# Real Mathematics, Real Learners, and Real Teachers

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*In this presentation, I consider the nature of **real mathematics**, in terms of its direct meaning with everyday life as compared with the underlying relationships that give power in making calculations and solving problems. This in turn takes us on to the nature of the "real learner" (or the wide spectrum of **real learners** in our mathematics class-rooms). Does such a learner develop through successive stages, adding more an more knowledge as represented by the increasing levels of the National Curriculum, or are there deeper subtleties in learning which need to be taken into account? Here we consider the spectrum of different ways in which children learn and the discontinuities in the curriculum which require significant reorganisation in knowledge to handle such topics as handling fractions, negatives, powers, algebraic expressions, limits, proof. This will bring us on to the need for **real teachers** who have an insight into the ways in which real learners are faced with the challenge of learning real mathematics.*

# Real Mathematics

# Everyday Mathematics

Different meanings of the word “real”

from HMIs, from Mathematicians, from Teachers, from Parents, from different individuals in these and other groups.

“real life”… “real numbers”.

… in most areas of East Anglia up to the end of the last century the farmer ploughed, reaped and harvested with men and animals as he had done since the beginning of history. … In the old prior culture the language of the ordinary unlettered people … was singularly concrete, free from most abstractions. … They would, for instance, rarely talk of early summer, but of beet-singling time … The old dialect speakers relate all states or qualities to objects or persons: and this concreteness gives the dialect a poetic quality that is full of images which capture and hold the interest of the listener. … The younger generation have to some degree ceased to use the images that a listener finds so easy to translate and give visual form to as he concentrates on what the speaker describes.

George Ewart Evans (1970),
*Where beards wag all:*
*the relevance of the oral tradition,*
London: Faber & Faber.
pp. 177,178.

Real (everyday) problem:

Divide 3 pizzas equally between 4 people.

Divide 3 pizzas equally between 5 people.

What is the connection between the everyday world and (real) mathematics?

1. Street Maths

2. Mental Arithmetic

3. Find the y-intercept of $3x+4y=12$.

$4y=-3x+12$

$y = -3/4x + 3$.

4. Amy's older brother Ben is now three times as old as her. In four years he will only be twice as old. How old is Amy now?

**$3x+4 = 2(x+4)$.**

| | |
|---|---|
| | $3x+4 = 2(x+4)$ |
| *expand brackets:* | $3x+4 = 2x+2\times 4$ |
| *simplify:* | $3x+4 = 2x+8$ |
| *subtract* 4 *from both sides:* | $3x+4-4 = 2x+8-4$ |
| *simplify:* | $3x = 2x+4$ |
| *subtract x from both sides:* | $3x-x = 2x+4-2x$ |
| *simplify:* | $x = 4$. |

Experience leads to compression of several steps into one, e.g.:

| | |
|---|---|
| | $3x+4 = 2(x+4)$ |
| *expand brackets:* | $3x+4 = 2x+8$ |
| *subtract* 4 *from both sides:* | $3x = 2x+4$ |
| *subtract 2x from both sides:* | $x = 4$. |

MATHEMATICS INVOLVES RELATIONSHIPS THAT MAY NOT BE CONNECTED TO THE REAL WORLD

MATHEMATICS INVOLVES RELATIONSHIPS THAT MAY NOT BE *RECOGNISED* AS BEING CONNECTED TO THE REAL WORLD

Keeping a real-world referent for our thinking processes is a primeval part of our nature. But the "real" world of mathematics grows beyond primitive human nature.

Great book to *read*: "The Number Concept" by Stanislaw Dehaene (1998, £20 hardback).

# Real Learners

Do learners naturally "grow" through successive levels of increasing sophistication?

Is it a *continuous* or a *discontinuous* experience?

Learners growing in experience through changing contexts. eg counting, performing arithmetic operations on whole numbers, then fractions, decimals, algebraic expressions, irrational numbers, etc.

Example 1: Fractions. Earlier example of pizzas and fractions. The wisdom of Solomon.

