Biological Brain, Mathematical Mind & Computational Computers

(how the computer can support mathematical thinking and learning)

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Brain, mind, computer

Biological brain … sensori-motor, visual, linguistic, process

Mathematical Mind … process, symbol, deduction, proof

Computational computer … algorithms, visual display

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THE BIOLOGICAL BRAIN

1990-2000 … the ‘decade of the brain’.

The Number Sense by Stanislas Dehaene (1997)
The Mathematical Brain by Brian Butterworth (1999).

Carter, R. (1998). *Mapping the Mind.* London: Weidenfeld & Nicholson.

Crick, F. (1994). *The Astonishing Hypothesis*, London: Simon & Schuster.

Pinker, S. (1997). *How the Mind Works.* New York: Norton.

Edelman, G. M. (1992). *Bright Air, Brilliant Fire*, NY: Basic Books, reprinted Penguin 1994.

Greenfield, S. (1997). *The Human Brain: A Guided Tour*. London: Weidenfeld & Nicholson.

Devlin. K. (2000), *The Maths Gene: Why everyone has it, but most people don’t use it*, London: Weidenfeld & Nicholson.

Freeman, J. F. (1999) *How the Brain makes up its Mind*. Pheonix.

Edelman, G. M. & Tononi, G. (2000). *Consciousness: How Matter Becomes Imagination*. New York: Basic Books.

> As a task to be learned is practiced, its performance becomes more and more automatic; as this occurs, it fades from consciousness, the number of brain regions involved in the task becomes smaller. (Edelman & Tononi, 2000, p.51)

Making complexity more manageable:

> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) "object(s)" can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once).
>
> (Crick, 1994, p. 61)

This is made more efficient by making the conscious elements as 'small as possible', using words or symbols:

> I should also mention one other property of a symbolic system – its compactibility – a property that permits condensations of the order $F=MA$ or $s = \frac{1}{2}gt^2$, … in each case the grammar being quite ordinary, though the semantic squeeze is quite enormous.
>
> (Bruner, 1966, p. 12.)

MATHEMATICAL MIND

Constructs to describe and explain the cognitive operation of the mathematical mind.

- the ***concept image***, which refers to the total cognitive structure in an individual mind associated with the concept, including all mental pictures, associated properties and processes (Tall & Vinner, 1981),
- a theory of ***cognitive units*** (the mental chunks we use to think with, and their related cognitive structure). (Barnard & Tall, 1997).

A particular type of cognitive unit:

- the notion of ***procept***, referring to the manner in which we cope with symbols representing both mathematical ***pro***cesses and mathematical con***cepts***. (Gray & Tall, 1994). Examples include

$$3+5,\ ax^2+bx+c,\ \frac{d}{dx}\left(e^x \sin x\right), \text{ or } \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

The brain is not configured for rapid & efficient arithmetic (Dehaene, 1997). It uses meaningful links between cognitive units.

Shaker Rasslan: a discussion about an algorithm for divisibilty by seven, e.g. 121, 131, 119.

Mathematical thinking is more than knowing procedures ‘to do’. It involves a knowledge structure compatible with the biological structure of the human brain:

- a huge store of knowledge and internal links,
- coping with many activities using a manageable focus of attention.

Consider, a ‘linear relationship’ between two variables. This might be expressed in a variety of ways:

- an equation in the form $y=mx+c$,
- a linear relation $Ax+By+C=0$,
- a line through two given points,
- a line with given slope through a given point,
- a straight-line graph,
- a table of values, etc

Successful students develop the idea of ‘linear relationship’ as a rich cognitive unit encompassing most of these links as a single entity.

Less successful carry around a ‘cognitive kit-bag’ of isolated tricks to carry out specific algorithms. Short-term success perhaps, long-term cognitive load and failure.

Crowley (2000)

COMPUTATIONAL COMPUTERS

– *complements* human activity.

- The brain performs many activities simultaneously and is prone to error, concept image as support.
- the computer carries out individual algorithms accurately and with great speed. No conceptual baggage.

Arithmetic – algorithms for four rules +, –, ×, ÷ (√ etc)

Algebra – rules for substitution, eg X*(–Y) may be replaced by –(X*Y) where X and Y are expressions.

Derive (an early version) simplified

```
      n   n                                n       n
(x+h)  -x                            (x+h)       x
---------          to give           -----  -  ---  .
    h                                  h         h
```

the limit option as h tends to 0, gave not nx^{n-1}, but

```
 n LN(x)-LN(x/n)
ê                .
```

The current version of *Derive* has added a new rule to give:

```
   n–1
n x    .
```

$\lim_{x\to 0}\frac{\sin x}{x}$ calculated mentally using graphic image.

Computer algebra system may use:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{D(\sin x)}{D(x)}=\frac{\cos 0}{1}=1\ .$$

Relationships between Brain, Mind and Computer

Given the constraints and support in the biological brain, the concept imagery in the mathematical mind can be very different from the working of the computational computer. A professional mathematician with mathematical cognitive units may use the computer in a very different way from the student who is meeting new ideas in a computer context.

Students using *Derive* on hand-held computers to draw graphs of functions were asked:

'What can you say about u if $u=v+3$, and $v=1$?'

None of seventeen students improved from pre-test to post-test and six successful on the pre-test failed on the post-test.

Hunter, Monaghan & Roper (1993)

A symbol manipulator replaces the mathematical procedures of differentiation by the selection of a sequence of procedures in the software. For instance *Derive* requires the user to take the following sequence of decisions carried out by touching the appropriate keys:

- select **Author** and type in the expression,
- select **Calculus**, then **Derivative**,
- specify the variable (e.g. **x**),
- **Simplify** the result.

A comparison of two schools in the UK, one following a standard course, one using *Derive* is as follows:

Please explain the meaning of $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

> … All the students in school *A* gave satisfactory theoretical explanations of the expression but none gave any examples. However, none of the *Derive* group gave theoretical explanations and only two students [out of seven] mentioned the words 'gradient' or 'differentiate'. Four of the *Derive* group gave examples. They replaced $f(x)$ with a polynomial and performed or described the sequence of key strokes to calculate the limit. (Monaghan, Sun & Tall, 1994.)

Students using the symbol manipulator saw differentiation as a sequence of keystrokes in a specific symbolic example rather than a conceptual idea of 'rate of change'.

Cognitive Development of Symbols

(1) ***arithmetic procepts***, $5+4$, 3×4, $\frac{1}{2}+\frac{2}{3}$, $1\cdot 54 \div 2\cdot 3$,

computational as processes and concepts. in the sum 8+6, the concept 6 can be linked to the operation 2+4, which can be combined in the sum 8+2+4 to give 10+4 which is 14.

