

David Tall
University of Warwick, UK
Presentations in Bogota, July 2-5, 2002
on the theme:

Biological Brain, Mathematical Mind, Technological Tools

1. Three worlds of Mathematics

How our biological sensory perception grows to develop EMBODIED, SYMBOLIC & FORMAL mathematical worlds. The role of the computer as a tool in this development.

Workshop: from embodied to formal geometry using Cabri.

2. An **Embodied** approach to the Calculus

supported by technology, leading to symbolism & formal proof.

Workshop: Visual Calculus.

3. Discontinuities in development of **Symbols**

Cognitive development in arithmetic, algebra, calculus.

Workshop: Programming in Calculus.

4. The Development of **Proof**

How proof develops cognitively through embodied, proceptual and formal stages.

Workshop: Problem-Solving and Proof.

Three Worlds of Mathematics

Bogota, Columbia, July 2–5, 2002.

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In this series of seminars, I am conscious that I am speaking to teachers who use the TI-92 as a major tool in their mathematics teaching. I intend to place its use in a wider perspective to reveal its strengths and weaknesses.

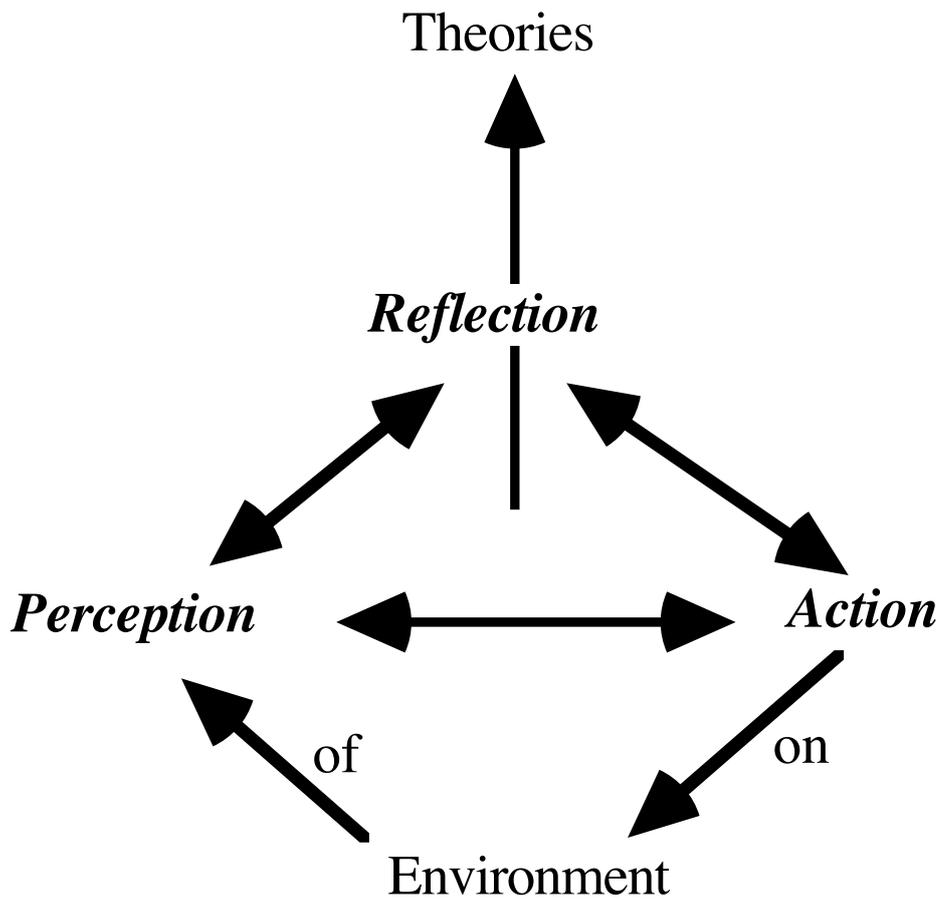
I shall contrast and compare three different kinds of mathematics that are so different that they seem to inhabit three distinct worlds:

*The **SENSORY-EMBODIED WORLD** of perception and action, including reflection on perception and action which develops into a more sophisticated Platonic form.*

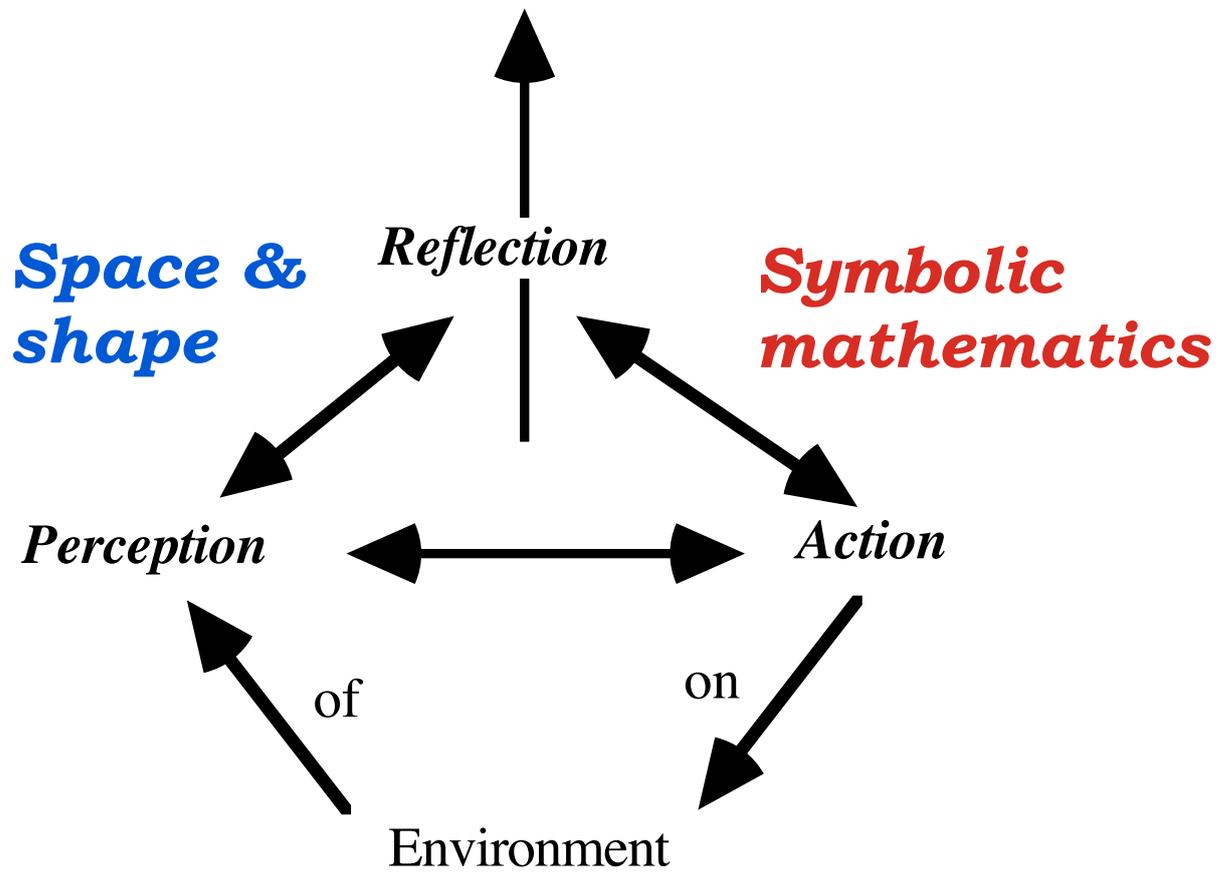
*The **SYMBOLIC-PROCEPTUAL WORLD** of symbols in arithmetic, algebra and calculus that act both as **PRO**cesses to do (eg $4+3$ as a process of addition) and **con**CEPTs to think about (eg $4+3$ as the concept of sum.)*

