Proof and the transition from Elementary to Advanced Mathematical Thinking

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Proof in the Mathematical Community

In the fall of 1982, Riyadh, Saudi Arabia … we all mounted to the roof … to sit at ease in the starlight. Atiyah and MacLane fell into a discussion, as suited the occasion, about how mathematical research is done. For MacLane it meant getting and understanding the needed definitions, working with them to see what could be calculated and what might be true, to finally come up with new "structure" theorems. For Atiyah, it meant thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs. This story indicates the ways of doing mathematics can vary sharply, as in this case between the fields of algebra and geometry, while at the end there was full agreement on the final goal: theorems with proofs. Thus differently oriented mathematicians have sharply different ways of thought, but also common standards as to the result.

(Maclane, 1994, p. 190–191.)

Concept image (of proof)

“The concept image the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.” (Tall & Vinner, 1981).

It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advanced guard.

[Poincaré, 1913]

Weierstrass leads everything back to the consideration of series and their analytic transformations; to express it better, he reduces analysis to a sort of prolongation of arithmetic; you may turn through all his books without finding a figure. Riemann, on the contrary, at once calls geometry to his aid; each of his conceptions is an image that no one can forget, once he has caught its meaning.

[ibid, page 212]

Logical thinkers have their own kind of intuition:

... When one talked to M. Hermite, he never evoked a sensuous image, and yet you soon perceived that the most abstract entities were for him like living beings. He did not see them, but he perceived that they are not an artificial assemblage and that they have some principle of internal unity.

[Poincaré]

We then have many kinds of intuition; first, the appeal to the senses and the imagination; next, generalization by induction, copied, so to speak, from the procedures of the experimental sciences; finally we have the intuition of pure number... [ibid., page 215.]

Mathematical proof for students

... Among our students we notice the same differences; some prefer to treat their problems ‘by analysis’ others ‘by geometry.’ The first are incapable of ‘seeing in space’, the others are quickly tired of long calculations and become perplexed. [Poincaré, 1913, page 212.]

Krutetskii (1976, p.178):

‘very capable’ (‘mathematically gifted’),

‘capable’,

‘average’

‘incapable’.

spectrum of performance from:

(incapable): incidental irrelevant detail, with long, often erroneous, inflexible solution procedures

(very capable); remembered general strategies rather than detail, focussed on essential elements and were able to provide alternative solutions.

Gifted also classified as
Analytic (6 children)
Geometric (5 children)
Harmonic (23 children) using both

(compare this with the analysis of Poincaré)

Experiences with symbols

- **count-all** as a process of counting 4, then counting 5 then counting all:

  4 + 5 count 4, count 5, count-all

  (all counting *processes)*

- **count-on**, starting from 4, to count-on 5:

  4 + 5 : number 4, count-on 5

  (one a *number* concept, one a counting *process*)

- **known fact** that 4+5 is 9:

  4 + 5 number 4, number 5, result 9

  (all three *number* concepts)

- **derived fact** using other facts:

  4+5 is "one more than 8".

  (because 4+4 is known to be 8)

Number is *both* process *and* concept.

The *counting* process is required initially to be able to calculate.

Thinking of a number as a *concept* is essential to be able to manipulate it as a mental object.

Those who flexibly move between *process* and *concept* have a more powerful method of thinking.

Those who see number only as counting procedures face greater problems.

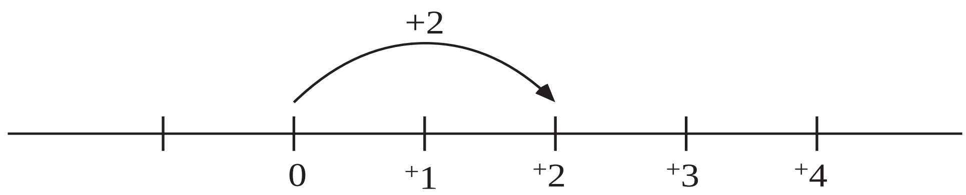
eg what is 19–16 ?

