What research into mathematical thinking tells us at university level

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The AMS mathematics subject catalogue now includes education as a recognized heading. This seminar will consider international developments in research in the thinking of math students at university, with a closer look at several recent studies at the University of Warwick.

References:

Erh-Tsung Chin & David Tall (2001), Developing Formal Mathematical Concepts over Time. Proceedings of PME 25, (2), 241-248.

Marcia Pinto & David Tall (2001). Following students' development in a traditional university classroom, Proceedings of PME 25, (4), 57-64.

Soo Duck Chae & David Tall (in press) Students' Concept Images for Period Doublings as Embodied Objects in Chaos Theory.

David Tall (in press), Natural and Formal Infinities, to appear in Educational Studies in Mathematics.

All available through the link *Birmingham Seminar, October 2001* at

www.warwick.ac.uk/staff/David.Tall (or via **davidtall.com**)

97-XX
Mathematics education

97-00 General reference works (handbooks, dictionaries, bibliographies, etc.)
97-01 Instructional exposition (textbooks, tutorial papers, etc.)
97-02 Research exposition (monographs, survey articles)
97-03 Historical (must also be assigned at least one classification number from Section 01)
97-04 Explicit machine computation and programs (not the theory of computation or programming)
97-06 Proceedings, conferences, collections, etc.

97Axx General
97Bxx Educational policy and educational systems
97Cxx Psychology of and research in mathematics education
97Dxx Education and instruction in mathematics
97Uxx Educational material and media. Educational technology

97Cxx
Psychology of and research in mathematics education

97C20 Affective aspects (motivation, anxiety, persistence, etc.)
97C30 Student learning and thinking (misconceptions, cognitive development, problem solving, etc.)
97C40 Assessment (large scale assessment, validity, reliability, etc.) [See also 97D10]
97C50 Theoretical perspectives (learning theories, epistemology, philosophies of teaching and learning, etc.) [See also 97D20]
97C60 Sociological aspects of learning (culture, group interactions, equity issues, etc.)
97C70 Teachers, research on teacher education (teacher development, etc.) [See also 97B50]
97C80 Technological tools and other materials in teaching and learning (research on innovations, role in student learning, use of tools by teachers, etc.)
97C90 Teaching and curriculum (innovations, teaching practices, studies of curriculum materials, effective teaching, etc.)
97C99 None of the above, but in this section

What is happening to research in mathematics learning in the USA, UK & elsewhere?

Rhetoric

How-to's

Theoretically driven empirical research.

Three Recent Doctoral Studies at Warwick

1. Student Conceptions of Relations, Equivalence Relations, and Partitions (Abe Erh Tsung Chin)

Method: Questionnaire for information on a spectrum of approaches, Clinical Interviews with selected students.

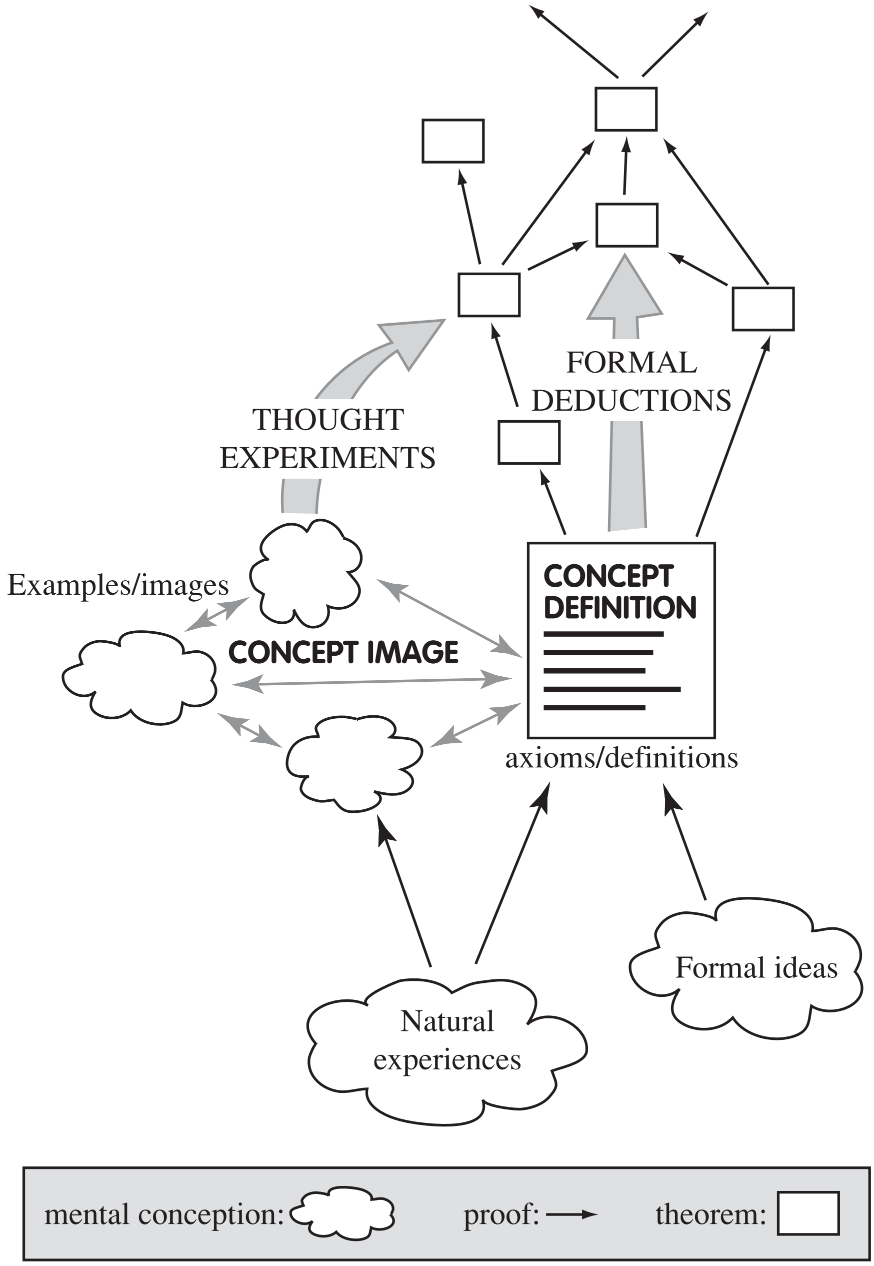
2. Natural and Formal Approaches to Analysis (Marcia Pinto.)