(2) ***algebraic procepts***, e.g $2+3x$, only a *potential process* (of numerical substitution), yet expressions are treated as *manipulable concepts* using usual algebraic rules.

Meaningful power operations such as

$$2^3 \times 2^2 = (2\times 2\times 2)\times(2\times 2) = 2^5$$

act as a cognitive basis for the power law

$$x^m \times x^n = x^{m+n}$$

(3) ***implicit procepts***, $x^{\frac{1}{2}}$, x^0, x^{-1}, where original meaning of x^n no longer applies, but is implicit in the generalised power law.

(4) ***limit procepts***,

e.g. $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a}$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$.

These have ***potentially infinite processes*** 'getting close' to a limit value, but this may not be computable in a finite number of computations. Limits are often conceived as 'variable quantities' which get arbitrarily close to a limiting value, rather than the limit value itself.

(5) ***calculus procepts***,

e.g. $\dfrac{d(x^2 e^x)}{dx}$, $\int_0^{\pi} \sin mx \cos nx \, dx$

These (may) have *finite operational algorithms* of computation (using various rules for differentiation and integration).

Should the limit procept be the first idea in calculus?

Formal definitions and proof	*defined* properties (axioms and definitions) *logical* processes (deductions) *formally constructed* concepts (theorems)
calculus	*computational* processes (rules) *manipulable* concepts (formulas)
(dynamic limit concept)	*potentially infinite* processes *arbitrarily small, close, or large* concepts (variable quantities)
(implicit algebraic concepts)	*implicit* processes (generalised properties) *manipulable* concepts (generalised expressions)
Algebra	*potential* processes (evaluating expressions) *manipulable* concepts (expressions)
Arithmetic	*computational* processes (arithmetic) *computational* concepts (numbers)

Changing meaning of symbols in arithmetic, algebra and calculus.

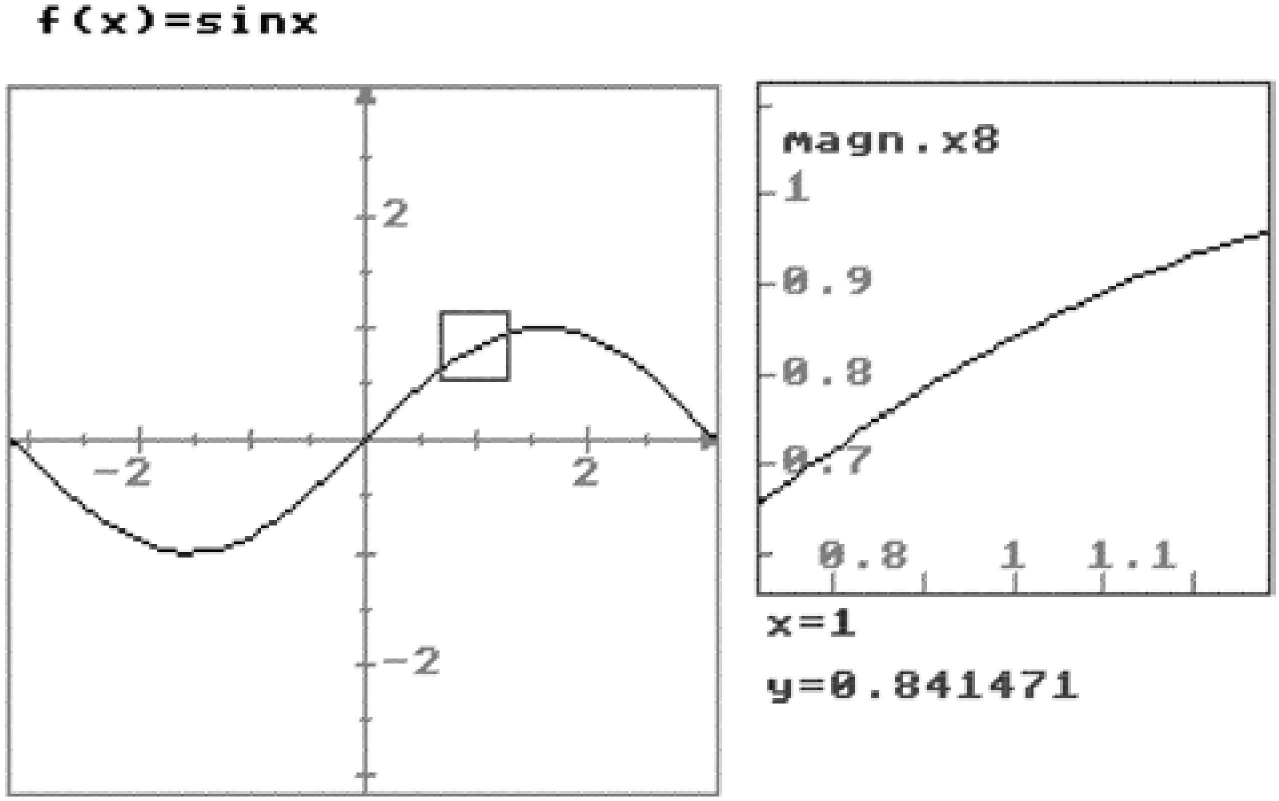
The development of symbol sense throughout the curriculum faces a number of major re-constructions causing increasing difficulties to more and more students as they are faced with successive new ideas that require new coping mechanisms.

For many it leads to the satisfying immediate short-term needs of passing examinations by rote-learning procedures. The students may therefore satisfy the requirements of the current course and the teacher of the course is seen to be successful.