Flexible: the ones (tens) cancel, so it’s 9–6, which is 3.

Procedural: count back … eighteen, seventeen, sixteen, …, (16 numbers in all) ... three!

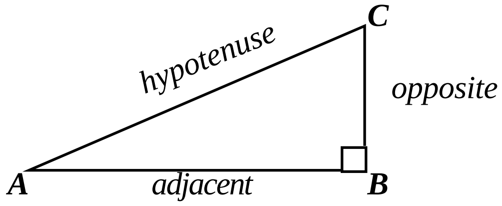
Mathematical symbols representing both process and concept:

- 3+2 represents both the process of addition and the concept of sum,
- 3x2 represents both the process of multiplication (repeated addition) & the concept of product,
- +2 represents both the procedure of "add two" (or shift two units to the right on the number line) and also the concept of a positive signed number,



- –2 represents both the procedure of "subtract two" (or shift two units to the left) and also the concept of negative number,
- $\frac{3}{4}$ represents both the process of division and the concept of fraction,

- $\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$ represents both the process of calculating the trigonometric ratio and also the concept of sine,



- π=3.14159… represents the concept π as a process of approximation.

The following represent both the process of tending to a limit and the value of that limit:

- $$\lim_{x\to 1}\frac{x^2-1}{x-1}$$
- $$\lim_{n\to\infty}\frac{1-x^n}{1-x}$$
- $$\sum_{n=1}^{\infty} a_n$$
- $$\lim_{\Delta x\to 0}\sum_{x=a}^{b} f(x)\,\Delta x.$$

The notion of procept

> The amalgam of *pro*cedure and con*cept* which is represented by the same notation is defined to be a *procept*.

I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation.

William P. Thurston, Fields Medallist, 1990.

Long-term difficulties with symbols

symbols operate in subtly different ways:

(1) ***arithmetic procepts***,

$5+4$, $3x4$, $\frac{1}{2}+\frac{2}{3}$, $1{\cdot}54 \div 2{\cdot}3$,

have explicit algorithms to obtain an answer.

(2) ***algebraic procepts***,

e.g. $2+3x$,

has ***potential process*** (numerical substitution), but can be manipulated. What about:

$$2^3 = 2\times2\times2 \qquad 2^{1/2}=???$$

$$x^m\, x^m = x^{m+n} \qquad x^{1/2}x^{1/2} = x^1$$

(3) ***limit procepts***,

e.g. $\lim_{x\to a}\frac{x^3-a^3}{x-a}$, $\sum_{n=1}^{\infty}\frac{1}{n^2}$

have a ***potentially infinite process*** “getting close, and may not be computable in a finite number of steps.

Formal definitions & proof	*defined* properties *logical* processes *formally constructed* concepts
calculus	*computational* processes (rules), *manipulable* concepts (formulas)
(dynamic limit concept)	*potentially infinite* processes, *arbitrarily small, close,* or *large* concepts (variable quantities)
algebra	*potential* processes (evaluating expressions) *manipulable* concepts (expressions)
arithmetic	*computational* processes *computational* concepts (numbers)

Different kinds of characteristics of processes and concepts in selected topics

Consider the cognitive reconstructions involved, particularly from symbol manipulation to definitions and proof.

Proof in elementary mathematics

"knowing" arithmetic facts are "true" from experience

eg 2+2 = 4 or $2^{10} > 10^3$

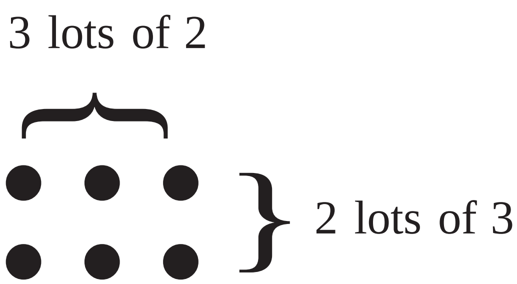
Various everyday notions of proof:

eg in law, by scientific experiment, by statistics,

Thought experiment: *Imagine assumptions are true and "see" if the conclusions follow.*

