Problem-solving

What do students *really* learn in mathematics?

I didn't understand all that stuff between "Good morning, class" and "That concludes my lecture for today."

MATHEMATICS EDUCATION RESEARCH CENTRE
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Maths education at university level, as it stands, is based like many subjects on the system of lectures. The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. The pressure of time seems to take away the essence of mathematics and does not create any true understanding of the subject. From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject.

Third (final) Year Mathematics Student

Since starting my university education, I have discovered that the key to advanced learning is persistence. Come the end of the year, everyone is faced with courses whose purpose they have failed to grasp, let alone its finer details. Faced with this problem, most people set about finding typical questions and memorising the typical answers. Many gain excellent marks in courses of which they have no knowledge. Most accept this as the norm, thinking that their stupidity is the problem and not considering it to have been a wasted 5 or 10 weeks of study.

Second Year Applied Mathematics Student

I for one suffered a confidence crisis as answers seemed to arrive from mid-air –

$$\text{“Oh, we’ll take } \varepsilon = \frac{2\delta}{\min(M,N)} \text{”}$$

or similar. This evoked the “panic” emotion … and, although frequently shown the solution, I often did not understand how to find it.

Third Year Mathematics Student

Professor's views of students and mathematics

To me mathematics is a tool for solving problems. One way of motivating the students is by showing them applications in the real world. In this way they get the knowledge and the skills for solving problems. … *I do not think the students are capable of creating new ideas on their own.*

We give them little room to do their own thinking. But we cannot change it because the system does not allow us to do so. So *we end up teaching them what they need to know.*

The system has been proven a failure. It has not been successful in producing good mathematicians, or engineers that can use mathematics effectively. They only know how to use procedures or computer packages without really understanding why they use them. …It's all down to the system. We are not training students to discover patterns, or how to prove a statement is true, for example. *What we teach them is mainly how to use the procedures.*

WHAT IS WRONG WITH MATHEMATICS TEACHING (ESPECIALLY AT UNIVERSITY?)

Skemp, *Psychology of Learning Mathematics, 1971*:

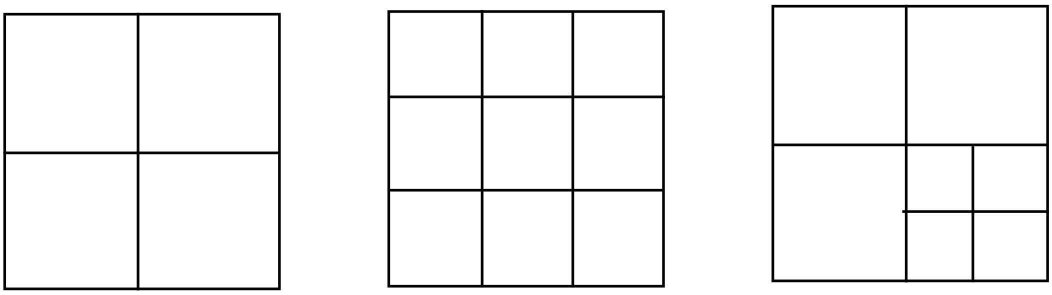
"We teach mathematical *thought*, not mathematical *thinking*."

even more:

We teach the *product* of mathematical thought, not the *process* of mathematical thinking!

Problems

Into how many squares can you cut a square?

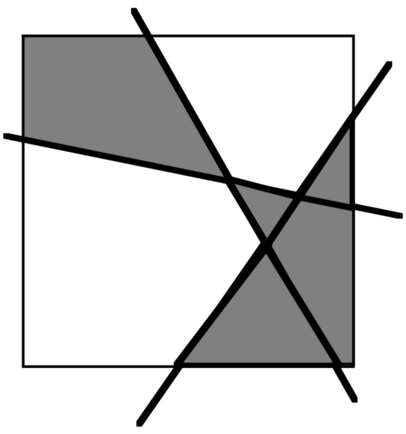


4 squares 9 squares 7 squares

What about 6? 253? 3? 5000234?

Proof?

Take a square and draw a number of straight lines across it. How many different colours are required so that no two touching areas have the same colour?



Proof?

Thinking Mathematically (John Mason, Leone Burton & Kay Stacey, Addison Wesley 1982)

✯✯✯✯✯

3 stages : **Entry, Attack, Review**

✯✯✯✯✯

1. **Entry**: What do I know?, What do I want?, what do I need to link the two?

Try specialization, generalization.

✯✯✯✯✯

2. **Attack**: Follow thinking started in entry.