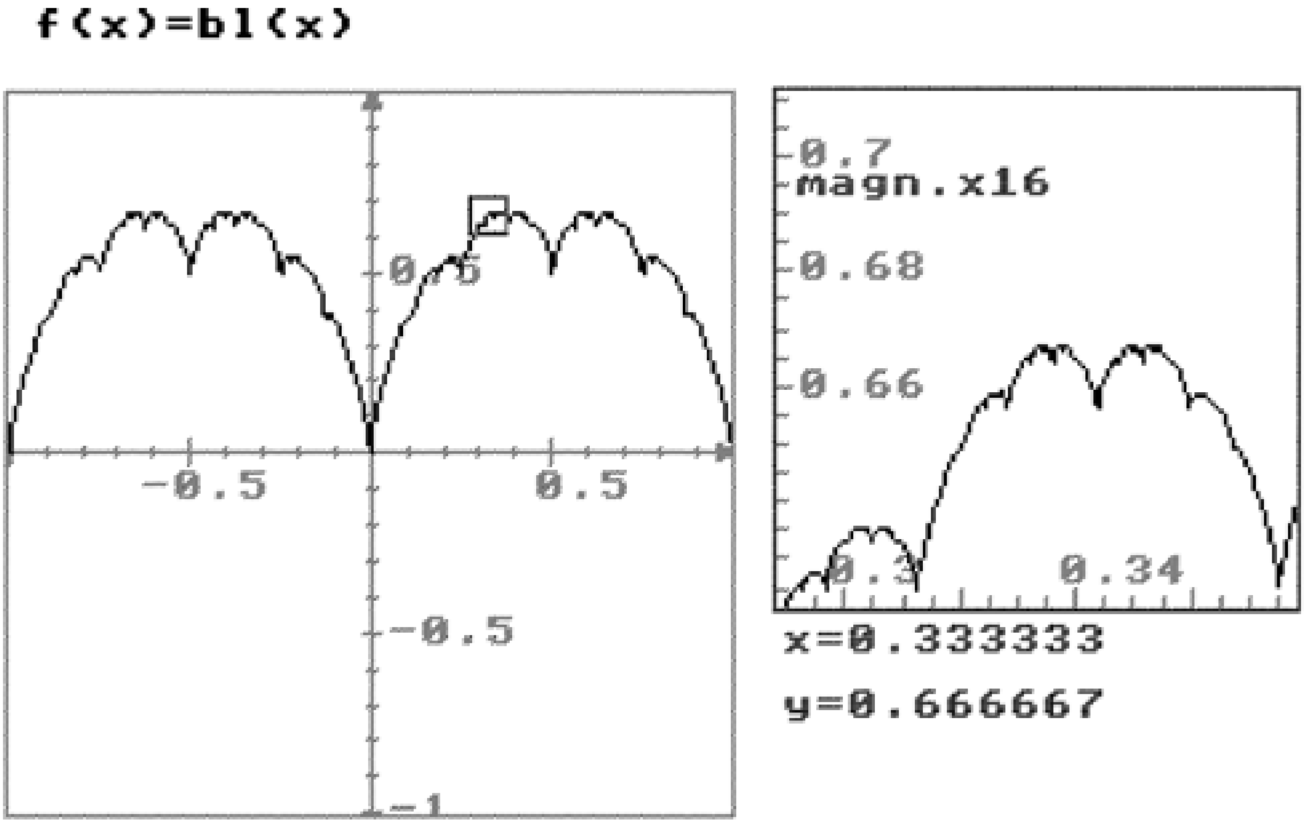
If the long-term development of rich cognitive units is not set in motion, short-term success may only lead to increasing cognitive load and potential long-term failure.

Computer Environments for Cognitive Development

- a *generic organiser* is an environment (or microworld) which enables the learner to manipulate *examples* and (if possible) *non-examples* of a specific mathematical concept or a related system of concepts. (Tall, 1989).
- a *cognitive root* (Tall,1989) is a cognitive unit which is (potentially) meaningful to the student at the time, yet contain the seeds of cognitive expansion to formal definitions and later theoretical development.



Local straightness is a cognitive root for differentiation. The program *Magnify* is a generic organizer for it.

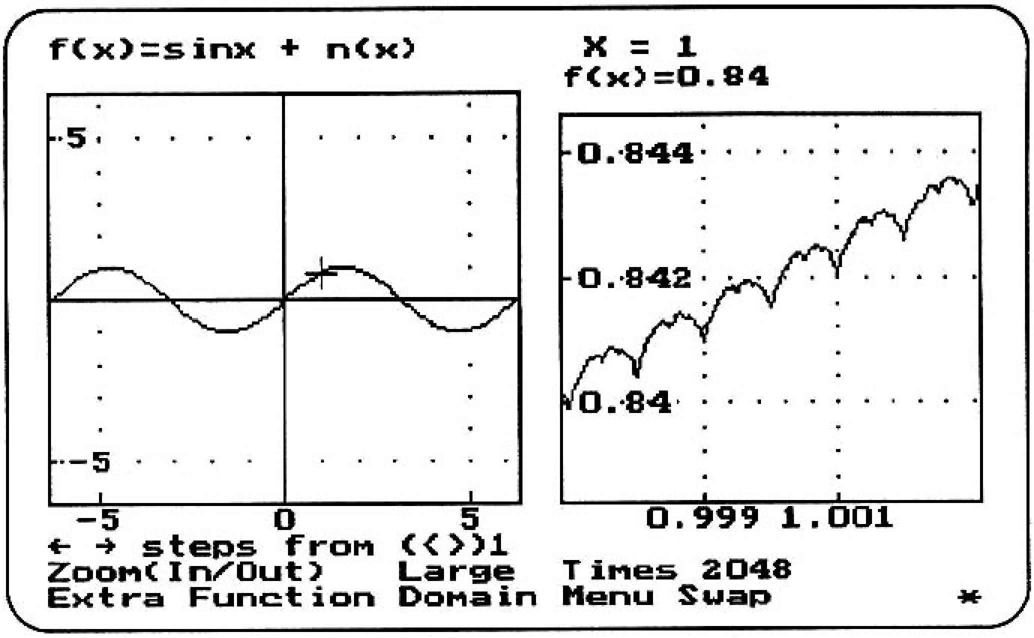


A graph which nowhere looks straight

It is the sum of saw-teeth
s(x)= min(d(x), 1– d(x)), where $d(x) = x$–INTx,

$s_n(x)$=s($2^{n-1}x$)/ 2^{n-1} .

$bl(x) = s_1(x)+ s_2(x)+ s_3(x)+ \ldots$



A ‘smooth-looking curve’ that magnifies ‘rough’.

