Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing

METSMaC 2007: "Active Teaching, Active Learning"

Abu Dhabi, U.A.E., March 17-19, 2007

Teachers as Mentors to encourage both power & simplicity in active mathematical learning

David Tall

THE UNIVERSITY OF WARWICK

Active Teaching, Active Learning - Why?

Teaching mathematics is under stress.

Imposed targets press teachers to train students to obtain higher marks on national tests.

‘Teaching to the test’ can produce higher marks on standard questions, but may not prepare students for more subtle mathematical thinking.

So how do we encourage students to think in more subtle powerful ways?

Active Teaching, Active Learning - Why?

For many individuals, mathematics is *complicated* and it gets more complicated as new ideas are encountered.

For others, by focusing on the *essential ideas*, it becomes possible to see mathematics in a more focused way that makes many ideas essentially more *simple*.

"Technical skill is mastery of complexity,
while creativity is mastery of simplicity."

(Sir Christopher Zeeman)

Active Teaching, Active Learning - Why?

Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches.

But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression.

You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process.

The insight that goes with this compression is one of the real joys of mathematics.

(William Thurston, (Fields Medallist))

Symbols as process and concept

Symbols in arithmetic, algebra, trigonometry, calculus, such as 4+3, $^3/_4$, and $\int \sin x \, dx$ so on, also usually have two complementary meanings:

- as a ***process*** to be carried out by one or more procedures of calculation or manipulation,
- as a ***concept*** in its own right, to be thought of as a mental object that can itself be manipulated.

Learned as a ***procedure***, it allows us to ***do*** mathematics, as a ***concept*** we can ***think about it*** and ***make connections*** to other concepts in flexible ways.

Symbols as process and concept

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- as a ***concept*** in its own right, to be thought of as a mental object that can itself be manipulated.

Using symbols flexibly as **pro**cess or con**cept** enables us *to do* mathematics and to *think about* mathematical concepts.

PROCEPT

Symbols as process and concept

Key idea:

It is not enough to learn procedures to *do* mathematics, the knowledge must be ***compressed* from *procedures to do*** mathematics to ***concepts* to *think about*** mathematics.

We should seek not only the ***technical skill*** to achieve **mastery of complexity**,

we should seek the ***compression of knowledge*** to *thinkable concepts* to achieve **mastery of simplicity**.

PROCEPT

Symbols as process and concept

Symbol	***Process***	***Concept***
3+2	addition	sum
–3	subtract 3	negative 3
3/4	division (sharing)	fraction
$3+2x$	evaluation	expression
$v = s/t$	ratio	rate
$\sin(A)$	opposite/hypotenuse	sine function
$f(x)$	evaluation	function f
$\lim \sum 1/n^2$	tend to limit	limit value
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\mathbf{v}$	translation	vector
$\sigma \in S_n$	permuting $\{1, 2, \ldots, n\}$	element of S_n

Symbols as process and concept

Do you personally see the following formulae being 'the same' or 'different'?

$$x(x+2), \qquad x^2+2x.$$

Think about it for a while.

They are *different* as procedures:

One takes x and adds 2, then multiplies x by the result.
The other squares x and adds it to twice x.

Symbols as process and concept

Do you personally see the following formulae being 'the same' or 'different'?

$$f(x) = x(x+2),\ g(x) = x^2+2x.$$

Think about it for a while.

As functions, they are *the same* ...

Symbols as process and concept

Four different stages:

A *single procedure*: $x(x+2)$

Several different procedures, same result: $x(x+2)$ x^2+2x

Several *equivalent* procedures, *same process*: $x(x+2)$ x^2+2x

A *single manipulable procept*, written in different ways: $x(x+2)$

Symbols as process and concept

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Symbols as process and concept

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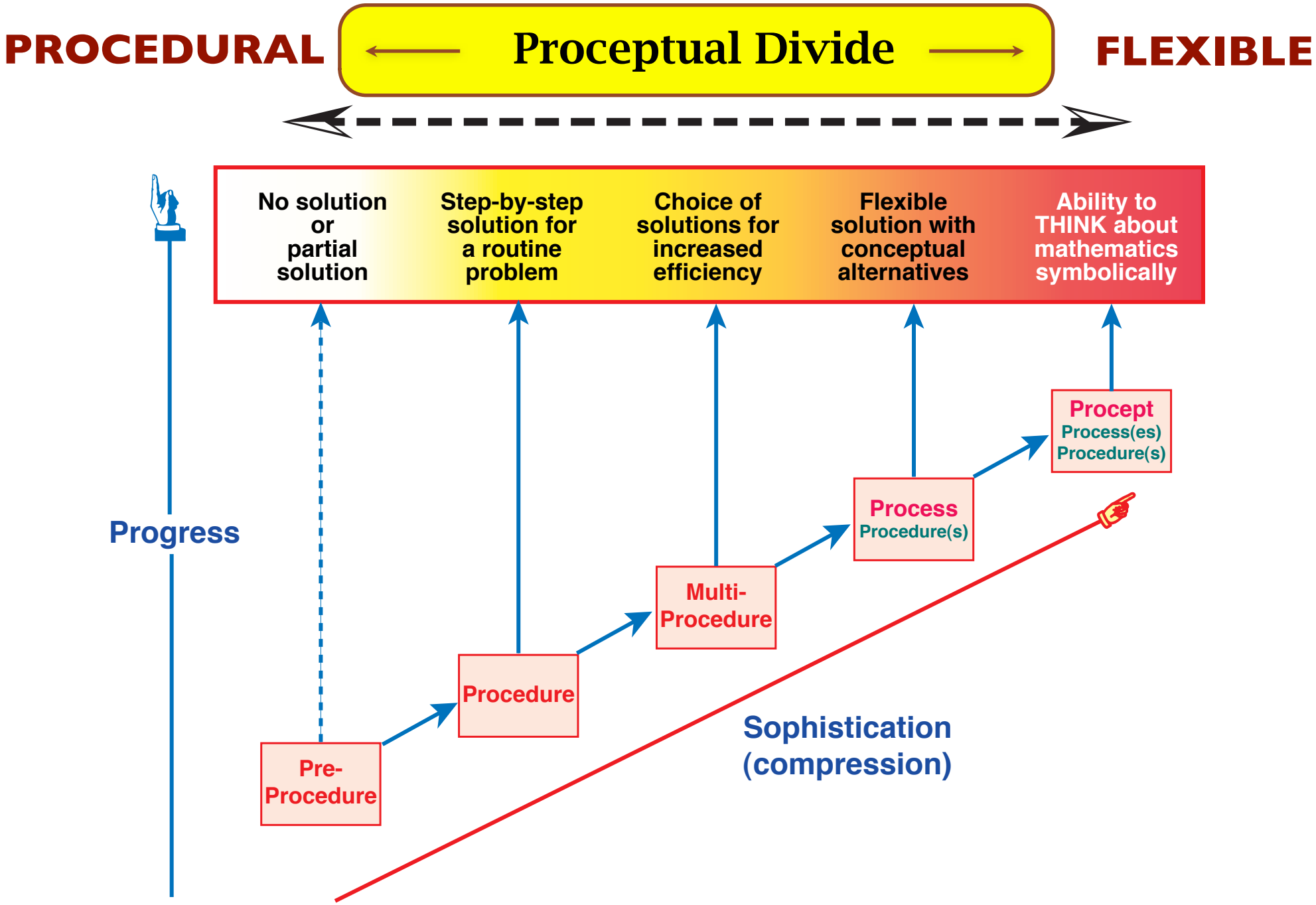
Symbolic Compression



