David Tall
University of Warwick, UK
Presentation in Taipei, October 14th, 2002

Three Worlds of Mathematics

David Tall
Mathematics Education Research Centre
University of Warwick
CV4 7AL, UK
e-mail: david.tall@warwick.ac.uk

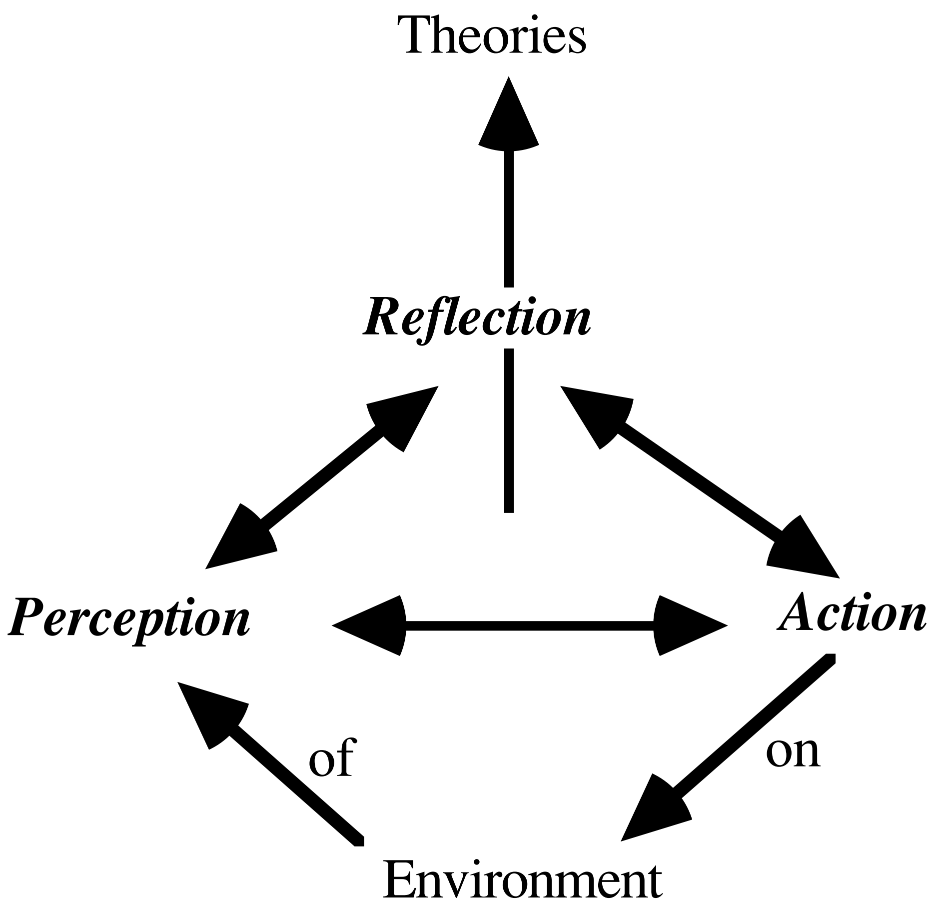
I shall contrast and compare three different kinds of mathematics that are so different that they seem to inhabit three distinct worlds:

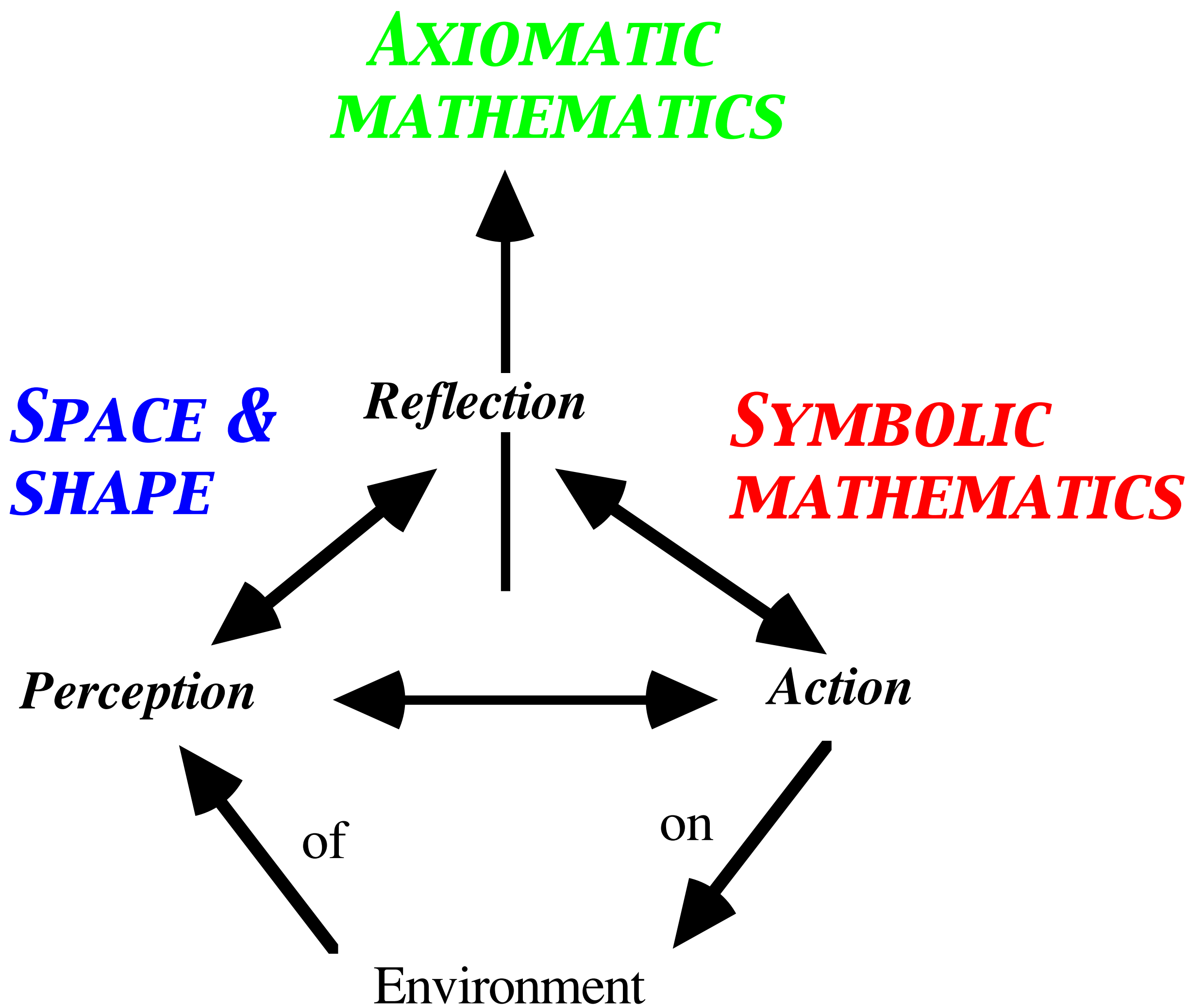
*The **EMBODIED WORLD** of perception and action, including reflection on perception and action which develops into a more sophisticated Platonic form.*

