REFLECTIVE INTELLIGENCE AND MATHEMATICS

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SUMMARY. This paper is an attempt to continue into the field of mathematics the kind of psycho-logical analysis which Piaget has made in that of number. Differences between arithmetic and mathematics are related to the difference between sensori-motor intelligence and reflective intelligence. The latter concept is developed further, and an experiment is described to test the hypothesis that the transition from mechanical arithmetic, via problem arithmetic, to mathematics, involves the transition from sensori-motor to reflective intelligence.

I.—INTRODUCTION.

INCREASING attention is now being given to the teaching of arithmetic in schools, and to the investigation of backwardness or learning difficulties in this subject. Some of this research is at the purely practical level ; e.g., that of finding what basic number facts a child does not know (Schonell and Schonell, 1957), or of comparing the results of teaching subtraction by different methods (Brownell, 1947). Other research is concerned with more general hypotheses. Into this category comes Piaget's *Child's Conception of Number* (1952). This is not easy to read : but the acuteness of his conceptual analysis, coupled with his experimental ingenuity, have led to some of the most important basic discoveries in the field.

For example, Piaget has demonstrated that a child may be able to count a set of objects correctly without having formed the concept of number. The formation of this concept is said to depend not on repeated practice of the verbal skill of counting, but on repeated sensori-motor experience of the invariance of the property ' five-ness ' during a number of changes of configuration, of the reversibility of these changes, and of the correspondence of two sets of objects having this property regardless of their having different configurations. (Five eggs, taken out of five egg-cups and bunched together, are eventually perceived as unchanged in number and capable of being returned whence they came.)

The application of this kind of discovery, both in the widening stream of research to which it has led (e.g., Churchill, 1956) and in schools, has broadened the links between researches into arithmetical achievement, concept formation, and teaching methods. Conversely, the limitations of the applicability of the Schonells' work, both for improving teaching methods and for indicating further research, result directly from their failure to distinguish between the verbal and conceptual meanings of ' learning the basic number facts.' The preliminary analysis did not go deep enough.

As exemplified by the foregoing, the value of experiments depends largely on that of the hypotheses which they test : which depends, in turn, on the thoroughness of the preliminary conceptual analysis of the problem. This paper, which is an attempt to begin in the field of mathematics what Piaget has done in that of number, therefore starts with an analysis of what is involved in the transition from arithmetic to mathematics, in a form which can yield hypotheses testable by experiment. One such experiment which has been done will then be described, and others which need to be done will be indicated.

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II.—Number, Arithmetic and Mathematics.

Since these terms are sometimes used as if they were interchangeable, it is desirable for our present purposes to indicate briefly what are the main differences between them. Each individual cardinal number is the concept of a certain common property of some groups of objects, which remains invariant throughout changes of configuration. So the concept of ' number ' is itself a superordinate concept, based on certain properties which are common to all individual numbers. ' Three ' represents a property held in common by three eggs, in three cups, on three plates, with three spoons ; and ignores other properties such as the colour of the eggs, etc. Similarly, ' number ' represents whatever properties are held in common by the *concepts* three, four, five.

Among the most important of the properties common to the cardinal numbers is their susceptibility to various operations such as seriation, addition, subtraction, multiplication, division ; and their inter-relatedness by these operations. Thus, ' addition ' is another superordinate concept, that which is common to all *operations* such as adding 2 and 3 to make 5, 4 and 6 to make 10, etc. ; and which ignores the individual numbers themselves. Subtraction, multiplication and division may be similarly defined. It is the totality of these concepts and operations which, I suggest, can usefully be taken as the starting point for our concept of arithmetic.

Continuing this development : arithmetic is first, the study of the properties of *numbers* (as opposed to *number*), their inter-relationships, the various operations which can be done on numbers, and the results of these operations. Second, it is the study of new classes of numbers arrived at by developing the original concepts of numbers to arrive at new classes such as negative numbers, fractions, and irrational numbers. Third, it is the study of the extension of these basic operations to the new concepts.