Alternative procedures and efficiency

Consider the following derivative:

$$\frac{d}{dx}\left(\frac{x^2+1}{x}\right)$$

It may be performed using several different procedures:

using the quotient rule $u = 1 + x^2, v = x$

using the product rule $u = 1 + x^2, v = \frac{1}{x}$

by simplifying the expression first: $\frac{d}{dx}\left(x + x^{-1}\right) = 1 - x^{-2}$

Alternative procedures and efficiency

Student Grade	0 or 1 methods Pre-procedure or **Procedure**	2 or 3 methods Multi-Procedure or **Process**
A	3	9
B	7	5
C	9	3
Total	19	17

Maselan bin Ali, working in Malaysia (Ali & Tall 1996)

Procedural and conceptual thinking

Procedures occur *in time*.
Alternate procedures may give *efficiency*.
But ...

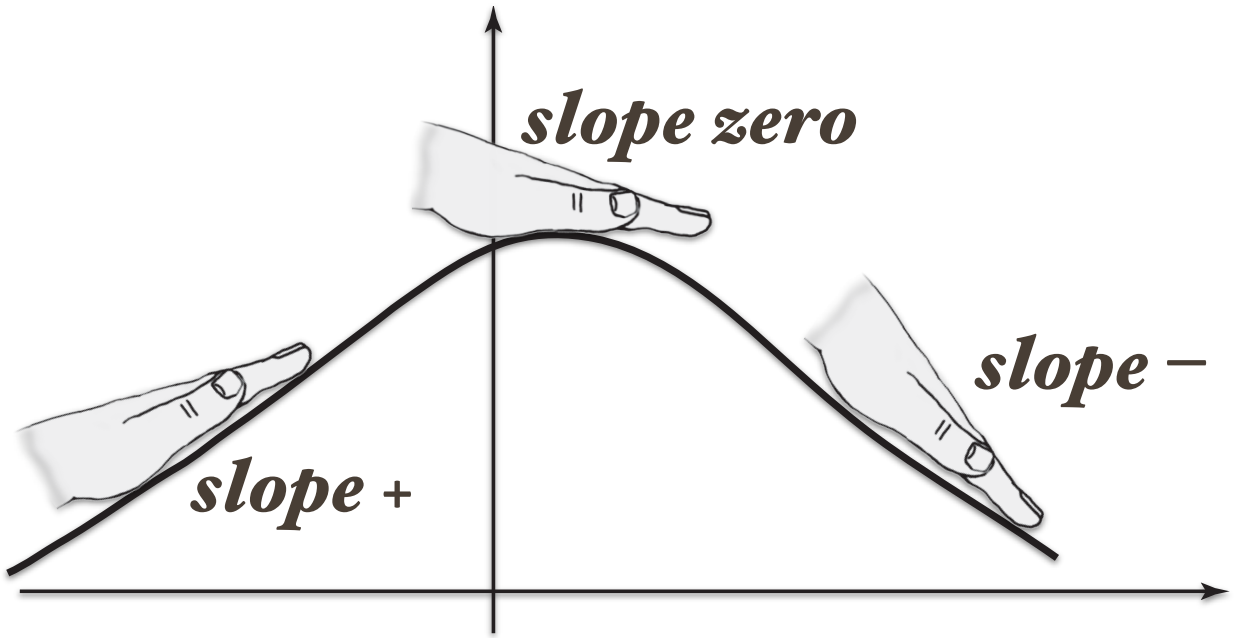
What has been noticed with some alarm is that there is a growing tendency for students to be able to solve 1-step problems requiring a single procedure, but a loss of ability in solving multi-step problems.

(London Mathematical Society, 1995).

... we need to think *about* the mathematics,
not just *do* it in time.
Let us *embody* mathematics in human action and perception to give it human meaning.

Embodiment in action

Look along a graph and see its slope change.
Imagine moving your hand along the curve.



