those changes in a schema which enable it to take in (or lead to) new experiences. It is these two processes separately, alternately, or in combination, which preserve the continuity of the schema from its early simple beginning to the final complexity which it may reach. They are therefore fundamental to our understanding of schematic learning, and it is regrettable once again to have to write that I have been able to find no experiments relating to these concepts in the literature of psychological or educational research. If however the foregoing ideas are applied to the field of mathematics in combination with those of the previous section on concept formation, and those of the next section on mathematical generalization, some very interesting results can be derived. These are given as hypotheses, in Section IV.

In a recent experiment of mine, two artificial schemata were devised, and used for a controlled experiment to compare the results of schematic learning with learning in which an appropriate schema had not previously been learnt. Schematic learning was twice as effective for immediate recall, which a month later seven times as much was recalled of the material schematically learnt as of the other. These results make it strikingly clear that the presence or absence of a suitable schema has a very great effect on the success of learning and recall, and indicate that further research is urgently needed into the nature, varieties, and mode of formation of those schemata. The brevity of this section is a measure, not of its importance, but of the lack of relevant research. For mathematics above all subjects a schematic learning theory is necessary. In its absence, the hypotheses of Section IV have therefore to do double duty; to suggest research leading to such a theory, and in the meantime to serve in its place as indications on which teaching methods ought probably to be based.

c) Mathematical exploration and generalization; Particularly characteristic of mathematics is the process of exploration and generalization whereby new class-concepts and operations are derived from existing ones, and existing class-concepts and operations are applied in different fields from those in which they were originally formed. In the natural sciences, there is constant interaction between theory and experiment; and new concepts are developed largely from effects obtained and observed in the sensory world of outside objects. In mathematics, however, this is in general not the case. It is true that its basic concepts arise from sensori-motor experiences, and that some branches of mathematics have their origins in practical problems -- e.g. trigonometry in surveying, calculus in movements of accelerating bodies. But these have been developed independently of their origins, and practical applications, perhaps, found afterwards. De Moivre's theorem was developed out of a complex scheme relating concepts from the calculus, infinite series, and complex numbers as well as trigonometry. Its usefulness in dealing with, for example, problems in alternating current theory results from its great power and generality, and not because it was based on experimental work with alternating currents. The development of new mathematical ideas is almost entirely conceptual, and herein lies another of its special problems.

Let us study the process in a simple case, such as indices. The pupil first encounters these in cases where the index is a positive integer. By examples such as  $x^2 = x.x.x.$ ,  $x^3 = x.x.x.x.x.$ ,  $x^5 = x.x.x.x.x.$ , the generalization is reached whereby  $x^a$  is defined as the product of  $a$  factors each equal to  $x$ . From first principles, products like  $x^2.x^5$  are then evaluated, leading to the general rule  $x^a.x^b = x^{a+b}$ . Similarly,  $x^a \div x^b = x^{a-b}$  and  $(x^a)^b = x^{ab}$  are arrived at. So far, the process fits exactly the description of concept formation described in III(a). In the next stage, however, something further takes place. As derived above, the rules apply only when  $a$  and  $b$  are positive integers such that  $a > b$ ; that is to say, the field of these generalized operations is a limited class of numbers. What happens when  $a = b$  or  $a < b$ ? Again returning to first principles, by simple algebra  $x^3 \div x^7 = \frac{1}{x^4}$ , while by the generalized operation we are considering,  $x^3 \div x^7 = x^{-4}$ . The two operations can be made to agree if the meaning  $\frac{1}{x^4}$  is assigned to  $x^{-4}$ , and in general if  $x^{-a}$  is given the meaning  $\frac{1}{x^a}$ .

In terms of the original definition, quantities such as  $x^{-4}$  have no meaning, and the process whereby meaning has been assigned to these is quite different from that whereby the earlier results were obtained. When a collection of results like  $x^7 \div x^3 = x^4$  led to the discovery that where  $a$  and  $b$  are positive integers,  $a$  being greater than  $b$ ,  $x^a \div x^b = x^{a-b}$ , this rule was derived as a superordinate concept from a number of particular examples. But when a meaning was assigned to terms such as  $x^{-4}$ , the result of a particular division  $x^3 \div x^7$  was determined was determined by a wish to make the results of this and similar

operations consistent with this rule. In the first case, the rule was derived from its exemplars; in the second, new exemplars were defined in such a way as to fit the rule after it was formed.

Having obtained one example of this new kind, it is of course in this case such a short further step to generalize it in the form  $x^{-a} = \frac{1}{x^a}$

that the mathematically minded person may well take both these steps as one. This immediate generalization takes place by the process of superordinate concept formation described earlier; and the new class of exemplars (negative indices) is added to the original class to give an extended field for the operation "dividing two powers of the same base by one another". Generalization thus includes the earlier process, in alternation with the new one. Both are examples of accommodation of the schema, the former spontaneous and often unconscious, the latter deliberate and necessarily conscious. It is on this new process, however, that attention must now be centred.