$n(x)=bl(1000x)/1000$

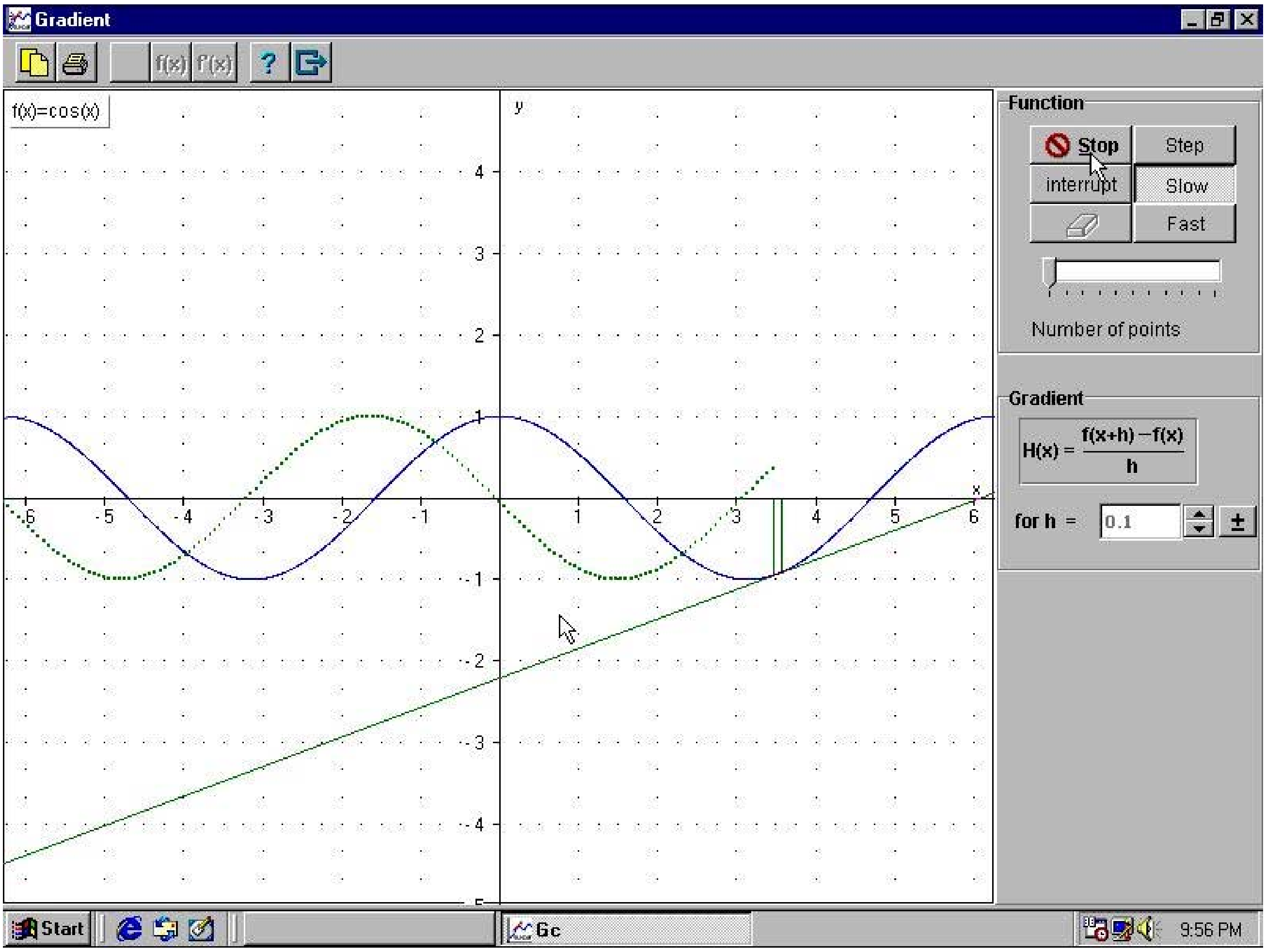
$\sin x$ is differentiable everywhere

$\sin x + n(x)$ is differentiable nowhere!

EMBODIED LOCAL STRAIGHTNESS AND MATHEMATICAL LOCAL LINEARITY

'Local straightness' is a primitive human perception of the visual aspects of a graph. It has global implications as the individual looks *along* the graph and sees the changes in gradient, so that the gradient of the whole graph is seen *as a global entity*.

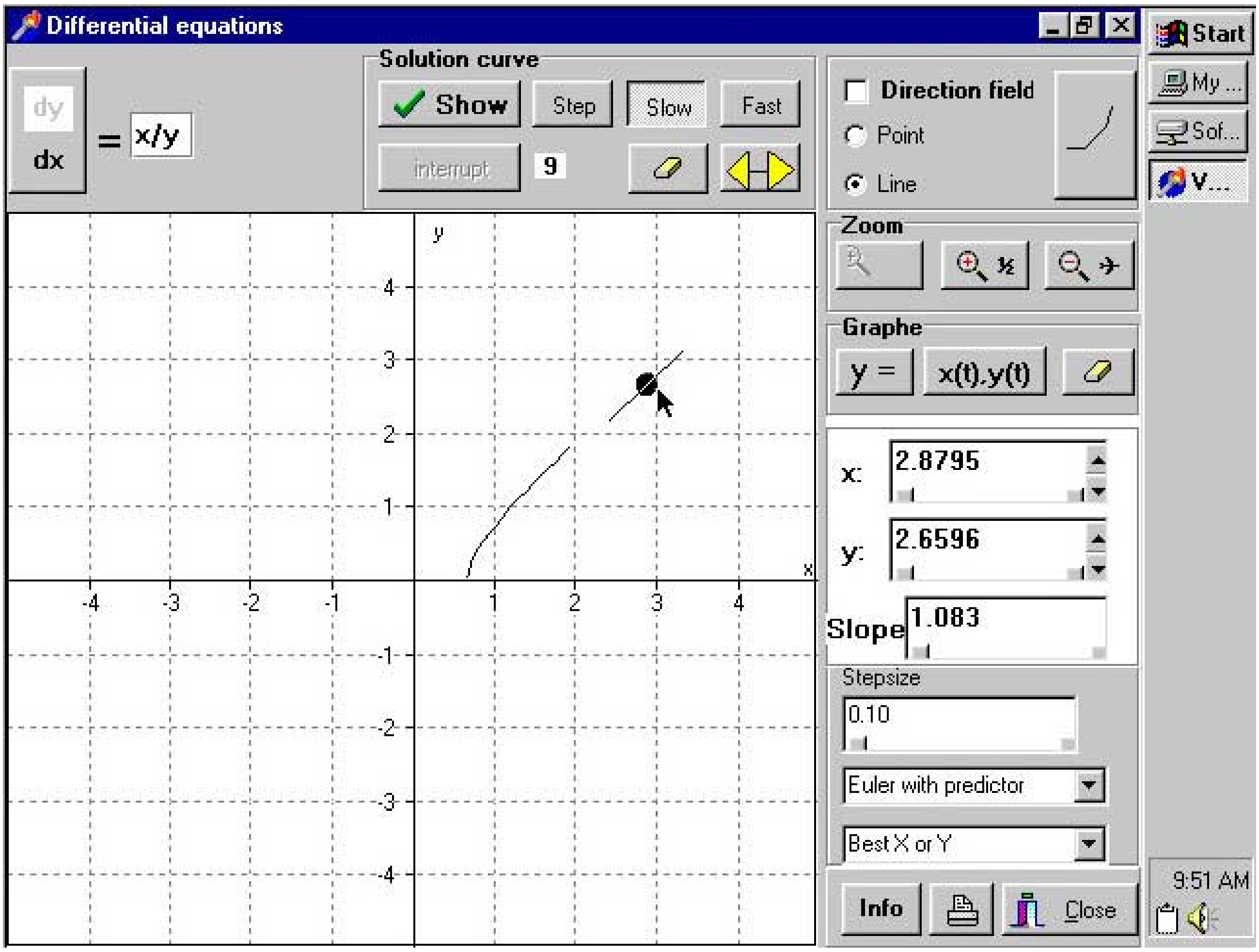
Local linearity is a *symbolic linear approximation* to the slope *at a single point* on the graph, having a linear *function* approximating the graph at that point. It is a *mathematical* formulation of gradient, taken first as a limit at a point x, and only then varying x to get the formal derivative. Local straightness *remains at an embodied level* and links readily to the global view.



