The Theory of PROCEPTs:

Flexible use of symbols as both PROcess and conCEPT in Arithmetic, Algebra, Calculus

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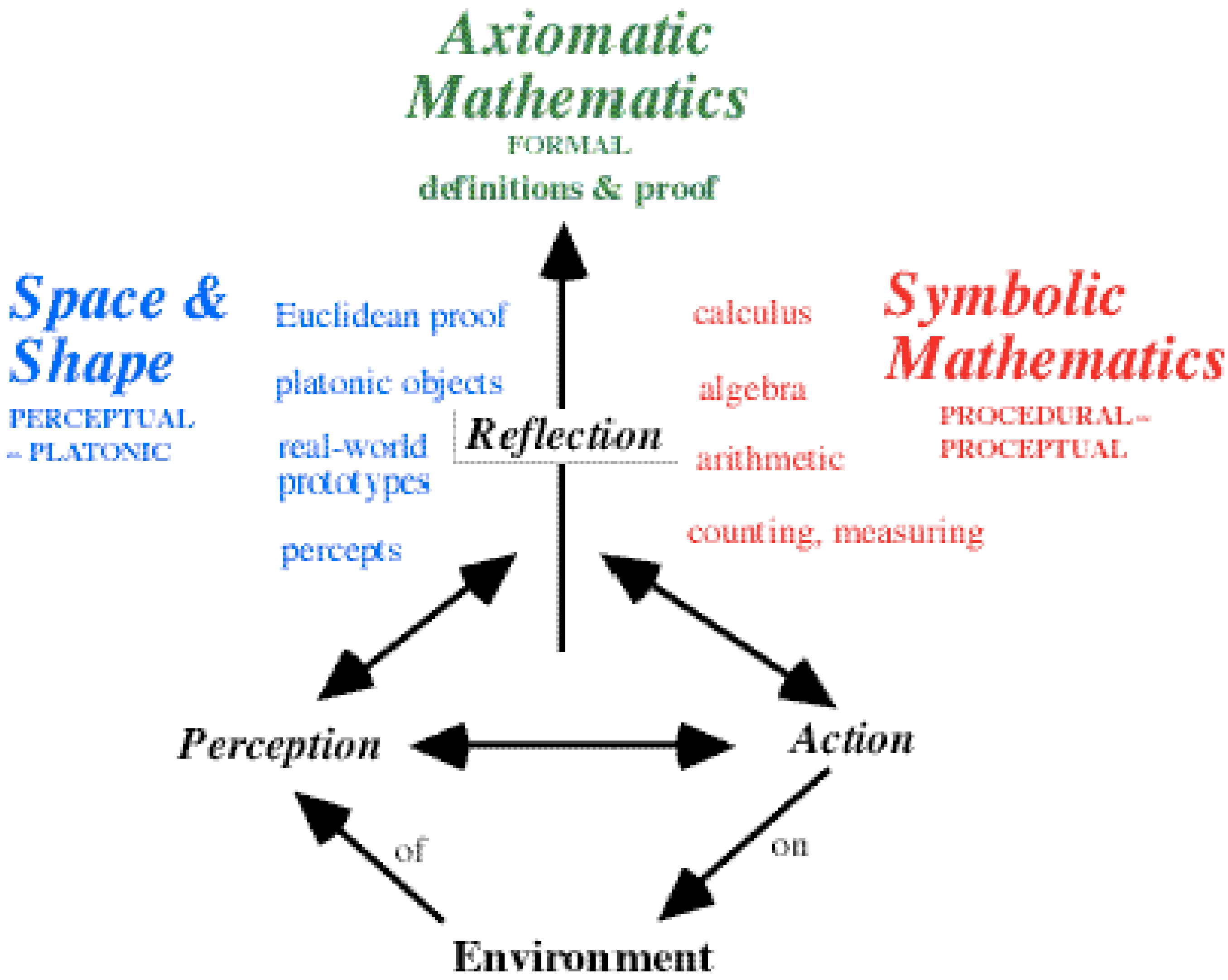
This talk is designed for all who are interested in teaching and using mathematics, from teachers of the young to teachers of mathematics and its applications at university. It focuses on the role of symbols in mathematics, specifically those which allow the human mind to switch effortlessly from "concepts to think about" to "processes to solve problems". For example, in arithmetic, the symbol 3+4 evokes both the process of addition (initally through counting) and also the concept of sum. Some students develop a flexible manner of relating such symbols with others, for instance, noting that "3+4 is 7", because it is ""one less than 4+4, which is 8". The flexible use of a symbol as either process or concept is called a procept. I hypothesise that those who focus on the limited skill of coping with step-by-step procedures have less chance of long-term success than those who use symbols flexibly as process and concept. I will reveal how procepts in arithmetic, algebra, calculus, and university mathematics operate in subtly different ways, causing obstacles in learning that are not overcome by everyone. This in turn causes a divergence between those who seek the limited security of step-by-step procedures and those who learn to use symbols in a more flexible, "proceptual" manner. It is my belief that these these theoretical ideas are extremely simple, yet have significant consequences for the learning and teaching of mathematics at all levels.



