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How do we *think* about axioms and proof in mathematics?

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Based on recent publications on the way that mathematicians use thought experiments and formal deductions to build up mathematical theories and how axiomatic theories are conceived by different students. In particular I refer to work from recent PhD theses at Warwick University.

Useful references for the talk are:

David Tall & Shlomo Vinner (1981). Concept Image and Concept Definition: with special reference to limits and continuity, *Educational Studies in Mathematics*, **12** 151–169. (Historical: aan oldie but goodie!)

Liz Bills & David Tall (1998). Operable Definitions in Advanced Mathematics: The case of the Least Upper Bound, *Proceedings of PME 22*, Stellenbosch, South Africa, **2**, 104–111.

Marcia Pinto & David Tall (2001). Following students' development in a traditional university classroom, Proceedings of PME 25, (4), 57-64.

Erh-Tsung Chin & David Tall (2001), Developing Formal Mathematical Concepts over Time. Proceedings of PME 25, (2), 241-248.

Erh-Tsun Chin & David Tall (in draft), University Students Embodiment of Quantifiers. Available from www.davidtall.com/papers

David Tall (in press), Natural and Formal Infinities, to appear in *Educational Studies in Mathematics*.

All of the above are available from my web-site: www.warwick.ac.uk/staff/David.Tall, or alternatively from www.davidtall.com/papers

Recent Studies at Warwick

0. Concept Definition & Concept Image.

(historical, 1981)

1. Operable Definitions in Analysis.

Method: Interviews with selected students every 3 weeks over a first year analysis course.

2. Student Conceptions of Relations, Equivalence Relations, and Partitions (Abe Erh Tsung Chin)

Method: Questionnaire for information on a spectrum of approaches, Clinical Interviews with selected students.

3. Natural and Formal Approaches to Analysis (Marcia Pinto.)

Method: Questionnaire for information on a spectrum of approaches, Select a spectrum of students to follow through 2 terms with Clinical Interviews every 3 weeks.

In all these pieces of work, the *concept image* plays a fundamental role, so ‘proof’ has cognitive aspects that are not purely logical. There is a difference that can be formulated in terms of ‘natural’ thinking, using informal concept images and ‘formal’ thinking involving only formal deduction.

Concept Image

The *concept image* is the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.

The portion of the concept image activated at a particular time is the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion.

The *concept definition* is the form of words used to specify that concept accepted by the mathematical community at large.

A *personal concept definition* (which may vary it from time to time) is the student's personal reconstruction of the definition.

Tall & Vinner 1981

Operable Definitions

A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

(Bills & Tall, 1998)

Empirical evidence shows, that even with good mathematics majors, many do not make the definition operable. New material is presented at great pace and the student can only focus on this week's task without consolidating earlier ideas.

Definition as given:

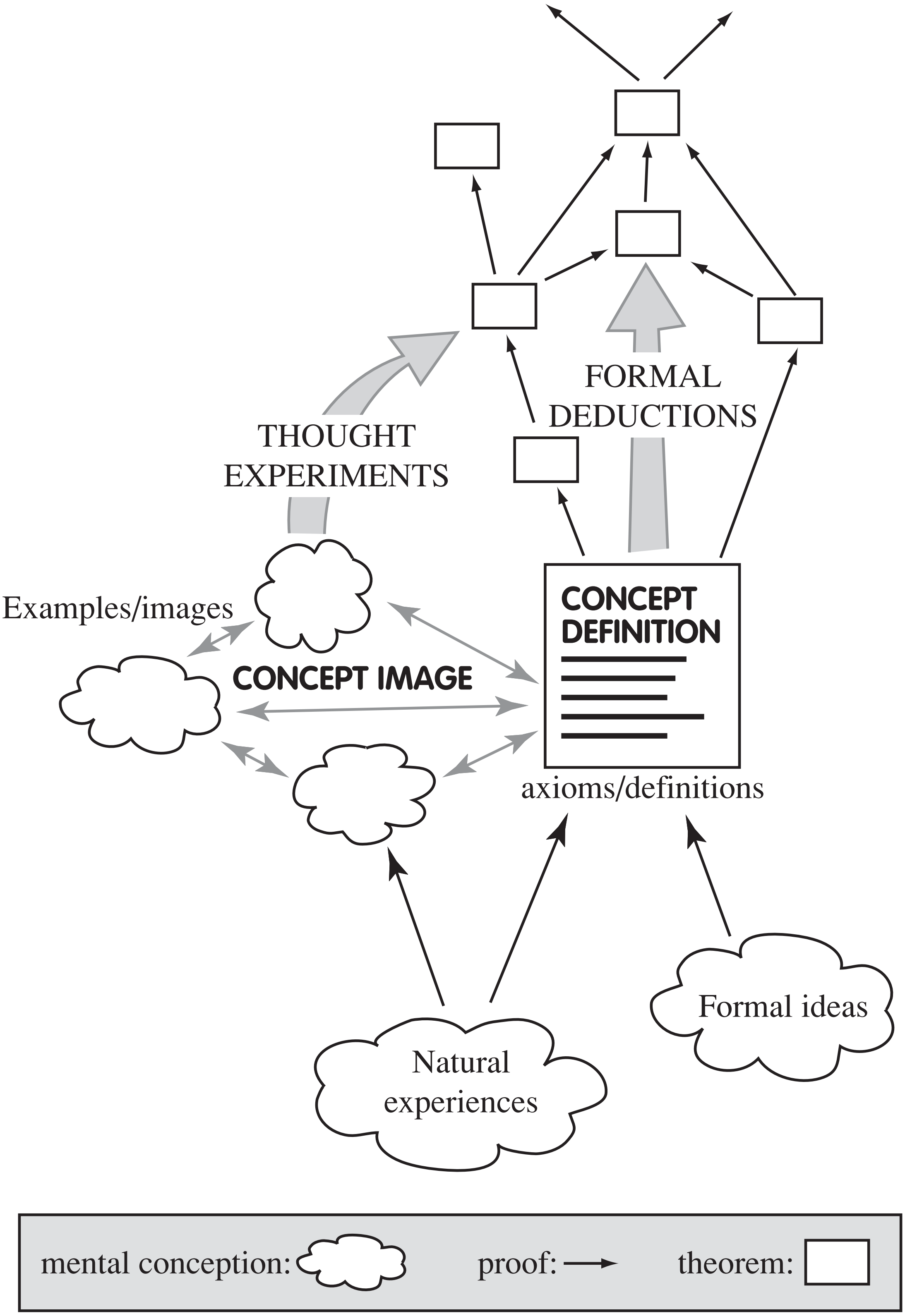
> An *upper bound* for a subset $A \subset \mathbb{R}$ is a number $K \in \mathbb{R}$ such that $a \leq K \ \forall a \in A$.
>
> A number $L \in \mathbb{R}$ is a *least upper bound* if L is an upper bound and each upper bound K satisfies $L \leq K$.

