Keynote presentation for RUME Conference
San Diego, February 24th 2007.

The overheads are laid out reading left to right along successive rows.

THE UNIVERSITY OF WARWICK

Embodiment, Symbolism and Formalism in Undergraduate Mathematics Education

David Tall

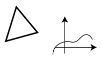
Undergraduate mathematics education includes the transition from the **geometric** and **symbolic** mathematics at school to the **formal** constructions of axiomatic systems in mathematics research.

To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"? Hilbert 1900 ICM lecture.

A simple framework that begins with the young child and extends to the research mathematician

From the embodiment and symbolism of school to the formal mathematics of axioms and proof

- **conceptual-embodied** (based on perception of and reflection on properties of objects);
- **proceptual-symbolic** that grows out of the embodied world through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts);
- **axiomatic-formal** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.



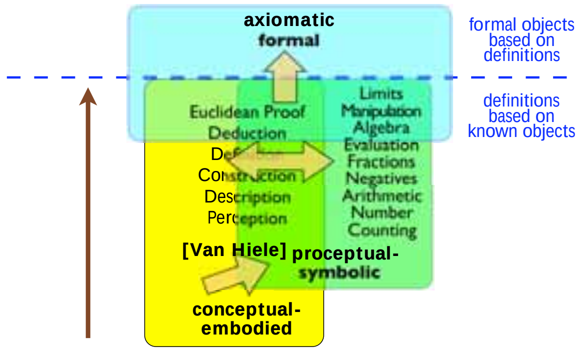
$3+4$

$\int \sin x \, dx$

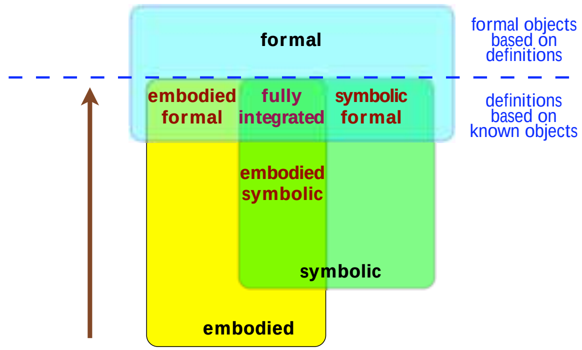
$\mathbb{R}$

$\aleph_0$

Three Worlds of Mathematics



Three Worlds of Mathematics



Compression into thinkable concepts

Categories, Prototypes (Lakoff)

Process-object (Piaget, Dubinsky, Sfard etc)

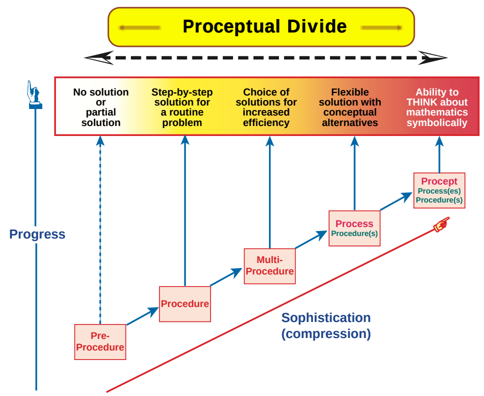
Axioms-structures (Hilbert, Bourbaki)

Structure of **O**bserved **L**earning **O**utcomes (Biggs, Collis)
unistructural, multi-structural, relational, extended abstract

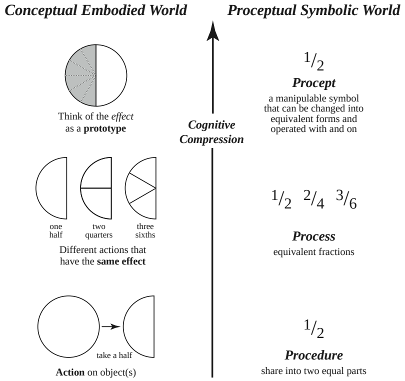
Symbolic Compression

SOLO	uni-structural	multi-structural	relational	extended abstract (uni-structural at next level)
procept theory	single procedure	multi-procedure	process	procept
APOS	Action		Process	Object