Proof by generic example:

Eg Generic visual proof that $m \times n = n \times m$



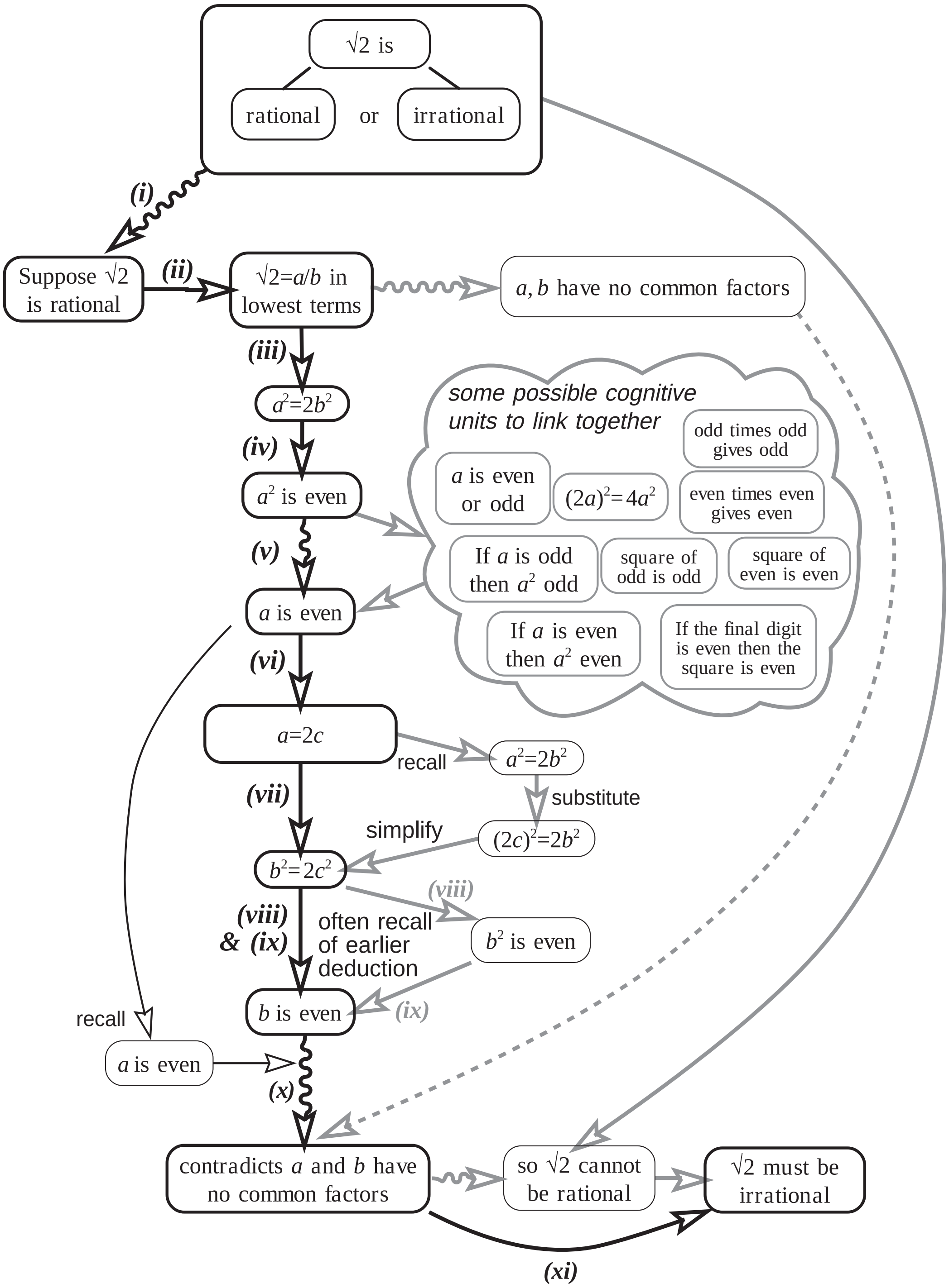
Euclidean verbal proof (with underlying generic visual examples):

Eg that a triangle with two equal sides is a triangle with two equal angles.