Lucy: Well, say *k* is an upper bound for the set, then we'll say that *m* [the least upper bound] is less than or equal to *k*.

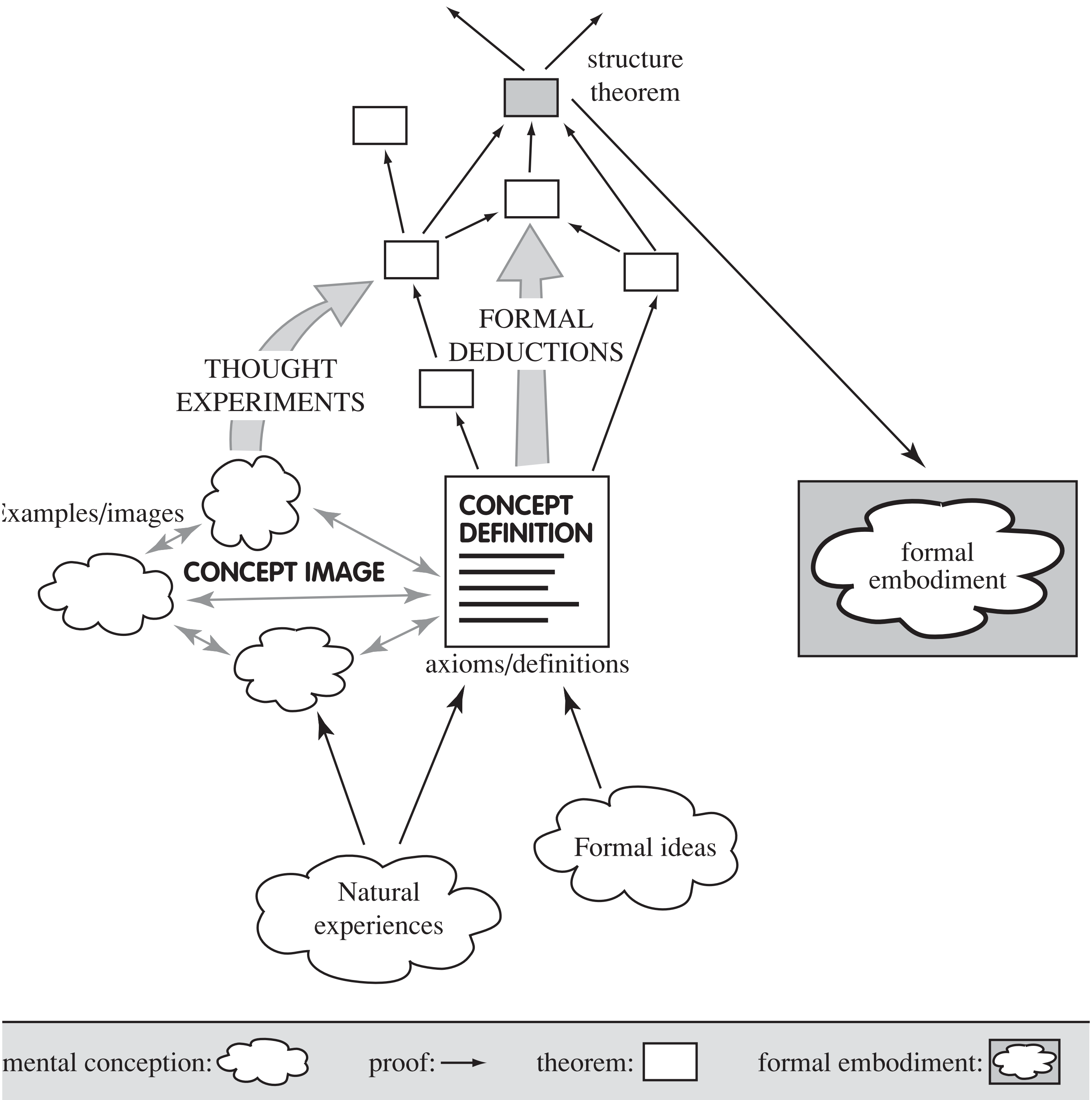
Alex: A least upper bound is the lowest number … that is an upper bound. Any number greater than it, no matter how little amount by, it's not going to be, you know it's not going to be, in the set.