Be prepared for Aha! ☺ and Stuck!☹ *and understand the emotional effects!*

Types of proof:

- Convince yourself,
- Convince a friend,
- Convince an enemy.

✯✯✯✯✯

3. **Review** : check, reflect, extend (to a wider class of problems)

What happens when problem-solving is taught?

Attitudes change!

Attitudes to Mathematics:

1. Mathematics is a collection of facts and procedures to be remembered.
2. Mathematics is about solving problems.
3. Mathematics is about inventing new ideas.
4. Mathematics at the University is very abstract.
5. I usually understand a new idea in mathematics quickly.
6. The mathematical topics studied at University make sense to me.
7. I have to work very hard to understand mathematics.
8. I learn my mathematics through memory.
9. I am able to relate mathematical ideas learned.

Attitudes to Problem-Solving

1. I feel confident in my ability to solve mathematics problems.
2. Solving mathematics problem is a great pleasure for me.
3. I only solve mathematics problems to get through the course.
4. I feel anxious when I am asked to solve mathematics problems.
5. I often fear unexpected mathematics problems.
6. I feel the most important thing in mathematics is to get correct answers.
7. I am willing to try a different approach when my attempt fails.
8. I give up fairly easily when the problem is difficult.

Students responded on a 5-point scale:

Y, y, –, n, N

(definitely **Y**es, **yes**, no opinion, **n**o, definitely **N**o)

What did the professors *expect* the students would say and what did they *prefer* the students to say?

Yes = Definitely Yes + yes, (Y= Definitely Yes).

No = Definitely No + no, (N = Definitely No).

				Expect			Prefer		
	Attitude	desired change		Yes (Y)	–	No (N)	Yes (Y)	–	No (N)
Mathematics	***facts and procedures***	$\downarrow^{+++}_{+}$	<1%	20 (8)	0	2 (0)	13 (4)	0	9 (2)
	solving problems	$\uparrow^{+++}_{+++}$	n.s.	19 (9)	0	3 (0)	22 (9)	0	0 (0)
	inventing new ideas	$\uparrow^{+}_{-}$	n.s.	8 (2)	0	14 (1)	11 (3)	0	11 (1)
	abstract	$\downarrow^{+++}_{--}$	<1%	20 (6)	0	2 (0)	7 (0)	0	15 (4)
	understand quickly	$\uparrow^{+}_{---}$	<1%	3 (0)	0	19 (6)	15 (1)	0	7 (1)
	make sense	$\uparrow^{++}_{-}$	<1%	8 (0)	0	14 (2)	19 (3)	0	3 (0)
	work very hard	$\downarrow^{+++}_{++}$	n.s.	21 (13)	0	1 (0)	18 (4)	0	4 (0)
	memorisation	$\downarrow^{++}_{---}$	<1%	15 (5)	0	7 (1)	2 (1)	0	20 (6)
	ability to relate ideas	$\uparrow^{+++}_{--}$	<1%	5 (0)	0	17 (5)	22 (5)	0	0 (0)
Problem Solving	**confidence**	$\uparrow^{+++}_{-}$	<1%	10 (1)	0	12 (0)	22 (3)	0	0 (0)
	pleasure	$\uparrow^{+++}_{+}$	n.s.	15 (0)	0	7 (2)	21 (4)	0	1 (0)
	only to get through	$\downarrow^{+++}_{-}$	<1%	21 (9)	0	1 (0)	7 (2)	0	15 (3)
	anxiety	$\downarrow^{++}_{---}$	<1%	16 (5)	0	6 (0)	2 (0)	0	20 (5)
	fear unexpected	$\downarrow^{++}_{--}$	<1%	15 (7)	0	7 (0)	3 (0)	0	19 (5)
	correct answers	$\downarrow^{++}_{-}$	<1%	19 (3)	0	3 (0)	6 (2)	0	16 (2)
	try new approach	$\uparrow^{+++}_{+}$	<1%	12 (1)	0	10 (0)	22 (4)	0	0 (0)
	give up	$\downarrow^{++}_{--}$	<5%	16 (2)	0	6 (0)	2 (0)	0	20 (2)