Method: Questionnaire for information on a spectrum of approaches, Select a spectrum of students to follow through 2 terms with Clinical Interviews every 3 weeks.

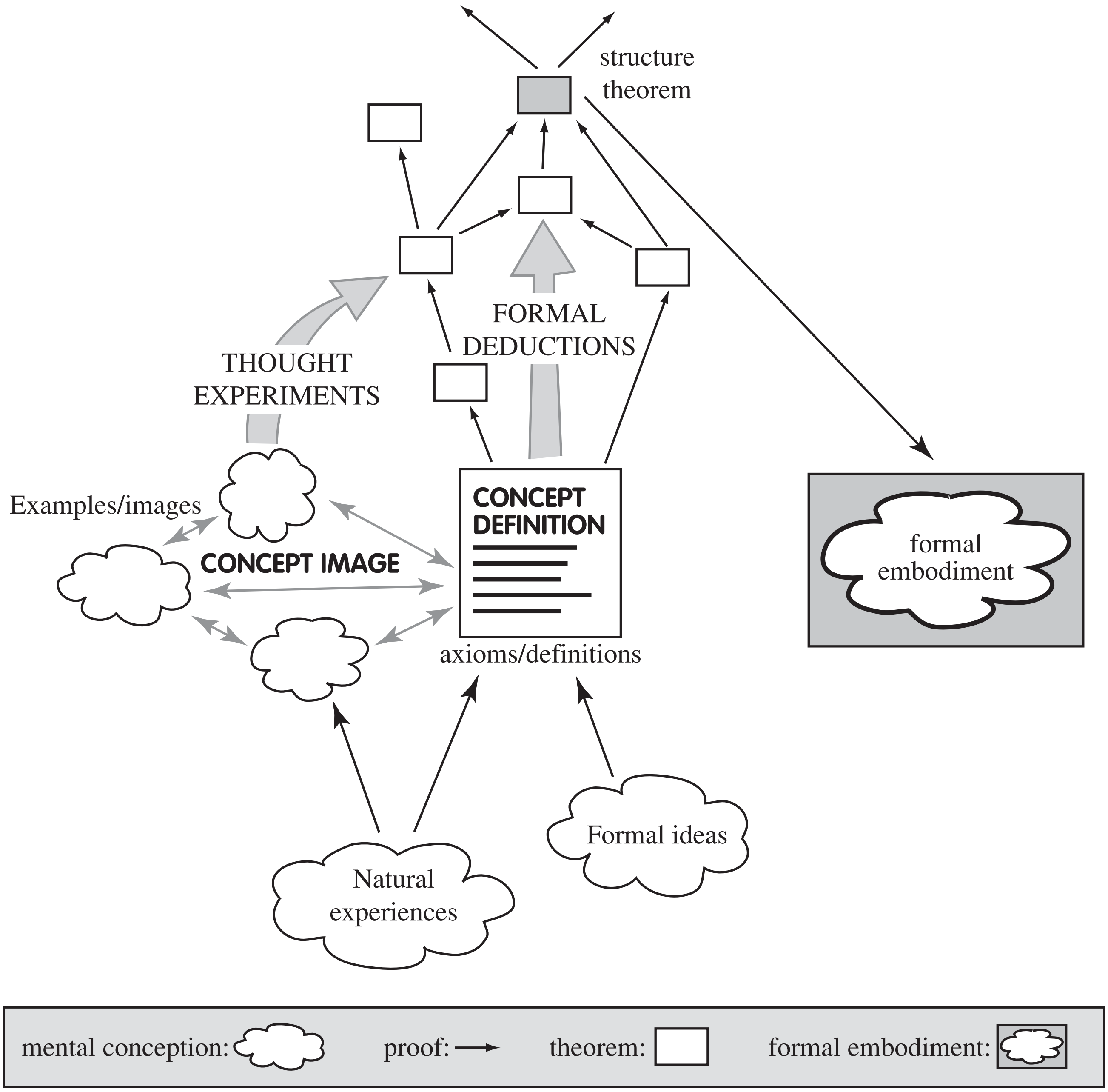
3. Students conceptions of period-doubling using interactive visual software (Soo Duck Chae.)

Method: Study several classes of students doing computer experiments into $x=f(x)$ iteration of $y=\lambda x(1-x)$, finding numerical values of $\lambda=\lambda_1, \lambda_2, \ldots$ for which period doubling occurs and comparing the initial numeric values in terms of geometric convergence, to find the limit point.

In all of these pieces of research, there is a difference that can be formulated in terms of ‘natural’ thinking, using informal concept images and ‘formal’ thinking involving only formal deduction.