The process of logical deduction exemplified above is of course already well known, and the laws by which it operates have frequently been formulated in general terms. But these are only yet another example of superordinate concept formation, being a set of generalizations about a class of acts of inference which come into the category "logical" (as distinct from "illogical", "wishful", etc.). What they do not make explicit is the concept of this actor or agent as an entity in itself, capable of these acts of inference but also of other kinds of purely mental acts, such as day-dreaming, speculation, rationalization, etc. We can observe, compare, and arrange our thoughts in ways analogous to those in which we can observe and manipulate objects in the external environment; in both cases, either intelligently or not. For example, we can test whether an idea is consistent or inconsistent with our schemata, and decide whether to accept or reject it, in the same way as we can decide by physical trial whether a piece fits a space in a jig-saw puzzle, or hear whether a violin is in tune. We can consider a series of mental representations of acts, and re-arrange these to give a desired outcome; as when we mentally plan a car trip to make all our calls in the shortest journey. This is to say that we have within us a system which is distinct from the sensori-motor system whereby we perceive and act on physical objects. Though the former has no visible sense organs and no set of voluntary muscles, it is capable of both receptor and effector functions: but the objects of these activities are not outside, physical objects, but inside, mental objects. The system, whereby the mind turns inward on itself, I call the reflective system; and it is one particular mode of activity of this reflective system on a certain variety of concepts which together comprises mathematical exploration. As an example of this has already been given (development of concepts relating to indices); and since it involves perception of relationships, classification, deduction, and similar acts usually classed as intelligent, it is logical to include it in the same class by calling it intelligent reflection.

The study of concept development is now well under way; but the systematic study of reflective intelligence has only recently been begun (see 3, 4). Is it formed in each individual simply as a result of maturation, or has it to be learnt? If the latter, then details of the learning process are of the utmost importance. Some individuals clearly succeed in learning to reflect intelligently by their own efforts, just as some succeed in forming mathematical concepts. But given a better understanding of the process of concept formation, many of those who have hitherto failed at it can be helped to do so; and the same should be true for the development of mathematical and other forms of intelligent reflection.

From their studies of the growth of logical thinking in children, Piaget and Inhelder (8) believe that children are not capable of formal logical thinking before puberty. They are, however, capable of concept formation from an early age. Since both of these activities are necessary for the process of mathematical generalization described above, it would follow (if Piaget and Inhelder are right) that children before this age are not mentally ready to learn mathematics. Yet teachers try to begin algebra and formal geometry with their pupils several years earlier; and Dienes claims (reference 2) to have taught logarithms and indices to American second grade children (aged about 7 years), matrices and vectors to fourth grade (nine year old) children, and other advanced mathematical topics to pre-adolescent children.

Can such claims be justified? For it is important to know whether, following Piaget's view that pre-adolescents are not ready for formal logical processes, mathematics should be postponed till the secondary age group; or whether Dienes is right in considering that children can be successfully introduced much earlier to a much wider range of mathematical concepts than is at present customary.

I think that no contradiction is found if we examine more closely what Dr. Dienes has done, which has been possible to me through his courtesy in making available many of his methods and results in advance of publication. He has given the mathematical concepts embodiment either in physical material or in a game, so that by using the material in various ways suggested to them by the experimenter, or by playing the games, the children are led to the formation of the concepts in the first of the two ways described. Vectors, for example, are introduced first by putting objects on a table -- say cups, boxes, and gloves. The children are told that an upside cup cancels one right way up, an open box cancels a closed one, and an inside-out glove cancels one right-way out. Any collection of such objects on a table he defines as a vector; and by the cancellation rule, 3 right-way up cups plus 1 upside-down cup plus 7 shut boxes plus 4 open boxes plus 2 right-way out gloves plus 3 inside-out gloves is to be considered as the same vector as 2 right-way up cups plus 3 shut boxes plus 1 inside-out glove. Dienes specifically states that the world thus created is a vector space, in which the number of dimensions is equal to the number of the varieties of objects we choose to consider. By learning to add and subtract such collections of objects the children are supposed to be learning to add and subtract vectors; and by putting

objects on a second table whose number and variety are calculated by certain rules from the collections on the first table, they are supposed to be transforming their vector space.

The reader will recognize that these situations and activities are isomorphic with vector algebra; but does this isomorphism justify actually calling them vector algebra? And is the thinking involved mathematical thinking, of the same kind as that described in the preceding section?

So far as the children are concerned, the answer to the latter question seems to me negative. It was the deviser of the games who had to do the mathematics; for this entailed first, formulation of the concepts involved in the vector algebra, and then the invention of an assortment of embodiments or exemplars of these concepts. By using the latter the children are led to form the concepts for themselves: but they do so without reflection or formulation, and so are reaching them by a different and easier path. Whether or not they are learning mathematics therefore depends on how the latter is defined. If it involves simply the formation of mathematical concepts, they are; but on the view here put forward, that mathematical thinking involves also the exercise of reflective intelligence on mathematical concepts, they are doing only the first half of it.

However, even if one does not accept the claim that ten year olds are being taught vector algebra, there may well be value in this sort of preparation for mathematics proper, if thereby a set of concepts can be developed ready for the time when the pupil's reflective intelligence shall be ready to make use of them. But care will need to be taken that suitable embodiments are chosen; and for this the criterion is surely that they should be true exemplars of the concept to be learnt. In the example given, since a vector is a concept developed from the common properties of directed quantities, it may be objected that games with groups of objects on a table are not suitable embodiments of this concept.