Example 2. Powers. The meaning of:

$$2^3, 2^{3+2}, 2^0, 2^{1/2}, 3^{-2}.$$

Example 3. Algebraic expressions, the meaning of

$$2+3x, x^n, e^{x^2}.$$

Example 4. Limits

The use of symbols as process to *do* mathematics and as concepts to *think about* mathematics.

(Flexibility and the notion of *procept*).

Compression from process to concept. (eg counting, addition, multiplication, fraction, algebraic expression, powers, limits etc)

Getting some of it done – the procedural trap.

Changing contexts and meaning – the trap of learning to *do* but not to *think about*.

Learning How and Learning Why: procedural and conceptual learning.

Procedures both as an introduction and a deficit fallback.

Procedural bugs: $\sqrt{9a^{16}} = 3a^4$, $\frac{a^{12}}{a^3} = a^4$, $3^2 \times 4^5 = 12^7$

Do we need to learn tables? Yes ... but … 121, 56.

The conceptual divide, seemingly forcing students who start to fail in personal "sense-making" to resort to rote-learning without relating one procedure to another.

One teacher's success may be the next teacher's burden.

# Real Teachers

| Teacher's beliefs | | |
|---|---|---|
| **Transmission** | **Discovery** | **Connectionist** |
| **about pupils *being numerate*** | | |
| primarily the ability **to perform calculations** by **standard procedures** | **finding the answer** to a calculation **by any method** | **calculation methods** which are both **efficient and effective** |
| heavy reliance on **paper&pencil methods** | heavy reliance on **practical methods** | confidence & ability in **mental methods** |
| **how pupils *learn* to become "numerate"** | | |
| **individual activity** based on **following instructions** | **individual activity** based on **actions on objects** | **interpersonal activity** through purposeful **interaction with others** |
| **pupils vary in their *ability* to become numerate** | **pupils vary in the *rate* at which their numeracy develops** | ***most* pupils are able to become numerate** |
| pupils learn through being introduced to one mathematical routine at a time and remembering it | pupils need to be 'ready' before they can learn certain mathematical ideas | pupils learn through being challenged and struggling to overcome difficulties |
| **how to *teach* pupils to become "numerate"** | | |
| **Teaching has priority over learning** | **Learning has priority over teaching** | Teaching and learning are **complimentary** |
| teaching is based on ***verbal explanation*** so that pupils understand the teacher's methods | teaching is based on ***practical activities*** to enable pupils to discover methods for themselves | teaching is based on ***dialogue*** between teacher and pupils to explore understandings |
| application is best approached through 'word problems' | application is best approached through using practical equipment | Application is best approached through challenges that need to be reasoned about |

Teacher orientation, selected from Askew, M., Brown, M., Rhodes, V., Johnson, D., Wiliam, D. (1997) *Effective Teachers of Numeracy, Final Report of a study carried out for the Teacher Training Agency 1995–96 by the School of Education, King's College, London*. London: King's College.

| | **Highly Effective** | **Effective** | **Moderately Effective** |
|---|---|---|---|
| Strongly Connectionist | Anne<br>Alan<br>Barbara<br>Carole<br>Faith | | |
| Strongly Transmission | | | Beth<br>Cath<br>Elizabeth |
| Strongly Discovery | | | Brian<br>David |
| No Strong Orientation | Alice | Danielle<br>Dorothy<br>Eva<br>Fay | Erica |

The relation between teacher orientation and effectiveness, adapted by selecting items from Askew *et al*, 1997, pp.31, 32

| Question type | Reflective | Prescriptive | Open-ended |
|---|---|---|---|
| Graphical | 68% | 31% | 39% |
| Non-graphical | 23% | 17% | 16% |

The relation between presentation style and effectiveness
(% correct on post-test of those who were incorrect on pre-test)

(selected from Beare. R. A. (1996): An investigation of different approaches to using a graphical spreadsheet model of population dynamics, *International Journal of Mathematics Education in Science and Technology*, 27 (4), 583–598.