Algebraic computations to represent general arithmetic statements:

$$(n-1)(n+1)+1 = n^2-1^2+1 = n^2.$$

The irrationality of √2 (outline and detail)



Student S:

I'd take the case where I assumed it was a rational and fiddle around with the numbers, squaring, and try to show that … if it was rational then you'd get the two ratios *a* and *b* both being even so they could be subdivided further, which we'd assumed earlier on couldn't be true so our assumption it was rational can't be true.

He wrote:

$$\left(\frac{a}{b}\right)^2 = 2,$$

$$a^2 = 4b^2,$$

"I think that's what he did, but he did it in one step whereas normally I would've taken two.

(Barnard & Tall, 1997)

Sub-proof by contradiction:

a^2 is even implies a is even,

Student S could remember the step but not the argument

"the root of an even number is even, he [the lecturer] just assumed it."

The square root of 4 and the square root of 16 are even. fact was a *consequence* of the other.

The contradiction proof conflicts with previous experience.

Generic proof:

Factorise numerator and denominator in lowest terms:

$$\frac{9}{40} = \frac{3^2}{2^3 \times 5}$$

squaring *doubles* the number of each prime factor:

$$\left(\frac{9}{40}\right)^2 = \frac{3^2}{2^3 \times 5} \times \frac{3^2}{2^3 \times 5} = \frac{3^4}{2^6 \times 5^2}$$

The square of any fraction in never equal 2=2/1 which has an *odd* number of 2's in the numerator.

Students prefer generic proof to standard proof. It is more "explanatory" and easier to generalise, eg to showing √(5/8) is irrational.

Consider appropriateness of generic proof and standard proof by contradiction …

The transition from thought experiment to formal proof

1 DEFINITIONS,

2 DEDUCTIONS,

5 SYSTEMATIC THEORY.

???

The mutual interplay:

DEFINITIONS <—> DEDUCTIONS

Bills & Tall (1998) defined:

A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

Observations of research:
Many students do not make definitions operable, they often work with their concept image.

Eg ***Sean*** thought the supremum of a set was the "biggest element in the set".

For instance, the biggest element in the set S of $x < 1$ is 0.999 recurring…

Lucy was very successful in discussing the concept definition, but at the end of the course she could not always write it out precisely.

Concept Image and Deduction of Structure Theorems

Mathematicians construct properties of concepts by proving structure theorems about them, giving the concept a ***formal (concept) image.***

Eg a complete ordered field is isomorphic to the real numbers,
Any finite dimensional real vector space is isormorphic to R^n,
Any finite group is isomorphic to a subgroup of a permutation group.

This gives concept imagery to the formally deduced image, and enables the mathematical individual to build on it.

But how do students cope?

Students giving and extracting meaning

Research by Marcia Pinto (1998) using methods of “grounded theory” (Strauss, 1987).

Data collected under headings that arise and are modified using ongoing data obtained.

Selected 15 students to interview at 3 week intervals.
Take into account spectrum of success & failure and also types of student (particularly “harmonic” in Krutetskii and procedural symbolic).

Initial headings using earlier mathematical analysis:

- DEFINITIONS,
- DEDUCTIONS,
- SYSTEMATIC THEORY.

The were inappropriate for the pilot study group. (trainee teachers grade around C in A-level.)
18 out of 20 used concept imagery not definition.

Revised headings *(based on data analysis)*

- DEFINITIONS,
- DEDUCTIONS,
- MISCONCEPTIONS.