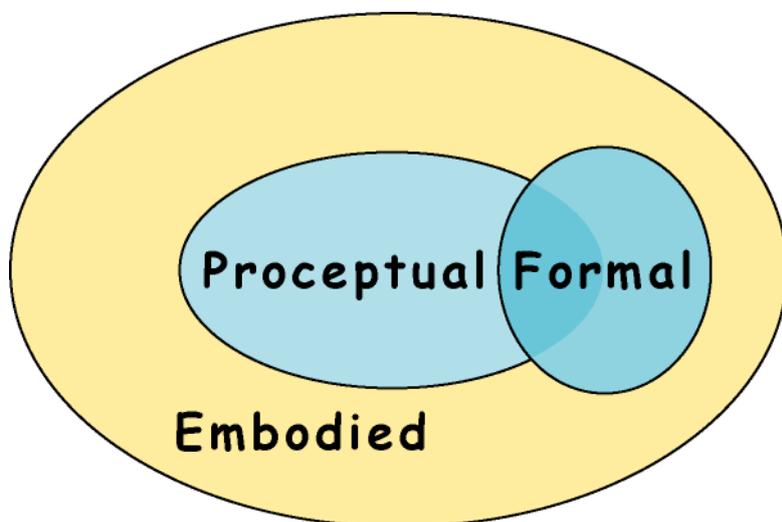
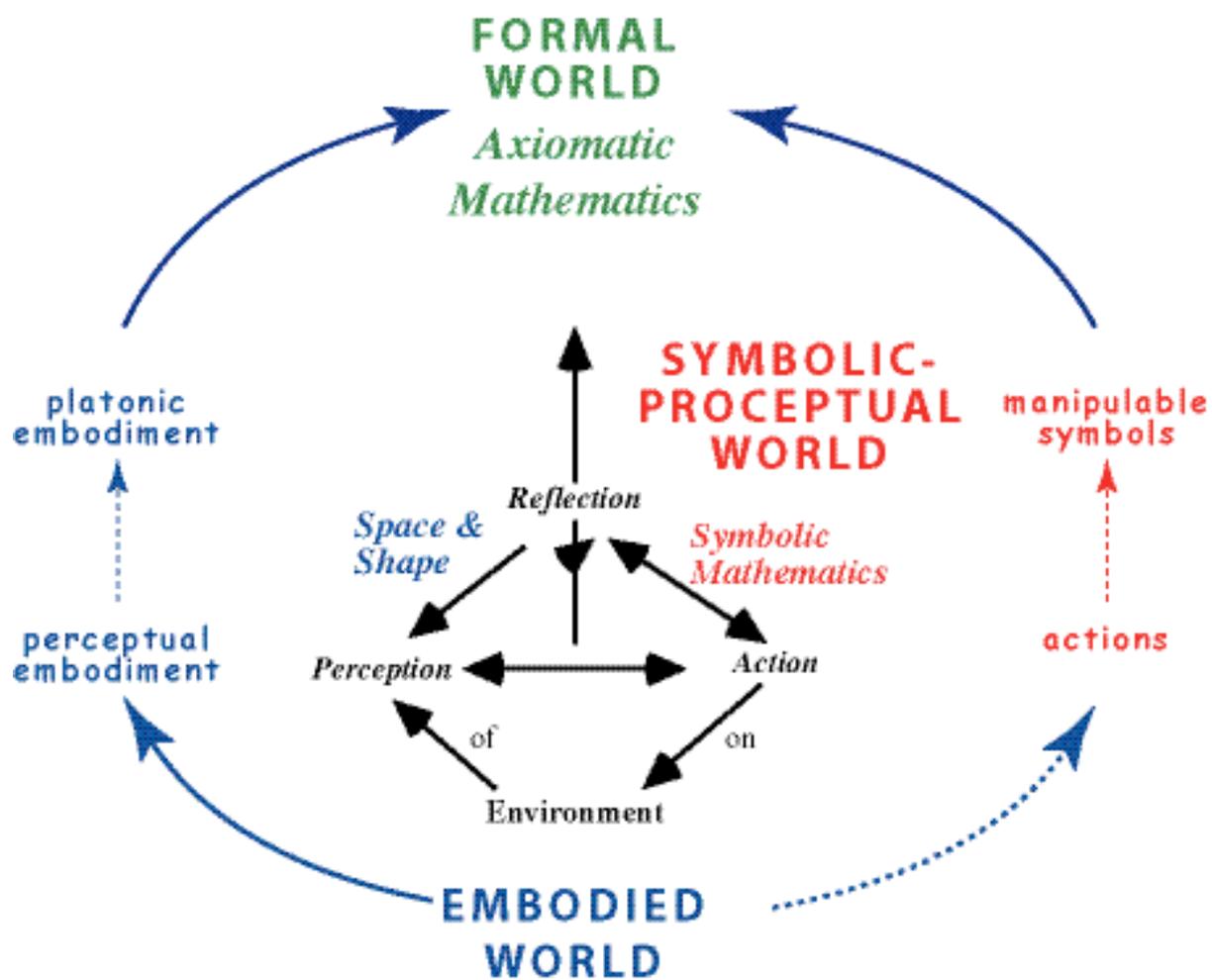
*The **FORMAL-AXIOMATIC** world of formal definitions and formal proof.*