*The **(SYMBOLIC) PROCEPTUAL WORLD** of symbols in arithmetic, algebra and calculus that act both as **PRO**cesses to do (eg 4+3 as a process of addition) and con**CEPT**s to think about (eg 4+3 as the concept of sum.)*

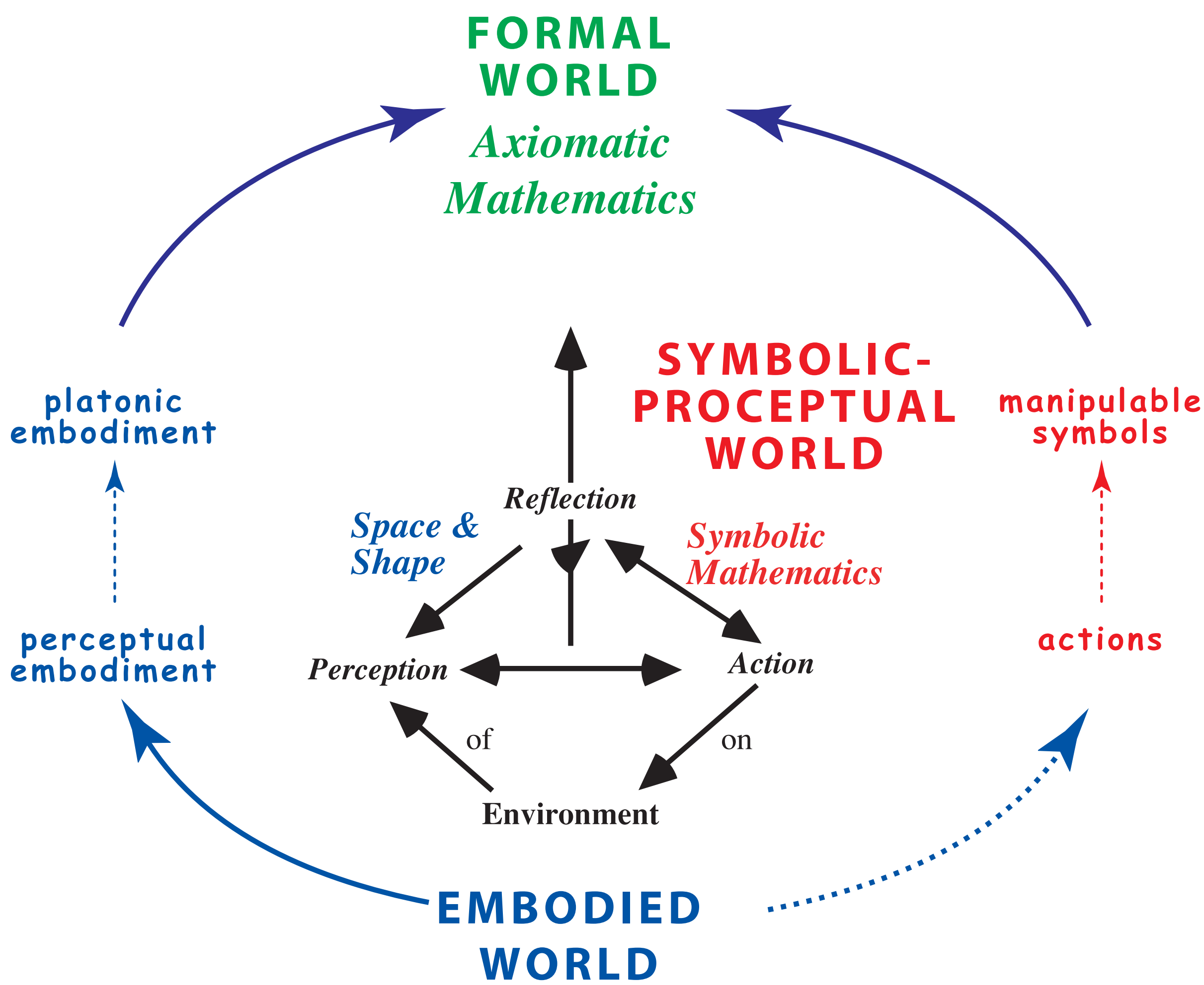
*The **FORMAL WORLD** of formal definitions and formal proof.*

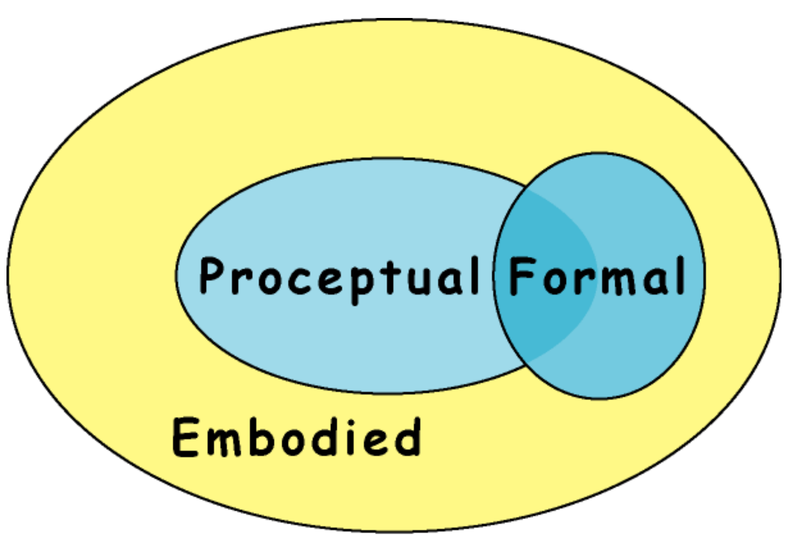
Human Development from embodied interaction to formal theory