***Final modification*:**

- DEFINITIONS,
- DEDUCTIONS,
- IMAGES.

organised like this:

DEFINITIONS <—> DEDUCTIONS

↕

IMAGES

Students building operable definitions and corresponding deductions

There is not a single way of building a coherent definition, given that it is built upon individual concept images. Pinto (1998) found the data suggesting two recurring possibilities:

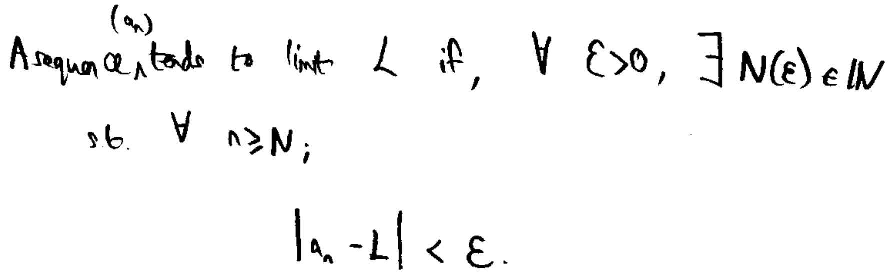
- *giving meaning to* the concept definition from concept image (using *thought experiments, perhaps translated into formal proof),*
- *extracting meaning from* the concept definition through using it to make formal deductions (using *formal proof*).

Each piece of data under

DEFINITIONS-DEDUCTIONS/IMAGES

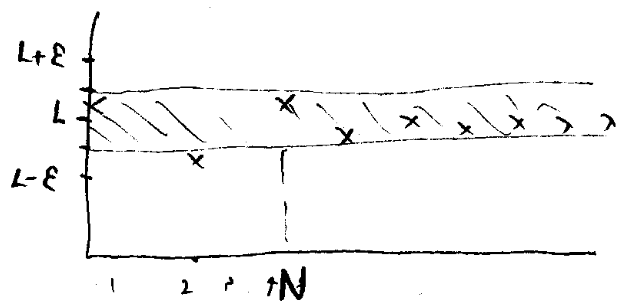
Is analysed to see if it involves giving or extracting meaning.

ROSS: EXTRACTING MEANING FROM THE DEFINITION



(Ross, first interview)

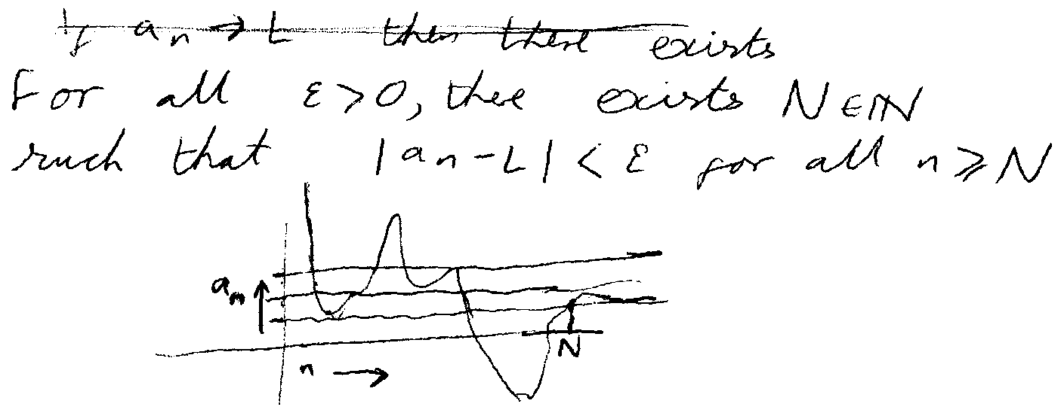
"**Just memorizing it**, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by **writing it down over and over again it get imprinted** and then I remember it."