The gradient of cos$x$ (drawn with Blokland et al (2000)

- an ‘embodied approach’.
- it can be linked directly to numeric and graphic derivatives, as required.
- it fits exactly with the notion of local straightness.
- it uses enactive software to build up the concept in an embodied form.

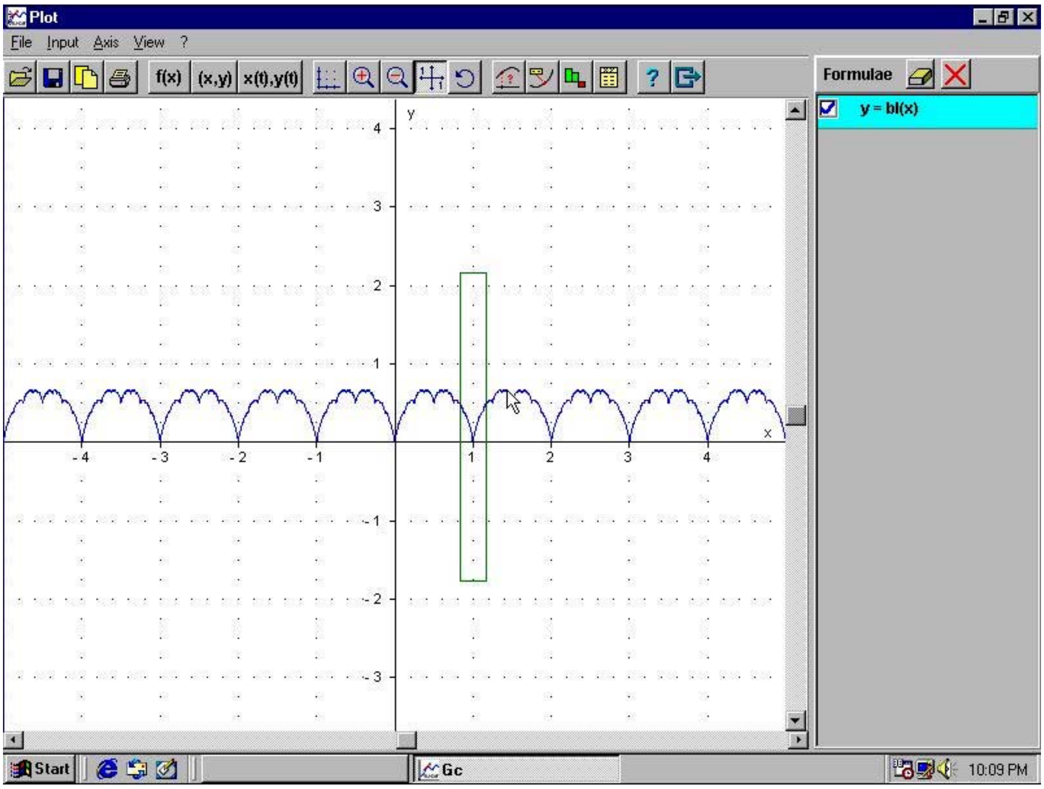
LOCAL LINEARITY AND THE SOLUTION OF DIFFERENTIAL EQUATIONS

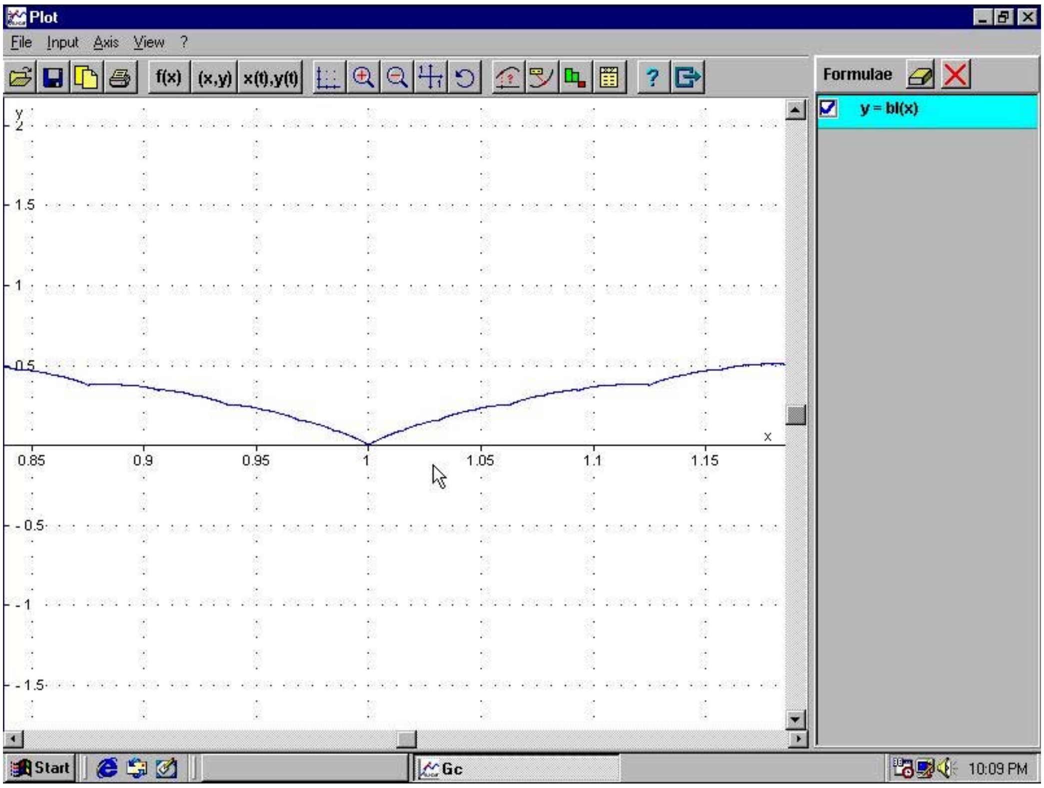


A generic organiser to build a solution of a first order differential equation by hand, (Blokland *et al*, (2000)).