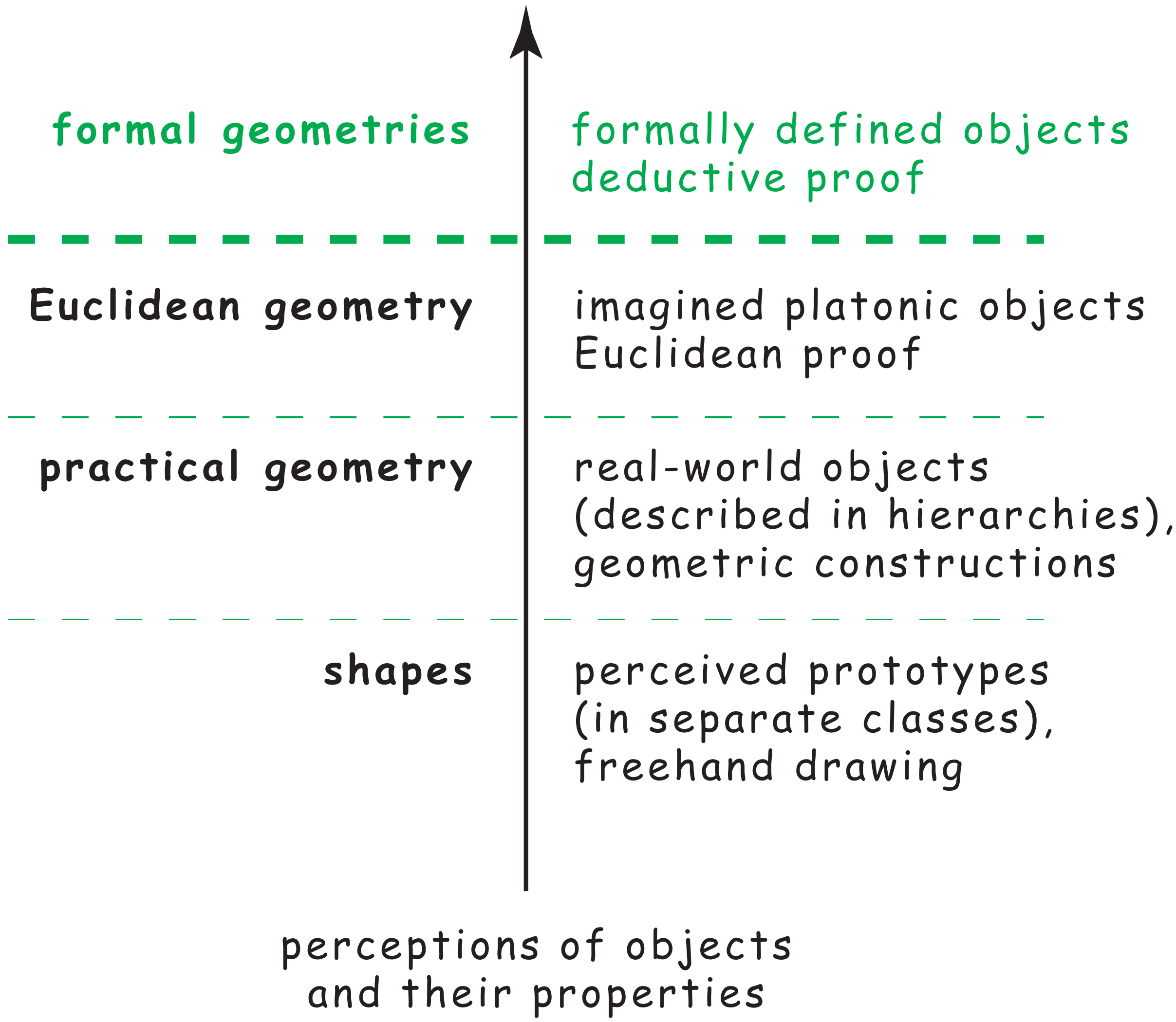


AXIOMATIC MATHEMATICS
Reflection
SPACE & SHAPE
SYMBOLIC MATHEMATICS
Perception
Action
of
on
Environment


FORMAL
WORLD
Axiomatic
Mathematics
SYMBOLIC-
PROCEPTUAL
WORLD
platonic
embodiment
manipulable
symbols
Reflection
Space &
Shape
Symbolic
Mathematics
perceptual
embodiment
actions
Perception
Action
of
on
Environment
EMBODIED
WORLD


Proceptual
Formal
Embodied

The Geometric Journey from the Embodied to the Formal World



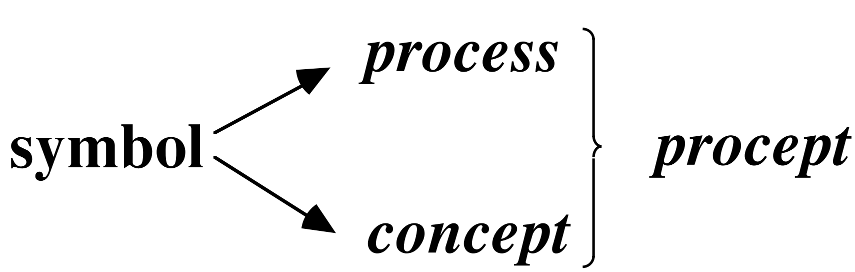
Shapes are visualised, named (in separate classes so that a square is not a rectangle), then defined (so that a square *is* a rectangle) then imagined as platonic objects satisfying the core definition.

At the formal level, the objects are given *only* by their definitional properties.

[We will return to this later in this presentation.]