Human Development from embodied interaction to formal theory

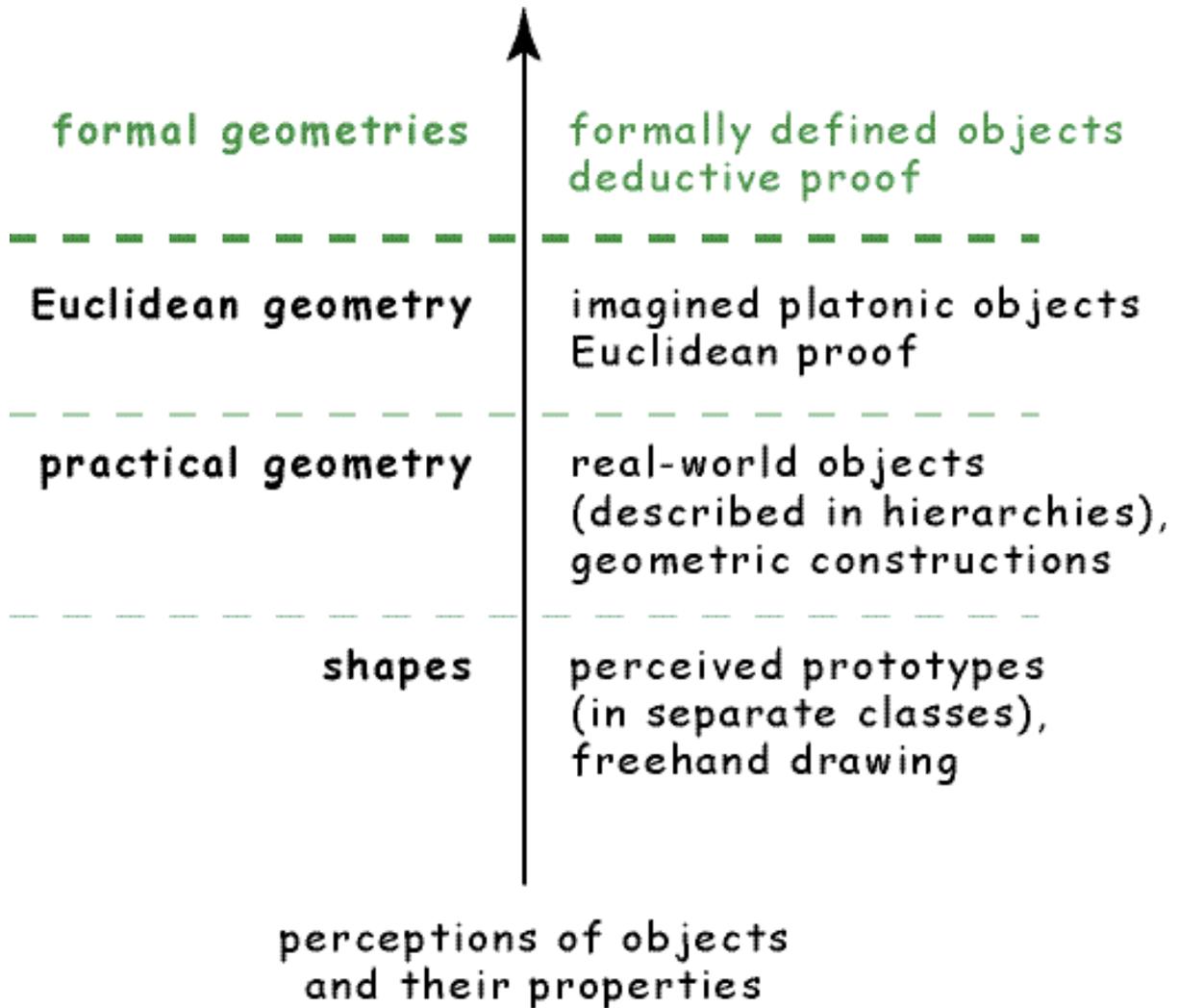


***Axiomatic
mathematics***





THE GEOMETRIC JOURNEY FROM THE EMBODIED TO THE FORMAL WORLD



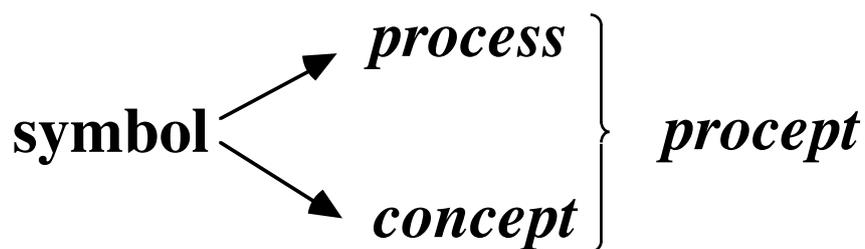
Shapes are visualised, named (in separate classes so that a square is not a rectangle), then defined (so that a square *is* a rectangle) then imagined as platonic objects satisfying the core definition.

At the formal level, the objects are given *only* by their definitional properties.

[We will return to this later in this presentation.]

COMPRESSION OF PROCESS INTO CONCEPT USING SYMBOLS

<i>symbol</i>	<i>process</i>	<i>concept</i>
$3+2$	addition	sum
-3	subtract 3 (3 steps left)	negative 3
$3/4$	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x) dx$	integration	integral
$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$	tending to limit	value of limit
$\sum_{n=1}^{\infty} \frac{1}{n^2}$		
σ_n	permuting $\{1, 2, \dots, n\}$	element of S_n



(Gray & Tall, 1994)