CONTINUITY

The blancmange graph and a rectangle to be stretched to fill the screen:

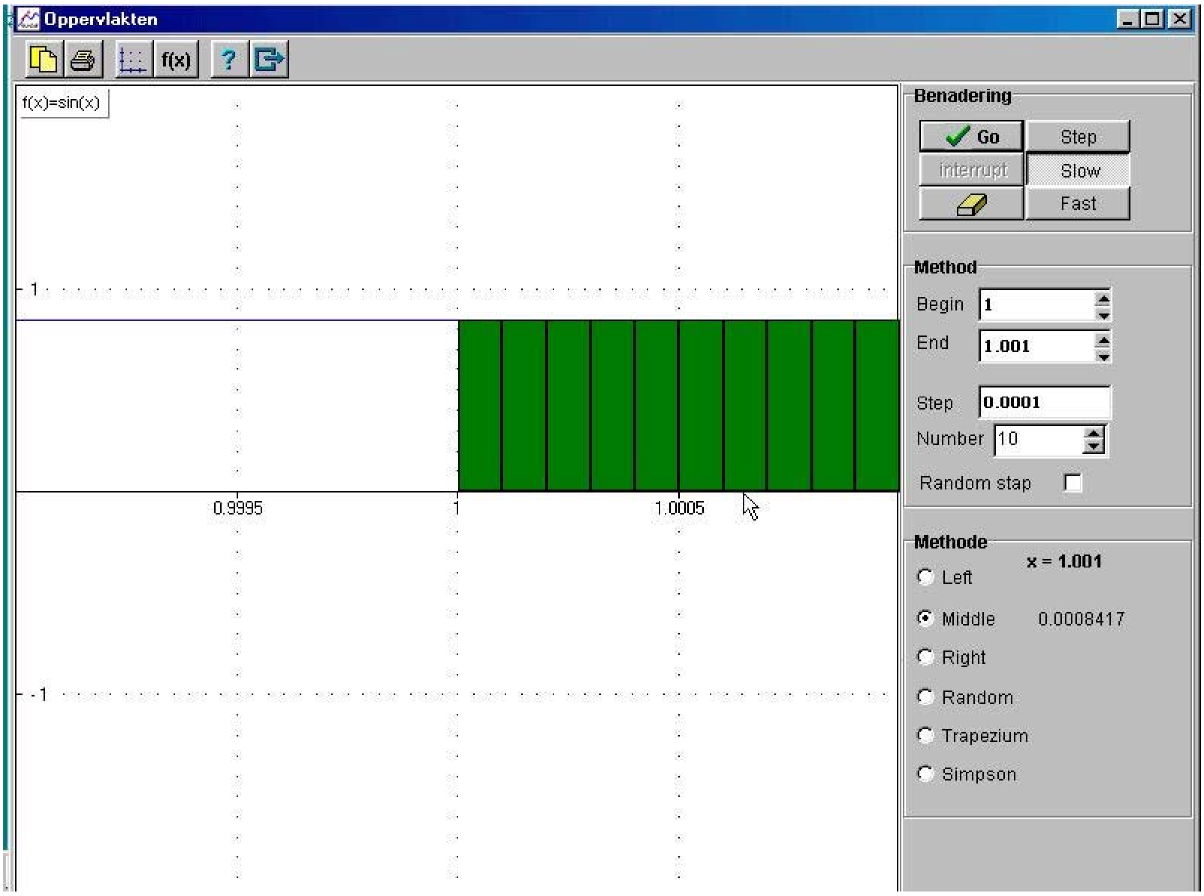




Embodied definition: A real function is continuous if it can be pulled flat.

Draw the graph with pixels height 2ε, imagine (a, $f(a)$) in the middle of a pixel. Find an interval a–δ to a+δ in which the graph lies inside the pixel height $f(a)$±ε.

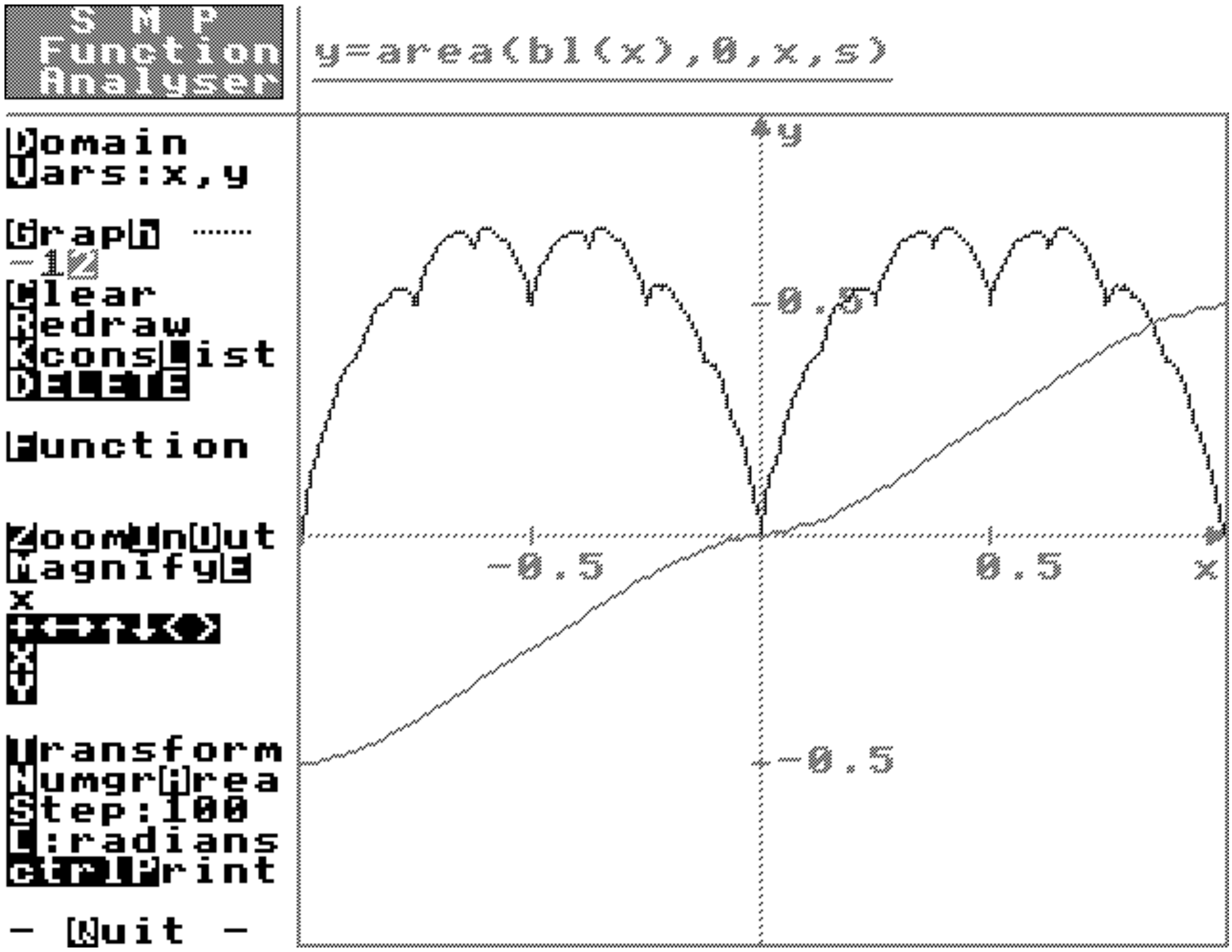
Example: f(x)= sin$x$ pulled flat from .999 to 1.001:



Area under sin$x$ from 1 to 1.001 stretched horizontally

The ***Fundamental Theorem of Calculus*** embodied. (Think about it!)

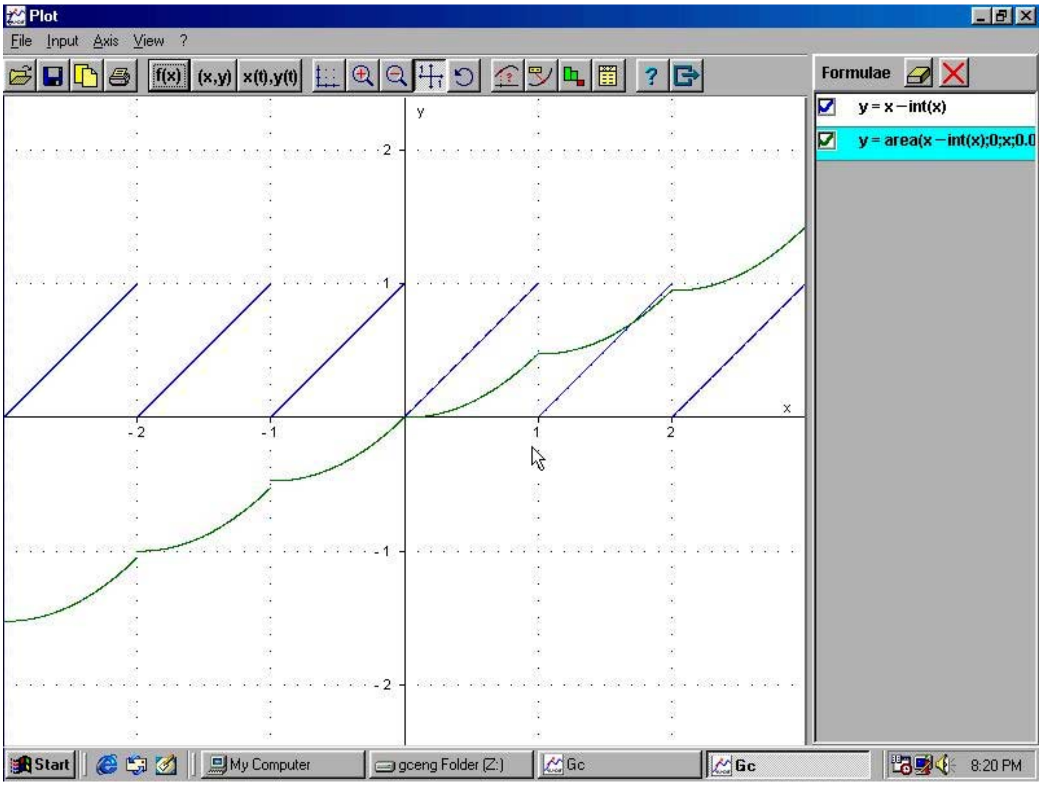
EMBODIED AREA AND FORMAL RIEMANN INTEGRATION



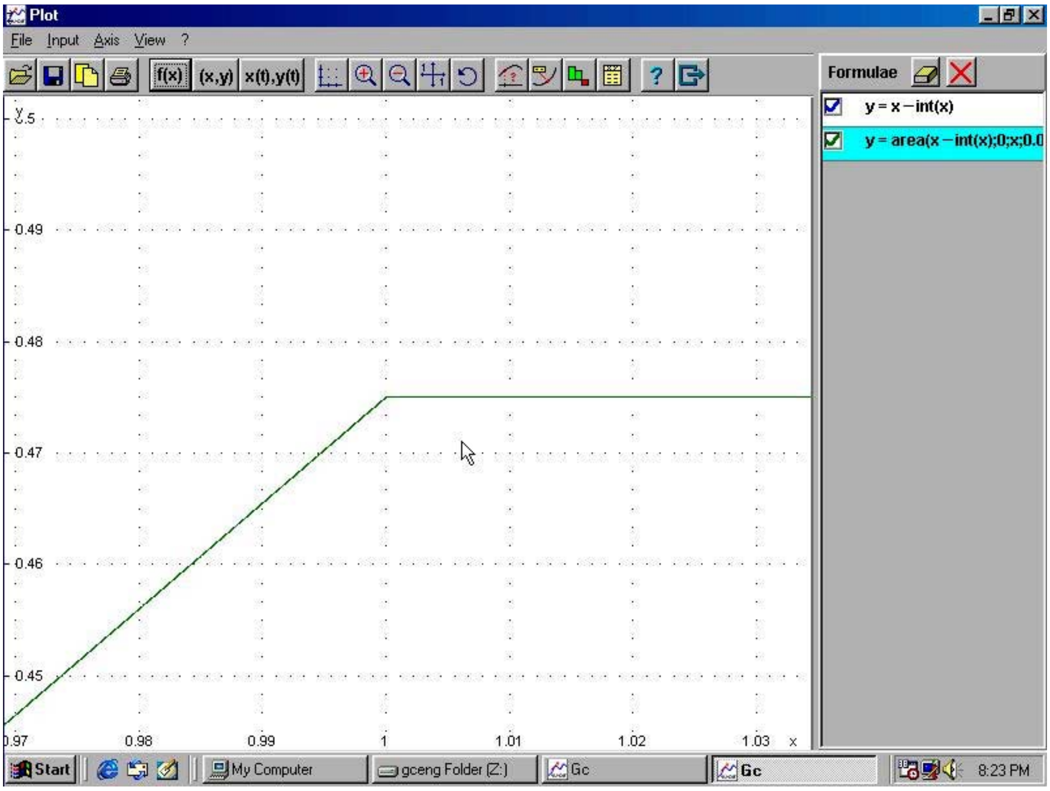
The area function under the blancmange and the derivative of the area (from Tall, 1991b)

The embodied notions of ‘area’ and ‘area-so-far’ as cognitive roots can support Riemann and even Lebesgue integration. For further detail, see Tall (1985, 1991a, 1992, 1993, 1995, 1997), These may be downloaded from the web-site:

http://www.warwick.ac.uk/staff/David.Tall



The area function for the discontinuous function $x – int(x)$ calculated from 0.