Another thing which has to be learnt in arithmetic is the abstraction of the numerical concepts from the total data of a given situation, application of the appropriate operations, and re-insertion into the situation to infer a result—as in problem-solving. For example, " If John has eighteen marbles and gives George five, how many has he left ? " The arithmetical processes are abstracting the concepts ' eighteen ' and ' five,' making a correspondence between the behaviour ' gives ' and the arithmetical operation of subtraction, doing the actual subtraction, and re-inserting the number 13 thus obtained into its context to get the verbal result—that John has left thirteen marbles. Into the total situation many other factors may enter, such as the personalities of the boys, whether the marbles were given out of friendship or won in play : and if the latter, what are the rules of the game. The arithmetical problem involves the ignoring of all these—which may well be more interesting to other investigators. If this point seems to be laboured, it should be observed that many children who can perform the other arithmetical processes of addition, multiplication, etc., experience difficulty when they come to ' problems ' ; that is to say, at just this very process of abstraction.

A similar analysis can now be applied to the transition from arithmetic to mathematics, of which algebra affords a typical example. Algebra is concerned with the study of the properties of *number* in general, that is, with statements which are true of any and every number, and not only of particular numbers. It is further concerned with the properties of arithmetical operations in general, irrespective of the particular numbers on which the operations are performed. Thus, while arithmetic is concerned with the uses and results of its operations on numbers, algebra turns its attention to the operations themselves and to the properties held in common by all operations of a particular kind.

The parallel may be continued. For the algebraist will continue, in due course, to develop concepts of new classes of numbers (e.g., complex numbers) and new functions (e.g., gamma functions) by generalising the field of application of certain operations (taking square roots, taking the factorial of a number); and will study the application of the existing set of operations to the new concepts. These correspond to the second and third of the headings given for arithmetic.

The fourth parallel is the task of abstracting from the total data the relevant algebraic concepts, suitably arranging them, application of the appropriate operations, and interpretation of the result. An example of the first two of these is what is commonly called by pupils ' getting the equation ' ; and as in problem arithmetic, it is found by most children to be the most difficult of their tasks.

Though algebra has been chosen as example, similar developments can be described for geometry, trigonometry, etc. Number, arithmetic and mathematics may thus be seen to form a hierarchical structure, of which the most important feature, for the present argument, is that of progressive abstraction and generalisation.

III.—SENSORI-MOTOR INTELLIGENCE AND REFLECTIVE INTELLIGENCE.

Before the foregoing consideration of the differences between arithmetic and algebra can be applied to the psychological problem of learning these subjects, it is necessary first to distinguish between sensori-motor intelligence and reflective intelligence. In the course of so doing it will be necessary to take for granted a certain minimum about the concept of intelligence, a full discussion of which is beyond the scope of this article.

The perception of numbers involves the awareness of certain properties of groups of outside objects, as already discussed.

These are not the properties most closely related to the sensory stimuli. It is the latter which govern the responses of the younger children in Piaget's now classical experiments. When a child can transcend these, and give responses which indicate his perception of the number property independently of configuration, we may say that he shows sensory intelligence—that is to say, awareness of certain relationships between sensory stimuli. It is this awareness of *relationships*, as distinct from (and often opposed to) *resemblance* between sensory stimuli, which will be taken as the criterion of intelligence as distinct from stimulus generalisation of a simpler kind.

This awareness can be inferred only from the responses which the subject makes, verbal or otherwise. Do these responses constitute the motor part of sensori-motor intelligence ? This would not afford a true parallel to sensory intelligence, which has been taken as awareness of relationships between receptor stimuli. Tentatively, I suggest that the concept of motor intelligence should involve awareness of relationships between *actions*, such as filling up and emptying out, putting together and taking apart, taking away and putting back. For a child learning arithmetic by manipulating objects, awareness of the relationships between (physical) addition and subtraction, successive addition and multiplication, multiplication and division would then all be described as manifestations of motor intelligence.

Learning ' practical arithmetic ' thus involves both sides of sensori-motor intelligence : sensory intelligence in the forming of concepts of the positive whole numbers, and motor intelligence in discovering the inter-relations of the basic operations just described : in both cases, with physical objects.