Sean: The supremum of a set is the highest number in the set.

None of these definitions were operable and only Lucy had an operable definition later.

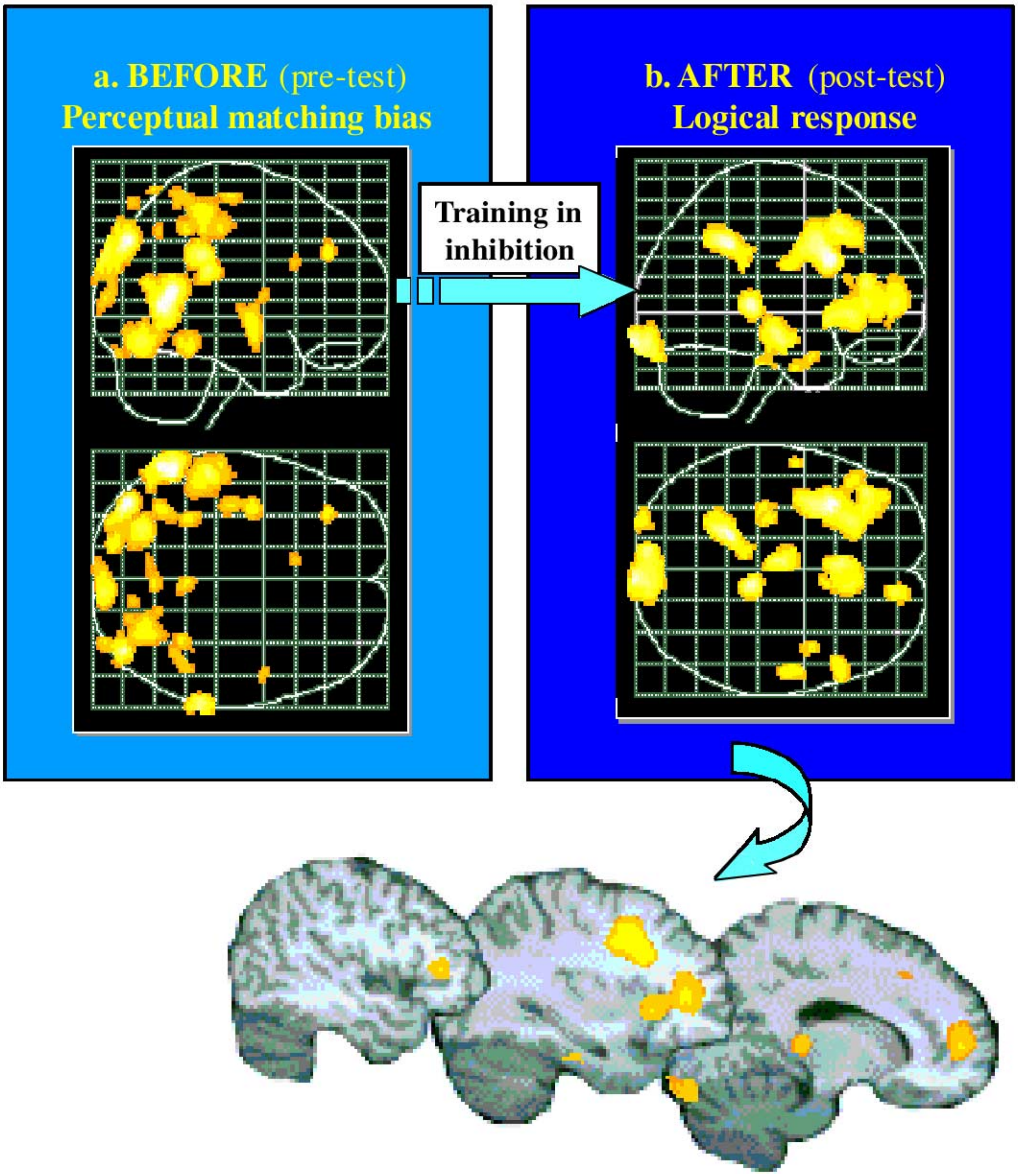


Some constituents in constructing a formal theory



Building new formal embodiments from a formal theory

WHAT HAPPENS IN THE BRAIN?



Houdé et al (2000). Shifting from the perceptual brain to the Logical Brain: The Neural Impact of Cognitive Inhibition Training. *Journal of Cognitive Neuroscience* 12:4 712–728.

Why are 'relations' difficult?

At Warwick, in the Foundations Course, the annual report commented that '*Euclid's algorithm* and *symbolic logic* were well understood, *basic set theory* and *functions* generally required extra work, but the topic on *relations* was often poorly understood.' On an average, only 20% of students declared that they understood relations well with nearly a third of students claiming that, even after extra study, they only understood the topic poorly.

What is the problem?

Theory : A relation is …

An equivalence relation is a relation s.t. …

A partition is …

Equivalence relation ⇔ Partition

Say what “equivalence relation’ means to you.

N=15	First Year	Second Year
Formal (quantified)	5	9
Formal (no quantifiers)	4	5
Outline (‘refl, symm, trans’)	5	1
Total definition	**14**	**15**
Example	0	0
Picture	0	0
Other	1	0
No response	0	0

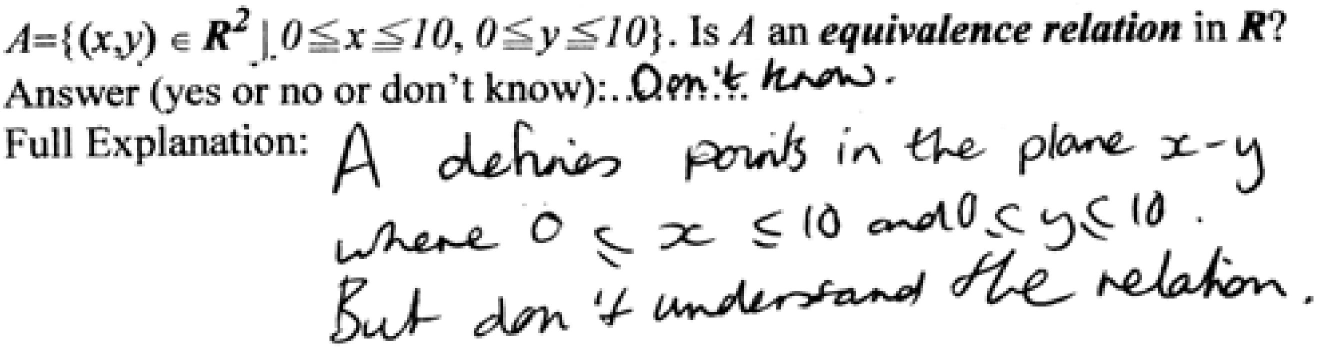
A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$.
Is this relation an equivalence relation?

N= 15		First Year	Second Year
Informal	Informal Definition	3	0
	Other	1	1
	No response	0	0
Formal perhaps with some informal language	Definition	7	2
	Theorem	3	12
	Partition	1	0

$$A = \{(x, y) \in \boldsymbol{R}^2 \mid 0 \le x \le 10,\ 0 \le y \le 10\}.$$

Is A an ***equivalence relation*** on $\boldsymbol{R}$?