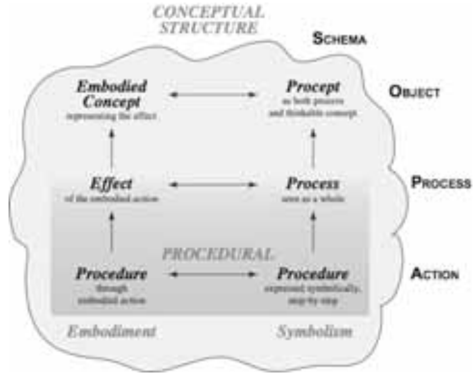
Symbolic Compression



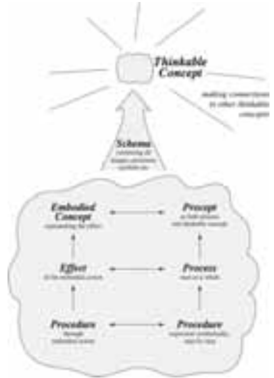
Embodied & Symbolic Compression



Embodied & Symbolic Compression



Embodied & Symbolic Compression



Set-befores & Met-befores

Long-term human learning is based on a combination of facilities **set-before** birth in the genes and builds on successive constructions based on conceptions **met-before** in development.

set-before: 'twoness' set in visual apparatus.

consistent met-before: 2+2=4 from whole numbers is a *consistent* when complex numbers are encountered.

inconsistent met-before: 'take away makes smaller' from everyday objects and whole numbers is an *inconsistent* when negative numbers and infinite cardinals are encountered.

Illustrations of the Framework in Action

College Algebra

Calculus

Proof

College Algebra

For some, audits and root canals hurt less than algebra. Brian White hated it. It made Julie Beall cry. Tim Broneck got an F-minus. Tina Casale failed seven times. And Mollie Burrows just never saw the point. This is not a collection of wayward students, of unproductive losers in life. They are regular people […] with jobs and families, hobbies and homes. And a common nightmare in their past.

Deb Kollar, *Sacramento Bee* (California), December 11, 2000.

College Algebra

Arithmetic builds on *conceptual embodiment:*
collecting into sets to count
Putting sets together to add
Taking away elements to subtract
Sharing into equal subsets to divide
& all arithmetic expressions can be calculated

Algebraic expressions cannot be calculated.
What are the embodiments for
2a+3b+4a, 3a–4b, 2+3x, 4x–2 = 2x+3

Algebra can be a minefield of dysfunctional met-befores

College Algebra

5x+3 = 13 as a process of evaluation
5x+3 = 9x-5 as two processes?
The 'didactic cut' (Filloy & Rojano)

As a balance? (a conceptual embodiment)
Makes sense for positive values (Vlassis 2002)
Negative values?

'Do the same thing to both sides'
'change sides, *change signs*'
'put the 3 in 3x=6 over the other side and *put it underneath*.'

A *functional embodiment* (in the sense of Lakoff?) moving terms around without conceptual meaning.

College Algebra

A study of students taught to 'do the same thing to both sides' (Lima & Tall 2006):

The students performed similarly on 5t–3 = 8, 3x–1 = 3+x.

They did not see the first equation as a process equals number.

They did not imagine the equation as a balance.

Instead, they moved the terms around using rules which were a combination of symbol shifting and 'magic' through a sequence of actions to 'get the answer'.

Consider this in APOS theory,
Lakoff's theory of embodiment, (conceptual/functional),
met-befores in conceptual embodiment & symbolism.

Calculus

Reform Calculus focuses on 3 (or 4) different representations: graphic, symbolic, analytic, (verbal).

A 'three worlds view':

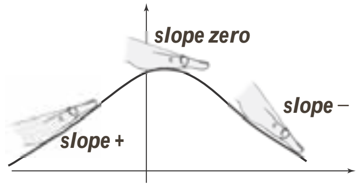
embodied: locally straight
symbolic: locally linear
formal: epsilon-delta limit

Harvard calculus: locally straight = locally linear.

But there is a *huge* difference between the two...