Mathematical generalization is thus a two-part process, in which concept formation alternates with reflection on these concepts. Children are capable of the first of these activities at an early age; but the second comes only later -- perhaps not in its complete form until adolescence. Before this time, teachers can probably do much to help children build up a useful set of mathematical concepts; but at this stage, the children are still dependent on the teacher for direction and for the choice and grouping of material. It is not until the children can themselves formulate and reflect on mathematical concepts that they are beginning to become mathematicians.

d). Problem solving and the correction of errors: So far, we have mainly been considering the building up of a schema of mathematical concepts and operations, as distinct from the application of it to a particular task. More attention has been given to the formation of a mental skill than to its use, except inasmuch as use plays a part in its formation. It is now time to consider the functioning of the skill in the many tasks to which it may be put.

Of particular importance, both practical and theoretical, is the solution of problems. By a problem is meant a task which cannot be done by routine application of methods already known, but which requires a new modification or combination of existing methods. Much of the practical value of any skill lies in its adaptability to new situations; and the less capable an individual of tasks which fall outside a narrow range of routine, the less his value in any position. This is particularly true for mathematics, since its special importance to science and daily life is its generality and applicability to a wide range of tasks. The ability to apply knowledge in a different situation is a widely used criterion for understanding, as distinct from having learnt something 'parrot fashion'. Theoretically, problem-solving appears as a case in which a schema has to accommodate before it is capable of the task; a rapid accommodation to a particular task, as distinct from the slow and permanent enlargement of the schema which was considered in part (c).

Successful problem solving requires first, that the necessary concepts shall be available. (This will depend on the building-up process described in (a)). Second, that the pupil is able to choose and modify them as required. Particularly necessary here is the ability to modify the method in the light of information gained by earlier attempts. (Readers who are familiar with the cybernetic concept of feed-back will recognize this as a particular case.) If the first attempt is successful, the task was for that pupil an easy problem. The essence of solving harder problems lies in the use of information gained from each unsuccessful attempt to direct the next, -- that is, in the correction of errors. This can be a random process or an intelligent one. In the first case, the only information used is that the attempt was unsuccessful, and the way in which it is used is simply to try a different method from the available repertoire. This may be further away from the right one than before. Intelligent correction of errors uses also the information wherein an attempt was unsuccessful, to direct the next attempt; obviously a more efficient method of attack. This involves, once again, reflective awareness of the method itself. Suppose that a pupil has been getting a particular kind of problem wrong, is told his mistake, amends his method at this point, and thereafter solves these problems correctly. To do this he has had to reflect on his sequence of operations, modify a particular part of it, and replace the former by the modified sequence. This entails both receptor and effector aspects of reflective activity. This reflection will be intelligent if and only if he understands his mistake; that is, if he can see the relationships between the correct and incorrect operations and the desired result. Otherwise he may be able thereafter to do correctly tasks resembling the earlier ones, but the change will only be rote-learnt, and there will be no real accommodation of his conceptual schema.

The need for greater knowledge about the development of reflective intelligence has been stated in the preceding section, and the foregoing discussion emphasises this. For it is in the matter of problem solving that many children find their greatest difficulty. This is no doubt partly due to a failure to learn schematically, with consequent lack of transfer to new tasks. The other cause of failure is to be sought somewhere in the



reflective system, which may either not be functioning at all, or be functioning otherwise than intelligently.

For there is certainly a variety of reflective activities, of which some have already been mentioned; and teachers are all too well aware that pupils often indulge in the wrong ones, as when they day-dream instead of thinking about their work. What causes an individual to reflect in one fashion rather than another on any particular occasion, and why do different individuals develop their reflective abilities in one direction rather than another? In the present context, what determines the development of reflective intelligence, necessary adjunct of mathematical concept development? And indeed, why does one ever reflect at all instead of, or before, simply responding in overt behaviour to an environment stimulus? To these and many other questions on the same topic, we have at present few answers. This is perhaps partly because of the climate of psychological thinking already referred to; but also I think largely because of the nature of the entity itself, which is to perceive and act, not to be perceived and acted upon. That is to say, the mind has (by some means) developed within itself an organization (call it R) capable to a limited extent of observing and acting upon other parts of itself (call these SM). This does not appear to have entailed the development of the ability of SM to perceive and act on R, and the latter has as a result remained largely un-noticed.

The first to point out explicitly the existence of the reflective system was, appropriately, the pioneer of the unconscious; Freud. It was however quite a different mode of reflective activity that he described; a restraining, self-punitive function which he called the super-ego. I have elsewhere (3) discussed at some length the relation between super-ego activity and reflective intelligence. The main conclusion was that these two modes of reflective activity are antagonistic, and that interference with reflective intelligence (and consequent inability at learning tasks and problems dependent on reflective intelligence, particularly mathematics) could be caused by excessive or inappropriate super-ego activity. This is however an over-simplification, and here serves mainly to make the point that the further study of reflective intelligence will necessitate also the study of other reflective activities which are related to it, and of the development and interaction of these in the course of the individual's development.

e) Symbolism: Some of the uses of concepts do not require the ability to become conscious of them. They can be formed by the process described in III(a), and to some extent used, while they remain unconscious. Their most effective use, however, demands that they become objects of awareness in themselves, on which the reflective processes described in III (c) and (d) can operate. In this often difficult process, symbolisation plays an important part which is not yet fully understood.