Mathematics begins with:
Perception OF the Environment
(leading to space, shape, geometry)

Action ON the Environment
(sorting, counting, measuring, arithmetic … **symbolic maths**)

My talk today focuses on the use of *symbols* in **arithmetic, algebra, calculus and formal mathematical theories.**

Examples of children doing arithmetic:

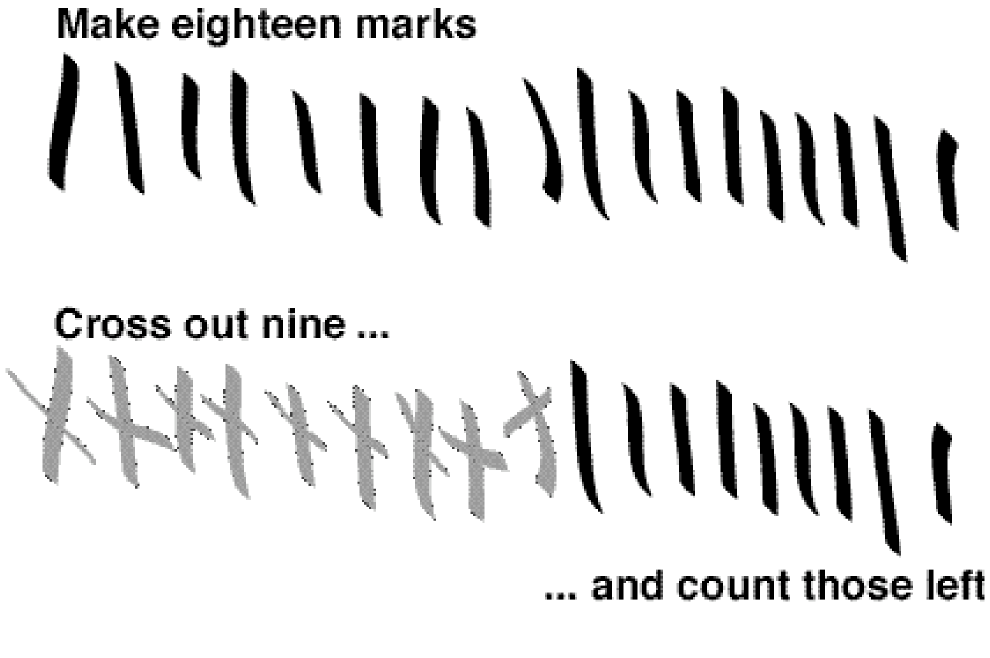
Michael (aged 9), what is 18–9?

He writes: 

and, as in the decomposition process, put a ‘little one’ by the eight …

“*This is the easy way of working it out. I can’t take nine from eight but if I put a little one it makes it easier because now its nine from eighteen*”.

He didn’t seem to realise that this was just the same problem all over again. After some time he resorted to his usual procedure for subtraction from teens. Eighteen marks were placed from left to right on his paper and then starting from the left hand side he crossed out nine marks, counting from one to nine as he crossed out. Recounting from the left those original marks not crossed out, he was able to provide the correct solution.



Gavin (aged 9)

“I like counting with my fingers
– that is what they are made for.”

but for problems up to twenty he assigned numbers in the teens to various parts of his body in a clockwise fashion from left shoulder, to waist, to thigh, to calf and ankle, then up his right side.

“I’ve only got ten fingers; I count as if I had a never-ending load”.

* * *

Philip (aged 8) counted using his toes to supplement his fingers, though he had difficulty moving his middle toes.

* * *

Jay (aged 10) rejected concrete materials:

“I’m too old for counters”,

but neither did he like using his fingers,

“my class don’t use counters or fingers”.

For numbers up to twenty he casually splayed his ten fingers on the edge of his desk and imagined another ten fingers to extend his counting techniques.

Michelle **(aged 10)**,what is **18–7**?

"*ten from eighteen leaves eight, seven from ten leaves three, eight and three makes eleven.*"

Michelle sees 18 as 8 and 10, but takes the 7 from the 10 rather than from the 8. Is this because the number bonds adding to 10 are stronger than others?

Stuart **(aged 10)**, what is 8+6?

"*I know 8 and 2 is 10, but I have a lot of trouble taking 2 from 6. Now 8 is 4 and 4; 6 and 4 makes 10; 10 and another 4 makes 14*".

Stuart is successful, but knows few number bonds, and has to search through his small repertoire to try to solve the problem. He is extremely creative in the mathematics that he is doing, but his methods are arduous and likely to come under considerable strain when he tries more complicated tasks.

Fundamental Questions :

Why are some highly successful with arithmetic, but others have great difficulty ?

Why do some think flexibly and others rely on routine procedures that may break down ?

What are the mental activities going on in the brain when the children DO arithmetic and THINK about it ?

Children first operate in the actual world on physical objects, later on imagined objects, then they use not only spoken, but written symbols.

Counting is performed by coordinating several activities, including:
Looking at a collection of objects
Pointing to each in turn (once and once only)
Saying the verbal number sequence.
(This takes several months to coordinate.)

Mental objects can be imagined to perform the operations on (counting fingers or parts of the body, or other imaginary objects).