What is the distinction between sensori-motor and reflective intelligence ?

The latter term is used by Piaget in his *Psychology of Intelligence* (1950). The most explicit formulation he gives is the following (page 121) : " There are thus three essential conditions for the transition from the sensori-motor level to the reflective level. Firstly, an increase in speed allowing the knowledge of the successive phases of an action to be moulded into one simultaneous whole. Next, an awareness not simply of the desired results of an action, but its actual mechanism, thus enabling the search for the solution to be continued with a consciousness of its nature. Finally, an increase in distances, enabling actions affecting real entities to be extended by symbolic actions affecting symbolic representations and thus going beyond the limits of near space and time."

This contains part, but not all, of what can be shown to be necessary conditions for reflective intelligence.

First, let us consider a simple act of reflective thought, such as may be done by a pupil who has been getting wrong a certain kind of algebraic problem, is told his mistake, amends his method at this point, and thereafter solves similar problems correctly. What he has done is to:

(1) reflect on what he did,

(2) modify a particular part of the sequence of operations, and

(3) replace the former sequence by the modified sequence.

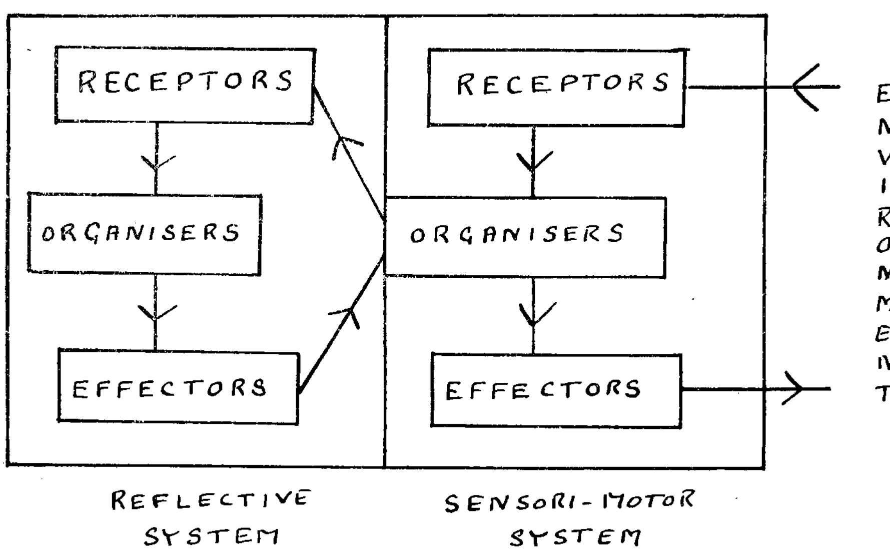
To do this he must be capable :

(1) of knowing what he did. There must be not only mental representations of his operations, but also a *system capable of becoming conscious of these.*

(2) (3) of changing some of these mental representations.

This requires a system capable also of acting upon the operations of sensori-motor intelligence.

Reflective thought may thus be regarded as a second order system, aware of and acting on the mental representations of the sensori-motor system in ways which resemble the receptor and effector activity of that system. The following diagram may be found helpful in thinking about the two systems.



Piaget's description refers to the mental representations, and to the receptor activity of the secondary system, but not explicitly to its effector activity. Moreover, another condition must be added to the foregoing description of reflective thought before it can qualify for the description of reflective intelligence. Sensori-motor activity is not necessarily intelligent. What makes it so, according to the definition here suggested, is the perception of relationships between objects and groups of objects presented to the senses, and between the individual's own various motor activities. Application of these criteria to the reflective system leads to the following provisional definition of reflective intelligence :

Reflective intelligence is the functioning of a second order system which :

(i) can perceive and act on the concepts and operations of the sensori-motor system ;

(ii) can perceive relationships between these concepts and operations, and

(iii) can act on them in ways which take account of these relationships and of other information from memory and from the external environment.