COMPRESSION OF PROCESS INTO CONCEPT USING SYMBOLS

symbol	***process***	***concept***
$3+2$	addition	sum
-3	subtract 3 (3 steps left)	negative 3
$3/4$	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x\to 2}\frac{x^2-4}{x-2}$ $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n



(Gray & Tall, 1994)

Discontinuities in the development of symbols

Arithmetic procept 2+3

built-in computational process of addition to give result

Algebraic procept 2+3*x*

potential process of evaluation, manipulable concepts

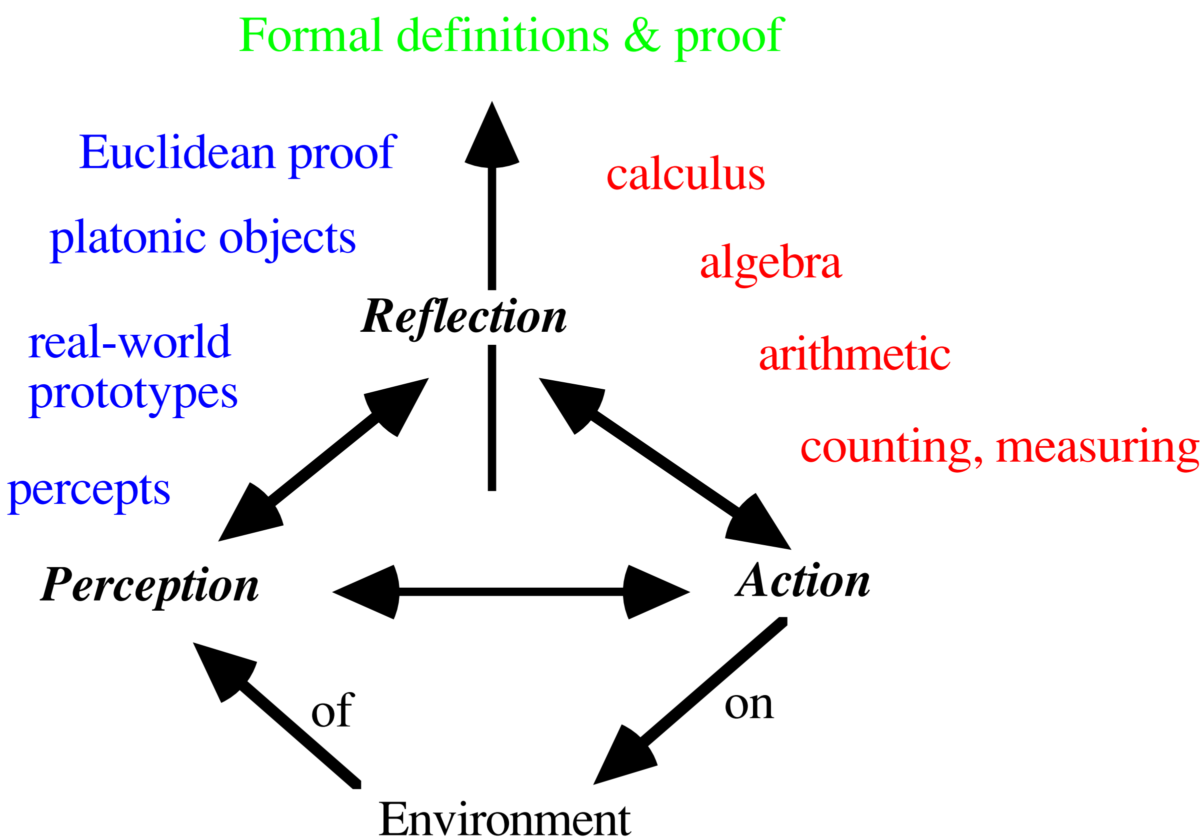
Limit procept

$\lim_{x\to 2}\frac{x^2-4}{x-2}$, $\sum_{n=1}^{\infty} 1/n^2$ have *potentially infinite processes*

Some, eg

$$\frac{d}{dx}(\sin x\cos x)$$

have finite computational processes.



Conceptual development of selected mathematical concepts

How does a biological creature do mathematics?

We humans have a BIOLOGICAL BRAIN.

Biological brain – Mathematical Mind

(+Technological Tools)

Biological brain … sensori-motor, enactive, visual, linguistic, intuitive, problem-solving capabilities.

Mathematical Mind: uses a variety of different ways of thinking.

Visual embodied Geometry becoming more sophisticated, leading to Euclidean proof and on to formal proof. **Embodiment** also underlies all our thinking, for instance in graphs, rate of change, conceptions of area, following a solution of a differential equation drawn pictorially etc.

Symbols in arithmetic, algebra, calculus etc,

The development of **Mathematical Proof**…

Technological Tools … algorithms, symbol manipulation, visual display for graphs, enactive manipulation of visual ideas, etc, etc.

THE BIOLOGICAL BRAIN

1990-2000 … the ‘Decade of the Brain’.

Stanislas Dehaene (1997). *The Number Sense*.

Brian Butterworth (1999). *The Mathematical Brain*.

Carter, R. (1998). *Mapping the Mind.* London: Weidenfeld & Nicholson.

Crick, F. (1994). *The Astonishing Hypothesis*, London: Simon & Schuster.

Pinker, S. (1997). *How the Mind Works*. New York: Norton.

Edelman, G. M. (1992). *Bright Air, Brilliant Fire*, NY: Basic Books, reprinted Penguin 1994.

Greenfield, S. (1997). *The Human Brain: A Guided Tour*. London: Weidenfeld & Nicholson.

Devlin. K. (2000), *The Maths Gene: Why everyone has it, but most people don’t use it*, London: Weidenfeld & Nicholson.

Freeman, J. F. (1999). *How the Brain makes up its Mind*. Pheonix.

Edelman, G. M. & Tononi, G. (2000). *Consciousness: How Matter Becomes Imagination*. New York: Basic Books.

Lakoff, G. and Johnson, M. (1980). *Metaphors we live by*.

Lakoff, G. (1987). *Women Fire and Dangerous Things*.

Lakoff, G. and Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Lakoff & Nunez (2000): Where Mathematics Comes From. New York: Basic Books.