Written symbols can be used to represent the numbers and, having realised that the count can be performed on a set in any order, the *number* of elements in the set may be conceived.

Some children continue to focus on objects, either real or imagined, others use symbols or both.

The way the symbols are used successively is interesting…

Counting …

There are “**one, two, three**, ***four***.”

There are “[one, two, three,] … ***four***.”

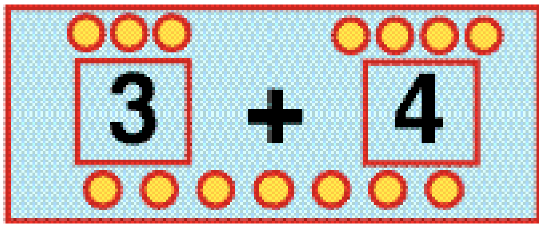
“**…** ***four***.”

“***four***”

Compression from counting process to number concept …

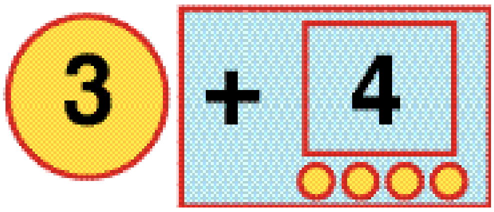
Different ways of viewing the sum of two numbers, 3, 4:

- **count-all**:



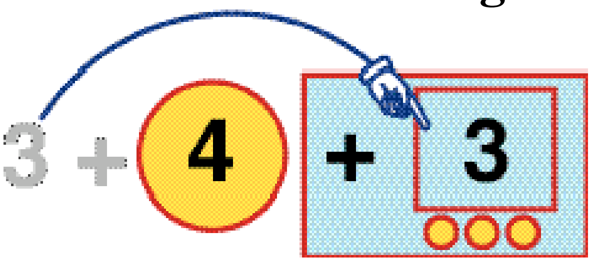
Count 3, count 4, count-all.
(all counting processes)

- **count-on**:



Starting from 3, count-on 4.
One concept, one process

- **count-on from larger**:

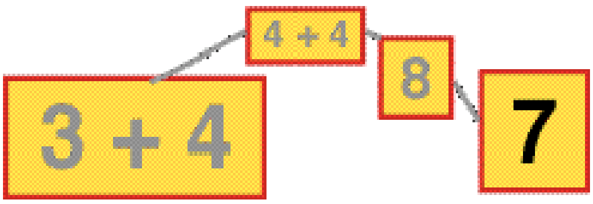


Starting from 4, count-on 3.
More efficient count-on.

- **known fact**:

3+4 is **7**
("flash" from problem to answer)

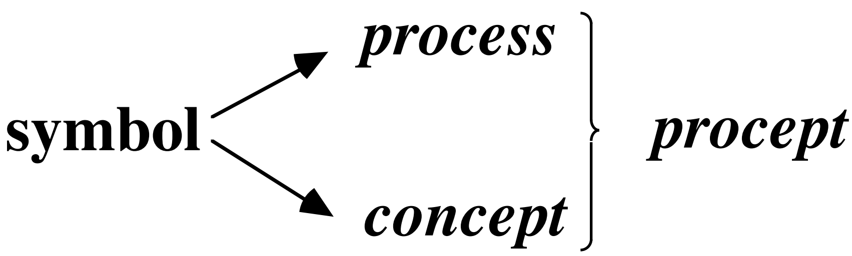
- **derived fact**:



3+4 is…[one less than 8]… **7**.
(using internal links)

Compression of process into concept using symbols

symbol	*process*	*concept*
3+2	addition	sum
–3	subtract 3 (3 steps left)	negative 3
3/4	division	fraction
3+2x	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$, $\sum_{n=1} 1/n^2$	tending to limit	value of limit
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n



(Gray & Tall, 1994)

The notion of procept

Preliminary definition:

> The combination of ***pro*****cess** and **con*****cept*** represented by the same symbolism is defined to be a ***procept***.

I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation.

William P. Thurston, Fields Medallist, 1990.

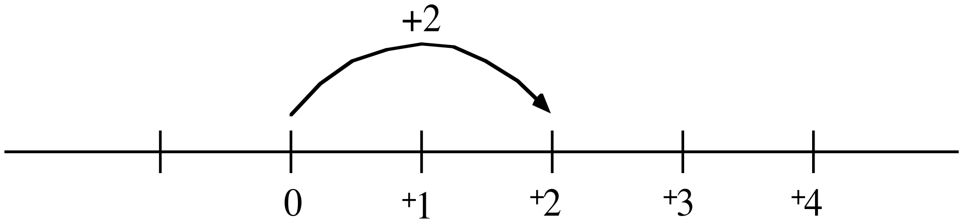
More sophisticated definition:

> An *elementary procept* is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object.