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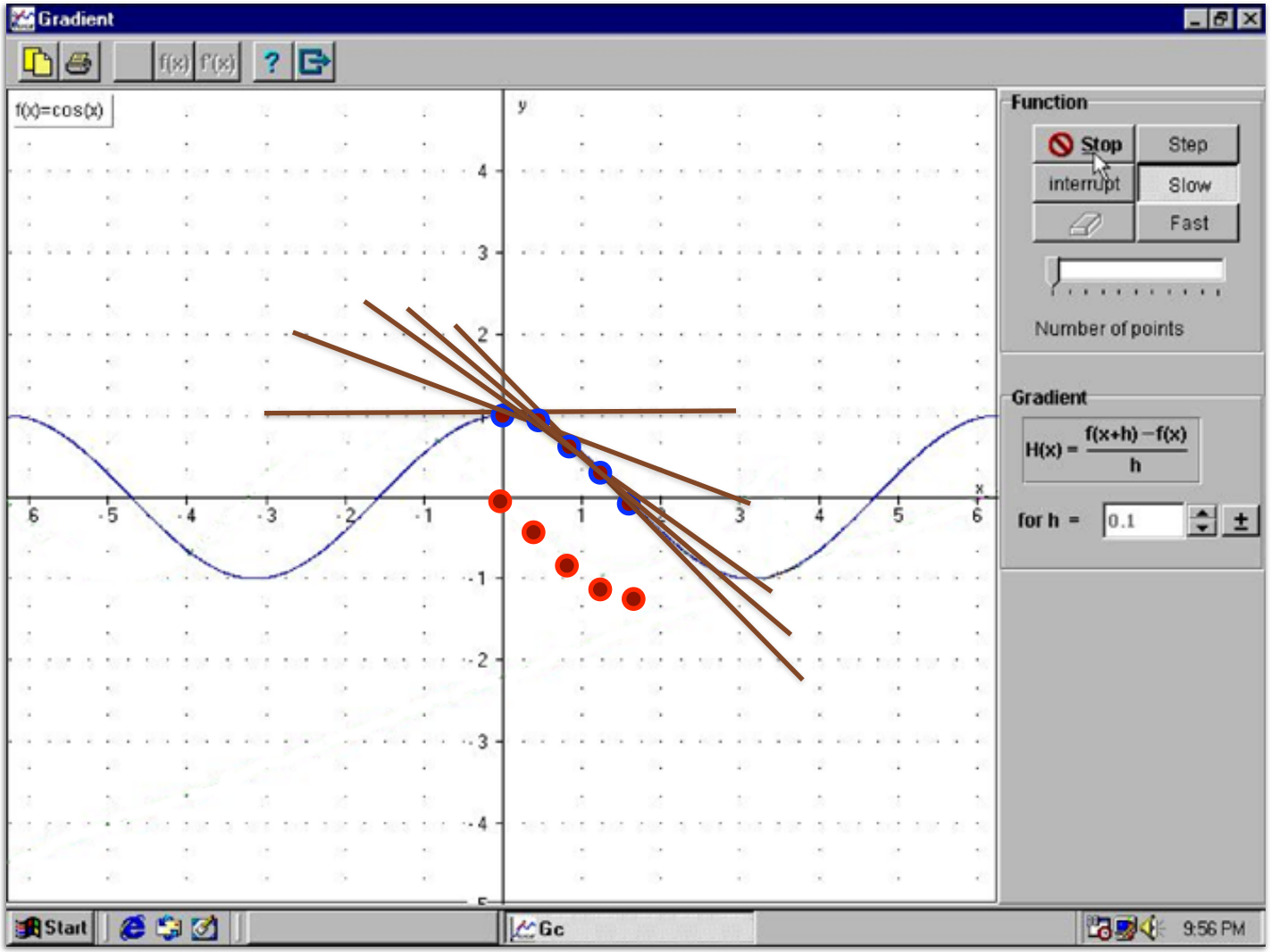
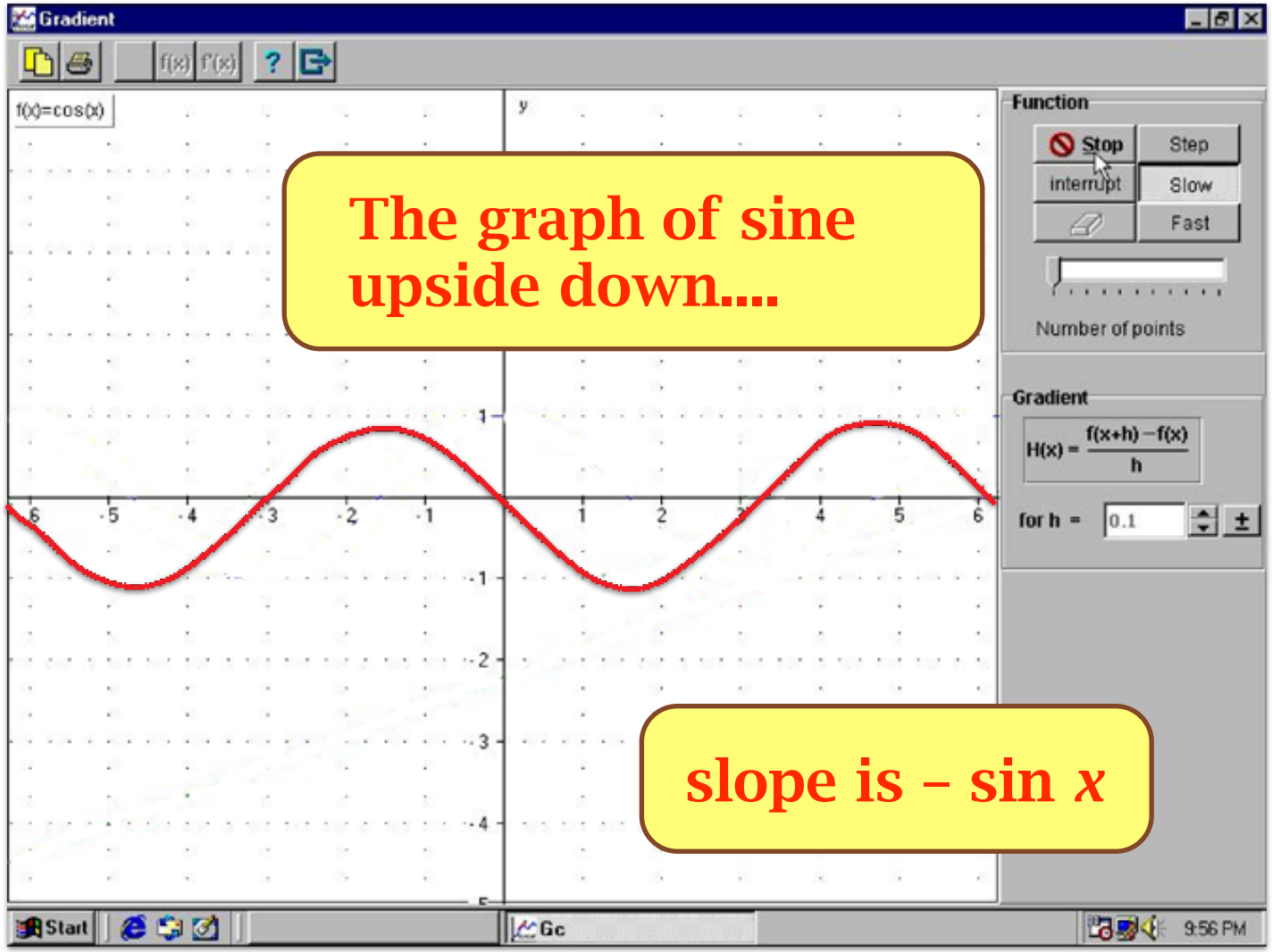


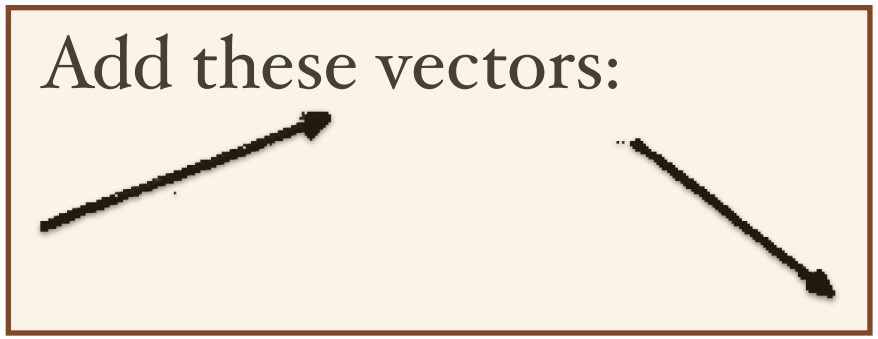
Embodiment in action

You can see *why* the derivative of cos is *minus* sine



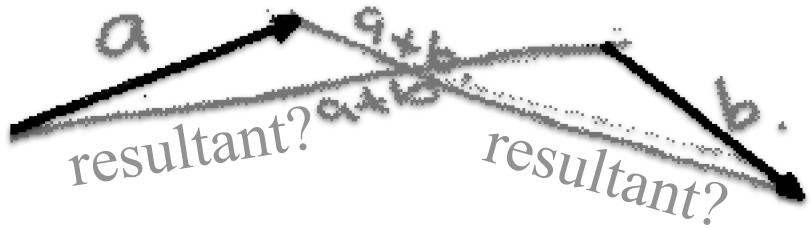
Embodiment in action
Vectors

Procedurally, one may teach students the rules to add vectors by the triangle law or by the parallelogram law, but the knowledge may be fragile and lead to error.