(Ross, first interview)

"Well, before, I mean before I saw anyone draw that, it was just umm ... thinking basically as n gets larger than N , a_n is going to get closer to L so that the difference between them is going to come very small and basically whatever value you try to make it smaller than, if you go far enough out then the gap between them is going to be smaller. That's what I thought before seeing the diagrams and something like that."

CHRIS : GIVING MEANING FROM IMAGERY
(THOUGHT EXPERIMENT LEADING TO FORMAL)



(Chris, first interview)

“**I don’t memorize that** [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down.”

“**I think of it graphically** ... you got a graph there and the function there, and I think that it’s got the limit there ... and then ε once like that, and you can draw along and then all the ... points after *N* are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that’s the *n* going across there and that’s a_n. ... This shouldn’t be a graph, it should be points.”

NEGATING BY MANIPULATION (EXTRACTING MEANING FROM DEFINITION)

The definition of a sequence that doesn't tend to a limit

Ross wrote,

$$\forall \varepsilon > 0, \ \exists N(\varepsilon) \in \mathbb{N} \text{ s.t. } \forall n \geq N.$$
$$|a_n - L| < \varepsilon.$$

then negated it by using the formal negation of quantifiers:

$$\forall L, \ \exists \varepsilon > 0 \text{ s.t. } \forall N(\varepsilon) \in \mathbb{N} \ \exists n \geq N, \text{ s.t.}$$
$$|a_n - L| \geq \varepsilon$$

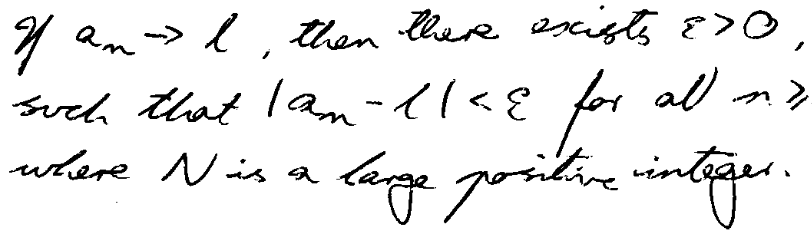
(Ross, second interview)

CHRIS NEGATED BY THINKING IT THROUGH MEANINGFULLY, GIVING MEANING.

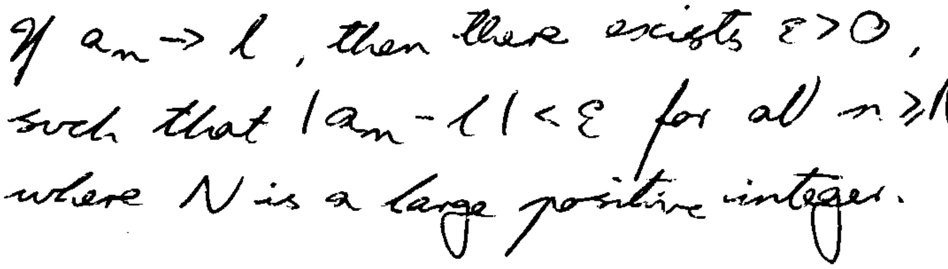
"A sequence (a_n) does not tend to a limit if for any L , there exists $\varepsilon > 0$ such that $|a_n - L| \geq \varepsilon$ for some $n \geq N$ for all $N \in \mathrm{N}$."

(Chris, second interview)

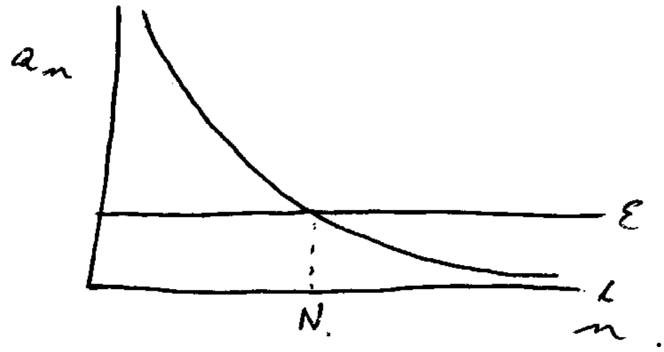
Less successful students



(Robin, first interview)



(Colin, first interview)



He had a restricted mental image of a decreasing function or sequence:

“.. umm, [I] sort of imagine the curve just coming down like this and dipping below a point which is ε ... and this would be N. So as soon as they dip below this point then ... the terms bigger than this [pointing from N to the right] tend to a certain limit, if you make this small enough [pointing to the value of ε].”