Gray & Tall, JRME, 1994

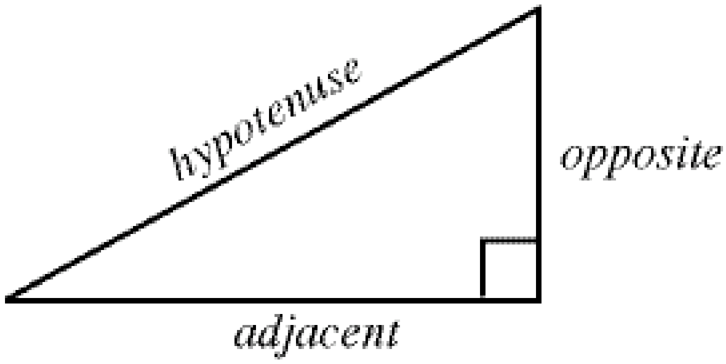
Mathematical procepts: symbols representing both process and concept:

- 3+2 represents both the **process of addition** and the **concept of sum**,
- 3x2 represents both the **process of multiplication** (repeated addition) & the **concept of product**,
- +2 represents both the **process "add two"** (or shift two units to the right on the number line) and also the **concept of a positive signed number**,



- –2 represents both the **process of "subtract two"** (or shift two units to the left) and also the **concept of negative number**,
- $\frac{3}{4}$ represents both the p **rocess of division** and the **concept of fraction**.

- $\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$ represents both the **process of calculating the trigonometric ratio** and also the **concept of sine**,



- π=3.14159… represents the **concept π** as a **process of approximation**.

The following represent both the **process of tending to a limit** and the **value of that limit**:

- $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

- $\lim_{n \to \infty} \dfrac{1 - x^n}{1 - x}$

- $\sum_{n=1}^{\infty} a_n$

- $\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\,\Delta x$.

Typical (traditional) symbolic development

(a) ***procedure***: finite succession of decisions and actions is built up into a coherent sequence,

(b) ***process***, increasingly efficient ways become available to achieve the same result, now seen as a whole,

(c) ***procept***, where the symbols are conceived flexibly as processes to *do* and concepts to *think about*.

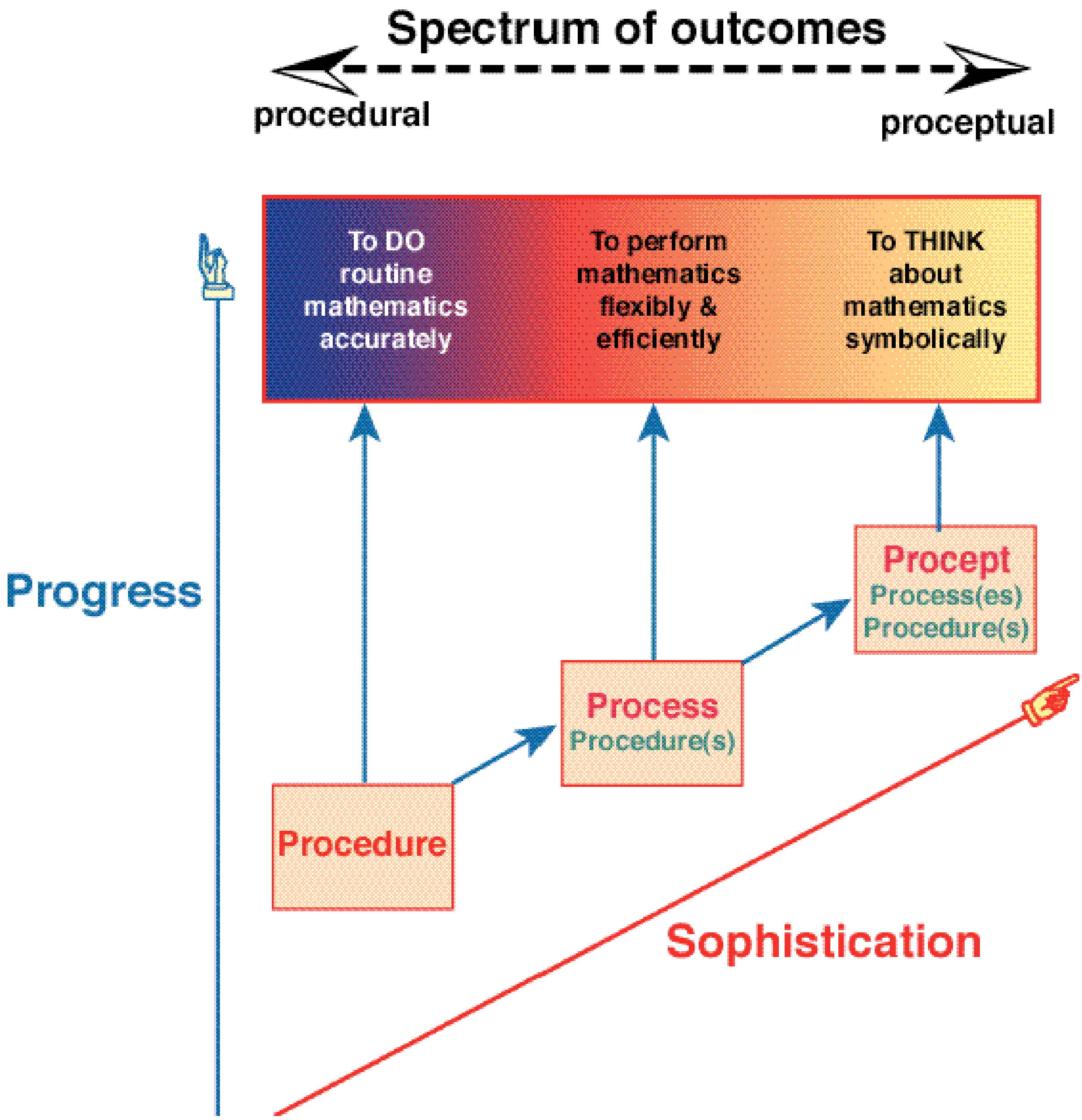
Small conscious focus of attention has the following effects:

Procedures occur *in time* and take up mental space.