(i)(a)

(i)(b)



Anna Poynter, PhD, Warwick 2004

Embodiment in action
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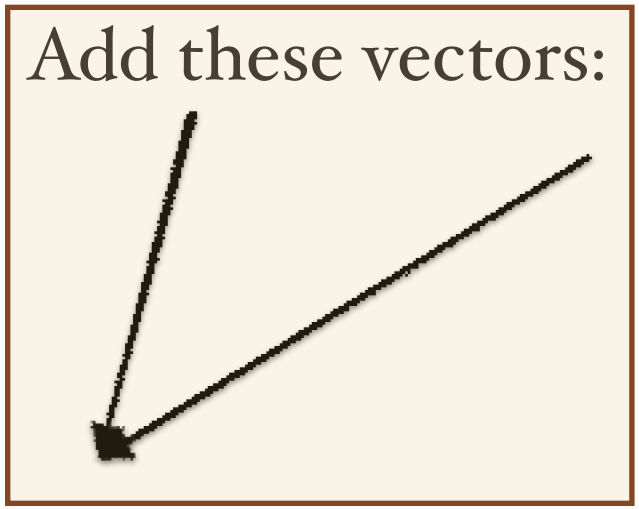
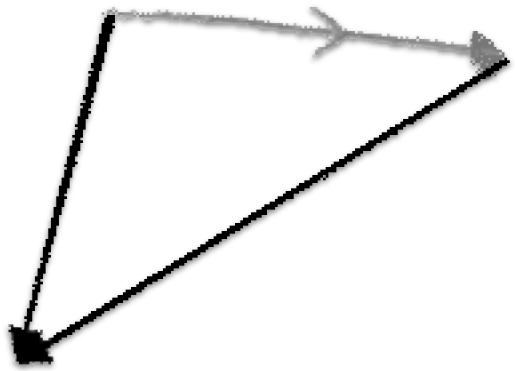
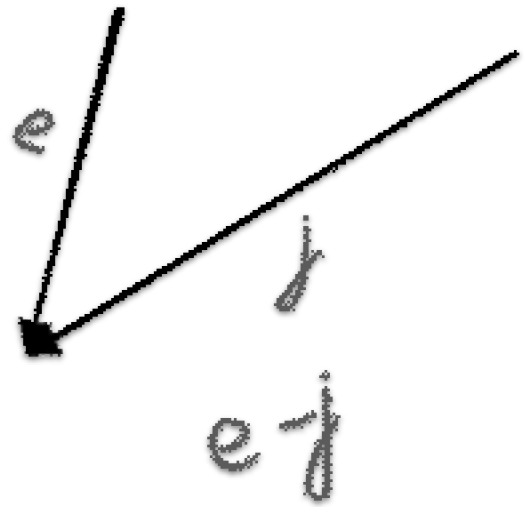


(iii)(a)



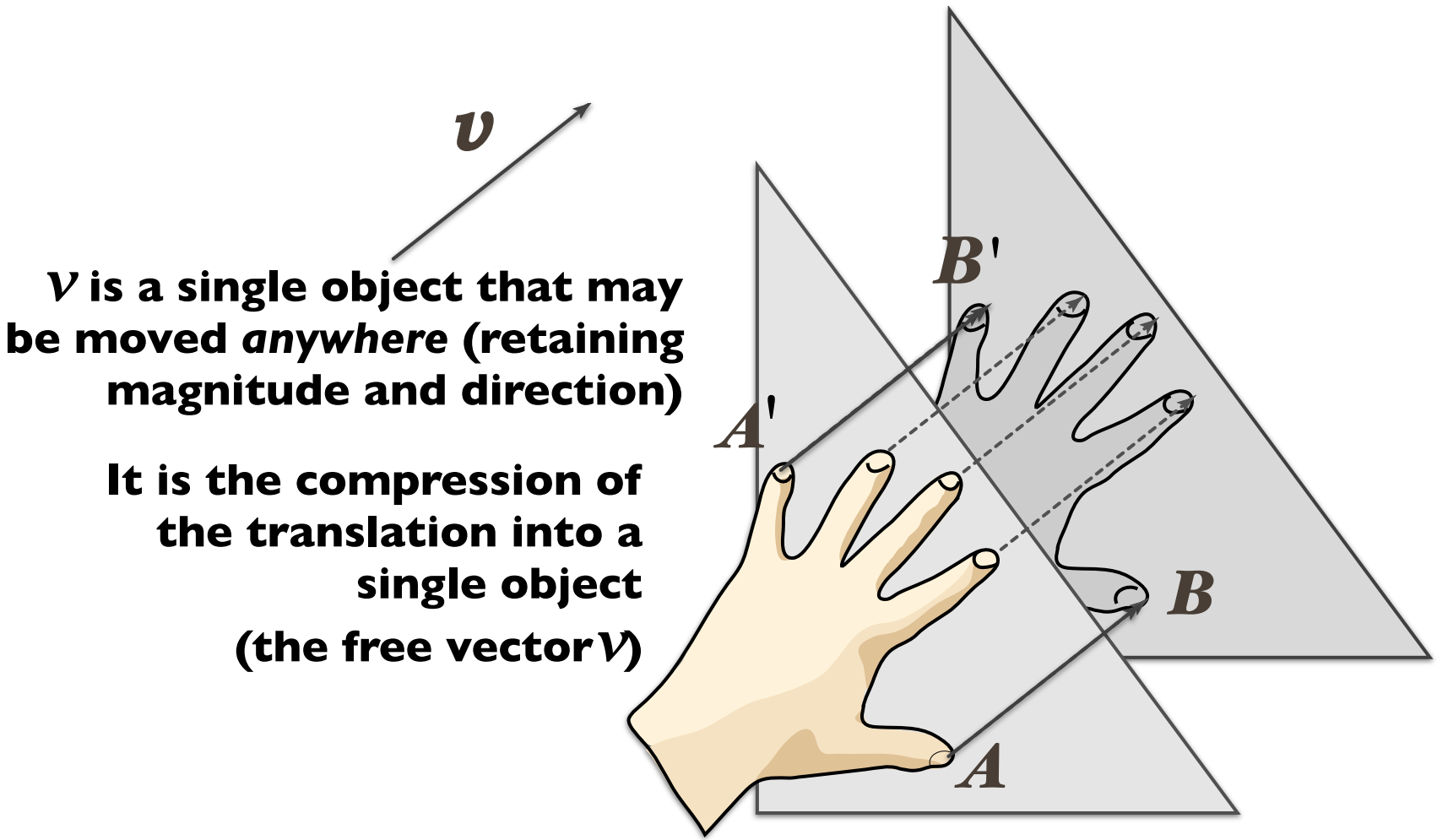
(iii)(b)



Instead of giving rules, begin with a physical act of translating an object on a flat table.