Discontinuities in the development of symbols

Arithmetic procept 2+3

built-in computational process of addition to give result

Algebraic procept $2+3x$

potential process of evaluation, manipulable concepts

Limit procept

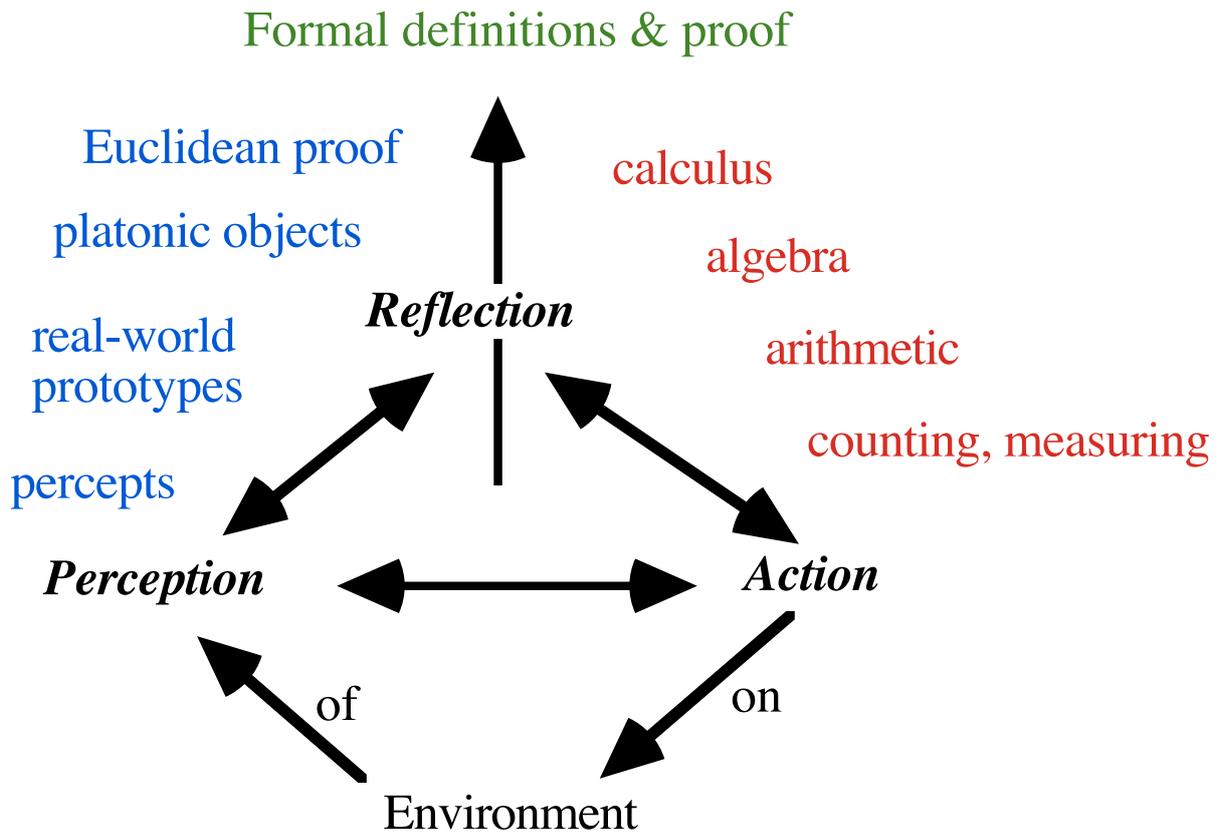
$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have *potentially infinite processes*

Some, eg

$$\frac{d}{dx}(\sin x \cos x)$$

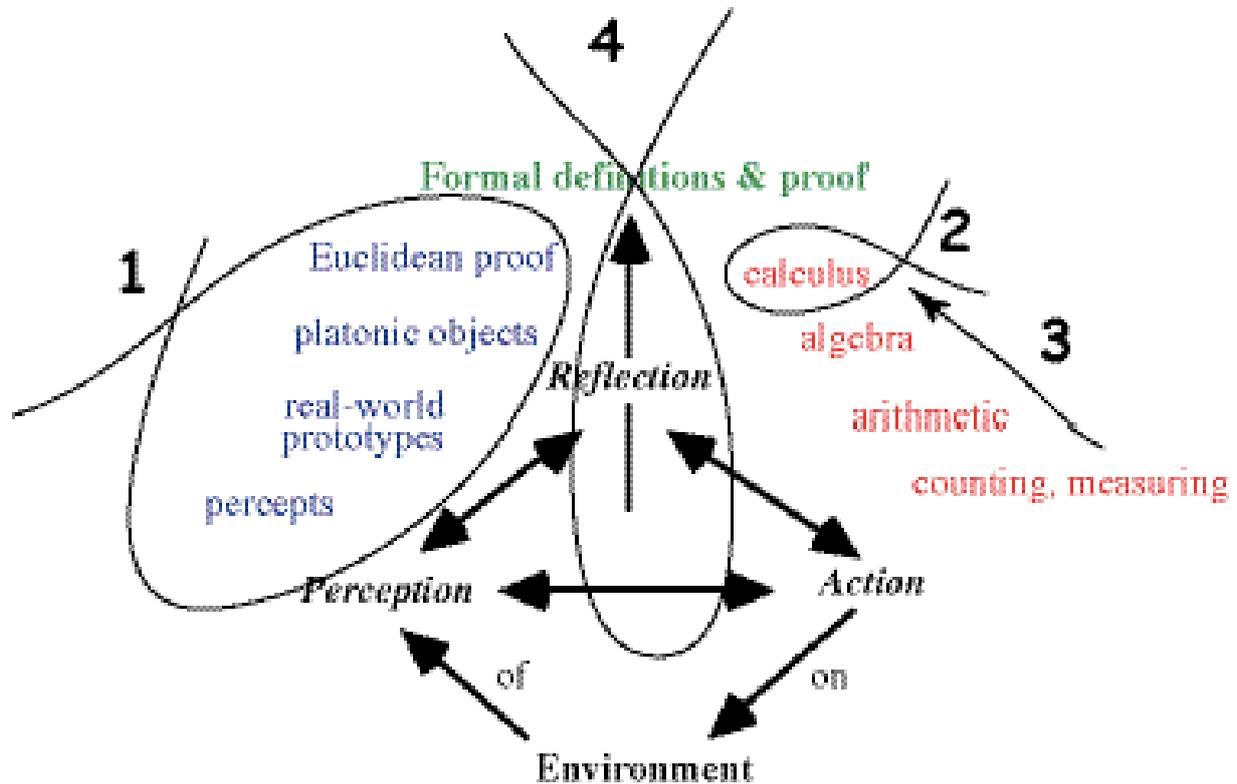
have finite computational processes.

[We will return to this in more detail in lecture 3.]



Conceptual development of selected mathematical concepts

PLAN OF THE WEEK



1. (today) Further discussion on the whole plan followed by a focus on geometry from perceptual embodiment via platonc embodiment and then Euclidean proof.
2. An **embodied** approach to the calculus (and links to symbolism and proof).
3. The **proceptual** development of **symbols**.
4. The development of **proof** via **embodied, proceptual & formal**.