Procepts can be conceived and manipulated as mental concepts.
Procepts are *easier* to manipulate for those who are flexible thinkers.

Procedures are more primitive ways of thinking which may give short-term success but prove burdensome for long-term thinking *about* mathematics.

Procepts act as a vital pivot between symbols to THINK ABOUT and processes to DO mathematics.



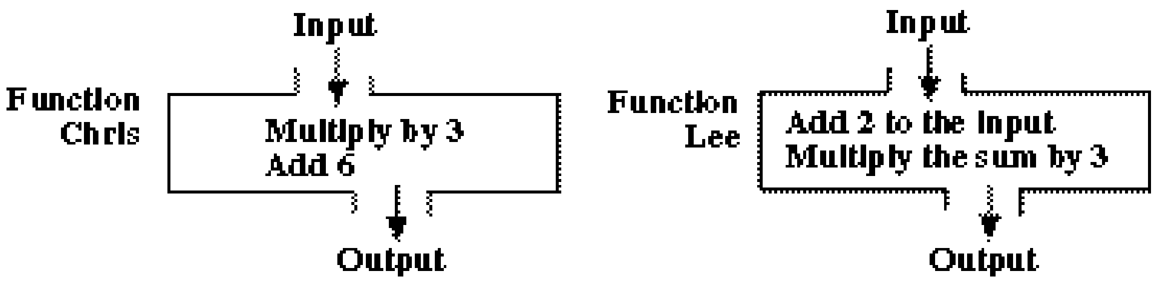
Procedure allows a student to *DO* a routine problem.

Process: one or more procedures which for the same input give the same output; thought about without being performed.

Procept: symbolized process representing output as a mental concept gives flexibility for building higher levels.

Each of these contains the previous ones within it…

Example from College Algebra:



Write the outputs of these two function boxes and say if they are the same function.

- *Student 1 (grade A)*: **3x+6, 3(x+2)**

Yes, if I distribute the 3 in Lee, I get the same function as Chris. **(procept)**

- *Student 2 (grade B)*: **x3+6, (x+2)3**

Yes, but different processes. **(process)**

in our terminology

- ***Student 3 (grade C)*: 3x+6, x+2 (3x)**

No, you come up with the same answer, but they are different processes. **(procedure)**

(Phil DeMarois, PhD Thesis, University of Warwick, 1998)

Procedure and flexible process in calculus

Determine the derivative of $\dfrac{1+x^2}{x^2}$

Procedural use of quotient rule:

$$y = \frac{1+x^2}{x^2},$$

$$\frac{dy}{dx} = \frac{(2x)(x^2) - (2x)(1+x^2)}{(x^2)^2} = \frac{2x^3 - 2x - 2x^3}{x^4} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

Flexible use of process (by conceptual preparation):

simplify to $x^{-2} + 1$ so derivative is $-2x^{-3}$,

Students' grade	Procedural rule **[PROCEDURE]**	Conceptual preparation **[PROCESS]**
A	2	10
B	6	6
C	8	4
Total	16	20

Difference between A and C grade students significant at the 5% level (χ^2 test).

(Maselan Bin Ali, PhD Thesis, University of Warwick 1997)

Available methods for students find the derivative:

Product rule,

Quotient rule,

Simplification first,

Differentiate $yx^2 = x^2+1$.

Students' grade	0 or 1 methods **[PROCEDURE]**	2 or 3 methods **[PROCESS]**
A	3	9
B	7	5
C	9	3
Total	19	17

Difference between A and C grade students significant at the 5% level (χ^2 test).

Maselan Bin Ali (PhD Thesis 1997)

This research suggests a possible refinement of

procedure/process/procept

to

procedure/multi-procedure/process/procept

where multi-procedure means that the student is still highly procedural but has a choice of procedures available.

Development of procepts in algebra

Is an expression **2+3*x*** seen as a process or a concept?
The process "add 2 to 3 times x" cannot be calculated until x is known. If x is known, why use algebra?

Other problems: children may write x and y next to each other as $x\ y$ and think of it as "x and y". But they are told that xy is "*x times y*".
Read from left to right $2+3x$ says "two plus three times x".
Because 2+3 is 5, children may think that $2+3x$ is $5x$. But it is not.

Algebra soon becomes a meaningless manipulation of meaningless symbols, each week of study bringing a new procedure to carry out: "collect together like terms", "do operations inside brackets first", "do multiplication before addition", "do the same thing to both sides", "change sides, change signs", …

Is algebra viewed procedurally or flexibly (proceptually!) ?