CONCEPTUAL COMPRESSION, FORMING COMPACT CONCEPTS AND POWERFUL MENTAL LINKS

How the brain makes complexity more manageable:

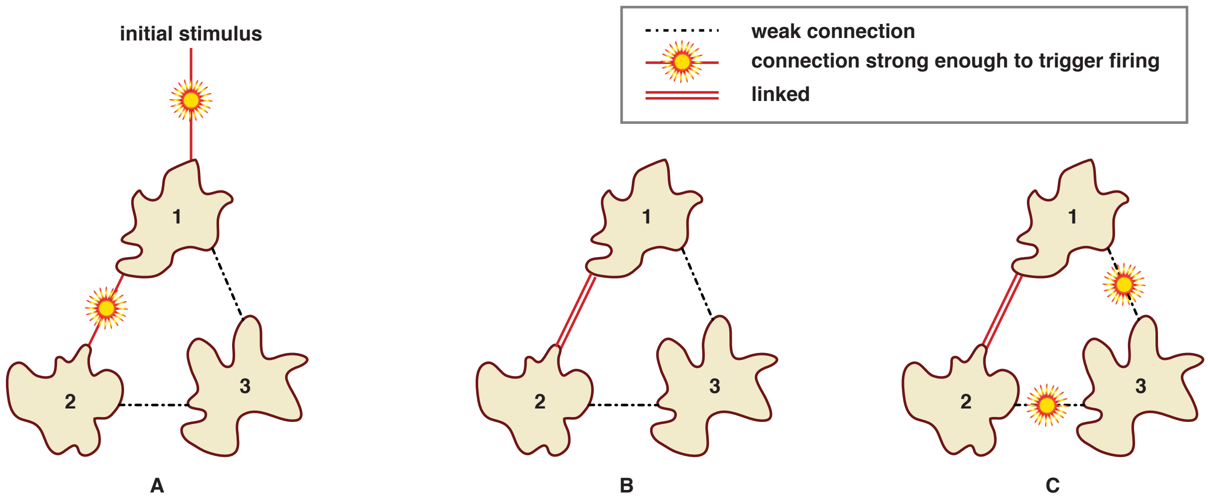
> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) "object(s)" can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once).
>
> (Crick, 1994, p. 61)

This is made more efficient by making the conscious elements as 'small as possible', using words or symbols:

> I should also mention one other property of a symbolic system – its compactibility – a property that permits condensations of the order $F = MA$ or $s = \frac{1}{2}gt^2$, … in each case the grammar being quite ordinary, though the semantic squeeze is quite enormous.
>
> (Bruner, 1966, p. 12.)

> As a task to be learned is practiced, its performance becomes more and more automatic; as this occurs, it fades from consciousness, the number of brain regions involved in the task becomes smaller.
>
> (Edelman & Tononi, 2000, p.51)

Long-Term Potentiation



Building memories in the brain by long-term potentiation (Carter, 1999, p. 160)

A : external stimulus to neuronal group 1 sufficient to fire neuronal group 2 but not group 3.

B: The firing causes the link between 1 and 2 to be more sensitive for a time and, if reactivated, it becomes more easily fired until any excitation of 1 also fires 2. Now 1 and 2 fire together.

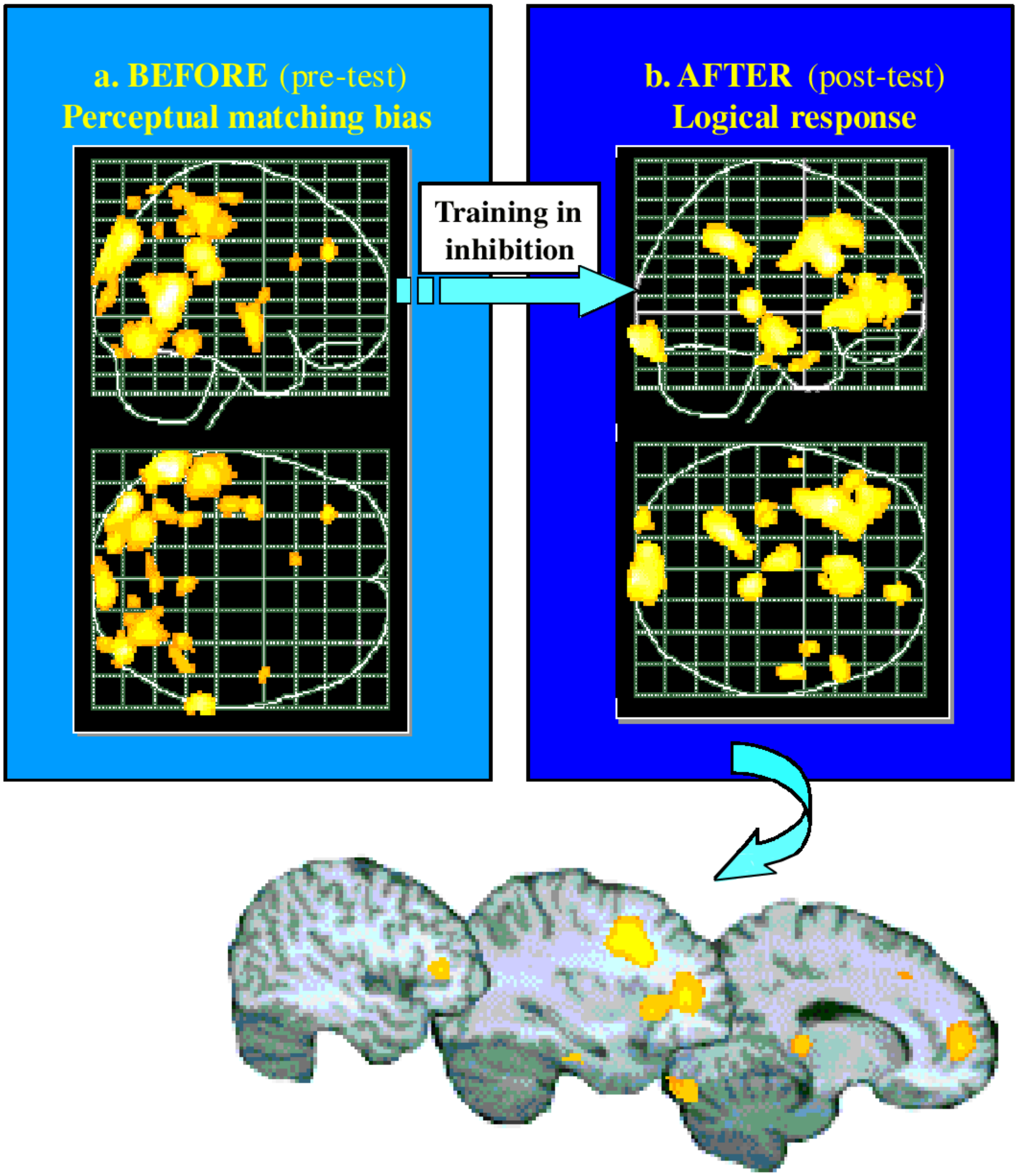
C: The combined strength of 1 and 2 now cause group 3 to be excited, forming a completely new link.