Embodiment in action

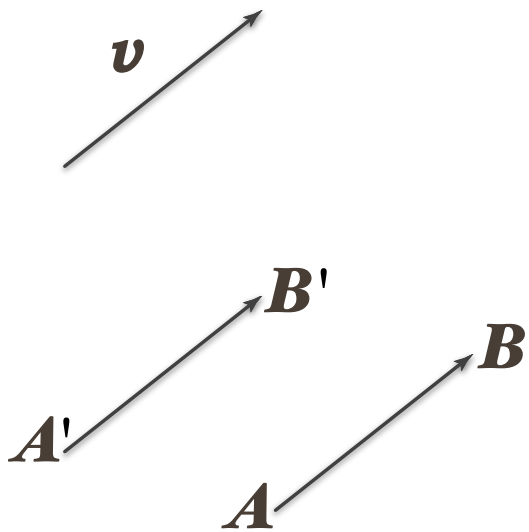
Vectors

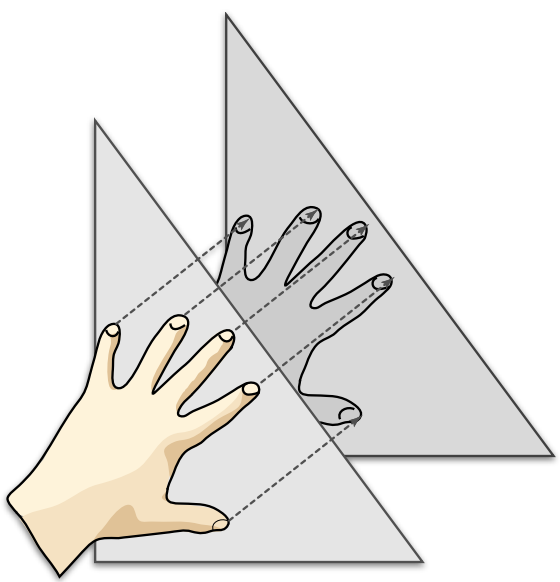


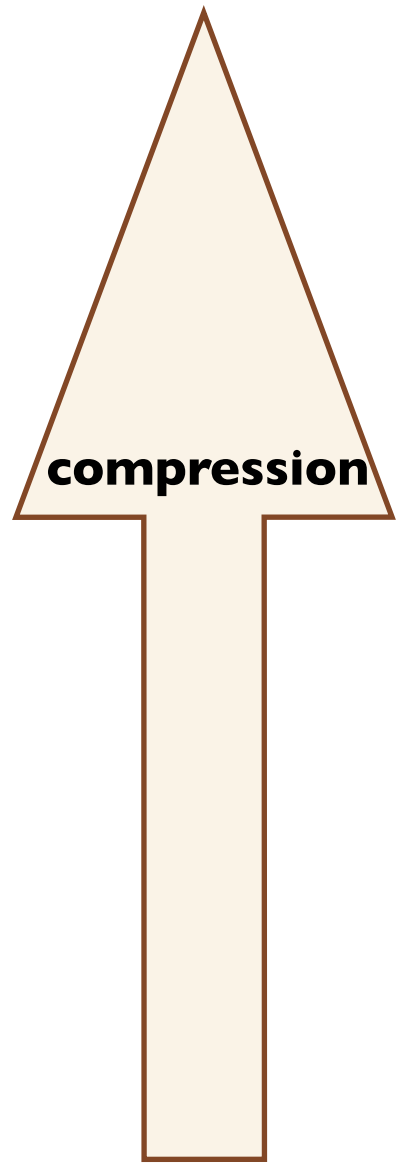
Consider the physical action of a translation and focus not on the *action* carried out but on the *effect* of the shift, represented by any of the arrows AB, $A'B'$ or even a single free vector v.

Embodiment in action

Vectors







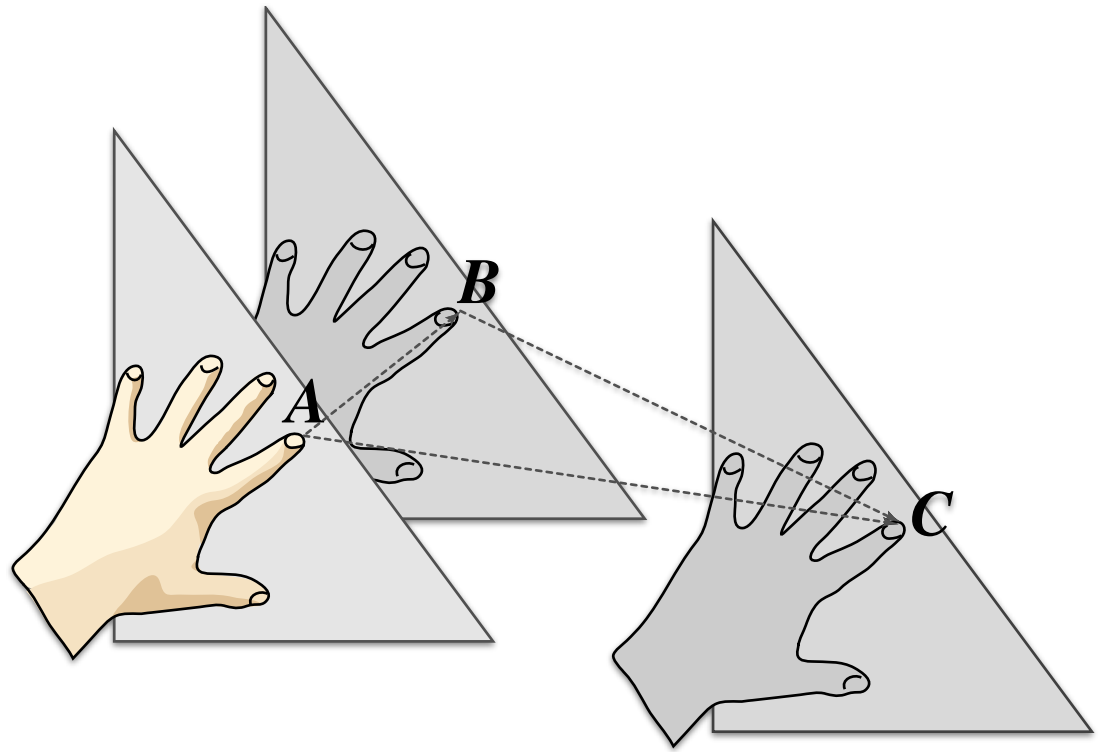
A *single free vector* $\boldsymbol{v}$ as a *mental object*, embodying the effect

The effect of the action as a *process* represented by any of the *equivalent* arrows $\boldsymbol{AB}, \boldsymbol{A'B'}\ldots$

the *action* of a translation

Embodiment in action
Vectors

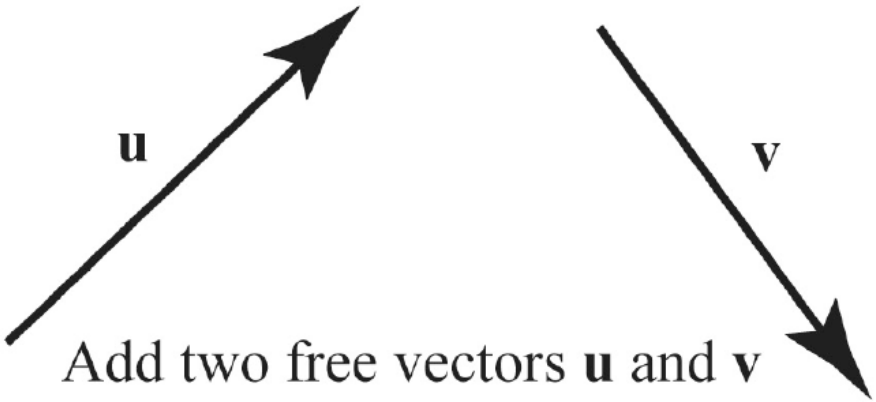
Adding free vectors together by following one translation by another.