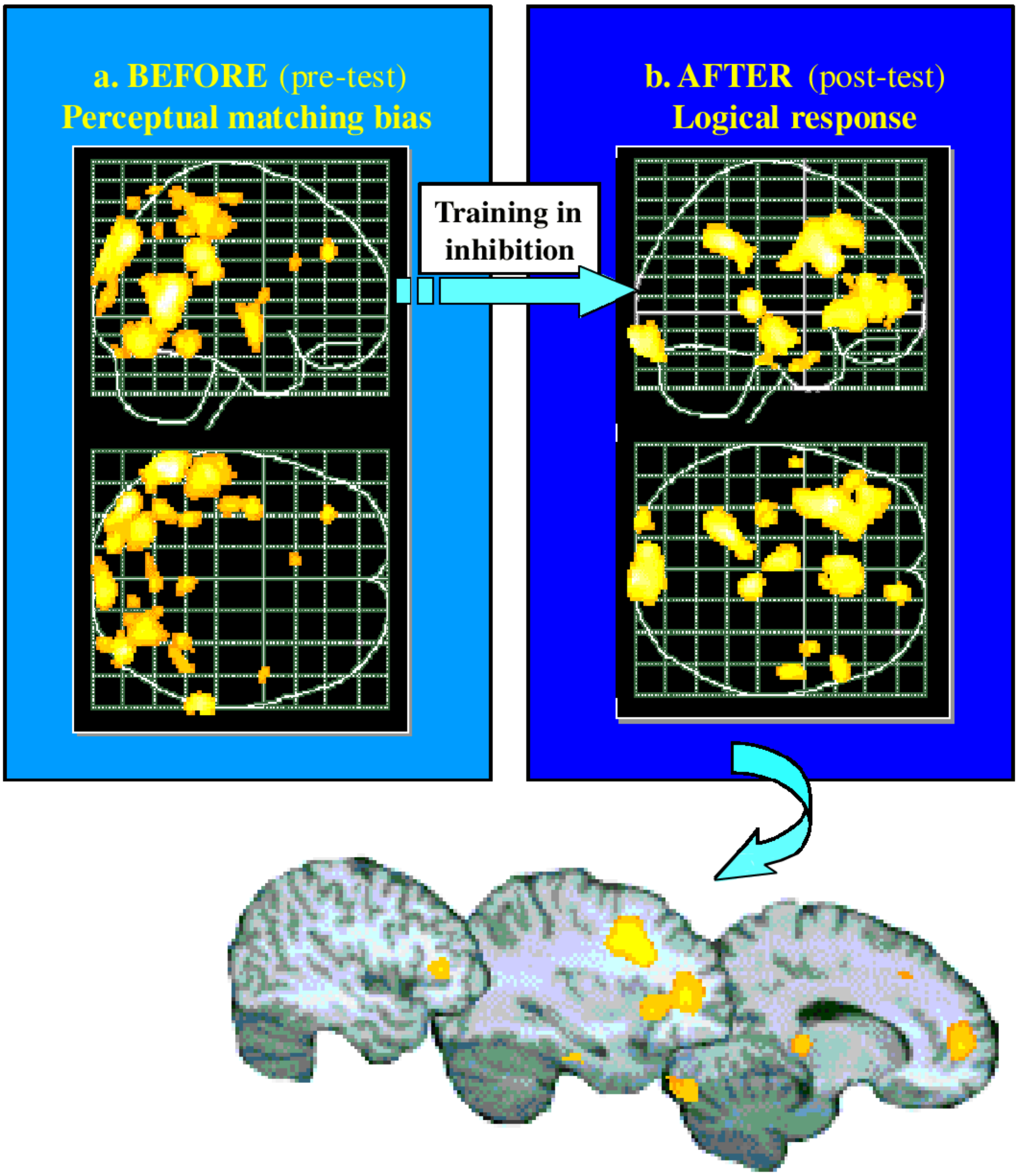


Some constituents in constructing a formal theory



Building new formal embodiments from a formal theory

WHAT HAPPENS IN THE BRAIN?



Houdé et al (2000). Shifting from the perceptual brain to the Logical Brain: The Neural Impact of Cognitive Inhibition Training. *Journal of Cognitive Neuroscience* 12:4 712–728.

Why are 'relations' difficult?

At Warwick. in the Foundations Course, the annual report commented that '*Euclid's algorithm* and *symbolic logic* were well understood, *basic set theory* and *functions* generally required extra work, but the topic on *relations* was often poorly understood.' On an average, only about 20% of students declared that they understood relations well with nearly a third of students claiming that, even after extra study, they only understood the topic poorly.

What is the problem?

Theory : A relation is …

An equivalence relation is a relation s.t. …

A partition is …

Equivalence relation ⇔ Partition

Say what “equivalence relation’ means to you.

N=15	First Year	Second Year
Formal/detailed	5	9
Formal/partial	4	5
Informal/outline	5	1
Total definition	**14**	**15**
Example	0	0
Picture	0	0
Other	1	0
No response	0	0

A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$.
Is this relation an equivalence relation?

N= 15		First Year	Second Year
Informal	Informal Definition	3	0
	Other	1	1
	No response	0	0
Formal perhaps with some informal language	Definition	7	2
	Theorem	3	12
	Partition	1	0

$$A = \left\{ (x, y) \in \boldsymbol{R}^2 \mid 0 \le x \le 10,\, 0 \le y \le 10 \right\}.$$

Is A an ***equivalence relation*** on $\boldsymbol{R}$?

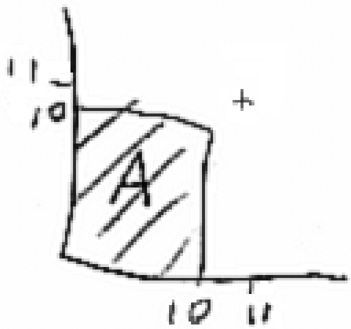
In the first year *no* student (out of the whole class) responded positively to this question.

$A=\{(x,y) \in \boldsymbol{R}^2 \mid 0 \leq x \leq 10,\, 0 \leq y \leq 10\}$. Is A an ***equivalence relation*** in $\boldsymbol{R}$?

Answer (yes or no or don't know):...Don't know.

Full Explanation: A defines points in the plane x-y where $0 \le x \le 10$ and $0 \le y \le 10$. But don't understand the relation.

In the second year, only Simon responded as follows:



Consider $11 \in R$, as $(11, 11) \notin A$, not reflexive.

(N=15)	First Year	Second Year
Formal/detailed	2	8
Informal/outline	6	3
Total definition	8	11
Example	0	0
Picture	1	1
Other	4	3
No response	2	0

Say what "partition" means to you.

The majority of students tried to use *their own language* to interpret the definition of 'partitions' so that their answers were highly varied.