Further, though it is not part of reflective intelligence itself, we also require the ability to change from sensori-motor to reflective activity and back ; and to put into (sensori-motor) action a sequence of operations which has been formed as a result of reflective activity.

IV.—REFLECTIVE INTELLIGENCE AND MATHEMATICS.

The argument of this section will be that the transition from elementary number work to mathematics involves the transition from sensori-motor to reflective intelligence.

The ability to do simple arithmetic requires the ability to perceive numbers and their relationships, and make correctly the appropriate responses ; but not necessarily the ability to become consciously aware either of the relationships which determined the response, or of the method by which the correct answer was obtained. Piaget (1952) has shown that a child can give the correct answer to a simple problem, but be unable afterwards to describe the method by which he arrived at this correct answer. This is a very important experimental result, since it demonstrates the dependence of simple arithmetic on sensori-motor but not on reflective intelligence.

The transition to algebra, however, involves deliberate generalisation of the concepts and operations of arithmetic, as has been discussed in detail in an earlier section. Such a process of generalisation does require awareness of the concepts and operations themselves. Since these are not physical objects, perceivable by the external senses, but are mental, this transition requires the activity of reflective thought. And further, the generalisation requires not only awareness of the concepts and operations but perception of their inter-relations., This involves true reflective intelligence.

A further requirement, as soon as the pupil has progressed beyond simple routine algebraic (etc.) processes, is deliberate modification of method in the light of experience. One example of this has already been given. Another is given by what are commonly called ' problems.' These may be defined as any questions which cannot be answered by routine application of already known methods, but which require a new combination or modification of existing methods. To do this effectively requires conscious awareness of the methods in one's repertoire, and the ability to try various combinations and modifications till the right approach is found. Both the receptor and effector aspects of reflective intelligence are thus involved.

If the foregoing argument is correct, then it is not only for mathematics that reflective intelligence is required, but also for some parts of arithmetic. These include arithmetical problems, and all concepts and operations which depend on generalisations, not of sensori-motor experience, but of other concepts and operations. Negative numbers are concepts in this category ; and examples of such operations are the subtraction of a larger from a smaller mumber, and multiplication and division of and by fractions. Part of the difficulty which many children have in learning these may, therefore, be due to their not yet having developed far enough in reflective intelligence, as distinct from the sensori-motor intelligence which is adequate for some of the other work.

We are now in a position to formulate a series of hypotheses, all of them testable by experiment, whose implications for teaching mathematics are parallel to those of Piaget's work for teaching number.

Hypothesis 1.—For mathematical achievement a necessary, though not sufficient, condition is the presence of reflective intelligence as well as sensori-motor intelligence.

If the former is to be used on mathematical concepts and operations, these must have been developed to a state in which reflective intelligence can act on them ; that is to say, they must be available not only as repeatable responses but *as information.* Hypotheses 2 and 3 relate to this development.

Implicit in Piaget's work is the suggestion that at the arithmetical stage they are the result of repeated sensori-motor experience. Trying to make this more explicit, one might suggest that from memory traces left by the repeated sensori-motor experiences there arises a cumulative trace. This will contain only those properties which are common to all the experiences of a certain group, for which, therefore, repetition has a cumulative effect, the individual or accidental elements being eliminated. This trace may eventually become independent of outside objects and of motor activity. At this final stage it has become a true concept or operation. Earlier, it may be evoked by, and projected on to, sensory or motor activity, the combination of sensation and concept resulting in perception.

Hypothesis 2 (*a*). The development of number concepts and arithmetical operations arises partly as a cumulative trace representing the invariant properties and relationships of groups of repeated sensori-motor experiences.

(*b*) The projection of these traces on to subsequent sensori-motor experiences results in perception.

Transposing hypothesis *2* (*a*) to the context of mathematics and reflective intelligence, we obtain :

Hypothesis 3.—The development of mathematical concepts and operations arises partly as a cumulative trace representing the invariant properties and relationships of groups of repeatedly employed arithmetical concepts and operations.