The chief functions of a symbol appear to be as a label for the concept with which it is associated, and as a handle by which to manipulate it. As concepts become of higher and higher orders, it becomes impossible to keep in mind their many attributes while at the same time operating on and with them. Verbal symbols, and mathematical figures and characters, are much easier to think of by virtue of their unitary nature; and by

temporarily letting go of their meanings (i.e. of the concepts for which they stand), they can be manipulated much more easily than the concepts themselves. Afterwards, the result of the manipulation -- solution of the equation, result of the integration, or whatever it may be -- is interpreted by passing back again from the symbols to their meanings.

From this it follows that a pupil needs to form a concept before learning the symbol for it. Otherwise, the symbol will be empty of meaning; and though he may learn to manipulate it correctly, he will do so by rote and not by assimilating the concept for which it stands to an intelligent schema.

However, when a concept is formed, but not formulated, symbolisation does seem to help in the latter process. Especially is this so if the symbolisation is of a particularly apt nature for evoking the concept as well as just labelling it. Thus  $x^a$  as a label for the concept of the product of a factors each equal to  $x$  represents more than just an agreement among mathematicians. By including  $x$  and  $a$ , the association is preserved with the concept in a way which facilitates transitions both ways. It is by these transitions to and from first principles that  $x^a \cdot x^b = x^{a+b}$  is arrived at. While preserving the association closely enough to be useful, however, the symbolism is also loose enough to make possible the detachability from its origins necessary for generalization of a concept. Extension to fractional, zero and negative indices would not be possible while the concept was written in some such form as " $x \cdot x \cdot x \dots$  (to  $a$  factors)". But again, it is the presence of the separate identity of base and index, the index being denoted by a symbol in customary use for numbers of all kinds, which greatly facilitates the transition of thought to the possibility of using all kinds of numbers for indices. The relation between concepts and their symbols is thus one of great importance, and this is true not only for those universally current among mathematicians (such as indices, differentials, matrices) but for those which one invents newly for a particular problem, using small, capital, and Greek letters, prefixes and suffixes, to symbolise the likenesses and differences of the concepts involved. "Let  $m_1$  and  $m_2$  be the masses of the particles,  $u_1$  and  $u_2$  their initial velocities and  $v_1$  and  $v_2$  their final velocities....") As my old mathematics tutor used to say, "A good notation is half the battle".

An even more interesting function of symbols is that of facilitating the reflective process. When symbols are written, they become visible; which is to say, accessible to the external senses, and able to be manipulated by pen or pencil. Further, when mathematical work has been written down, all the individual stages remain visible, and the whole agreement can be re-examined with the greatest ease. Thus by the use of written symbolism, a much easier sensori-motor process partly replaces the reflective process. Symbolism thus relieves the cognitive strain in two stages: by allowing the attention to let go temporarily of the concepts themselves, and work with the symbols for them instead; and then by allowing this symbolic activity to be done

partly by a sensori-motor activity instead of a reflective one. In both cases it is essential that the process shall be at any time reversible, from writing to reflection, and from symbol to concept, in order that the written symbolic work shall always correspond to the conceptual schema. What the written symbols do is to allow the attention to be concentrated on one part at a time of this complex activity, while preserving and keeping available the rest of it.

Finally, there is the part played by symbols in communication. Provided that a set of concepts exists in the mind of both speaker and hearer, or writer and reader, and that these are associated with the same symbols, then the symbols can be used to communicate information in terms of the concepts. If, however, some of the concepts do not exist in the mind of the hearer or reader, to that extent there will be no communication. Applied to teaching, this means once again that the formation of the concepts has to come first. If these are not in the mind of the pupil, the teacher's words and writing on the blackboard will have no meaning for him.

Yet in the building up of the concepts themselves, symbols have an important function. For the concepts of mathematics are themselves based on other concepts, and it is only by the use of symbols that the latter can be grouped together in the right ways for the formation of the superordinate concepts.

The symbolism of mathematics appears from the foregoing to play an indispensable part in the mental processes involved; yet once again, I have been unable to trace any research on its effect. By way of starting point, three hypotheses are offered in section IV.

f) Motivation: What are in general the motivations for learning school lessons is too complex and difficult a problem to attempt here. I have selected a single aspect which is of special interest for mathematical learning: a possible difference between the motivations for rote and for schematic learning.

Though reinforcement learning theories differ in detail, a central feature of them all is that learning takes place because certain acts are followed by certain favourable results, described variously as need satisfaction, drive reduction, anxiety reduction, etc.. That is, the acts are learnt not for their own sake but because they lead to results which are of value to the organism. This accords well with the biological aspects of learning, as a means of acquiring behaviour helpful to the survival of the organism in a given environment. Many experiments support these theories, and I have no doubt of their validity in the field of rote learning.