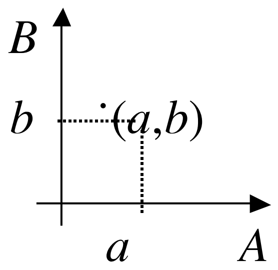
All ten students interviewed said they had a mental picture of a partition.

Nine thought they understood 'partitions' better than 'equivalence relations'. Arthur understood 'partitions' better than 'equivalence relations' because he could visualise 'partitions' but not 'equivalence relations'.

Yet the students were actually better at answering problems about equivalence relations than partitions!

Relation

(on $A\times B$)



Let A and B be sets. A *relation* between A and B is a subset of $A\times B$.

⇒ Theorems

Equivalence relation

(on A)

Examples like "*the similar triangles*" and "*the integers modulo n*".

A relation ~ on a set A is an *equivalence relation* if for all a, b, $c \in A$

1. $a\sim a$ (reflexive)

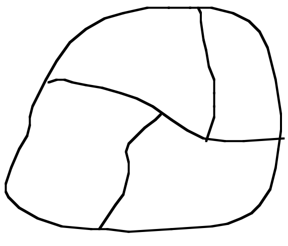
2. if $a\sim b$ then $b\sim a$ (symmetric)

3. if $a\sim b$ and $b\sim c$ then $a\sim c$ (transitive)

⇒ Theorems

Partition

(of A)

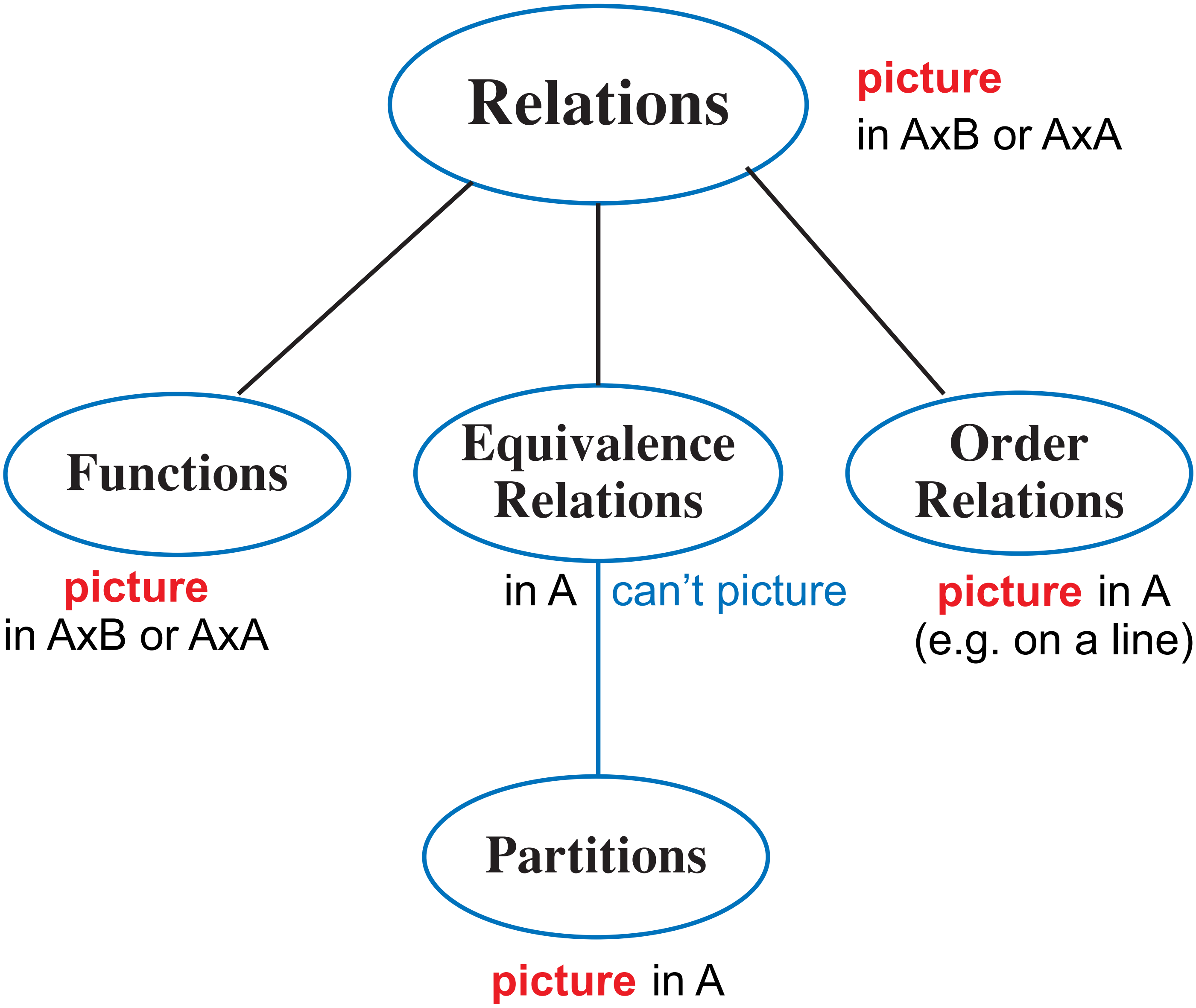


A partition of a set A is a set $\wp$ whose members are non-empty subsets of A satisfying

1. each $a\in A$ belongs to some $X\in \wp$

2. if $X, Y\in \wp$, and $X\neq Y$, then $X\times Y=\emptyset$

⇒ Theorems


Relations
picture
in AxB or AxA
Functions
picture
in AxB or AxA
Equivalence Relations
in A
can't picture
Order Relations
picture in A
(e.g. on a line)
Partitions
picture in A

Natural and formal learning

(Pinto 1998, Pinto and Tall, 2001)

Formal learners essentially construct the theory by deduction, coping with the great cognitive strain as best they can, producing a deductive formal theory. Natural learners—working from their concept imagery—reconstruct it taking account of more general ideas met in the course. They must then develop the formal theory from their reconstructed imagery, producing a formal theory integrating both imagery and deduction.