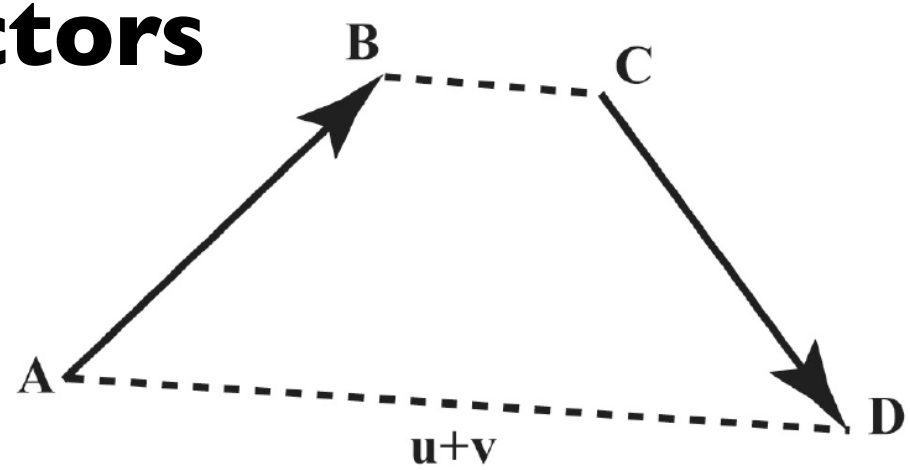
Student Joshua explained that the combination of one translation followed by another was the single translation that *had the same effect*.

Embodiment in action

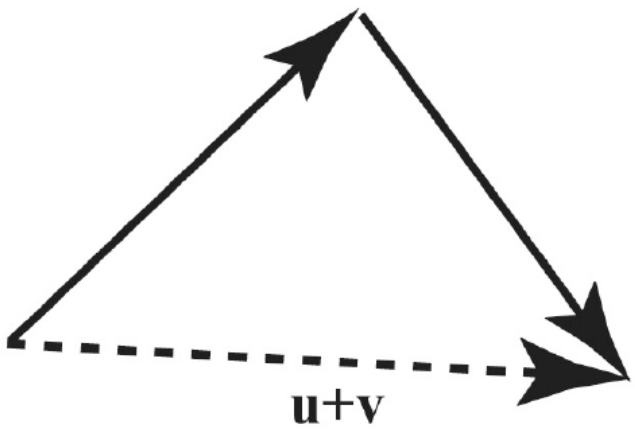
Vectors



Add two free vectors **u** and **v**



(i) as *journeys* travel along from A to B, jump to C then along CD to get **u+v** as AD



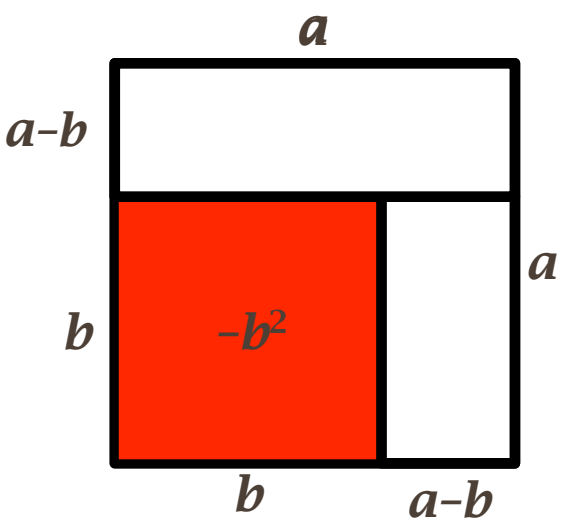
(ii) as *free vectors*, shift **v** to follow after **u** to get **u+v**

Embodiment gives initial meaning to the concept of vector and, in particular, to the flexible concept of ***free vector***.

Compression of Knowledge: Embodiment gives meaning to Symbolism

Example:

The difference between two squares:



Compression of Knowledge: Embodiment gives meaning to Symbolism

Example:

The difference between two squares:

$$a^2 - b^2 = (a-b)\,(a-b)$$

Compression of Knowledge: Embodiment as action gets more complicated

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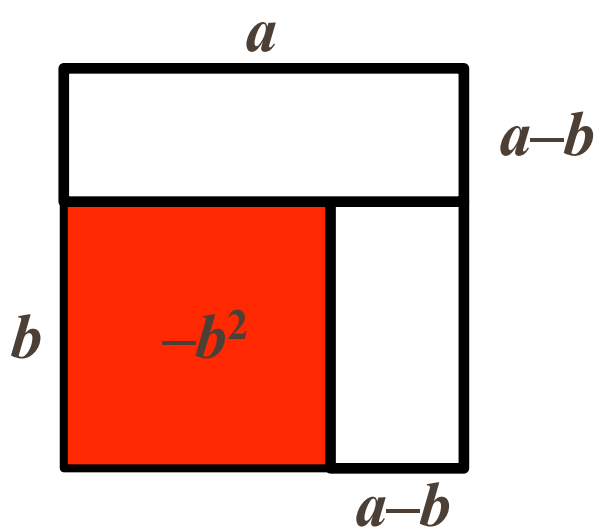
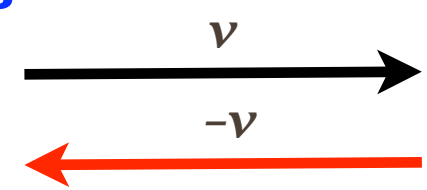
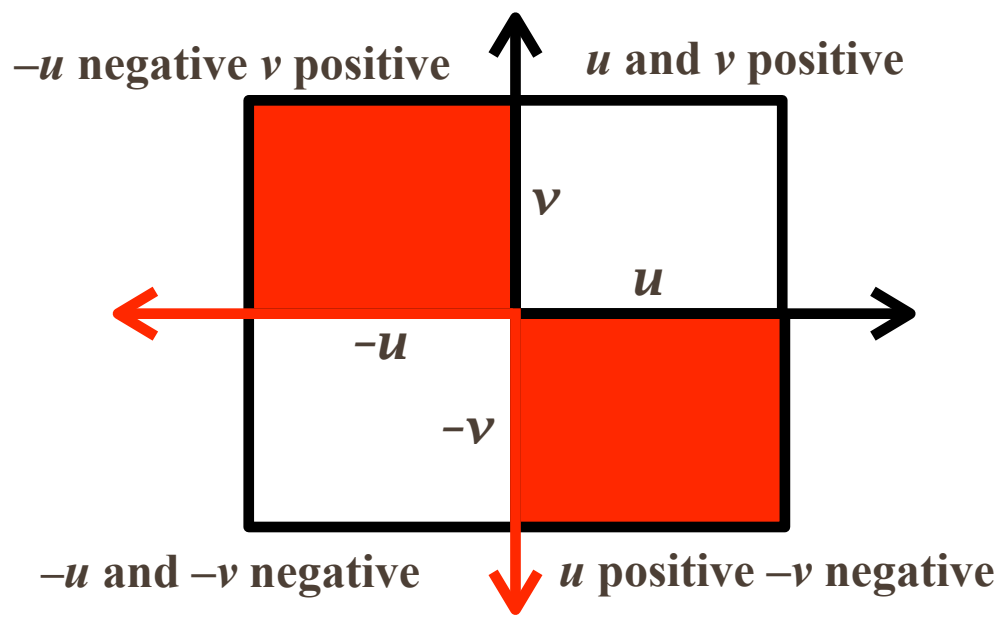


$$a^2 - b^2 = (a-b)\,(a-b)$$

But what happens when, say, *a* or *b* is negative?