HOW DOES A BIOLOGICAL CREATURE DO MATHEMATICS?

We humans have a BIOLOGICAL BRAIN.

How do we do mathematics with a biological brain and how can the computer (eg a TI-92) help in the *learning* and *understanding*, as well as the *doing* of mathematics.



**Biological brain – Mathematical Mind
– Technological Tools**

Biological brain ... sensori-motor, enactive, visual, linguistic, intuitive, problem-solving capabilities.

Mathematical Mind: uses a variety of different ways of thinking.

Visual embodied Geometry becoming more sophisticated, leading to Euclidean proof and on to formal proof. **Embodiment** also underlies all our thinking, for instance in graphs, rate of change, conceptions of area, following a solution of a differential equation drawn pictorially etc.

Symbols in arithmetic, algebra, calculus etc,

The development of **Mathematical Proof**...

Technological Tools ... algorithms, symbol manipulation, visual display for graphs, enactive manipulation of visual ideas, etc, etc.

THE BIOLOGICAL BRAIN

1990-2000 ... the 'Decade of the Brain'.

Stanislas Dehaene (1997). *The Number Sense*.

Brian Butterworth (1999). *The Mathematical Brain*.

Carter, R. (1998). *Mapping the Mind*. London: Weidenfeld & Nicholson.

Crick, F. (1994). *The Astonishing Hypothesis*, London: Simon & Schuster.

Pinker, S. (1997). *How the Mind Works*. New York: Norton.

Edelman, G. M. (1992). *Bright Air, Brilliant Fire*, NY: Basic Books, reprinted Penguin 1994.

Greenfield, S. (1997). *The Human Brain: A Guided Tour*. London: Weidenfeld & Nicholson.

Devlin, K. (2000), *The Maths Gene: Why everyone has it, but most people don't use it*, London: Weidenfeld & Nicholson.

Freeman, J. F. (1999). *How the Brain makes up its Mind*. Pheonix.

Edelman, G. M. & Tononi, G. (2000). *Consciousness: How Matter Becomes Imagination*. New York: Basic Books.

Lakoff, G. and Johnson, M. (1980). *Metaphors we live by*.

Lakoff, G. (1987). *Women Fire and Dangerous Things*.

Lakoff, G. and Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Lakoff & Nunez (2000): *Where Mathematics Comes From*. New York: Basic Books.

CONCEPTUAL COMPRESSION, FORMING COMPACT CONCEPTS AND POWERFUL MENTAL LINKS

How the brain makes complexity more manageable:

The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) “object(s)” can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once).

(Crick, 1994, p. 61)

This is made more efficient by making the conscious elements as ‘small as possible’, using words or symbols:

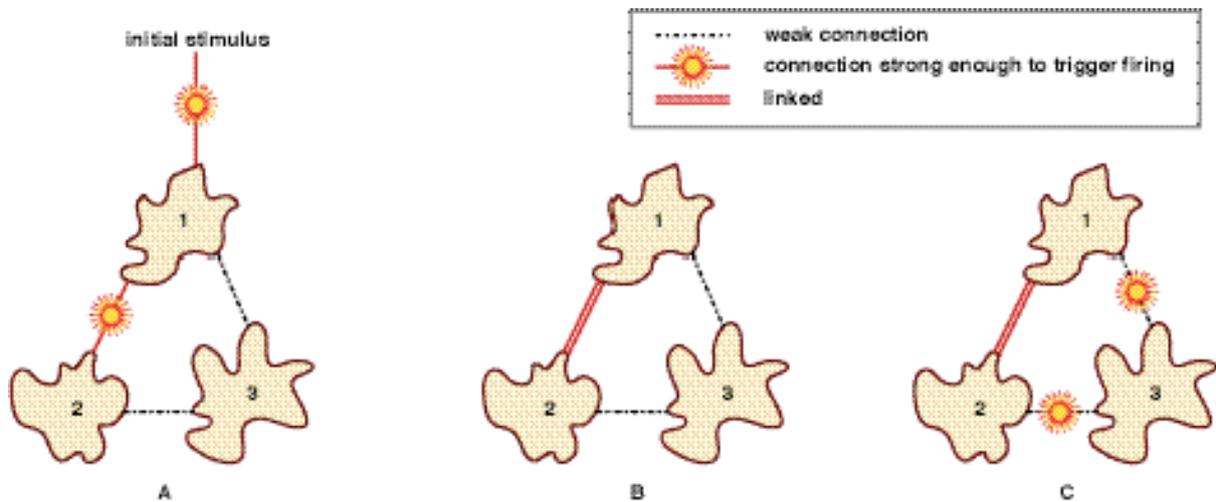
I should also mention one other property of a symbolic system – its compactibility – a property that permits condensations of the order $F = MA$ or $s = \frac{1}{2}gt^2$, ... in each case the grammar being quite ordinary, though the semantic squeeze is quite enormous.

(Bruner, 1966, p. 12.)

As a task to be learned is practiced, its performance becomes more and more automatic; as this occurs, it fades from consciousness, the number of brain regions involved in the task becomes smaller.

(Edelman & Tononi, 2000, p.51)

LONG-TERM POTENTIATION



Building memories in the brain by long-term potentiation (Carter, 1999, p. 160)

A : external stimulus to neuronal group 1 sufficient to fire neuronal group 2 but not group 3.

B: The firing causes the link between 1 and 2 to be more sensitive for a time and, if reactivated, it becomes more easily fired until any excitation of 1 also fires 2. Now 1 and 2 fire together.