	Formal learning	**Natural learning**
1 Initial obstacles	Based on concept definition, may be problematic either (a) unsatisfactory defns, eg problems with quantifiers, disjoint from images (b) defns **conflict**ing with images	Informal (based on concept image) so may (a) reject formal, retain images (b) relate formal to informal knowledge, with **conflict**
2 Theory Building	**Formal construction** of theory	**Formal *re*construction** (with some conflict) (a) Thought experiments, reconstructing images (b) Deductions reconstructing formal theory
3 Formal theory	**Formal (deductive)**	**Formal (integrated)**

Natural and formal routes to learning formal mathematics

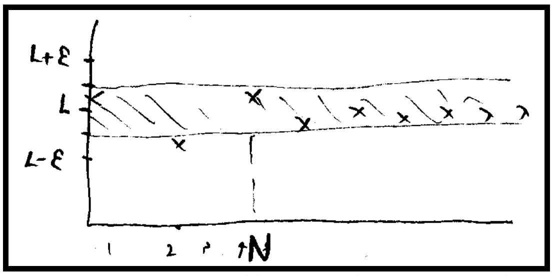
ROSS: THE FORMAL LEARNER

A sequence (a_n) tends to limit L if, $\forall\ \varepsilon>0,\ \exists\ N(\varepsilon)\in\mathbb{N}$ s.t. $\forall\ n\geq N$;

$$|a_n - L| < \varepsilon.$$

(Ross, first interview)

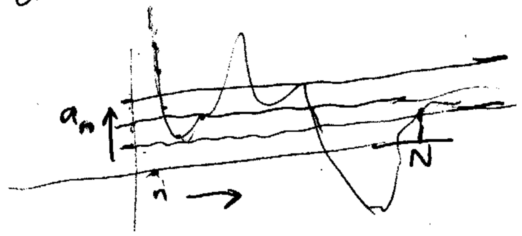
"Just memorizing it, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it get imprinted and then I remember it."



(Ross, first interview)

"Well, before, I mean before I saw anyone draw that, it was just umm ... thinking basically as n gets larger than N , a_n is going to get closer to L so that the difference between them is going to come very small and basically whatever value you try to make it smaller than, if you go far enough out then the gap between them is going to be smaller. That's what I thought before seeing the diagrams and something like that."

~~If $a_n \to L$ then there~~ exists
For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$
such that $|a_n - L| < \varepsilon$ for all $n \geq N$



(Chris, first interview)

"I don't memorize that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down."

"I think of it graphically ... you got a graph there and the function there, and I think that it's got the limit there ... and then ε once like that, and you can draw along and then all the ... points after *N* are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that's the *n* going across there and that's a_n. ... Err this shouldn't really be a graph, it should be points."

	Sequences	Series	Continuity	Derivative	Final Interview
al obstacles	Rolf (a) Robin (a& b)	Rolf (a) Robin (b)	[Rolf withdrew] Robin (b)		
nal uction				Robin	Robin
	Ross		**Ross**		
nal tive)		**Ross**		**Ross**	
					Ross

Students following an essentially formal route

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Cliff (a) Colin (b)	Cliff (a) Colin (b)	Cliff (a)	Cliff (a)	Cliff (a)
2. Formal Reconstruction			Colin (a)	Colin (b)	Colin (b)
		Chris (a&b)	**Chris (a&b)**	**Chris (a&b)**	
3. Formal (deductive)	**Chris**				
					Chris

Students following an essentially natural route

Students' Concept Images for Period Doublings as Embodied Objects in Chaos Theory

Soo D. Chae and David Tall

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Research using computers and oscillators at university as students explore iteration of $f(x)=\lambda x(1-x)$ for increasing values of λ.

Students' concept images focus on embodied objects in the form of graphic representations of the 'final orbits'; period doubling is seen not symbolically, but in terms of one orbit bifurcating visually into a 'doubling' of the preceding orbit.

Based on the empirical evidence we propose a cognitive development that occurs with many students as a result of linking visual orbits with the underlying symbolic theory.

The logistic function $f(x) = \lambda x(1-x),\ 0 \le x \le 1$

Start with some x_1 and iterate: $x_2 = f(x_1)$

$x_3 = f(x_2)$

etc

What happens to a sequence $x_1, x_2, x_3, \ldots, x_n \ldots$

$\mathbf{x_{n+1}} = f(\mathbf{x_n})$

$\mathbf{x_{n+2}} = f(\mathbf{x_{n+1}}) \quad \Rightarrow f(x_{n+1}) - f(x_n) = x_{n+2} - x_{n+1}$