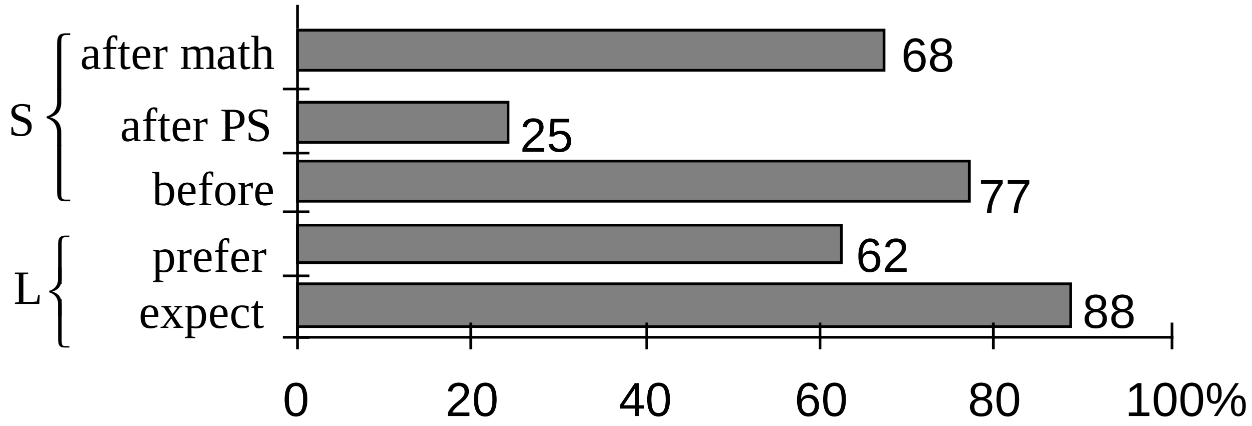
An attitude in **bold** is a desired change,
in ***bold italic*** is not desired.

		desired change		After P S		After math		Total change	
	facts and procedures	$\downarrow^{+++}_{+}$	<1%	$\downarrow^{++}_{--}$	<1%	$\uparrow^{++}_{--}$	<1%	$\downarrow^{++}_{++}$	n.s.
	solving problems	$\uparrow^{+++}_{+++}$	n.s.	$\uparrow^{+++}_{+}$	<1%	$\downarrow^{+++}_{++}$	<1%	$\uparrow^{++}_{+}$	n.s.
	inventing new ideas	$\uparrow^{+}_{-}$	n.s.	$\uparrow^{+++}_{-}$	<1%	$\downarrow^{+++}_{+}$	<1%	$\uparrow^{+}_{-}$	n.s.
	abstract	$\downarrow^{+++}_{--}$	<1%	$\downarrow^{+}_{-}$	n.s.*	$\uparrow^{+}_{-}$	n.s.	$\downarrow^{+}_{+}$	n.s.
Mathematics	**understand quickly**	$\uparrow^{+}_{---}$	<1%	$\uparrow^{o}_{--}$	<1%	$\downarrow^{o}_{-}$	n.s.	$\uparrow^{-}_{--}$	n.s.
	make sense	$\uparrow^{++}_{-}$	<1%	$\uparrow^{++}_{-}$	<1%	$\downarrow^{++}_{+}$	n.s.	$\uparrow^{+}_{-}$	n.s.*
	work very hard	$\downarrow^{+++}_{++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.*	$\uparrow^{++}_{++}$	n.s.	$\downarrow^{+++}_{++}$	n.s.
	memorisation	$\downarrow^{++}_{---}$	<1%	$\downarrow^{+}_{--}$	<1%	$\uparrow^{-}_{--}$	<5%	$\downarrow^{+}_{-}$	n.s.*
	able to relate ideas	$\uparrow^{+++}_{--}$	<1%	$\uparrow^{++}_{+}$	<5%	$\downarrow^{++}_{++}$	n.s.	$\uparrow^{++}_{+}$	n.s.
	confidence	$\uparrow^{+++}_{-}$	<1%	$\uparrow^{++}_{+}$	<5%	$\downarrow^{++}_{++}$	n.s.	$\uparrow^{++}_{+}$	<5%
	pleasure	$\uparrow^{+++}_{+}$	n.s.	$\boxed{\downarrow^{+++}_{+++}}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.
	only to get through	$\downarrow^{+++}_{-}$	<1%	$\downarrow^{-}_{---}$	<1%	$\uparrow^{--}_{---}$	<1%	$\downarrow^{-}_{-}$	n.s.
Problem	***anxiety***	$\downarrow^{++}_{---}$	<1%	$\downarrow^{-}_{--}$	<5%	$\boxed{\downarrow^{--}_{--}}$	n.s.	$\downarrow^{-}_{--}$	n.s.
Solving	***fear unexpected***	$\downarrow^{++}_{--}$	<1%	$\downarrow^{++}_{--}$	<1%	$\uparrow^{-}_{--}$	n.s.	$\downarrow^{++}_{-}$	<1%
	correct answers	$\downarrow^{++}_{-}$	<1%	$\downarrow^{+}_{--}$	<1%	$\uparrow^{-}_{--}$	<1%	$\downarrow^{+}_{-}$	n.s.
	try new approach	$\uparrow^{+++}_{+}$	<1%	$\uparrow^{+++}_{+++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.
	give up	$\downarrow^{++}_{--}$	<5%	$\downarrow^{-}_{---}$	<1%	$\downarrow^{--}_{---}$	n.s.	$\downarrow^{-}_{--}$	<5%