A by-product of my research into schematic learning, however, has been to form a strong impression that the children found it pleasurable in itself, quite apart from any end which it might serve. Indeed, for then the experiments served no purpose at all. Colleagues and students to whom I showed the artificial schemata often began playing with them for their own amusement. Another example -- a far from clever child to whom I gave an

intelligence test at his boarding school was disappointed at my next visit that there were no more of these interesting puzzles.

These experiences are confirmed by some of the writings of mathematicians themselves (e.g. 13, 14), and have led me to the hypothesis that schematic learning may be self-reinforcing, and independent of any particular need which it may be instrumental in satisfying.

This does not contradict the biological view of the adaptive value of learning, since knowledge acquired for its own sake can often be put to uses which could never have been thought of in advance, and which therefore could never have served as goals. General scientific curiosity has great adaptive value; and for the techniques of mathematics are found practical applications unsuspected by those who originated them (e.g. differential equations and electromagnetic theory). What it does mean is that the elaborate system of external pressures (marks, examinations and tests, form orders, reports) which are considered necessary at many schools to make children work may be doing more harm than good, if they interfere with the pursuit of mathematical skill as a pleasurable end in itself.

I know of no real research into this, in the form of controlled experiments. Dienes' verbatim reports of experimental lessons with children, however, also support the possibility outlined above; so I have considered it worth adding to the set of hypotheses in part IV.

#### S U M M A R Y

This summary is presented chiefly in the form of a set of hypotheses, a method which has several advantages for the present purpose. By formulating concisely the principles which were in the previous sections extracted, synthesized and extrapolated from contemporary thinking and research, these principles can be offered as a basis for devising methods of teaching mathematics which are likely to be sounder psychologically than most of those in current use. By labelling them as hypotheses, their provisional nature is acknowledged, and the need emphasised for an extensive research programme to test and develop them into theory. But the value of an experimental investigation depends largely on that of the hypotheses which it tests; and it is my hope that by offering this systematic set of hypotheses, mostly of great generality, some of the future individual researches in this field may be led to take forms such that the results can be integrated into a comprehensive theory of mathematical learning.

a) Concept formation

It is desirable first to formulate the idea of a simple concept. This may be described as a mental abstraction of common properties of a group of experiences or phenomena -- of objects (class-concepts), actions (operational concepts), comparisons (relational concepts), etc. Implicit in Piaget's work, and explicit in that of Dienes, is the suggestion that concepts of the simplest orders are the result of repeated sensory and/or motor experiences. I have extended this to apply to the higher order concepts of mathematics.

Hypothesis 1. A simple concept is a cumulative memory trace embodying only those elements which are common to a certain group of and/or motor experiences. For these, repetition has a cumulative effect; individual or accidental properties of the separate elements do not accumulate, and so do not form part of the concept. Memory traces of the individual experiences can however co-exist with that of the concept.

Hypothesis 2. Concepts can also be formed from groups of other concepts, in a way corresponding to that described in hypothesis 1.

The former will be called superordinate or higher order concepts, to denote their relationship with the (subordinate) concepts from which they are formed.

Concepts can also be formed by combining existing concepts: and perhaps in other ways too, -- these hypotheses are not exhaustive.

Hypothesis 3. Concepts in a new field, or of a higher order than those already in possession in that field, cannot be communicated directly by definition; but only indirectly, by arranging for the recipient to form them in his own mind.

Hypothesis 4. The way to help a person to form a particular concept is to arrange for him to have a group of suitable experiences such that the attributes of the concept are present in all, but with variability in the properties which do not form part of the concept.

The latter component of the experience has been termed "noise". (What is to be regarded as noise will of course depend on the concept which is to be formed.) There is much scope for experiment to find out the best noise levels. Too much noise, and the concept is not formed; too little, and some of the noise may become part of the concept. Assuming that intelligence constitutes, or at least includes, the ability to form concepts, then it follows: -

Hypothesis 5. The more intelligent the person, the greater the proportion of noise which can be tolerated and is useful.

This gives a criterion for fitting teaching methods to children. The less intelligent the child, the less noise. Further, as the concept becomes

gradually more securely formed, it becomes stronger relative to a given amount of noise. From this comes: -

Hypothesis 6. Early lessons on a particular concept should contain less noise than the later ones.

Hypothesis 7. Once formed, a concept is evoked by experiences which have the relevant attributes, and forms part of the experience. In the case of external objects, it is perceived as part of the object.

The foregoing hypotheses are entirely general, though particularly relevant to mathematics because of its specially conceptual nature. Particularizing now for arithmetic and mathematics;

Hypothesis 8. The development of number concepts and arithmetical operations takes place mainly by abstraction from repeated memory and motor experiences with physical objects.

This predicts that verbal and blackboard teaching, and written exercises, are not by themselves a sufficient basis for learning; while allowing, by the word 'mainly', room for them in their functions of directing and clarifying pupils' experiences.

Indications for choosing and grouping suitable experiences are contained in hypotheses 1 to 6.