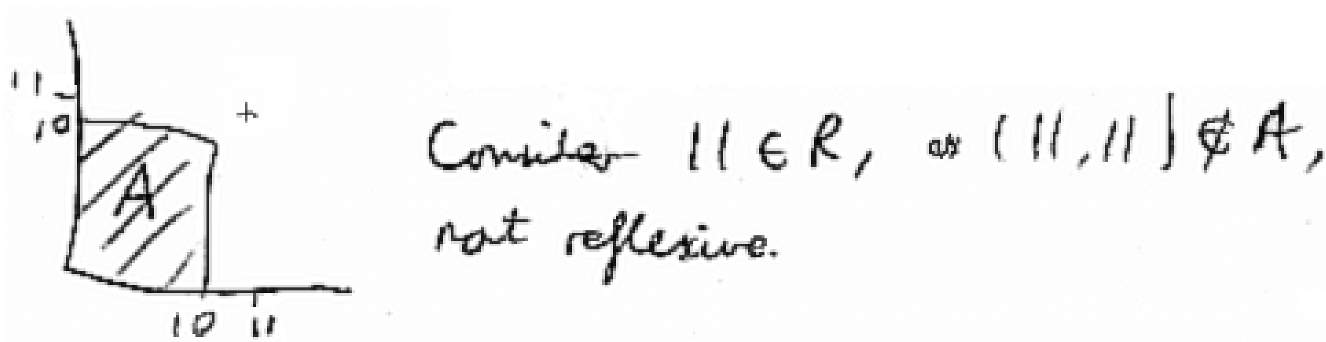
In the first year *no* student (out of the whole class) responded positively to this question.


$A=\{(x,y) \in \boldsymbol{R}^2 \mid 0 \leq x \leq 10,\ 0 \leq y \leq 10\}$. Is A an ***equivalence relation*** in $\boldsymbol{R}$?

Answer (yes or no or don't know):..Don't know.

Full Explanation: A defines points in the plane x-y where $0 \leq x \leq 10$ and $0 \leq y \leq 10$. But don't understand the relation.

In the second year, only Simon responded as follows:



(N=15)	First Year	Second Year
Formal/detailed	2	8
Informal/outline	6	3
Total definition	8	11
Example	0	0
Picture	1	1
Other	4	3
No response	2	0

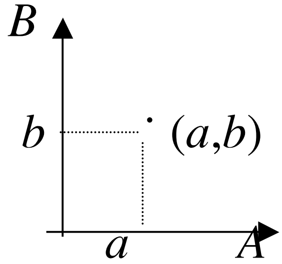
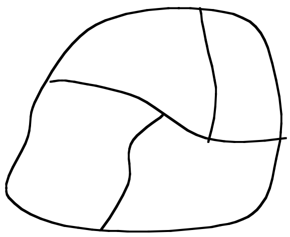
Say what "partition" means to you.

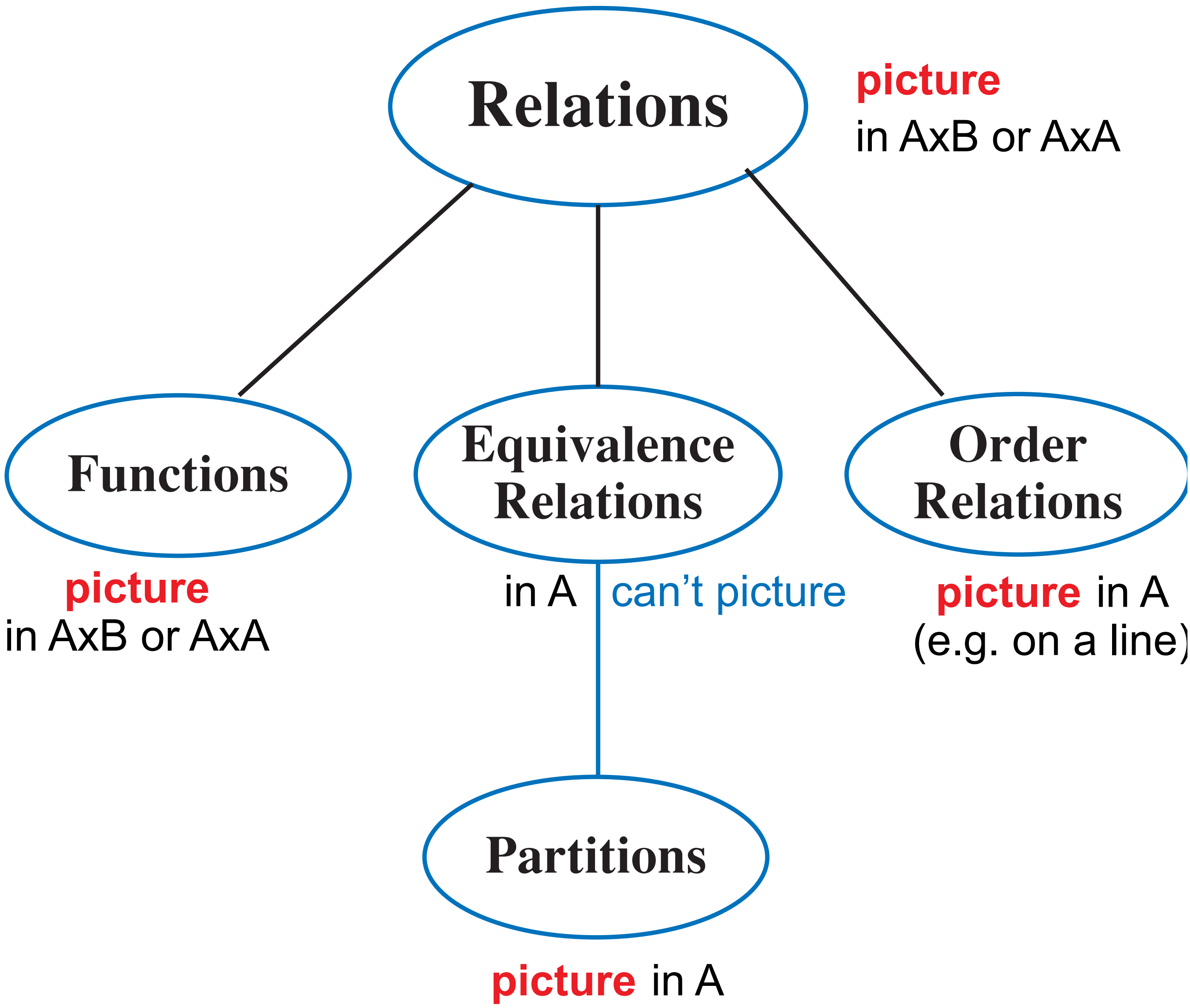
The majority of students tried to use *their own language* to interpret the definition of 'partitions' so that their answers were highly varied.