Unsuccessful Negation

"A sequence a_n does not tend to the limit L if for any $\varepsilon > 0$, there exists a positive integer N s.t. $|a_n - L| > \varepsilon$, whenever $n \geq N$"

(Robin, second interview)

(original quantifiers for the definition of limit unchanged, negates the inner inequality $|a_n–L| < \varepsilon$ to (incorrectly) get $|a_n–L| > \varepsilon$.
Unable to treat the whole definition as a meaningful unit, and focuses on the inner statement as something he can handle.

- -

"Umm ... I would just say there doesn't exist a positive integer because we can't work it out ... no ... you cannot find an integer N ...".
& writes:

There exists a term where $|a_n - L| \geq \varepsilon$ where $n \geq N$, where N is a positive integer.

(Colin, second interview)

Further along the spectrum

Other students remembrances of the definition of limit of a sequence after 2 weeks.

(a) Given $\varepsilon > 0$, $\exists$ a δ s.t. ......

(b) given $\varepsilon > 0$ $\exists l \in \mathbb{R}$ s.t. $|l - x_n| \leq \varepsilon$

(c) Gn $\varepsilon > 0$ $\lim_{n \to \infty} S_n = l$

~~[illegible]~~ series ---- tend to a limit

$\varepsilon + \lambda$ $\varepsilon - \lambda$.

$S_n \to$ limit s't $\varepsilon + \lambda$, $\varepsilon - \lambda$

where λ

$$\lim_{n \to \infty} s_n = l$$

(d) as $n \to \infty$ the terms approach and get closer + closer to l (or may reach it) but l is not exceeded

ie $S_n - S_{n+1} \to 0$ as $n \to \infty$

I can do examples with numbers and things like that but I can't do things with definitions. I just don't know what they are about.

I mean it's not as if these things are real. You need to swat them up to pass exams but you are never going to use them again.

When I try to prove things in my teaching, I show examples and do it in particular cases. In school I'll never have to teach stuff like this.

I work at the example sheets but after a while I get so mad, all I want to do is throw the papers all over the kitchen.

Two ends of the spectrum responding to the question:

"Why do you think the lecturer introduced the axioms for the real numbers?"

Well, when you prove things properly you need to say exactly where to start, what it is you are assuming, and that is what the axioms are for.

I dunno really. I've seen most of it before. I knew most of this stuff when I was about five.

The second student lives in the real world where his pragmatic grasp of economics will probably earn him a higher salary than a research mathematician; he has no conception of the world of formal definitions distinct from his powerful concept imagery.

Summary

- mathematicians use different cognitive techniques to generate new theorems.
  E.g. Geometric/Symbolic etc
- depends on individual concept images.
- Student concept image includes manipulation, & generic proof.
- transition from elementary maths to formal proof is a huge transition for most students.
- Success comes in (at least) two ways: *giving* meaning working from concept image, *extracting* meaning working formally from the definition.
- Many students fail. Some at least have a concept image which allows some kind of generic proof. Others can make no sense of the complex quantified statements.
- A fall-back strategy to attempt to pass exams is to learn proofs by rote.
- The teaching and learning of formal proof remains an important component of theory building in advanced mathematical thinking.
- many who intend to teach mathematics to young children are unable to appreciate the full extent of mathematical thinking including proof.
- *Do all students require formal proof or only those who will be mathematics specialists? Is generic proof enough for some?*