Hypothesis 9. Mathematical concepts are not directly derived from sensory or motor experiences, but all are superordinate concepts.

There follow two related hypotheses about how these superordinate concepts are formed. These partly anticipate later ones about mathematical generalization.

Hypothesis 10. The superordinate concepts of mathematics are formed by two processes, which may occur separately or in combination;  
(a) by a continuation of the abstracting process already described, in which new concepts are now formed from the common properties of a group of concepts;  
(b) by processes of logical inference from existing concepts, in which the latter are made objects of consideration in themselves, having become independent of the instances in which they first were formed.

From Hypothesis 10(a), the teaching of mathematics requires that the pupil be given suitably grouped experiences which embody instances of the concept which he is to form. Hypothesis 10(b) implies intelligent reflection, and this forms the subject of a later group of hypotheses.

(b) Schematic learning

Hypothesis 11. There are (at least) two different kinds of learning: rote learning, and schematic learning.

Rote learning has already been much studied, and there is a considerable literature about it (see 11). It is therefore not a hypothesis to describe it as the learning of a set of habits as a result of certain acts being repeatedly followed by satisfaction of primary needs. Those who are familiar with Hull's 'habit-family hierarchy' will recognize that the next hypothesis is but a slight extrapolation.

Hypothesis 12. Habits which are the result of rote learning remain isolated, have no tendency to become an integrated system, and show little or no transfer to new situations.

In contrast to the foregoing:

Hypothesis 13. Schematic learning is the formation of an integrated system of knowledge as a result of certain experiences (sensory, motor, and also conceptual) being grouped together in the mind by the formation of concepts.

Hypothesis 14. These concepts may be evoked by any new situation which contains exemplars of the existing set of concepts. This makes possible maximum transfer of the relevant parts of the existing schema.

Hypothesis 15. The learning process appropriate to arithmetic and mathematics is schematic learning, not rote learning.

Though the repetition typical of rote learning also has its place in schematic learning, its function is quite different: namely, the building-up of concepts. Repetition should therefore embody sufficient variability to ensure this (see 1). Too little variability results in a tendency to rote-learning, with failure of transfer (as Katona's experiment shows: reference 9, page 150).

Closely linked by Piaget to his concept of the schema are two other concepts, assimilation and accommodation. By assimilation he means the fitting of new sensory or motor activities into an existing schema without modification of the latter; and by accommodation he means any modification of a schema which may be necessary before new activities can be fitted into it. It is clear that only if these are achieved can schematic learning be continued. If accommodation is unsuccessful, schematic learning breaks down and the pupil cannot understand the new task. Further understanding of how accommodation can be aided is therefore of great importance for teaching. Two hypotheses are offered bearing on these two related processes.

Hypothesis 16. When new material contains only exemplars of existing concepts, and new tasks presented require only existing operations, then the new situation can be assimilated to existing schemata.

("Existing" implies, of course, "in the mind of the subject.")

Hypothesis 19. When new material cannot be fitted into an existing schema in the way just described, accommodation of the schema may sometimes be achieved by the formation of new concepts which (a) are closely related to the concepts of the existing schema (b) include as exemplars the new material.

An example of this process was the accommodation of the schema of indices, by the formation of the new concepts of as negative indices, to enable it to assimilate new tasks such as  $x^4 \div x^7$  (of which it was previously incapable). But this has already been used in section III as an example of mathematical generalization; which may therefore be regarded as one kind of accommodation.

Further hypotheses about mathematical generalization form the subject of the next section.

### c) Mathematical generalization

In contrast to the natural sciences, the development of new mathematical concepts is mainly from existing ones, rather than from new data derived from exploration of the physical environment.

Hypothesis 20. Mathematical generalization takes place partly by the formation of superordinate concepts from existing ones.

Hypothesis 21. New concepts are also formed in the course of deliberately accommodating schemata, either because they have been found incapable in their present form of assimilating new tasks and data, or to resolve a contradiction.

In the course of this, the schema itself has to be made the object of study.

Hypothesis 22. The kind of generalization described in hypothesis 21 is only possible in its fully developed form when the person doing it can become aware of the concepts as objects of attention in themselves, independent of any particular embodiment.

This implies that the person can be conscious of the concepts when there is no presentation to the external senses, and leads to



Hypothesis 23. Also required is a mental organization capable of becoming aware of purely mental objects, of acting on them in various ways, and of observing the results of these mental operations.

This mental organization has been named the reflective system; and by analogy with the sensori-motor system one can distinguish in it receptor and effector functions, though it has no visible sense organs and no muscles. Continuing the analogy, reflective activity, like sensori-motor activity, can be of many varieties.

Hypothesis 24. The reflective activity necessary for mathematical generalization is reflective intelligence.

We would therefore like to know more about the development of reflective intelligence. According to Piaget and Inhelder:

Hypothesis 25. Reflective intelligence is not fully developed until after puberty.

If this development is not solely a process of maturation, that is, if environmental influences exert a contributory effect, then some teaching methods should be more favourable to its development than others.