This is *long-term potentiation*.

It leads to the building of highly connected neuronal groups that act in sophisticated ways.

Initially mental activities are sensory based, but then, through the development of language and communication and by reflection on mental activity, cognitive development moves from the purely sensory areas to higher brain functions.

FROM PERCEPTUAL TO LOGICAL THINKING



Houdé et al (2000). Shifting from the perceptual brain to the Logical Brain: The Neural Impact of Cognitive Inhibition Training. *Journal of Cognitive Neuroscience* 12:4 712–728.

MATHEMATICAL MIND

Constructs to describe and explain the cognitive operation of the mathematical mind.

- the ***concept image***, which refers to the total cognitive structure in an individual mind associated with the concept, including all mental pictures, associated properties and processes (Tall & Vinner, 1981),

This is biologically a complex connection of neuronal groups. The plan is to use experiences to turn broad intuitive links into powerful units.

- a ***cognitive unit*** (is a mental chunks we use to think with, and their related cognitive structure). (Barnard & Tall, 1997).

A particular type of cognitive unit:

- the notion of ***procept***, referring to the manner in which we cope with symbols representing both mathematical ***pro***cesses and mathematical con***cepts***. (Gray & Tall, 1994). Examples include

$$3+5,\ ax^2+bx+c,\ \frac{d}{dx}\left(e^x \sin x\right),\ \text{or}\ \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

The brain is not configured for rapid & efficient arithmetic (Dehaene, 1997). It uses biological links between cognitive units. (As an example I will discuss Shaker Rasslan and the algorithm for divisibilty by seven, e.g. 121, 131, 119.)

Mathematical thinking is more than knowing procedures 'to do'. It involves a knowledge structure compatible with the biological structure of the human brain:

- a huge store of knowledge and internal links,
- coping with many activities using a manageable focus of attention.

Consider, a 'linear relationship' between two variables. This might be expressed in a variety of ways:

- an equation in the form $y = mx+c$,
- a linear relation $Ax+By+C = 0$,
- a line through two given points,
- a line with given slope through a given point,
- a straight-line graph,
- a table of values, etc

Successful students develop the idea of 'linear relationship' as a rich cognitive unit encompassing most of these links as a single entity.

Less successful carry around a 'cognitive kit-bag' of isolated tricks to carry out specific algorithms. Short-term success perhaps, long-term cognitive load and failure.

Crowley (2000)

MENTAL TOOLS FOR MATHEMATICS

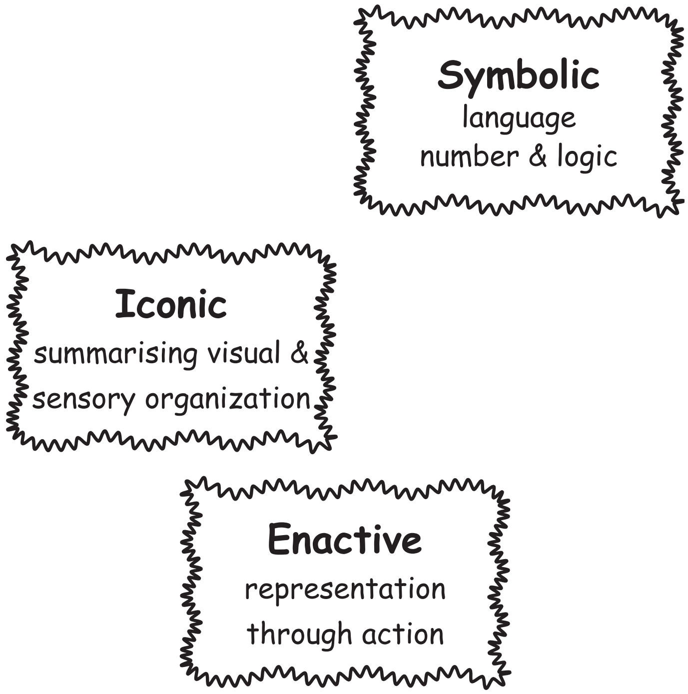
Physical & mental tools and modes of representation

Man's use of mind is dependent upon his ability to develop and use "tools" or "instruments" or "technologies" that make it possible to express and amplify his powers. His very evolution as a species speaks to this point. It was consequent upon the development of bipedalism and the use of spontaneous pebble tools that man's brain and particularly his cortex developed. It was not a large-brained hominid that developed the technical-social life of the human; rather it was the tool-using, cooperative pattern that gradually changed man's morphology by favoring the survival of those who could link themselves with tool systems and disfavoring those who tried to do it on big jaws, heavy dentition, or superior weight. What evolved as a human nervous system was something, then, that required outside devices for expressing its potential.

(Bruner, *Education as Social Invention*, 1966, p. 25.)

What does it mean to translate experience into a model of the world. Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. […] There is a second system of representation that depends upon visual or other sensory organization and upon the use of summarizing images. […] We have come to talk about the first form of representation as **enactive**, the second is **iconic**. […] Finally, there is a representation in words or language. Its hallmark is that it is **symbolic** in nature.

(Bruner, 1966, pp. 10–11)



Bruner's three modes of representation

"… any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in recognizable form" (Bruner 1966. p. 44).

Modern computer interfaces and Bruner's philosophy:

- Enactive interface,
- Icons as summarizing images to represent selectable options,
- Symbolism through keyboard input and internal processing.

'Symbolism' in mathematics requires further subcategories.

Bruner (1966, pp. 18, 19)

- "language in its natural form"
- the two "artificial languages of number and logic."

To these categories we must add not just *number*, but *algebraic* and other *functional symbolism* (trigonometric, exponential and other functions in calculus) *and the huge range of symbolism in axiomatic mathematics*.