The word ' partly ' in the last two hypotheses is to leave room for the activity of the teacher. If the hypotheses are correct, his first task is to provide the pupil with the kinds of repeated experience which will best lead to the formation of the concept or operation. But we do not know whether these will then arise spontaneously, or whether there is still a need for communication and demonstration by the teacher when the ground has been prepared. (Without this preparation, verbal and blackboard teaching will lead to no more than rote memorisation. The concepts which are required as the bases of later superordinate concepts are then not formed, and the pupil is incapable of ever really understanding mathematics).

Hypothesis 4. The successful use for solving problems, and for further generalisation, of mathematical concepts and operations will be aided by any teaching method which increases pupils' awareness of the concepts and operations which they use.

This hypothesis predicts that explanation and demonstration by the teacher are still useful additions to the directed activities of the pupils. It also predicts that requiring the pupil to explain and describe what they do, and thereby directing their reflective awareness to their methods, will help their progress.

V.—EXPERIMENT.*

(i) *Introduction.* Hypothesis 2 is already supported by Piaget's experiments, out of which it arose, and is being further investigated as in the researches already quoted. Hypothesis 4 is in accordance with most teachers' experience, but still requires to be made the subject of controlled experiments, as also does hypothesis 3. These are the hypotheses which have the most direct implications for teaching. But the first hypothesis is the crucial one, since two of the other three hypotheses depend on it. It is also the most general in its implications outside the context of mathematics, since it suggests the possibility of tests which discriminate between reflective intelligence and sensori-motor intelligence. This is, therefore, the subject of the experiment here to be described, based on the foregoing analysis and argument.

(ii) *Method.* The experiment consisted of four parts: two preliminary parts, concerned respectively with the formation of concepts and operations; and the two on which the test of the hypothesis chiefly depends, concerned with the (mental) manipulation of these concepts and operations.

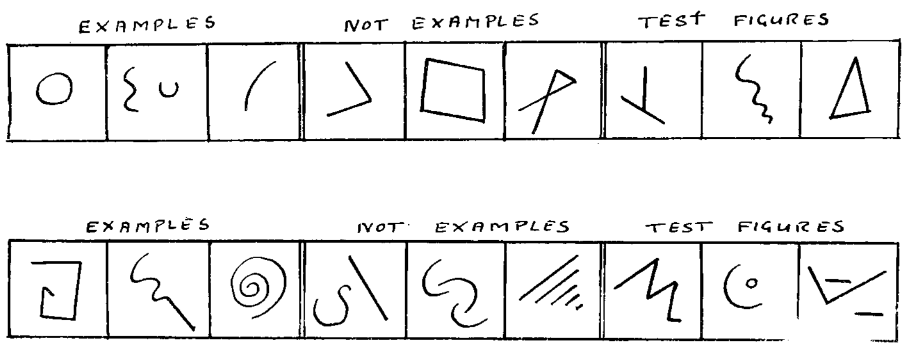


Fig. 1

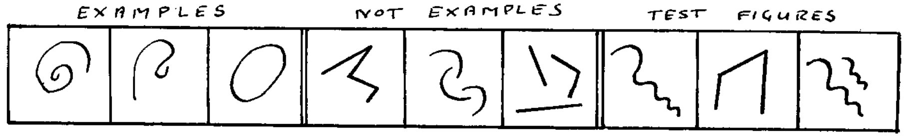


Fig. 2

* The writer wishes here to acknowledge the much valued co-operation given by the headmaster, staff and pupils of Cheadle Hulme School and Stockport Grammar School.