Hypothesis 26. The development of reflective intelligence will be favoured by requiring pupils to explain and describe their methods, and by other teaching method which directs pupil's awareness towards the concepts they use.

d) Problem solving and correction of errors

Problems were defined as tasks requiring a new modification or combination of existing methods. This rapid accommodation may occur spontaneously where the gap between the available methods and the particular ones necessary for the problem is not too great, i.e. for easy problems. In general, however, accommodation will be more efficient if the method can be deliberately modified in the light of information gained from unsuccessful attempts.

Hypothesis 27. Successful problem solving requires the ability to reflect on the relevant mathematical schemata.

This requirement applies also to the simpler case, where pupils correct their errors in more routine tasks as a result of correction by the teacher, if the change is to be based on an understanding of their mistakes and not on rote learning.

Hypothesis 28. Failure at mathematics by children of good general intelligence may be due to two possible causes, separately or in combination;

- (a) Failure to learn schematically
- (b) Inability to reflect intelligently on the mathematical schema.

And to account for the second of these effects in children of an age when their reflective intelligence should be fully operative:

Hypothesis 29. Normal functioning of reflective intelligence may be blocked by other reflective activities which are antagonistic to it.

e) Symbolism

Hypothesis 30. The process of formulating a concept, and also its manipulation, are both helped by associating the concept with an appropriate symbol.

Hypothesis 31. An appropriate symbol is one which can act as a reminder of the concept without being too closely tied to specific exemplars. It can also imply a relationship with other concepts.

Hypothesis 32. In the building-up of the superordinate concepts of mathematics by the processes described in hypotheses 2, 3, 4, 10, the way in which the contributory concepts are evoked and grouped is by means of the symbols for them.

Symbolism thus plays an essential part in the building-up of new concepts; but it can only be effective if the learner has the concepts for which the symbols stand. If he has not, the result can at best be only rote learning.

f) Motivation

Hypothesis 33. Schematic learning is experienced by the learner as pleasurable in itself, quite apart from any need which it may serve.

On this hypothesis, the learning of mathematics can be enjoyed for its own sake.

Appendix: Factor analysis

There have been a certain number of investigations into the psychology of mathematics based on the method of factor analysis, of which no mention has yet been made. There are two reasons for this omission. The first is that I believe the method to be invalid (though highly ingenious!) on both psychological and mathematical grounds which would take too much space to explain here. The second is that even if the results obtained thereby (that certain factors, such as general intelligence, verbal, spatial, numerical, collectively and in certain proportions account for

mathematical ability) are valid, this information leaves us little or none the wiser as to how to teach mathematics. On this view, it seemed wrong to use space and occupy the reader's time with material which did not contribute to our understanding. It would however be equally improper to omit also the information that the factor analytic approach is still in favour with many psychologists, or to fail to give the reader the opportunity to form his own opinion of the matter. To this end I have given section (V) of the bibliography to some representative factor analytic investigations of mathematical ability.

BIBLIOGRAPHY

Sections (i) and (ii) contain only works which deal directly with the psychology of mathematics as here presented, or which bear directly on it. A short list of selected works was considered more useful for the present purpose than a comprehensive unselected list from which the reader would not know where to choose. An additional reason for brevity is the fact that most of the existing research has as its basis either reinforcement learning theory or factor analysis. Some of my reasons for rejecting these approaches have already been given; but to allow the reader to form his own judgement if he wishes, I have listed in section (iv) six references to papers of the review type, which give very full bibliographies, and in section (v) a selection of works on factor analysis.

(i) Works on the psychology of mathematics.

1. Dienes, Z.P. Building up Mathematics

London, Hutchinson Educational, 1960.

Essential reading for the teacher who wishes to know how the principles of concept formation described earlier can be applied to the learning and teaching of mathematics. It only deals with the elementary stages, but a teacher who has grasped the principles (which are made very clear) may be able to devise more advanced applications.

2. An Experimental Study of Mathematics Learning, part I.

To be published shortly by the Harvard University Press, in collaboration with Bruner, J.S. (who contributes part II).

Further development of his ideas at the theoretical level, alternating with verbatim accounts of their experimental application with children at the Harvard Centre for Cognitive Studies. Again, essential reading as soon as it becomes available; not necessarily to accept every word, but to help build up one's concepts about the learning of mathematics.

3. Skemp, R.R. An investigation into difficulties in learning mathematics by children of good general intelligence.

Unpublished doctoral thesis (1958), Manchester University Library.

A full statement of the development of the ideas given in sections III and IV about reflective intelligence and its relation to the learning of mathematics, with particular consideration of interfering factors. English readers can obtain it by inter-library loan.

4. Reflective Intelligence and Mathematics.

British Journal of Educational Psychology, Vol.XXXI (1961) pp. 45 - 55.

A summary of part of (3), with an account of one of the experiments.

(ii) Works which have a close bearing on the psychology of mathematics.

5. Piaget, J. The Child's Conception of Number.<sup>\*</sup>

London, Routledge and Kegan Paul, 1952.

This is about the earliest foundations of arithmetic, and not the kind of mathematics with which we are concerned. But it is valuable for the study of the growth of concepts, and some of the experiments related are already regarded as classical.

6. The Psychology of Intelligence.<sup>\*</sup>

London, Routledge and Kegan Paul, 1950.