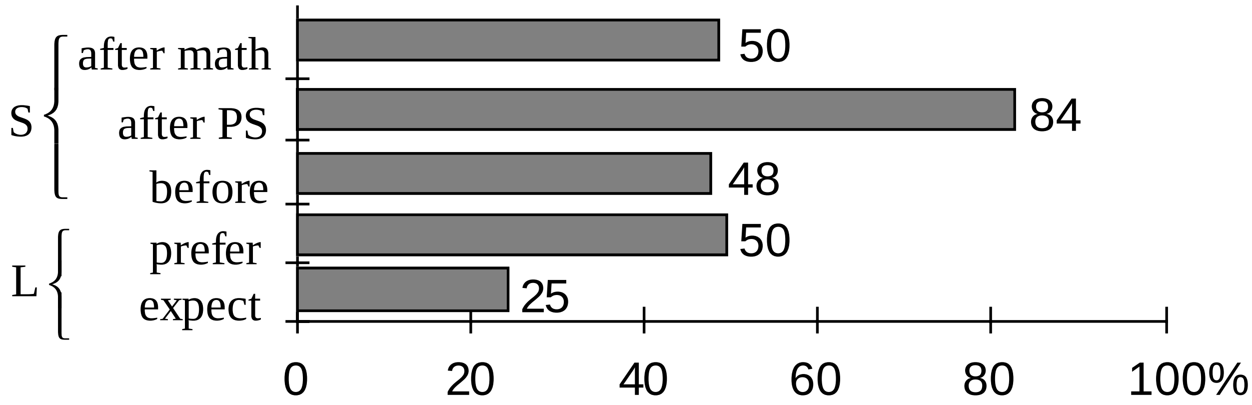
Changes after problem-solving & after mathematics lectures

Yudariah Binte Mohammad Yusof:

Mathematics is remembered facts



Mathematics is inventing new ideas



Student Attitudes to Mathematics:

L: How Lecturers *expect* their students to respond, and how they *prefer* their students to respond.

S: How Students respond *before* Problem-Solving, *after* a Problem-Solving course,

% of lecturers expecting/preferring a positive response

% of students giving a positive response

Students’ comments on mathematics

Since following the course I know mathematics is about solving problems. But whatever mathematics I am doing now doesn’t allow me to do all those things. They are just more things to be remembered.

I believed mathematics is useful in that it helps me to think. Having said that it is hard to say how I can do this with the maths I am doing. Most of the questions given can be solved by applying directly the procedures we had just learned. There is nothing to think about.

At the moment I am finding difficulty with maths because I am just not enjoying it. Too much emphasis is put on getting the right answer and not on method and understanding.

I did not enjoy most of the maths courses—too dependent on the lecturers. I don’t find the way most of them teach particularly inspiring. We find ourselves hurrying through to keep up. *There is no time to think about the mathematics we are doing.*

Students’ Comments on Problem-Solving

The problem solving techniques help me come to terms with the abstract nature of the maths I am doing. I try to connect the ideas together and talk about them with my friends. It is not that easy though. But I felt all the effort worth it when I am able to do so.

I find the problem solving knowledge very useful in helping me understand the whys and the hows of advanced mathematics. It is much more satisfying than rote-learning. Furthermore it is actually easier to remember something that you understand.

Maths has always given me a lot of problems because I don’t have the ability for memorisation. … Now that I know about mathematical thinking, my interest and desire to learn maths have increased.

This is the first time that I have actually used maths to think. Before I just learnt maths to pass the exam.

Using problem-solving methods in formal mathematics.

1. Giving meaning to definitions by relating it to other ideas (what (other) things do I know that might be related, what do I want (in the definition), what links do I introduce to give it meaning).

2. Enrich the meaning of definitions by relating them to the results that can be proved (as theorems)

3. Use the results of the theorems (known) to build up other required ideas (wanted) by introducing links.

4. Reflect on the ideas and try to get them in a coherent package with easily handled concepts with a lot of internal links.

6. Get an overall grasp of the theory when revising.

In the exam:

7. Read the question carefully to find out *precisely* what is given (known) and what is wanted. Answer the question that you are asked as clearly and concisely as possible.