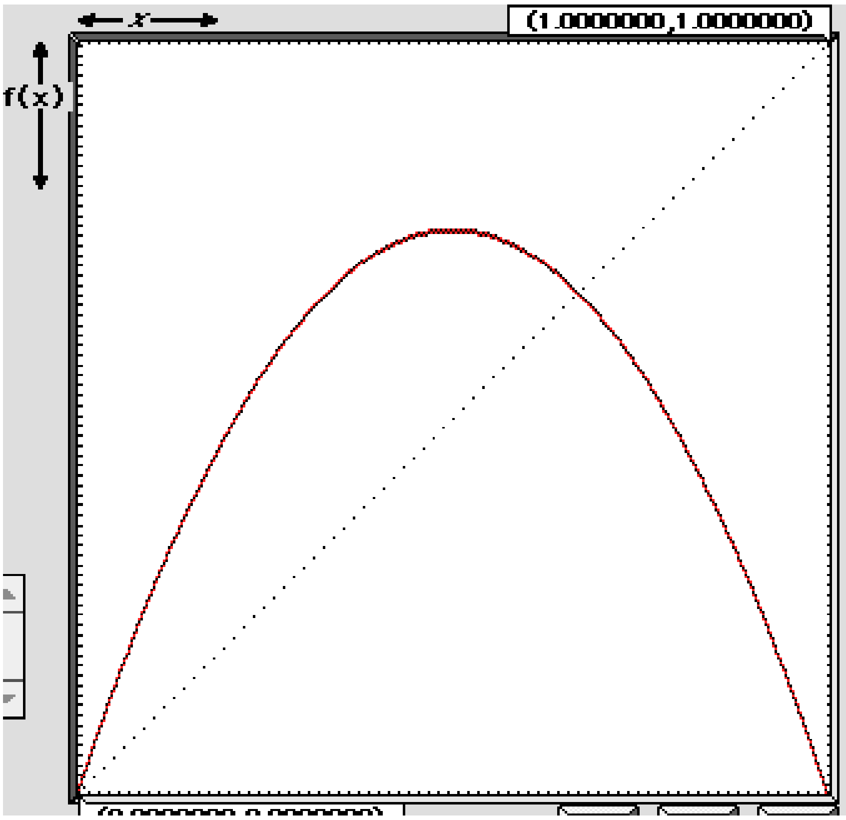
so

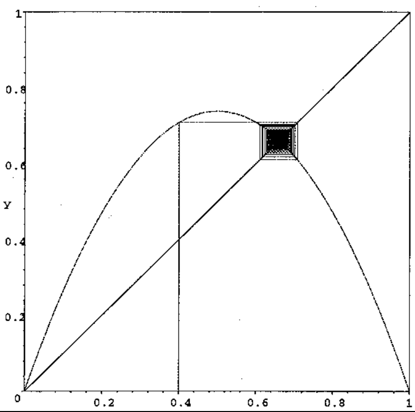
$$\left|\frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}\right| = \left|\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}\right| \approx \left|f'(x_n)\right|$$

Convergence if $|f'(\mathrm{x})| < 1$.

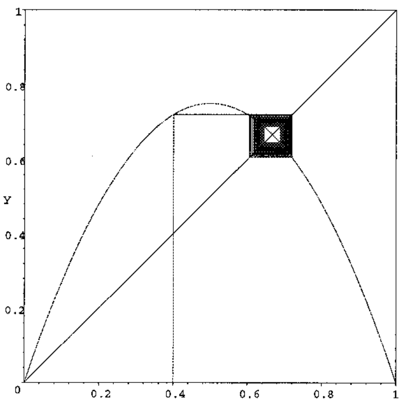
Divergence if $|f'(\mathrm{x})| > 1$.