Difficult, but important because of the new ideas which it pioneers. Not advised for those new to psychology.

7. How children form mathematical concepts. Scientific American, November 1953, p.74.

A useful concise introduction to Piaget's ideas.

8. Inhelder, Barbel and Piaget, J. The Growth of Logical Thinking <sup>\*</sup>

London, Routledge & Kegan Paul, 1958.

This describes experiments by which are traced the stages of development of logical thinking, the latter being formulated in the terminology of symbolic logic.

9. Katona, G. Organizing and Memorizing.

New York, Columbia University Press, 1940.

This book, which is about the difference between meaningful and non-meaningful learning, has received less attention than it deserves. The numerous experiments become even more significant when considered in relation to the principles of concept formation and schematic learning. Recommended.

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<sup>\*</sup> References are to the English translations

10. Wertheimer, M. Productive Thinking.

London, Tavistock Publications, 1961.

This is an enlarged edition of a work first published in 1943. Much of it is about methods of solving mathematical problems, and it is interesting to read as an early step forward from associationist psychology.

11. Hilgard, E.R. Theories of Learning.

London, Methuen, 1958.

A book on learning theory for psychologists and psychology students which is in America the standard work on this subject.

(iii) Some works about the nature of mathematics.

12. Russell, B. Introduction to Mathematical Philosophy.

London, Allen and Unwin; New York, MacMillan, 1919.

In the present context, this famous work may be read as a beautiful example of the systematization of high-order concepts, and also as a demonstration of how unhelpful this approach would be for the purpose of trying to communicate the concepts of mathematics to someone who did not possess them already.

13. Hardy, G.H. A Mathematician's Apology.

Cambridge, University Press, 1940.

Another famous mathematician emphasizes the autonomous nature of mathematical studies.

14. Sawyer, W.W. Prelude to Mathematics.

London. Penguin Books, 1955.

This is an excellent book for illustrating the almost playful manipulation and exploration of concepts as such which mathematicians enjoy for its own sake. I suggest that it be read partly as a study of the motivations of mathematicians.

(iv) Recent review papers.

15. Du Bois, P.G. and Feierabend, Rosalind L. (Ed.) Research Problems in Mathematics Education.

Washington, U.S. Dept. of Health, Education and Welfare (1960).

This begins with a review of completed research. The other two sections discuss problem areas, and proposals for further research. Since the first section alone is longer than the present monograph and has a 285 item bibliography, it is hardly possible to summarize this (itself a summary) herein; especially because also the approach is not easily assimilable to the schema which I have developed. It is therefore recommended as complementary reading. Being a U.S. Government publication, it is inexpensive and easily obtained even outside the U.S.A. Nine authors contribute.

16. Mialaret, G. The Teaching of Mathematics.

Education Abstracts, Vol. XI, No. 9. Paris Unesco, 1959.

More than half of this is an annotated bibliography of books about the teaching of mathematics.

17. Wall, W.D. and Biggs, J.B. Psychological and Educational Researches into the Teaching of Arithmetic and Mathematics. Paris, Organization for European Economic Co-operation, 1959.

Part I surveys the principal lines of investigation before 1950, and Part II those since 1950. 163 item bibliography.

18. Johnson, D.A. Implications of Research in the Psychology of Learning for Science and Mathematics Teaching. Review of Educational Research, Vol. XXVII (1957 No. 4: chapter VIII).

This author too laments "the paucity of research on learning". 76 item bibliography.

19. Hotyat, F. Teaching Mathematics. Paper given at meeting of European Centres of Educational Research, June 11 - 13, 1959.

Obtainable in duplicated form from National Foundation for Educational Research, London.

This author considers that "The research work at the secondary level year is much less in quantity and much less advanced in scope than the research for the elementary school". He formulates some of the chief issues in a useful way, and gives a 64 item bibliography.

20. Curr, W. and Peel, W. Summary of Research in Arithmetic and Mathematics.

Paper read at Eltham Conference, duplicated by National Foundation for Educational Research, London, 1959. 69 item bibliography.

(v) Factor analysis.

21. Thurstone, L.L. The Vectors of Mind.

Chicago, University of Chicago, 1955.

By no means all the books on factor analysis give enough of the mathematical theory for the reader to follow and evaluate the argument for himself. This does.

22. Wrigley, J. The factorial nature of ability in elementary mathematics. British Journal of Educational Psychology, vol. XXVIII (1958), p.61.

This contains a useful survey of earlier research in this field which used the method of factor analysis.

23. Lee, Doris M. A study of specific ability and attainment in mathematics. British Journal of Educational Psychology, vol. XXV (1955), p.178.

24. Barakat, M.K. Factors underlying the mathematical abilities of grammar school pupils. British Journal of Educational Psychology vol. XXI (1951), p.239.

25. Weber, H. Untersuchungen über die Faktorenstruktur numerischer Aufgaben. (An investigation into the factorial structure of numerical tasks.) Z. exp. angewand. Psychol., vol. 1 (1953), p.336.

26. Murray, J.E. An analysis of geometric ability. J. Educ. Psychol., vol. 40 (1949), p.118.