The area function magnified.

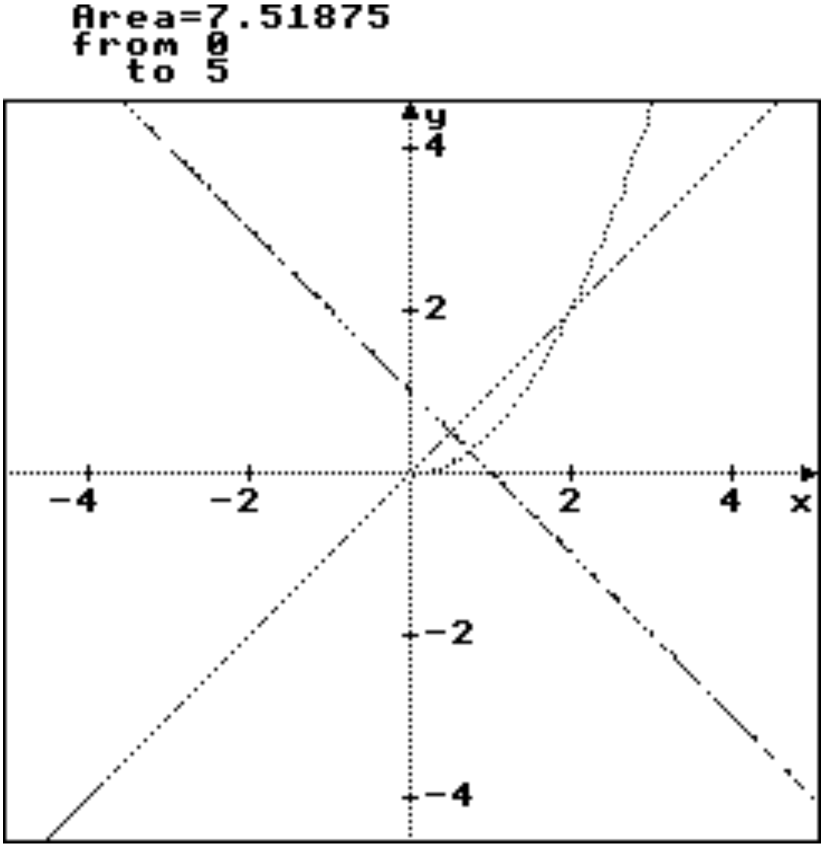
INTEGRATING HIGHLY DISCONTINUOUS FUNCTIONS

such as *f(x)=x for x rational, f(x)=*1*–x for x irrational*.

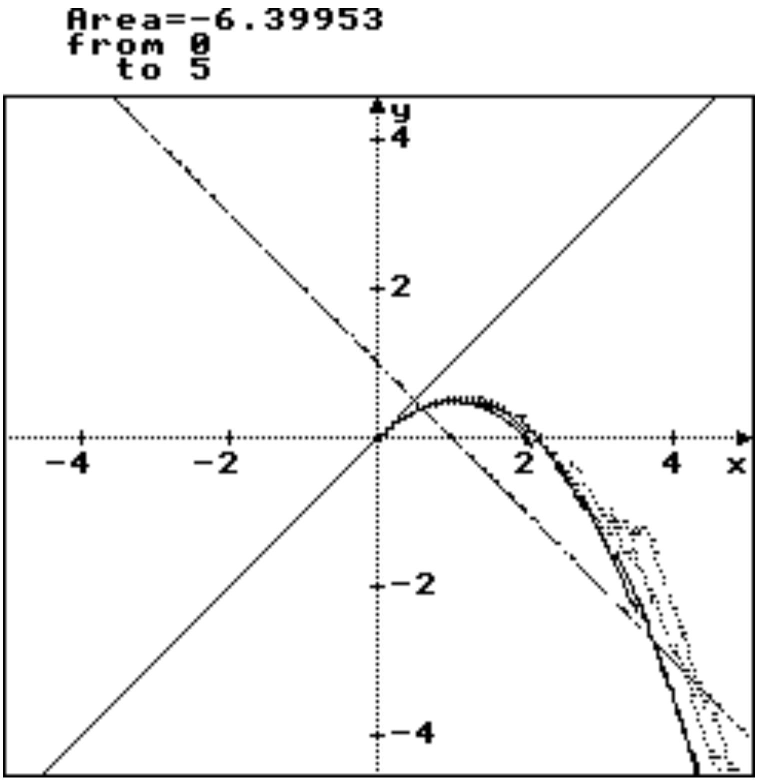
Idea: if (x_n) is a sequence of *rationals* $x_n = a_n/b_n$ tending to the real number x, then if x is rational, the sequence (x_n) is ultimately constant and equal to x otherwise the denominators b_n grow without limit.

Definition: x is ***(ε–N)-rational*** if the sequence of rationals is computed by the continued fraction method and, as soon as $|x - a_n/b_n| < \varepsilon$, then $b_n < N$.

Fix, say $\varepsilon=10^{-8}$, $N = 10^4$ and define x to be ***pseudo-rational*** if it is (ε–N)-rational, otherwise it is ***pseudo-irrational***.



The (pseudo-) rational area (rational step, midpt)



The (pseudo-) irrational area (random step)

Epilogue

- The human mind does not always do mathematics logically, but is guided by a concept image that can be both helpful and also deceptive.
- Symbolism is more precise and safe than visualisation, but cognitive development of symbols in arithmetic, algebra and calculus have many potential cognitive pitfalls.
- Local straightness and visual ideas of area can be cognitive roots to build an embodied understanding of the calculus. This can be motivated to link to the formal theory of differentiation, continuity and integration.
- This requires a special approach to mathematical thinking that supports the **concept imagery of the biological brain** by **interaction with a computational computer** to produce **a versatile mathematical mind**.