All ten students interviewed said they had a mental picture of a partition.

Nine thought they understood 'partitions' better than 'equivalence relations'. Arthur understood 'partitions' better than 'equivalence relations' because he could visualise 'partitions' but not 'equivalence relations'.

Yet the students were actually better at answering problems about equivalence relations than partitions!

Relation (on $A\times B$)		Let A and B be sets. A *relation* between A and B is a subset of $A\times B$.	⇨ Theorems
Equivalence relation (on A)	Examples like "*the similar triangles*" and "*the integers modulo n*".	A relation ~ on a set A is an *equivalence relation* if for all a, b, $c \in A$ 1. $a\sim a$ (reflexive) 2. if $a\sim b$ then $b\sim a$ (symmetric) 3. if $a\sim b$ and $b\sim c$ then $a\sim c$ (transitive)	⇨ Theorems
Partition (of A)		A partition of a set A is a set $\wp$ whose members are non-empty subsets of A satisfying 1. each $a\in A$ belongs to some $X\in \wp$ 2. if $X, Y\in \wp$, and $X\neq Y$, then $X\times Y=\varnothing$	⇨ Theorems


Relations
picture
in AxB or AxA
Functions
picture
in AxB or AxA
Equivalence Relations
in A
can't picture
Order Relations
picture in A
(e.g. on a line)
Partitions
picture in A

This research so far shows the influence of the concept image on mathematical thinking.
What about the underlying logic?

Consider the role of the quantifiers in the definition of equivalence relation:

An *equivalence relation* on a set S is a binary relation ~ on S that is

reflexive: $a \sim a$ for all $a \in S$

symmetric: if $a \sim b$ then $b \sim a$ for all $a, b \in S$

transitive: if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$

(Stewart & Tall, 1977)

Let $X=\{a, b, c\}$ and the relation ~ be defined where $a \sim b$, $b \sim a$, $a \sim a$, $b \sim b$, but no other relations hold. Is this an **equivalence relation**? If not, say why?

This is NOT and equivalence relation.
The reflexive law fails because c ~ c is missing.
It *is* symmetric and transitive.

Let $X=\{a, b, c\}$ and the relation ~ be defined where $a{\sim}b$, $b{\sim}a$, $a{\sim}a$, $b{\sim}b$, but no other relations hold. Is this an **equivalence relation**? If not, say why?

Classification of responses:

- Correct deduction & answer,

No because $c \not\sim c$ ⇒ not $\forall a,b,c \in X$ ⇒ not reflexive. ⇒ not equiv. relat

- Incorrect deduction with 'correct' answer,

does it follow rules? $a \sim a$ ✓ $b \sim b$ ✓
$a \sim b \Rightarrow b \sim a$ ✓ symmetry
$a \sim b,\ b \sim c \Rightarrow a \sim c$; but $b \sim c$ doesn't hold, so this is not an equivalence relation

- Incorrect deduction & answer,

1) $a \sim b,\ b \sim a$
2) $a \sim a,\ b \sim b$
3) No occurance Yes it is an equivalance relation.