The Reform movement in the calculus, for example the Harvard Calculus, focused on three representations: graphic, numeric and symbolic (or analytic):

> One of the guiding principles is the 'Rule of Three,' which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles. (Hughes Hallett 1991, p. 121)

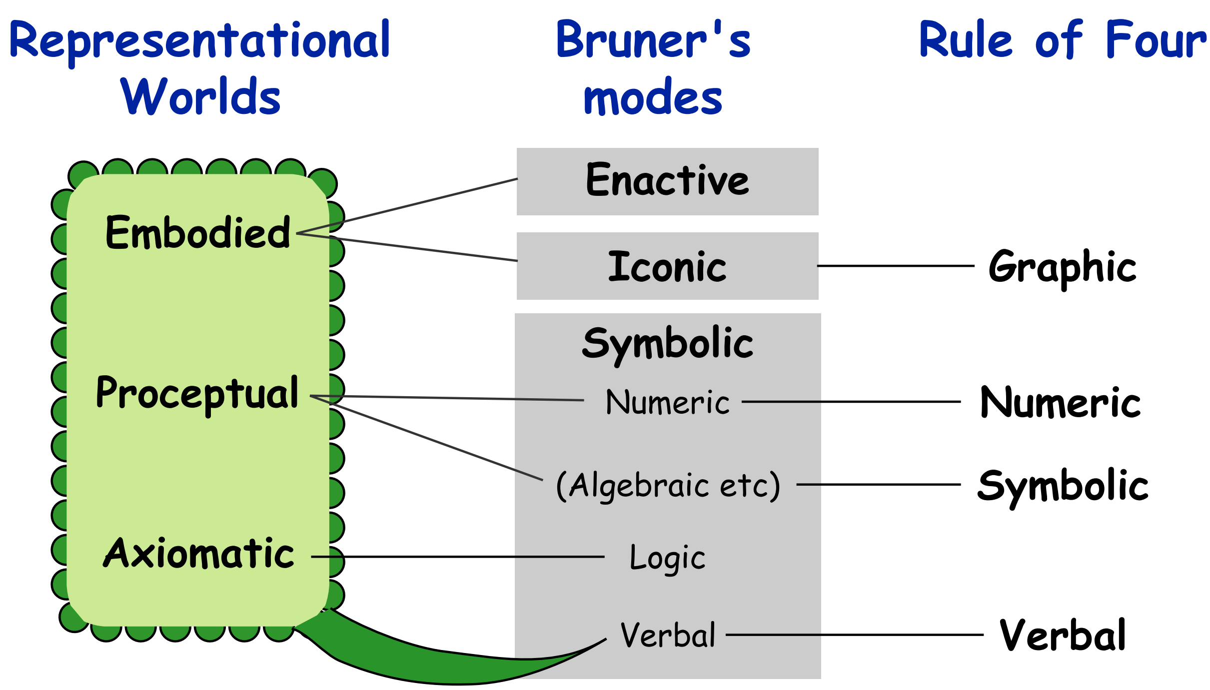
The 'Rule of Three' later became the 'Rule of Four', extending the representations to include the *verbal*.

Note:

- The enactive mode is completely omitted, presumably because it does not seem to be a central focus in the graphs and symbols of the calculus.
- The "verbal" mode was not seen as being important until late on in the development of the curriculum.
- Axiomatic formulations using logical deduction are not seen as part of calculus but of the later study of analysis.

My solution is to categorise representations into three different worlds of operation:

- **Embodied**: based on human perceptions and actions in a real-world context including but not limited to enactive and visual aspects.
- **Proceptual**: combining the role of symbols in arithmetic, algebra, symbolic calculus, based on the theory of these symbols acting dually as both process and concept (procept). (Tall *et al*, 2001, see below).
- **Axiomatic**: a formal approach starting from selected axioms and making logical deductions to prove theorems.



Three representational worlds

Relationships with other theories

Piaget

sensori-motor / preconceptual / concrete operational / formal

SOLO taxonomy (Structure of Observed Learning Outcomes)

Biggs & Collis (1982)

sensori-motor / ikonic / concrete-operational / formal / post-formal

The SOLO taxonomy is intended to provide a template for assessment. Within each mode the assessment of concepts is performed according to how the student handles the particular concepts and whether this is:

- pre-structural (lacking knowledge of the assessed component)
- unistructural (focusing on a single aspect)
- multi-structural (focussing on several separate aspects)
- relational (relating different aspects together)
- extended abstract (seeing the concept from an overall viewpoint)

---

i) *in each mode, increased sophistication occurs.*

ii) *As new modes develop, earlier modes may remain available.*

Why the *Three* Worlds of Operation?

(1) Embodied : takes enactive and visual/iconic *together* to focus on the physical senses of the individual. It emphasises the physical senses as a fundamental cognitive foundation of mathematics, almost absent from calculus reform movements.

(2) **Proceptual** :

Why not just 'symbolic' ? … (many meanings).

Why not subdivide into numeric/algebraic or other?
(because the full development needs to be considered).
(… the subcategories can also be considered…)

calculus	**computational** processes (rules) **computational** concepts (formulas)
	[problems with deriving rules for calculus]
(dynamic limit concept)	**potentially infinite** processes (limits) **arbitrarily small, close or large** concepts (numbers)
	[problematic meaning of negative & fractional powers]
algebra	**potential** processes (evaluating expressions) **manipulable** concepts (expressions)
	[problematic meanings of negatives, fractions]
arithmetic	**computational** processes (arithmetic) **computational** concepts (numbers)

Some different types of procept in mathematics

(3) **Axiomatic** :

Why *axiomatic* ? Why not just a *formal* presentation ?

(some 'formal' arguments are essentially proceptual)

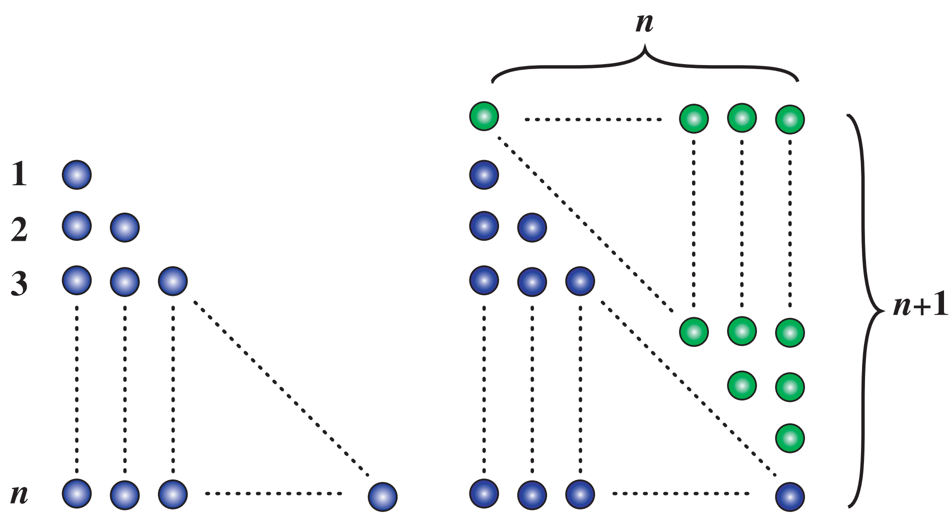
('axiomatic' means axioms for real numbers, formal definitions of limits, etc.)