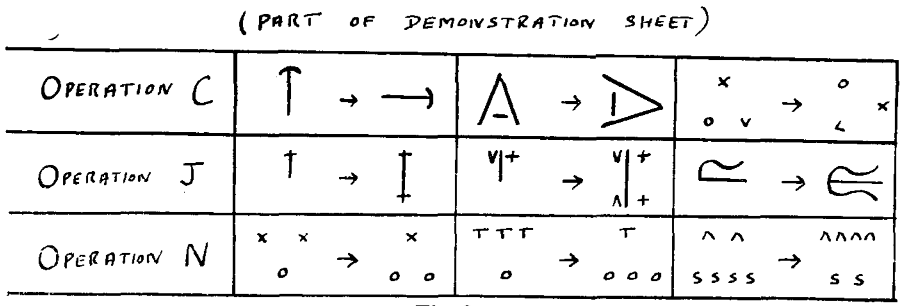


Fig. 3

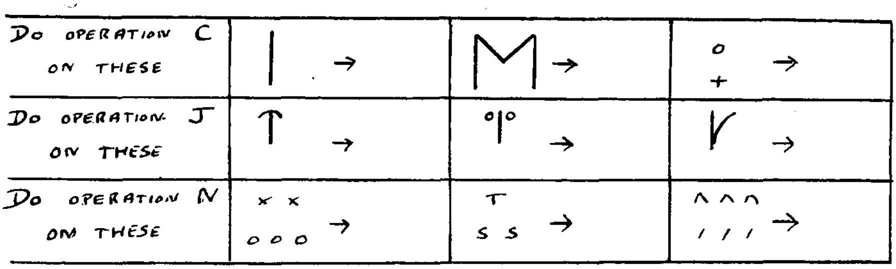


Fig. 4

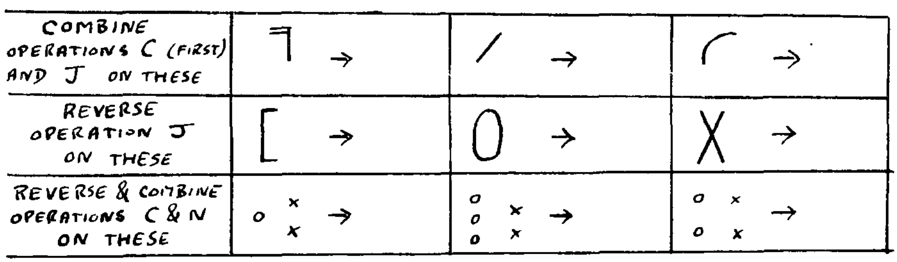


Fig. 5

Specimens from each of the four parts of the test are given in figures 1 to 4.

In the test of concept formation, three exemplars and three non-exemplars were given for each concept ; and the subject then had to indicate, for each of three test examples, whether it was or was not an exemplar. (See figure 1). The criterion of whether he had formed the concept was, therefore, whether he could use it, not whether he could put it into words—an important point in view of Piaget's result quoted above.

The reflective process chosen for concepts was that of logical multiplication, that is, of combining two of the concepts to form a new double concept.* The test material (see figure 2) gave three exemplars of the double concept, and three

* This experiment was done before a copy of Bruner, Goodnow and Austin (1956) had reached the present writer. This book, though it does not explicitly use the term reflective intelligence, might well be described as a detailed study of some aspects of its functioning.

non-exemplars having respectively only one, the other, or neither of the class-properties. The subject had, therefore, to test his idea of what was the double property by taking them separately and in combination—a reflective activity requiring awareness of the concepts, and deliberately combining and separating them. He had then, as before, to indicate which of the three test examples had the double property.

The operations were shown by three examples of each, on a demonstration sheet (see figure 3) ; and the subject then had to do these operations to the given figures (see figure 4).

Finally, the subject had to act reflectively on these operations. (To eliminate failures resulting from not having discovered, in the preliminary part of the experiment, what the operations were, these were all explained to the subjects before the final part). The reflective activities chosen were combining, reversing, and both combining and reversing. These effector activities were dependent on the prior receptor activity of awareness of the operations, which were not sensorily available to the subjects, by their nature. (Examples can show only the effects of an operation, not the operation itself). Examples of the final test are given in figure 5.

The subjects were the fifth and fourth form pupils at a mixed grammar school. The fifth forms had just taken a G.C.E. trial examination set by the school, and the combined marks of the three mathematics papers were used as the criterion of mathematical achievement.