- Don't know/ no response.

	Correct deduction & answer	*Incorrect deduction*		*Don't know/ No response*	*Total*
		'correct' answer	*incorrect answer*		
Pure mathematics	94	24	30	3	151
Other mathematics	45	44	18	19	126
Total	139	68	48	22	277

Responses to the "use of quantifiers" question

pure students give more correct responses (χ^2=19.34, p<0.0001)

Do students who give formal definition give a better response?

Compare responses on definition

quantified, unquantified, outline, other,

with

correct (including reason) v. other.

Pure maths, not significant (χ^2=10.94, p=0.28).

Other maths, highly significant (χ^2=29.39, p<0.001).

This arises because ‘other maths’ has a correlation between students responding ‘other’ in both categories.

If the ‘other’ data is removed, the correlation is
Pure maths, not significant (χ^2=3.11, p=0.54),

Other maths, not significant (χ^2=5.91, p=0.21).

There is no correlation between quality of definition and ability to solve a simple deduction from the definition applied to a set with three elements!

Reasons for failure:

Symmetry fails …

No, because if it was, for it to be symmetric
$a \sim c$ and $c \sim a$, and $b \sim c$ and $c \sim b$.
For it to be reflexive, $c \sim c$.

Transitivity fails …

$a \sim a$ hence reflexive
$a \sim b$ & $b \sim a$ hence symmetric
However not transitive as c is not involved.

Transitivity fails because it requires 3 elements …

No because $a \sim b$, $b \sim a \not\Rightarrow a \sim a$.
need 3 elements for transitivity to hold.

Question: What is the concept image of the transitive law?

if $a \sim b$ and $b \sim c$ then $a \sim c$

Who does not always use along with the double inequality

$$a > b > c$$

the picture of three points following one another on a straight line as the geometrical picture of the idea “between”?

(Hilbert, 1900)

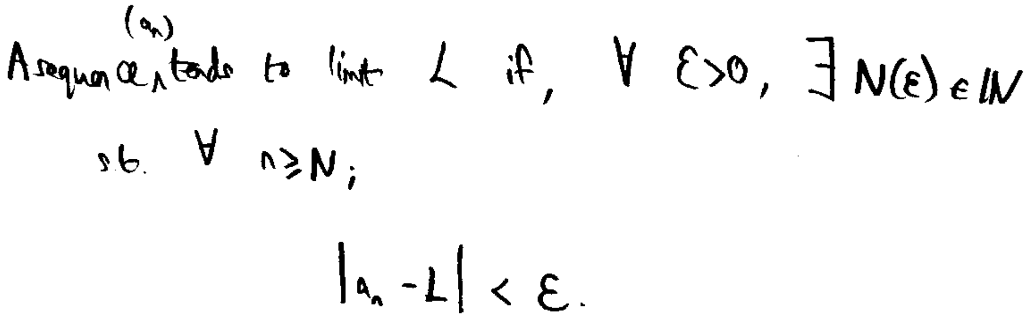
Natural and formal learning

(Pinto 1998, Pinto and Tall, 2001)

Formal learners essentially construct the theory by deduction, coping with the great cognitive strain as best they can, producing a deductive formal theory. Natural learners—working from their concept imagery—reconstruct it taking account of more general ideas met in the course. They must then develop the formal theory from their reconstructed imagery, producing a formal theory integrating both imagery and deduction.

	Formal learning	Natural learning
1 Initial obstacles	Based on concept definition, may be problematic either (a) unsatisfactory defns, eg problems with quantifiers, disjoint from images (b) defns **conflict**ing with images	Informal (based on concept image) so may (a) reject formal, retain images (b) relate formal to informal knowledge, with **conflict**
2 Theory Building	**Formal construction** of theory	**Formal *re*construction** (with some conflict) (a) Thought experiments, reconstructing images (b) Deductions reconstructing formal theory
3 Formal theory	**Formal (deductive)**	**Formal (integrated)**