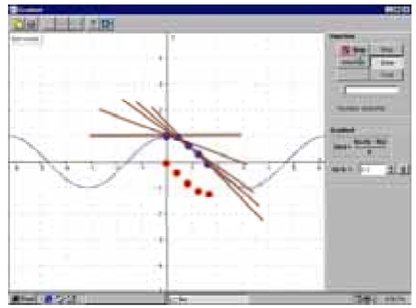
Calculus

Local straightness is *embodied:*
You can see *why* the derivative of cos is *minus* sine



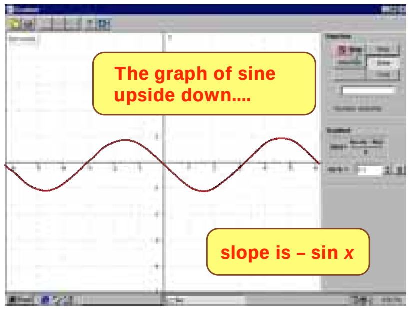
Calculus

Local straightness is *embodied:*
You can see *why* the derivative of cos is *minus* sine



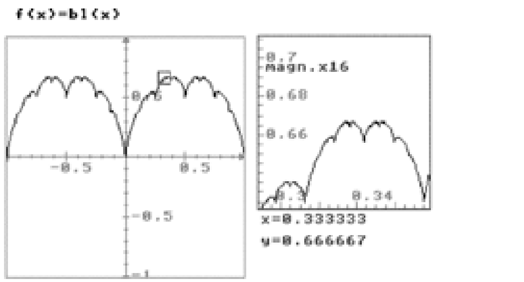
Calculus

Local straightness is *embodied:*
You can see *why* the derivative of cos is *minus* sine



Calculus

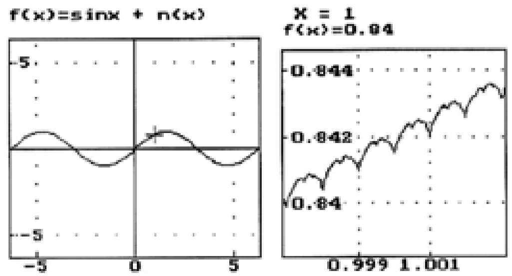
Local straightness is *embodied:*
It gives *non*-examples of differentiability of great insight



Calculus

Local straightness is *embodied:*
It gives reasons to question naïve ideas:

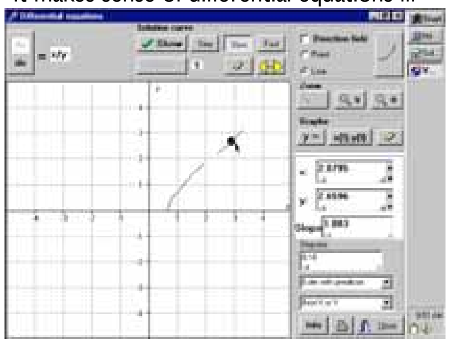
Let n(x)=bl(1000x)/1000



Calculus

Local straightness is *embodied:*

It makes sense of differential equations ...



Calculus

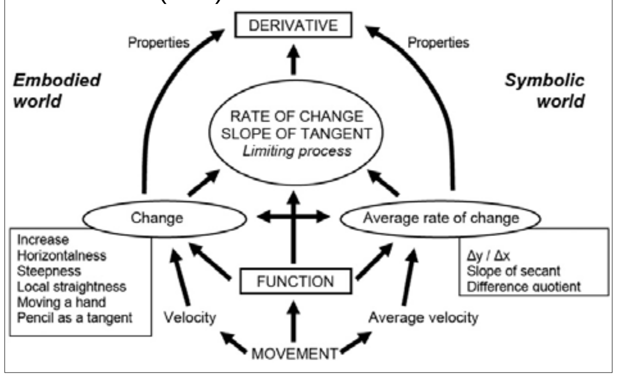
Hahkiöniëmi (2006) studied his own teaching in a framework of embodiment and symbolism.
He found:

- The embodied world offers powerful thinking tools for students.
- The derivative is considered as an *object* at an early stage.
- students followed different cognitive paths, including an embodied route, a symbolic route and a combination of the two.