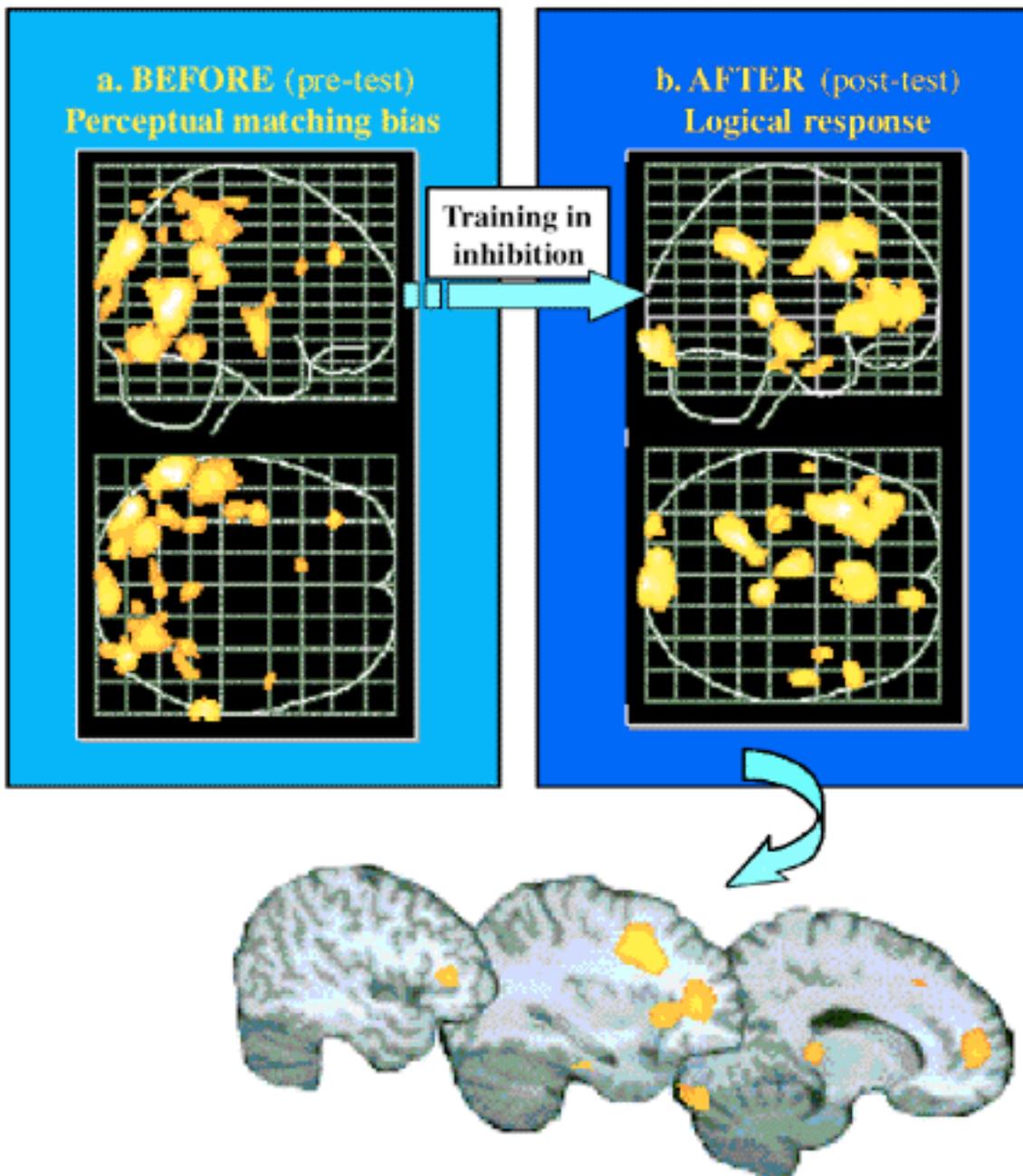
C: The combined strength of 1 and 2 now cause group 3 to be excited, forming a completely new link.

This is *long-term potentiation*.

It leads to the building of highly connected neuronal groups that act in sophisticated ways.

Initially mental activities are sensory based, but then, through the development of language and communication and by reflection on mental activity, cognitive development moves from the purely sensory areas to higher brain functions.

FROM PERCEPTUAL TO LOGICAL THINKING



Houdé et al (2000). Shifting from the perceptual brain to the Logical Brain: The Neural Impact of Cognitive Inhibition Training. *Journal of Cognitive Neuroscience* 12:4 712–728.

MATHEMATICAL MIND

Constructs to describe and explain the cognitive operation of the mathematical mind.

- the *concept image*, which refers to the total cognitive structure in an individual mind associated with the concept, including all mental pictures, associated properties and processes (Tall & Vinner, 1981),

This is biologically a complex connection of neuronal groups. The plan is to use experiences to turn broad intuitive links into powerful units.

- a *cognitive unit* (is a mental chunks we use to think with, and their related cognitive structure). (Barnard & Tall, 1997).

A particular type of cognitive unit:

- the notion of *procept*, referring to the manner in which we cope with symbols representing both mathematical *processes* and mathematical *concepts*. (Gray & Tall, 1994). Examples include

$$3+5, ax^2+bx+c, \frac{d}{dx}(e^x \sin x), \text{ or } \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

The brain is not configured for rapid & efficient arithmetic (Dehaene, 1997). It uses biological links between cognitive units. (As an example I will discuss Shaker Rasslan and the algorithm for divisibility by seven, e.g. 121, 131, 119.)

Mathematical thinking is more than knowing procedures ‘to do’. It involves a knowledge structure compatible with the biological structure of the human brain:

- a huge store of knowledge and internal links,
 - coping with many activities using a manageable focus of attention.
-
-

Consider, a ‘linear relationship’ between two variables. This might be expressed in a variety of ways:

- an equation in the form $y = mx + c$,
- a linear relation $Ax + By + C = 0$,
- a line through two given points,
- a line with given slope through a given point,
- a straight-line graph,
- a table of values, etc

Successful students develop the idea of ‘linear relationship’ as a rich cognitive unit encompassing most of these links as a single entity.

Less successful carry around a ‘cognitive kit-bag’ of isolated tricks to carry out specific algorithms. Short-term success perhaps, long-term cognitive load and failure.

Crowley (2000)

MENTAL TOOLS FOR MATHEMATICS

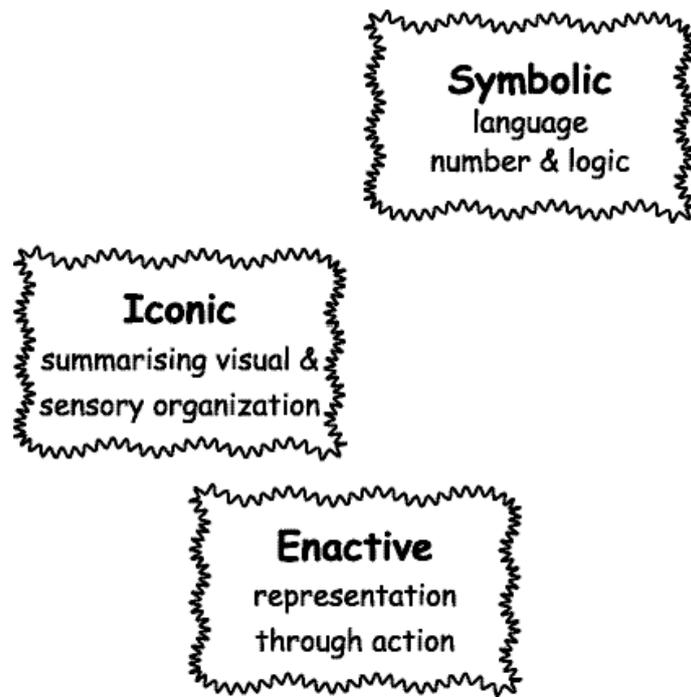
Physical & mental tools and modes of representation