Different 'warrants for truth' in embodied, proceptual and axiomatic worlds.

Example: The sum of the first n whole numbers is $\frac{1}{2}n(n+1)$.

Proof 1: (embodied). Lay out rows of stones with 1 in the first, 2 in the second, and so on. Then take an equal layout of pebbles, turn it round and fit the two together. Visibly the two together make a rectangle size n by n+1, giving $n(n+1)$ stones altogether, making $\frac{1}{2}n(n+1)$ in the original.

The validity of this proof is in the visual picture.



Embodied proof that the sum of the first n whole numbers is $\frac{1}{2}n(n+1)$

Proof 2: (proceptual). Write out the sum

$1+2+3+\ldots n$

backwards

as $n+\ldots+3+2+1$

and add the two together in order, pair by pair, to get

$(1+n)+(2+n-1)+\ldots+(n+1)$

to get n lots of n+1, ie. $n(n+1)$, so, again, the original sum is half this, namely $\frac{1}{2}n(n+1)$. ***Validity by calculation.***

Proof 3 : (axiomatic). By induction. ***Validity by proof.***

Axiomatic proof

Begins with definitions/axioms, and deduces theorems.

This has *logical* processes, and *formal* objects.

Axiomatic systems are (rarely) procepts.

A group G is not a symbol representing process/concept.

However:*elements* of a group can be both

processes (eg transformations)

and ***objects*** (elements of the group).

Two forms of Advanced Mathematical Thinking:

Technical (=embodied proceptual)

(involving computation, eg vectors as n-tuples and matrices), usually based on modelling real-world examples.

Formal

(involving definition, eg formal vectors satisfying axioms).

Different constructions of objects in Advanced Mathematical Thinking

Technical objects may include ***procepts***

(eg vectors, transformations, functions etc)

Formal objects are ***defined concepts***

The difference between "described objects" and "defined objects" …

A ***described objec****t* e.g. in a dictionary is given a description to *identify* it. i.e. **object→description**

Formal object *has specified properties to* ***define*** it.

i.e. **definition → object**

The move from elementary mathematics to formal mathematics often reverses the order of previously acquired knowledge. Eg A child learns subtraction b-a first, then the negative -a. But in an axiomatic theory the *negative* is defined first as -b and then *subtraction* is defined as a+(-b).

A story…

The Professor is teaching students undergraduates that one is bigger than zero, based on the field axioms.

Proof that 1 is bigger than 0…

Either 1 > 0 or 1 = 0 or –1 > 0

(“axiom of trichotomy”)

But **if –1 > 0** then, $(-1) \times (-1) > 0$

(because “ $a > 0$, $b > 0$ implies $ab > 0$”)

so, $(-1) \times (-1) = 1$

(by some previously deduced theorem)

giving $1 > 0$.

Hence both **–1 > 0 and 1 > 0**,

which contradicts “trichotomy”.

This proves 1 > 0

Note the proof uses “product of two minuses is a plus” which most students learn procedurally, to show 1 > 0 which most students have known since the age of two.

THREE DIFFERENT WORLDS AND THEIR WARRANTS FOR TRUTH

The **embodied world** is a world of sensory meaning. Its warrant for truth is that things behaves predictably in an expected way.

The **proceptual world** is a world of computation and manipulation. Statements are true because they are stated in symbols and can be verified by calculation or manipulation.

The **formal world** is a world of axioms, definitions and theorems. Statements are true because they can be proved from the axioms and definitions by formal deduction.

Example 2: The notion of **vector**.

Embodied world

A vector has magnitude and direction. Addition of vectors is by placing the tail of the second vector after the nose of the first. Addition is commutative by visual geometry.

Proceptual world

Vectors can be expressed using coordinates. Addition of vectors is by adding coordinates. Addition is commutative because addition of the coordinates is commutative.

Formal world

Vectors are elements of a formal system called a vector space. (They need no longer have magnitude or direction!) Addition of vectors is part of the definition of a vector space. Addition is commutative because this is part of the definition.

Example 3: The case of calculus:

The **embodied world**

A (locally straight) graph *has* a slope, because you can *see* it.

It has an area underneath because you can *see* it.

The **proceptual world** is a world where calculations can be made (both arithmetic and algebraic). A graph has a slope (derivative) or an area (integral) because you can *calculate* it.

The **formal world** is a world where explicit axioms are assumed to hold and definitions are given formally in terms of quantified set-theoretic statements. A function has derivative or integral because you can *prove* it.

DIFFERENT COMMUNITIES OF PRACTICE

In different communities, different emphases occur.

In school, emphasis is on embodied and proceptual worlds. As students grow older, links with the embodied are reduced and reliance on the proceptual is increased. 'Proof' is introduced via numeric checking or algebraic manipulation or through verbal descriptions of visual figures in geometry.

Pure mathematics retains implicit use of the embodied to varying degrees but focuses more on the proceptual and formal worlds. A statement is 'true' if it is has a formal proof.

Individuals in pure mathematics reach the formal level in different ways. Some are think in a 'natural' manner that builds embodied imagery into formal definitions and deductions. Others are 'formal' thinkers who reject embodiment and attempt to work only from the formal definitions using formal deduction. In practice, however, *all* mathematicians use a biological brain that depends, consciously or unconsciously, on embodied conceptions.

Applied Mathematics, Engineering, etc focuses more on increasingly sophisticated symbolic activities. It is embodied and proceptual rather than formal.