Consider these equations. Which is more difficult?

(i) $\square + 3 = 7$,

(ii) $3x + 1 = 4x - 1$,

(iii) $3 + x = 7$,

(iv) $3 + \square = 7$,

(v) $3x + 1 = 7$.

suggestion : (iv) < (i) < (?) (iii) < (?) (v) <<< (ii)

The procedural conspiracy

As teachers we want to help our student *do* mathematics. When a child cannot cope with mathematics the cry is “show me how to *do* it”. Showing “how to do it” has short-term gains:

- the child is happy to be able to “do it”
- the teacher is happy the child can do it
- parents and politicians are happy.

But there may be long-term draw-backs:

- the child may become increasingly procedural
- the child suffers increased cognitive load and eventual failure

Procepts in the theory of limits

The limit process can get "arbitrarily close" to the limiting value. Therefore the "limiting value" can be conceived as a variable quantity.

Most students do *not* view an infinite decimal as a *limit*. It is a means of approximating the limit as close as is desired. It is "infinitesimally close".

Complete the following sentences:

1, 1/2, 1/4, 1/8, … tends to ____________

The limit of 1, 1/2, 1/4, 1/8, is ____________

Tends to …	0	0	1/∞	0	2	0	0
Limit is …	0	1/∞	1/∞	?	2	2	1
Pre-test (*N*=25)	0	**11**	1	5	0	2	2
Post-test (*N*=23)	**8**	3	3	0	4	0	2

"2" may indicate the sum of the *series* $1+\frac{1}{2}+\frac{1}{4}+\ldots$.

the response "1" for the limit may be the largest term.

The most commonly occurring response changed from

"tends to 0, limit 1/∞" to "tends to 0, limit 0"

suggesting 1/∞ as an indefinite number, arbitrarily small, is being replaced by the numeric limit 0.

However, "0·9 repeating" did not change its image:

Is $\boldsymbol{0\cdot\dot{9}=1}$ ***?***	Y	**N**	?	no response
pre-test (*N*=25)	2	**21**	1	1
post-test (*N*=23)	2	**21**	0	0

Lan Li, MSc Thesis, University of Warwick, 1992

(A) Can you add $0\cdot 1+0\cdot 01+0\cdot 001+\ldots$ and go on forever to get an exact answer? (*Y*/*?*/*N*)

(B) Is 1/9 equal to $0\cdot 1+0\cdot 01+0\cdot 001+\ldots$? (*Y*/*?*/*N*)

The favoured response on both pre-test and post test is *No* to (A) and *Yes* to (B):

Responses to (A) & (B)	**Y Y**	**Y N**	**N N**	**N Y**	**N ?**	**? N**	***No response*** **Y**
pre-test (N=25)	4	0	1	**18**	0	1	1
post-test (N=23)	2	2	2	**14**	1	0	0

Observation: Many students believe in (variable) infinitesimal quantities, infinite or arbitrarily large quantities, variable arbitrarily close quantities.

Question:

Is there a smallest positive real number ?

Is there a first positive real number?

What are the implications for the concept image of the number line ???

A cauchy continuum???

Formal proof

Begins with definitions/axioms, and deduces theorems.

This has *logical* processes, and *formal* objects.

Axiomatic systems are (rarely) procepts.

A group G is not a symbol representing process/concept.

However: *elements* of a group can be both

processes (eg transformations)
and ***objects*** (elements of the group).

Two forms of Advanced Mathematical Thinking:
Technical (=embodied proceptual)
(involving computation, eg vectors as n-tuples and matrices), usually based on modelling real-world examples.

Formal
(involving definition, eg formal vectors satisfying axioms).

Different constructions of objects in Advanced Mathematical Thinking

Technical objects may include ***procepts***
(eg vectors, transformations, functions etc)

Formal objects are ***defined concepts***

The difference between "described objects" and "defined objects" …
A ***described objec****t* e.g. in a dictionary is given a description to *identify* it. i.e. **object□ description**

Formal object *has specified properties to* ***define*** it.
i.e. **definition □ object**

The move from elementary mathematics to formal mathematics often reverses the order of previously acquired knowledge. Eg A child learns subtraction *b-a* first, then the negative -*a*. But in an axiomatic theory the *negative* is defined first as -*b* and then *subtraction* is defined as *a*+(-*b*).

A story…

The Professor is teaching students undergraduates that one is bigger than zero, based on the field axioms.

Proof that 1 is bigger than 0…

Either 1 > 0 or 1 = 0 or –1 > 0

("axiom of trichotomy")

But **if –1 > 0** then, (–1) x (–1) > 0

(because " $a > 0$, $b > 0$ implies $ab > 0$")

so, (–1) x (–1) = 1

(by some previously deduced theorem)

giving 1 > 0.