Man's use of mind is dependent upon his ability to develop and use "tools" or "instruments" or "technologies" that make it possible to express and amplify his powers. His very evolution as a species speaks to this point. It was consequent upon the development of bipedalism and the use of spontaneous pebble tools that man's brain and particularly his cortex developed. It was not a large-brained hominid that developed the technical-social life of the human; rather it was the tool-using, cooperative pattern that gradually changed man's morphology by favoring the survival of those who could link themselves with tool systems and disfavoring those who tried to do it on big jaws, heavy dentition, or superior weight. What evolved as a human nervous system was something, then, that required outside devices for expressing its potential.

(Bruner, *Education as Social Invention*, 1966, p. 25.)

What does it mean to translate experience into a model of the world. Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. [...] There is a second system of representation that depends upon visual or other sensory organization and upon the use of summarizing images. [...] We have come to talk about the first form of representation as **enactive**, the second is **iconic**. [...] Finally, there is a representation in words or language. Its hallmark is that it is **symbolic** in nature.

(Bruner, 1966, pp. 10–11)



Bruner's three modes of representation

“... any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in recognizable form” (Bruner 1966. p. 44).

Modern computer interfaces and Bruner's philosophy:

- Enactive interface,
- Icons as summarizing images to represent selectable options,
- Symbolism through keyboard input and internal processing.

‘Symbolism’ in mathematics requires further subcategories.

Bruner (1966, pp. 18, 19)

- “language in its natural form”
- the two “artificial languages of number and logic.”

To these categories we must add not just *number*, but *algebraic* and other *functional symbolism* (trigonometric, exponential and other functions in calculus) *and the huge range of symbolism in axiomatic mathematics.*

The Reform movement in the calculus, for example the Harvard Calculus, focused on three representations: graphic, numeric and symbolic (or analytic):

One of the guiding principles is the ‘Rule of Three,’ which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles. (Hughes Hallett 1991, p. 121)

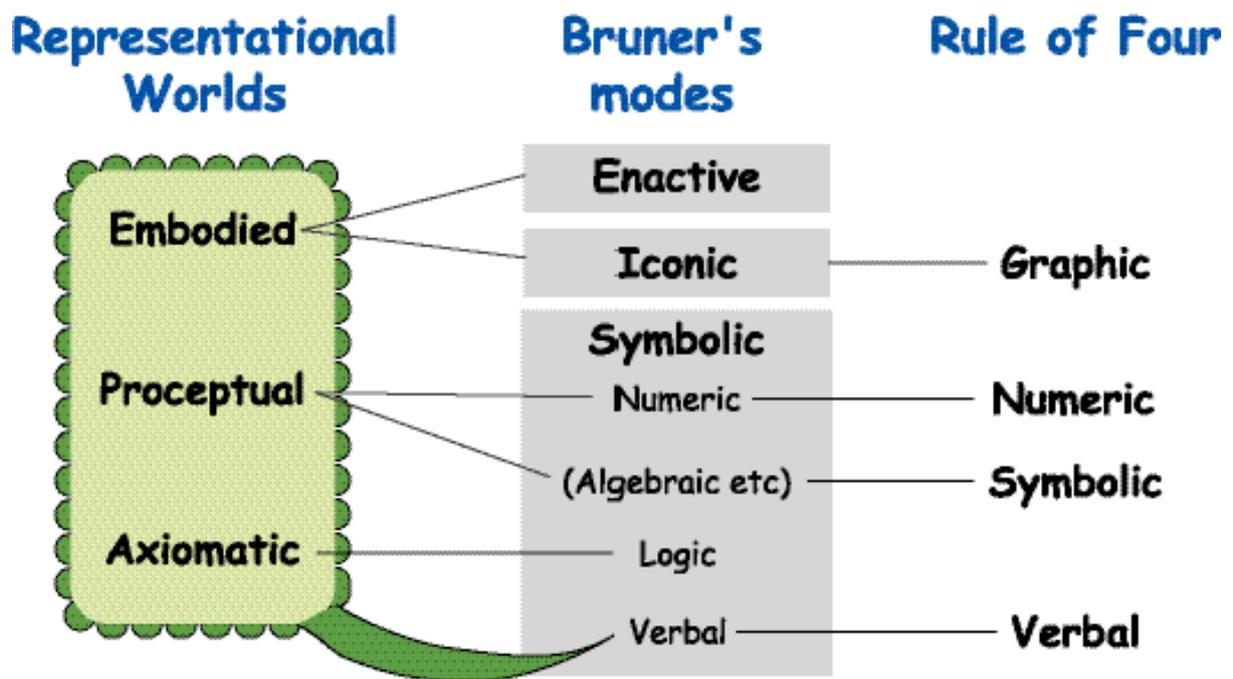
The ‘Rule of Three’ later became the ‘Rule of Four’, extending the representations to include the *verbal*.

Note:

- The enactive mode is completely omitted, presumably because it does not seem to be a central focus in the graphs and symbols of the calculus.
- The “verbal” mode was not seen as being important until late on in the development of the curriculum.
- Axiomatic formulations using logical deduction are not seen as part of calculus but of the later study of analysis.

My solution is to categorise representations into three different worlds of operation:

- **Embodied:** based on human perceptions and actions in a real-world context including but not limited to enactive and visual aspects.
- **Proceptual:** combining the role of symbols in arithmetic, algebra, symbolic calculus, based on the theory of these symbols acting dually as both process and concept (procept). (Tall *et al*, 2001, see below).
- **Axiomatic:** a formal approach starting from selected axioms and making logical deductions to prove theorems.