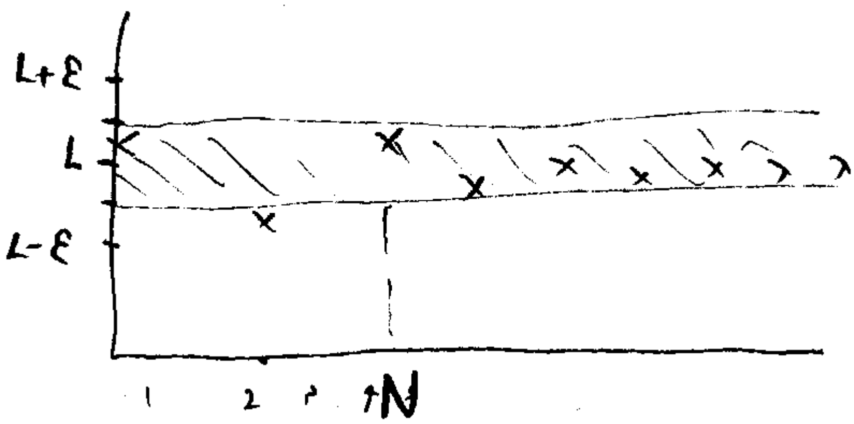
Natural and formal routes to learning formal mathematics


A sequence (a_n) tends to limit L if, $\forall\ \varepsilon>0,\ \exists\ N(\varepsilon)\in\mathbb{N}$ s.t. $\forall\ n\geq N$;

$$|a_n - L| < \varepsilon.$$

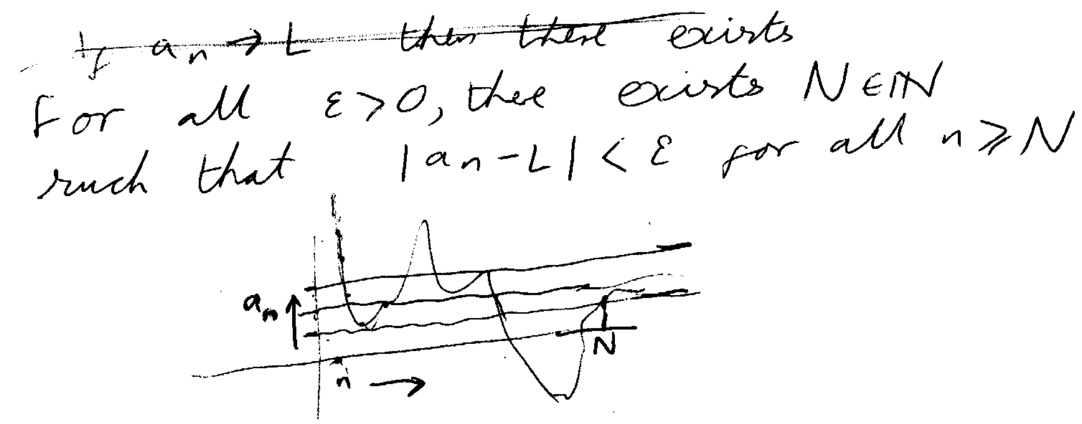
(Ross, first interview)

"Just memorizing it, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it get imprinted and then I remember it."



(Ross, first interview)

"Well, before, I mean before I saw anyone draw that, it was just umm ... thinking basically as n gets larger than N , a_n is going to get closer to L so that the difference between them is going to come very small and basically whatever value you try to make it smaller than, if you go far enough out then the gap between them is going to be smaller. That's what I thought before seeing the diagrams and something like that."



(Chris, first interview)

"I don't memorize that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down."

"I think of it graphically ... you got a graph there and the function there, and I think that it's got the limit there ... and then ε once like that, and you can draw along and then all the ... points after N are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that's the n going across there and that's a_n. ... Err this shouldn't really be a graph, it should be points."

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Rolf (a) Robin (a& b)	Rolf (a) Robin (b)	[Rolf withdrew] Robin (b)		
2. Formal Construction				Robin	Robin
	Ross		**Ross**		
3. Formal (deductive)		**Ross**		**Ross**	
					Ross

Students following an essentially formal route

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Cliff (a) Colin (b)	Cliff (a) Colin (b)	Cliff (a)	Cliff (a)	Cliff (a)
2. Formal Reconstruction			Colin (a)	Colin (b)	Colin (b)
		Chris (a&b)	**Chris (a&b)**	**Chris (a&b)**	
3. Formal (deductive)	**Chris**				
					Chris

Students following an essentially natural route

SUMMARY

- The concept image plays a powerful role in mathematical thinking
- Thought Experiments and Deductive Proof uses different parts of the brain
- Students often do not build operable definitions in their first encounter with formal mathematics
- Natural and Formal Thinking
- The theoretical structure, say of relations, may involve different natural and logical linkages with some concepts more predominantly verbal/logical and others more visual.
- A logical axiom such as the transitive law often builds on underlying imagery causing subtle logical difficulties
- Ongoing Cognitive Research is beginning to say things in a way that may be more amenable to being incorporated into teaching and learning.