The logistic function $f(x) = \lambda x(1-x)$

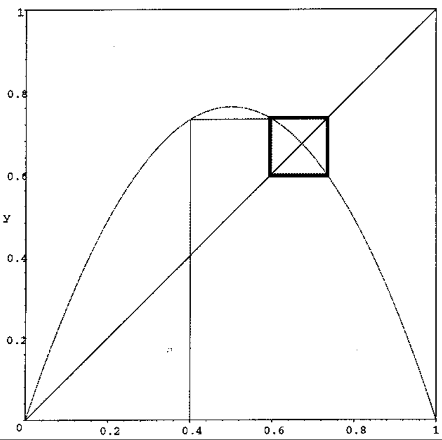




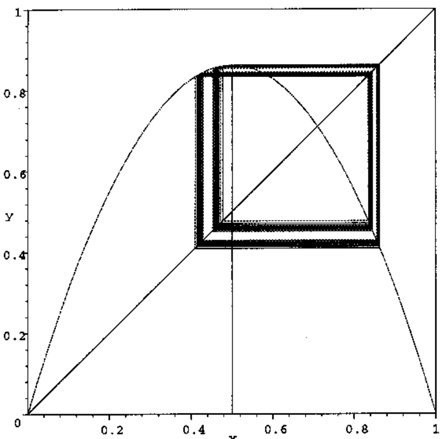
λ = 2.95



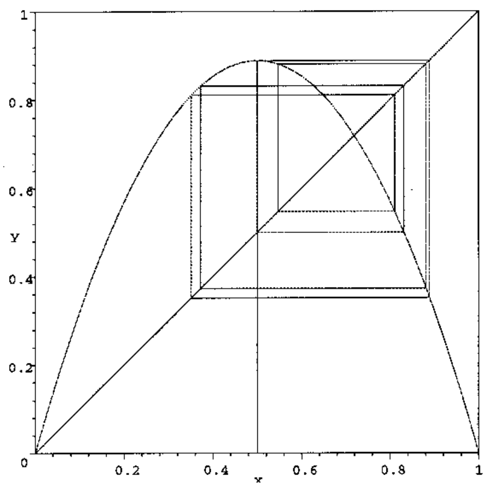
λ = 3.0



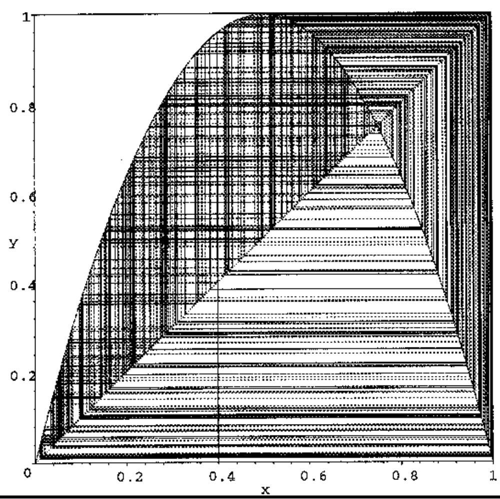
λ = 3.05



λ ≈ 3.4495

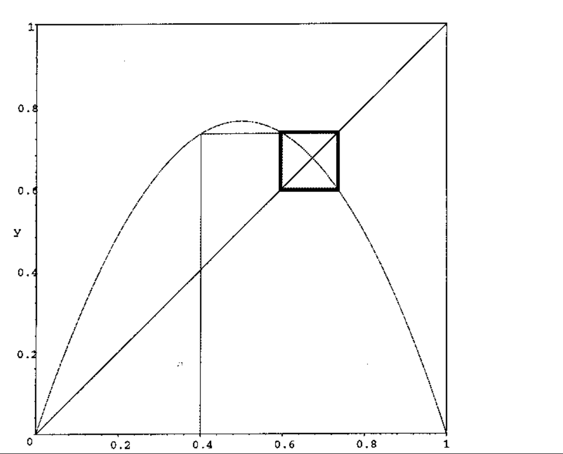


λ = 3.555



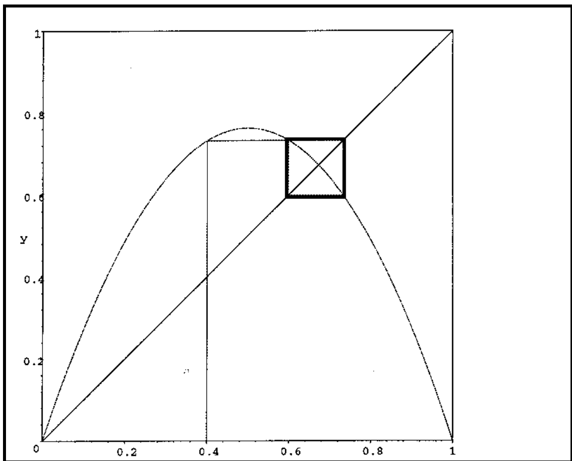
λ = 3.99

The ‘final state’ as a visual object. A ‘base object’.

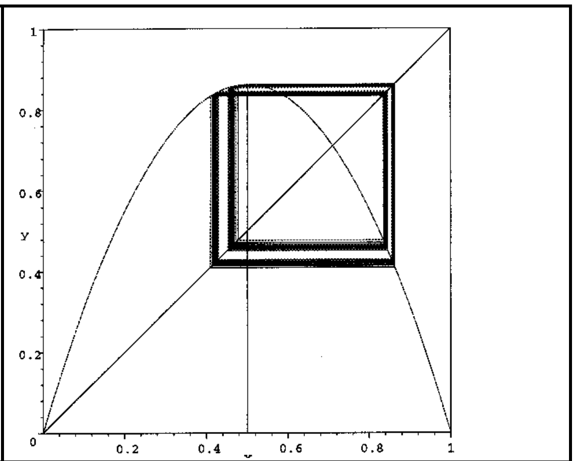


$\lambda = 3.05$

The *process* of period doubling as a change from one state to the next:

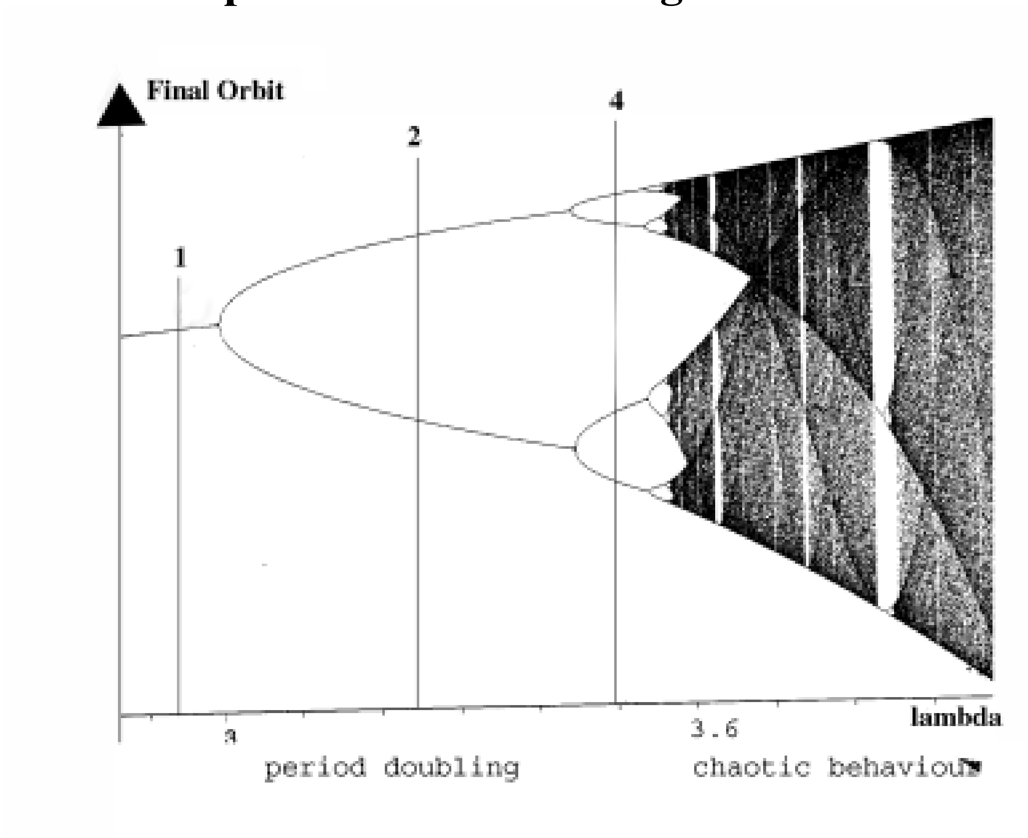


$\lambda = 3.05$



$\lambda \approx 3.4495$

The Concept of 'Period Doubling'



The Feigenbaum Diagram

Methodology

Subjects

Mathematics Students in the first year course at the University of Warwick in 1999.

Software

Logis

Pre-requisite test

Computer and Oscillator Experiment

1. Students explore x = f (x) iteration for $f(x) = \lambda x(1-x)$, $0 \le x \le 1$

2. Find $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots$ where period doubling occurs.

   3.0, 3.449, 3.545, 3.561, 3.569…………

3. Find the approximate limit of $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n, \ldots (\approx 3.6)$

   (using the idea of geometric convergence)

[4. An analogous problem using an oscillator]

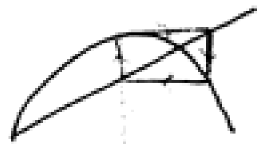
Post-test

- What **first** comes into your mind?
- What comes to your mind **most of all** when you think about ‘period doubling’? Please draw/describe it in your mind’s eye.