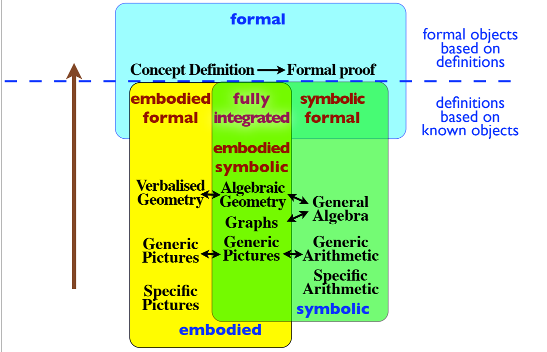
He questioned the use of a symbolic approach using a process-object theory and a too-simple application of Sfard's notion that operational precedes structural.
A structural embodiment gives a meaning for the object that is to be constructed accurately using symbols.

Calculus

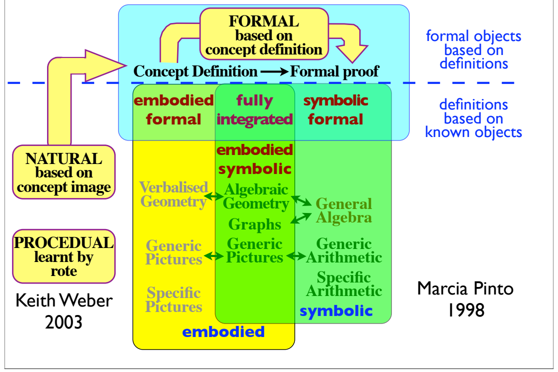
Hahkiöniëmi (2006):



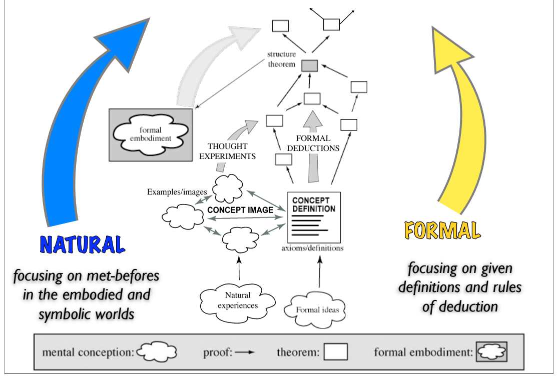
Proof



Proof



Proof



From formal proof back to embodiment & proceptual symbolism

Structure theorems take us back from axiomatic formalism to conceptual embodiment and proceptual symbolism

- *An equivalence relation on a set A corresponds to a partition of A;*
- *A finite dimensional vector space over a field F is isomorphic to F^n;*
- *Every finite group is isomorphic to a group of permutations;*
- *Any two complete ordered fields are isomorphic (to* ***R****).*

In every case, the structure theorem tells us that the formally defined concept has an embodied meaning, and (in 3 cases) a symbolic meaning for manipulation and calculation.

From formal proof back to embodiment & proceptual symbolism

Formal mathematical thinking is supported by met-befores from embodiment and symbolism.

e.g. the number line as an embodiment of ***R***.

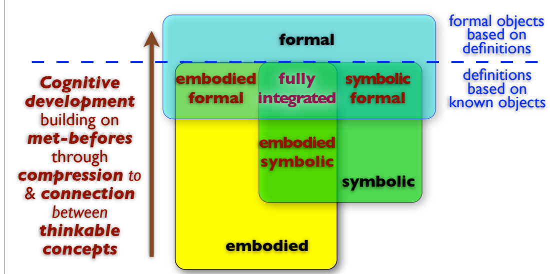
Using Dedekind cuts to 'fill in' the irrationals intimates there is 'no room' for infinitesimals.

Formal Theorem

Let ***K*** be any ordered field extension of ***R***, then ***K*** contains positive elements (positive infinitesimals) x smaller than all positive elements in ***R*** and every element of ***K*** is either infinite (> or < all x in ***R***) or of the form a+e where a is in ***R*** and e is infinitesimal.

Proof: Trivial.

Three Worlds of Mathematics



Mathematicians live in the three worlds building on met-befores, preferring different (combinations of) areas.
A broad framework for mathematical thinking.