Hence both **–1 > 0 and 1 > 0**,

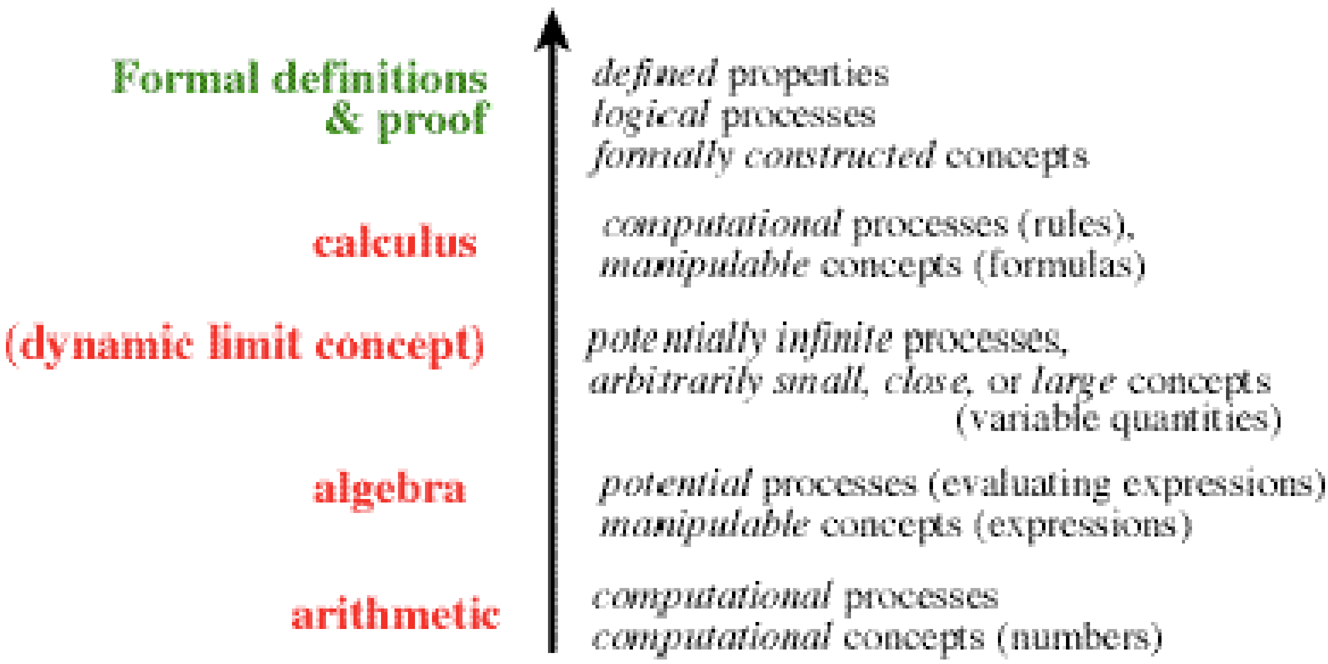
which contradicts "trichotomy".

This proves 1 > 0

Note the proof uses "product of two minuses is a plus" which most students learn procedurally, to show 1 > 0 which most students have known since the age of two.

Summary of procept development

The successive development of symbols in the curriculum operate in subtly different ways:



Are there solutions?
Arithmetic and the calculator

8–5	
	3
18–5	
	13
28–5	
	23

Representing numerical relationships
on a suitable calculator

Calculators:

- can be used to focus on *relationships* rather than *procedures* of counting,
- allow children to see the regularities of numbers above twenty rather than the verbal irregularities in the numbers from 10 to 20,
- Calculators enable children to get a sense of number size for a wider range of numbers, quicker,
- do *not* render children unable to do simple arithmetic,
- should be complemented by mental work and remembering of known facts.

Problems with traditional algebra teaching

Fruit salad algebra:

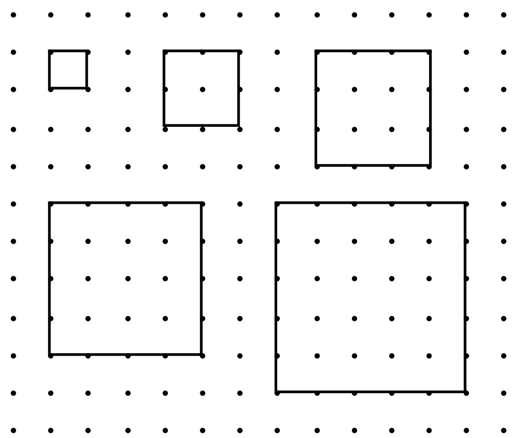
2a+3b is 2 apples plus 3 bananas,

2a+3b+4a is 2 apples, 3 bananas & 4 apples, so it is 6 apples and 3 bananas, that is 6a+3b.

What does $a^2 – b^2$ mean? What is a square banana?

* * *

Algebra through pattern?



How many unit slabs round a square pond?

1 by 1 has 8

2 by 2 has 12

3 by 3 has 16

etc

What is the pattern? (4 extra each time?)

What is the value for the *n*th square? What is *n*?