Computer-generated image:

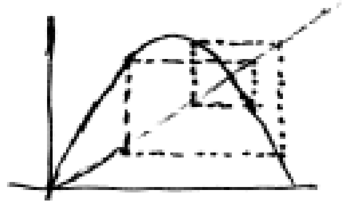
period 2 orbit

1 Student



period 4 orbit

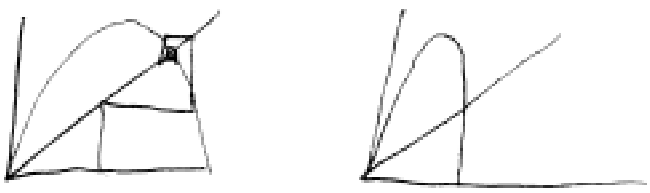
4 Students



Thinking about period doubling as an embodied object

Computer-generated image

from period1 to period 2 (period 2 picture incomplete) *(1 Student)*

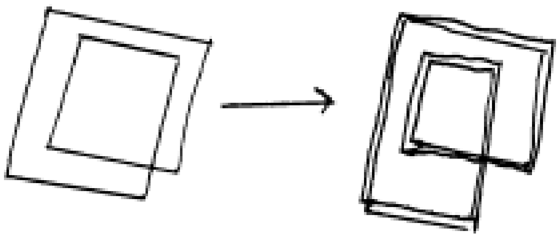


From period 2 to period 4

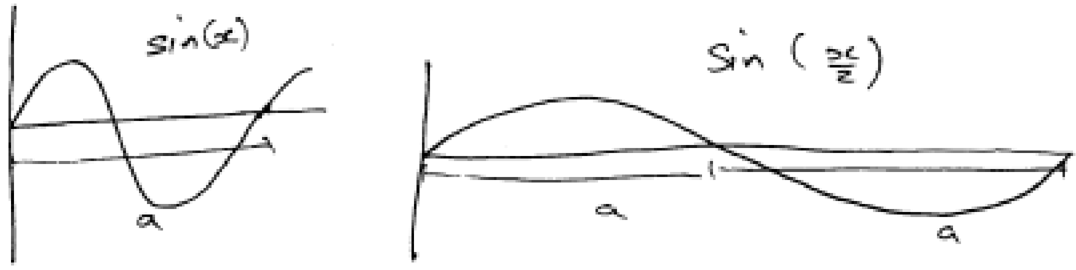
(3 Students)

From period 4 to period 8

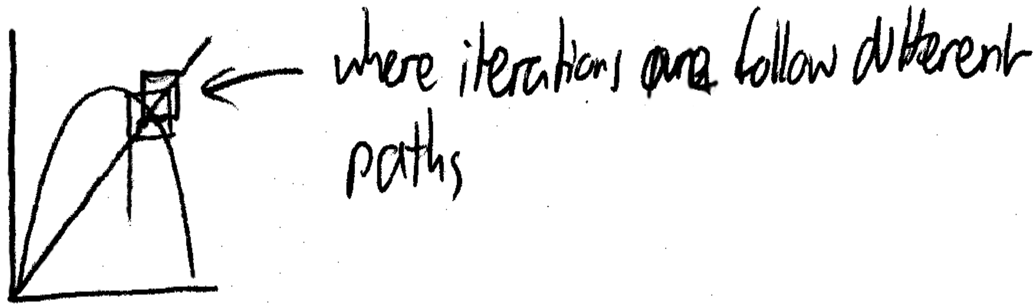
(1 Student)



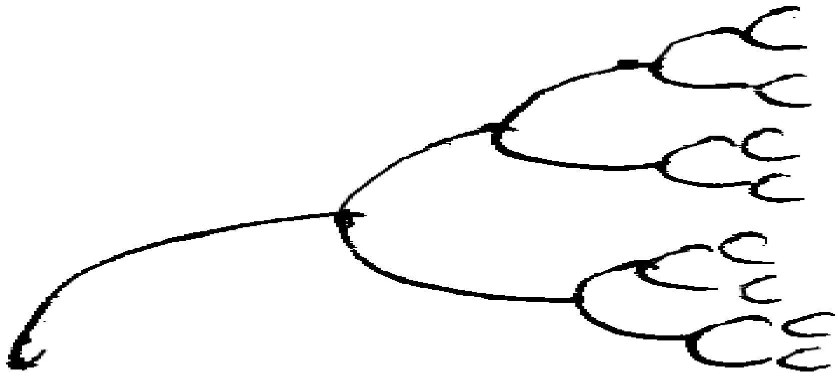
Thinking about period doubling as a process of change



concept image based on earlier knowledge



an orbit drawn as two separate squares



concept image based on the bifurcation tree (concept of period doubling)

Embodied object as a **base object**		**Process of change**		**Encapsulated object**	
Oscillator-generated circle	1	Oscillator image: period 1 to period 2	5	Bifurcation Tree	1
Oscillator-generated period 2 orbit	1				
Computer-generated image: period 2 orbit	1	Computer-generated image from 1 to 2	1		
period 4 orbit	4	From 2 to 4	3		
period 8 orbit	0	From 4 to 8	1		
		Other	1		
Total	**7**		**11**	**1**	

Classification of students’ concept images for period doubling

What does research tell us about mathematical thinking in undergraduates?

- Beginning to understand how different students think differently and have different problems in different places at different times
- Natural and Formal Thinking
- Thought Experiments and Deductive Proof uses different parts of the brain
- The theoretical structure, say of relations, may involve different natural and logical linkages with some concepts more predominantly verbal/logical and others more visual.
- Dynamic visual graphics can support natural visualization
- Ongoing Cognitive Research is beginning to say things in a way which may be more amenable to being incorporated into teaching and learning.