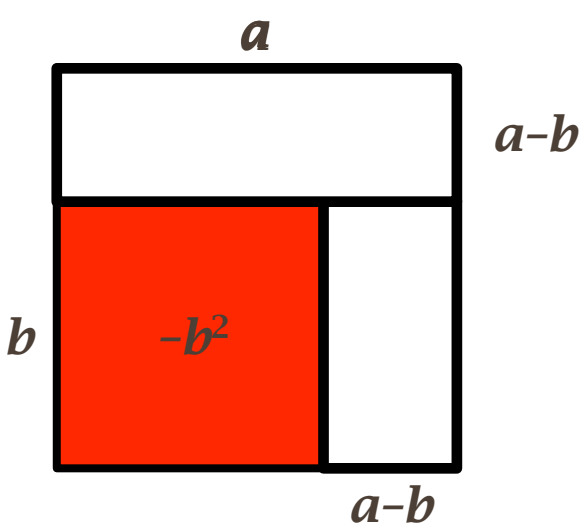


Compression of Knowledge: Embodiment as action gets more complicated

Example:

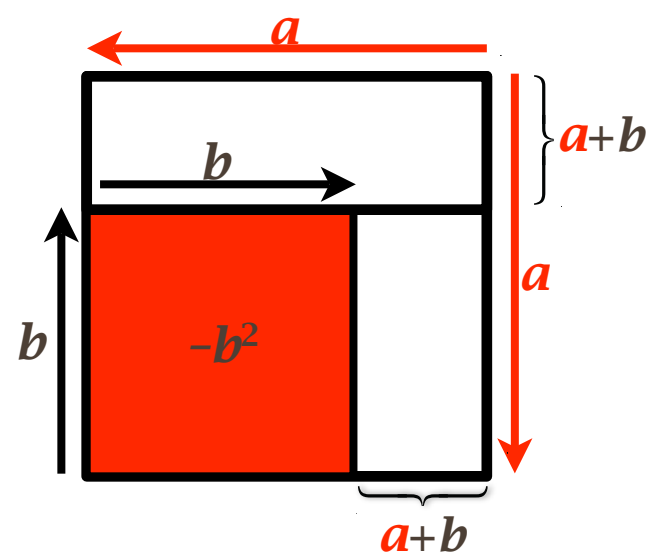
The difference between two squares:

a,b positive.



a negative

b positive, $b < |a|$.



$$a^2 - b^2 = (a-b)\,(a-b)$$

Compression of Knowledge: Embodiment as action gets more complicated

Example:

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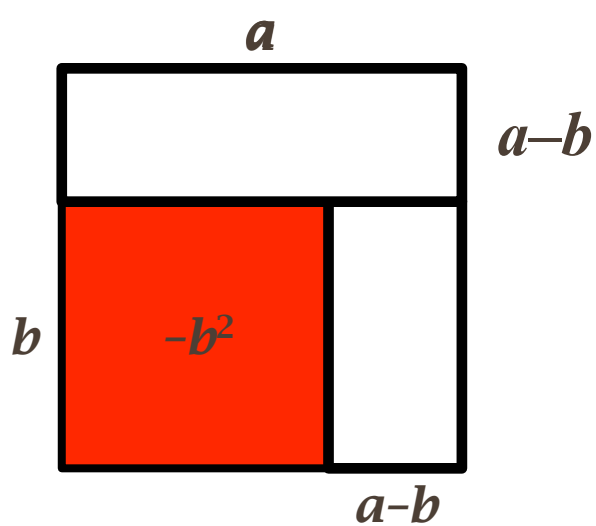
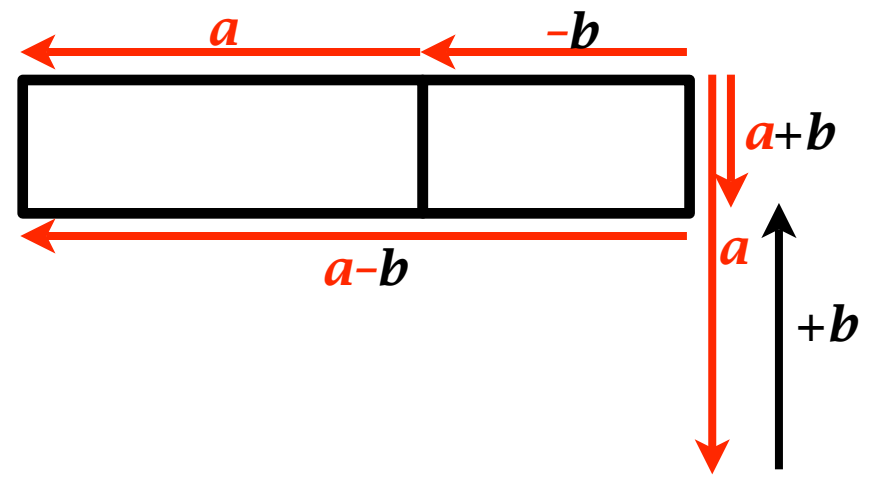


$$a^2 - b^2 = (a-b)\,(a-b)$$

a negative

b positive, $|b| < a$.



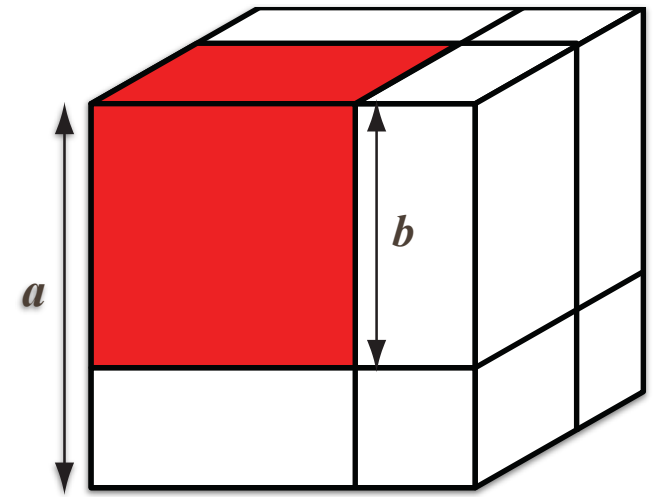
Seen as actions the situation is complicated

Compression of Knowledge: Embodiment as action gets more complicated

If we try to interpret

$a^3 - b^3$

$a^4 - b^4$



$a^3 - b^3$

as pictures in 3 and 4 dimensions,

the first is possible but more complicated,
the second involves imagining 4 dimensions.

Compression of Knowledge: Symbolism can become more simple

The difference of two squares by manipulation of symbols:

$$
\begin{aligned}
(a+b)\,(a-b) &= (a+b)a - (a+b)b \\
&= a^2 + ba - ab - b^2 \\
&= a^2 - b^2
\end{aligned}
$$

Compression of Knowledge: Symbolism can become more simple

The difference between two cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

The difference between two fourth powers:

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$$
$$= (a - b)(a + b)(a^2 + b^2)$$

The symbolism simply uses the difference between two squares twice over.