No similar criterion was available at the time for the fourth form pupils, since different papers were given to each of the four mathematics sets into which they were divided. Nearly a year later, however, when these pupils had progressed to the fifth forms and taken a similar mathematics examination, it became possible to calculate the correlations for this group also between the mathematical criterion and the test scores.

(iii) *Results*. These correlations are given below, together with the reliabilities of the test scores from which the correlations were calculated. (The figures for reliability are the correlations of odd/even items, corrected by the Spearman-Brown formula to the full length of the test).

TABLE 1

CORRELATIONS BETWEEN MATHEMATICAL ABILITY AND TEST SCORES.

x = mathematical criterion.
t = reflective activity on concepts.
u = use of operations, not requiring reflective activity.
v = reflective activity on operations.

	V forms (N = 50)	IV Forms (N = 88)
r_{tx}	·58	·56
r_{ux}	·42	·48
r_{vx}	·72	·73

RELIABILITY OF TEST SCORES for V and IV forms together (N = 138) :

Test	Reliability
r_{tt}	·76
r_{uu}	·94
r_{vv}	·95

The agreement between the two sets of correlations is remarkable, particularly in view of the fact that two different mathematical examinations, a year apart, were used as criteria. The figures support the experimental hypothesis, the two highest correlations being those between the two tests of reflective activities and mathematics.

It is interesting to note that reflective manipulation of operations appears to be the more important. This accords with the earlier discussion of the role of this activity in solving problems. Adaptive modification and combination of methods, for each new problem, and in the light of information gained while trying to solve them, requires the ability to reflect on one's operations. Reflection on concepts is necessary for their generalisation and further development, and so in mathematics for the forward progress of the pupil to new ways of thinking, as has also been discussed. But for these school children, any particular mathematical question is unlikely to require the formation of new concepts : whereas it will almost certainly come somewhere into the category of ' problem.'

No correlations are given for non-reflective use of concepts, since most of the subjects got all, or almost all, of these right. This was because an earlier try-out, with different subjects, had been used to eliminate any concepts which proved at all difficult. The aim was to make the test of reflection on concepts independent of the subjects' ability to form them. For the later test with operations, however, the simpler plan was adopted of telling the subjects what the operations were, after the preliminary test involving non-reflective use of operations ; so a fairly well scattered set of scores was obtained. The correlation of this test with the mathematical criterion is also far from negligible, and may be taken as indicating that the formation and simple application of operations is a pre-requisite of the reflective activites under discussion.

VI.—CONCLUDING DISCUSSION.

There are many consequences of the foregoing for teaching and for research, and a number of beginnings to be followed further. If a child does badly at mathematics, this paper indicates that it may be because he has not formed the necessary concepts and operations, or it may be *because he cannot reflect on them*. Research at present in progress suggests that it is this reflective ability which is particularly liable to blockage by certain kinds of emotional disturbance. and offers one possible solution to the problem of why some highly intelligent persons have great difficulty in learning mathematics.

If the understanding of mathematics (as here defined) is dependent on reflective ability, then the ages at which its various stages are taught should be related to the stages by which reflective intelligence develops. Inhelder and Piaget (1958) believe that it is not fully attained before puberty. If this is so, children before that age are not ready for formal mathematics and teaching should be devoted to providing the right preliminary experiences as a foundation for future progress (see Hypothesis 3). Just what kinds of experience are best will depend on further knowledge of those fore-runners of reflective intelligence which the child has before puberty ; for the transition from sensori-motor to reflective intelligence is obviously a gradual process, extending perhaps throughout childhood.

Further understanding of the relationship between sensori-motor intelligence and reflective intelligence is of particular importance while selection at 11 plus remains with us. If reflective intelligence is intended to be the basis of selection for grammar school education, then we may well be trying to select for an ability which is not fully developed at the age of testing. The success of selection will then depend largely on the size of the correlation between whatever can be measured at that age and reflective intelligence when it subsequently develops. But these considerations lead beyond the scope of this paper. Like the concepts and operations of higher mathematics, the ideas have shown a tendency to become independent of the particular circumstances in which they were first developed.

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