* * *

Introducing algebra with a computer/calculator

Giving meaning to variables and expressions through predicting and testing evaluations

In BASIC	Using a TI-92:	
x=3	3 sto x	3
PRINT x+1	x+1	4
3		
PRINT 2*x+1	2x+1	7
7		

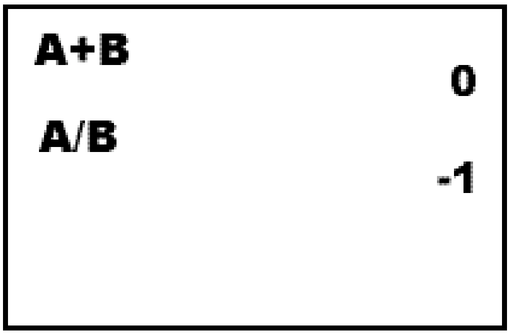
What do these activities show?

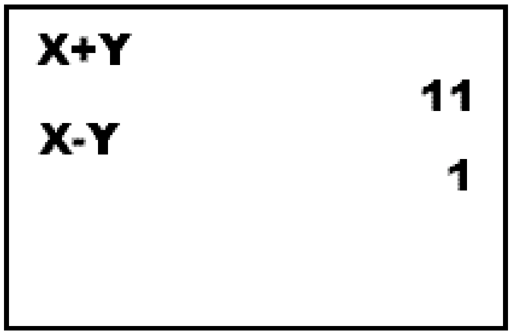
In BASIC	Using a TI-92:	
10 INPUT x	5 sto x	5
30 PRINT 2+3*x	2+3x	17
40 PRINT 5*x	5x	25
10 INPUT x	2 sto x	2
20 INPUT y	3 sto y	3
30 PRINT 2*(x+y)	2(x+y)	10
40 PRINT 2*x+2*y	2x+2y	10

The computer/calculator carries out the evaluations and allows the student to focus on the (similarity of) the results.

‘Screen Snaps’

Students are given a screen view and required to reproduce it on their own calculator screen:





Screen snaps to be reproduced,
requiring the student to find the values
of A, B in each case

(Graham & Thomas, Educational Studies in Mathematics, 2000,

See Thomas, M. O. J. & Tall, D. O. (2001). The long-term cognitive development of symbolic algebra, *International Congress of Mathematical Instruction (ICMI) Working Group Proceedings - The Future of the Teaching and Learning of Algebra, Melbourne*, 2, 590-597.

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Screen snaps have the advantage of encouraging beginning algebra students to engage in reflective thinking using variables. This is beneficial since, unlike experienced mathematicians, they do not attempt to reproduce them by using algebraic procedures, but by assigning various values to the variables and predicting and testing outcomes.

Three kinds of algebra

1) **Evaluation algebra**: algebraic expressions for specifying processes of calculation,
2) **Manipulation algebra**: standard algebra with expressions as manipulative procepts for solving equations
3) **Formal algebra**: as in group theory, rings, fields, vector spaces etc given by definitions and deductions.

Experiments with BASIC and variables on a calculator both significantly improved the conceptual solutions of the experimental students compared with a corresponding control group. This has important outcomes

1) Evaluation algebra (using spreadsheets and graphic calculators) has an important role in technology that was non-existent in traditional mathematics.
2) It has the potential to lay the foundations to equivalent expressions with different procedures representing the same process. Equivalence is an essential ingredient to understanding the manipulation of symbols.
3) Programming using evaluation algebra on a TI-92 can give important insight into the foundations of functions, algorithms, calculus, etc.

Programming on the TI-92

The TI-92 is incredibly flexible. Type

```
:f(x)
:Func
:sin(x)
:EndFunc
```

The area under **y=f(x)** from **a** to **b** in **n** steps:

```
:area(a,b,n)
:Func
:Local k, s, h, x
:0→s : (b-a)/n→h : a+h/2→x   [see note below]
:For k,1,n
: s+f(x)*h → s
:x+h →x
:EndFor
:s
:EndFunc
```

Note: this is the mid-ordinate rule. Replace by **a→x** for first ordinate, **a+h→x** for last ordinate. Consider the changes necessary for the trapezium rule.

Try plotting the values of area(0,x,10) as a graph! (To allow for the speed of the TI-92, do the following:

Select **WINDOW** and specify **Y=area(0,x,10)**.

Select **WINDOW** and specify a suitable x-range, y-range, say from **-10** to **10** for both, then set **xres: 10** so that it only plots once every 10 pixels. Use **F2:ZoomSqr** to reset the x-range to make the graph square.

You might like to generate investigations for the classroom. Eg. go to **HOME** and calculate

area(0,π/2,5)

area(0,π/2,10)

area(0,π/2,100)

and, if you have time,

area(0,π/2,1000)

What do you notice?

If you use **APPS** **7**, **Type: function**, **variable: t** and type:

```
t(x)
:Func
:area(0,π/2,n)
:Endfunc
```

Then you have a *sequence* t(1), t(2), … which tends to the limit 1! You could go to **HOME** and successively calculate

t(1), t(2), t(3), t(5), t(10), t(100), etc.

If, on the other hand, you define

```
appr(x)
:Func
:area(0,x,10)
:Endfunc
```

then you have an approximate area function **appr(x)** calculating the area under sin(x) from 0 to x!

What happens to the sign of the expressions area(0,π/2,10), area(π/2,π,10) ? What about area(π/2,0,10), area(π,π/2,10)? Etc., etc.