The symbolism here generalizes relatively easily while the embodiment increases in complication.

Compression of Knowledge: Long-term aspects

Embodiment: often gives simple initial meanings.

It may become increasingly complicated in new contexts.

Symbolism has its own difficulties as actions but become more flexible as procepts and may generalize more readily.

The same symbolism can be seen to be increasingly sophisticated.

1. a single procedure to solve a particular problem,
2. a variety of different procedures to choose most efficient,
3. a process that can be achieved by different procedures,
4. a procept to allow flexible thinking shifting effortlessly from a procedure to do to a concept to think about.

A teacher as mentor can help to encourage students to compress knowledge from procedural symbolism to flexible use of symbols as process and concept.

Compression of Knowledge: Symbolism can become more simple

How can a teacher help students to make sense of more sophisticated ideas?

Can it be done by focusing on *teaching* the ideas by transmission?

Can it be done by focusing on *students learning* by discovery?

Is there a middle way in which *teachers act as mentors to encourage students to make connections*?

Connectionist Teaching & Learning

Teacher Beliefs		
Transmission	**Connection**	**Discovery**
Pupils being numerate		
Perform calculations by standard procedures	Develop methods that are efficient and effective	Find the answer by any method
Paper and pencil methods	Confidence in mental methods	Reliance on practical methods
How pupils ***learn*** to become numerate		
Individual activity following instructions	Interaction with others to develop effective methods	Individual activity based on actions on objects
Pupils vary in their ability to become numerate	Most pupils can become numerate	Pupils vary in the rate at which their numeracy develops
Pupils learn one mathematical routine at a time	Pupils learn through challenge and struggle to face difficulties	Pupils need to be 'ready' before they can learn certain concepts
How to ***teach*** pupils to become numerate		
Teaching has priority over learning	Teaching and learning are complementary	Learning has priority over teaching
Verbal explanation by teacher for the pupil to understand	Dialogue between teacher and pupils and between pupils	Based on practical activities and pupil discovery
Solving 'word problems'	Through challenging problems	Using practical equipment

Askew et al, Kings London, 1997

Connectionist Teaching & Learning

	Highly Effective	Effective	Moderately effective
strongly transmission			Beth Cath Elizabeth
strongly connectionist	Anne Alan Barbara Carole Faith		
strongly discovery			Brian David
no strong orientation	Alice	Danielle Dorothy Eva Fay	Erica

Askew et al, Kings London, 1997

Teachers as Mentors to encourage both power & simplicity in active mathematical learning

Embodiment: can give simple initial meanings
but may become increasingly complicated in new contexts.

Symbolism has its own difficulties but becomes more flexible as our focus of attention switches from *performing the steps* of the procedures to thinking about the *effects* of the procedures.

Teaching procedures may give short-term success and may be the basis for later conceptual development,
but it is not enough to build power & simplicity in mathematical thinking.

A teacher as mentor can help to encourage students to build powerful connections that enable simple thinking:

by building initial meaning from ***embodiment***
transferring it to symbolism to carry out ***procedures***,
then changing the focus to symbols which are dually ***processes*** *to do* mathematics and ***concepts*** *to think about* mathematics (***procepts***).

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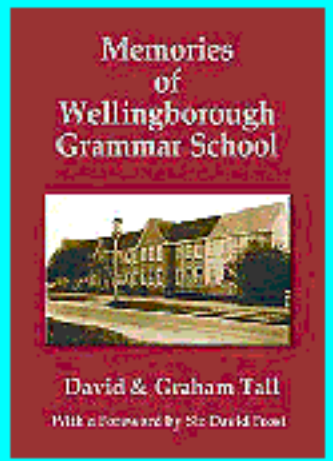

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Research Students
Overheads
Conferences
PME 25 Photographs
PME 26 Photographs
Proceedings of Malt I

References to my papers may be made directly to www.davidtall.com/papers.

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THE UNIVERSITY OF WARWICK

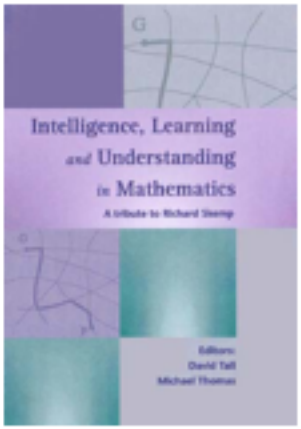
MERC MATHEMATICS EDUCATION RESEARCH CENTRE

David Tall - Professor in Mathematical Thinking

Welcome to the **HOME** page of my website. Feel free to use information about my **research** as a resource, or **download** a paper. There is NEW **news** about recent changes on this site (made on **Tuesday, February 20, 2007**), drafts of current papers and **links** to other sites of interest.
See below for more information.

- information on several **research themes** with links to relevant papers:
  - cognitive development | concept image | cognitive units | cognitive roots | generic organisers
  - procepts | algebra | limits, infinity & infinitesimals
  - visualization | calculus (& computers) | computers in school and college
  - problem solving | advanced mathematical thinking | proof
  - *three worlds of mathematics* NEW
- **a glossary of terms**
- **published books**
- **research students** and joint publications.

Downloads: **Research Papers** in PDF format, **Current writing in draft**, **Selected Lectures**, **Curriculum Vitae**.



News: A continual list of information on updates to focus on the most recent additions, including new papers, information on books including **Fermat's Last Theorem** (3rd Edition with Ian Stewart) and Intelligence, Learning and Understanding: **A Tribute to Richard Skemp** (ed. with Michael Thomas).

David retired in September 2006 and is now Emeritus Professor of Mathematical Thinking at Warwick His retirement conference RETIREMENT AS PROCESS AND CONCEPT, shared with Eddie Gray at Charles University Prague, is available for download and pictures of the conference are here. From now

Published Articles on Mathematics and Mathematics Education

See also other pages for **Drafts**, **Selected Lectures**, **Curriculum Vitae**.

The downloads below include all my papers available in electronic form. Any problems: e-mail david.tall@warwick.ac.uk.
Go directly to papers published in 1975, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 2000, 01, 02, 03, 04, 05, 06, 07
The same papers are also organised under themes or, where appropriate, with research students.

See also: **Draft papers & articles under review**. UPDATED | **Selected Lectures** | **Curriculum Vitae**

2007y Rosana Nogueira de Lima and David Tall. Procedural embodiment and magic in linear equations. Submitted for publication. NEW

2007x Eddie Gray & David Tall (2006). NEW Abstraction as a natural process of mental compression. Submitted to *Mathematics Education Research Journal*. [Reflections and developments of our work together.]

2007b Teachers as Mentors to encourage both power and simplicity in active mathematical learning. *Plenary at The Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing*, 17–19 March 2007, Abu Dhabi.

2007a David Tall (2007). Embodiment, Symbolism and Formalism in Undergraduate Mathematics Education, Plenary at *10th Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*, Feb 22–27, 2007, San Diego, California, USA. [Overheads]