Three representational worlds

Relationships with other theories

Piaget

sensori-motor / preconceptual / concrete operational / formal

SOLO taxonomy (Structure of Observed Learning Outcomes)
Biggs & Collis (1982)

sensori-motor / ikonik / concrete-operational / formal / post-formal

The SOLO taxonomy is intended to provide a template for assessment. Within each mode the assessment of concepts is performed according to how the student handles the particular concepts and whether this is:

- pre-structural (lacking knowledge of the assessed component)
- unistructural (focusing on a single aspect)
- multi-structural (focussing on several separate aspects)
- relational (relating different aspects together)
- extended abstract (seeing the concept from an overall viewpoint)

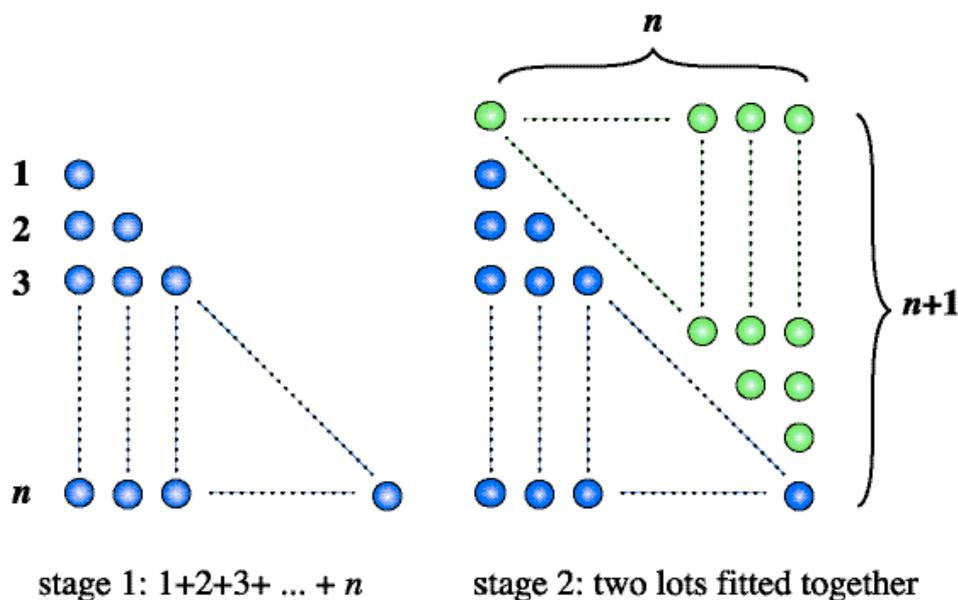
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- i) in each mode, increased sophistication occurs.*
 - ii) As new modes develop, earlier modes may remain available.*

Different ‘warrants for truth’ in embodied, proceptual and axiomatic worlds.

Example: The sum of the first n whole numbers is $\frac{1}{2}n(n + 1)$.

Proof 1: (embodied). Lay out rows of stones with 1 in the first, 2 in the second, and so on. Then take an equal layout of pebbles, turn it round and fit the two together. Visibly the two together make a rectangle size n by $n+1$, giving $n(n+1)$ stones altogether, making $\frac{1}{2}n(n + 1)$ in the original.

The validity of this proof is in the visual picture.



Embodied proof that the sum of the first n whole numbers is $\frac{1}{2}n(n + 1)$

Proof 2: (proceptual). Write out the sum

$$1+2+3+\dots+n$$

backwards

$$\text{as } n+ \dots +3+2+1$$

and add the two together in order, pair by pair, to get

$$(1+n) + (2+n-1) + \dots + (n+1)$$

to get n lots of $n+1$, ie. $n(n+1)$, so, again, the original sum is half this, namely $\frac{1}{2}n(n + 1)$. **Validity by calculation.**

Proof 3 : (axiomatic). By induction. **Validity by proof.**

THREE DIFFERENT WORLDS AND THEIR WARRANTS FOR TRUTH

The **embodied world** is a world of sensory meaning. Its warrant for truth is that things behaves predictably in an expected way.

The **proceptual world** is a world of computation and manipulation. Statements are true because they are stated in symbols and can be verified by calculation or manipulation.

The **formal world** is a world of axioms, definitions and theorems. Statements are true because they can be proved from the axioms and definitions by formal deduction.

Example 1: The notion of **vector**.

Embodied world

A vector has magnitude and direction. Addition of vectors is by placing the tail of the second vector after the nose of the first. Addition is commutative by visual geometry.

Proceptual world

Vectors can be expressed using coordinates. Addition of vectors is by adding coordinates. Addition is commutative because addition of the coordinates is commutative.

Formal world

Vectors are elements of a formal system called a vector space. (They need no longer have magnitude or direction!) Addition of vectors is part of the definition of a vector space. Addition is commutative because this is part of the definition.

Exampe 2: The case of calculus:

The **embodied world**

A (locally straight) graph *has* a slope, because you can *see* it.

It has an area underneath because you can *see* it.

The **proceptual world** is a world where calculations can be made (both arithmetic and algebraic). A graph has a slope (derivative) or an area (integral) because you can *calculate* it.

The **axiomatic world** is a world where explicit axioms are assumed to hold and definitions are given formally in terms of quantified set-theoretic statements. A function has derivative or integral because you can *prove* it.

DIFFERENT COMMUNITIES OF PRACTICE

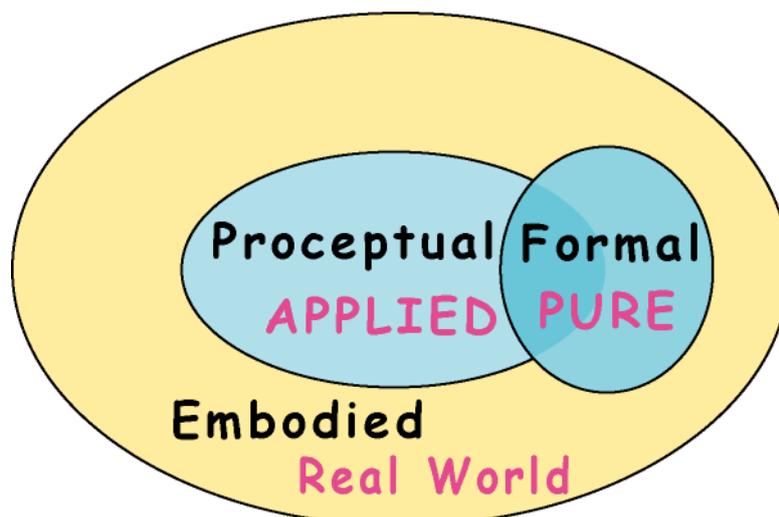
In different communities, different emphases occur.

In school, emphasis is on embodied and proceptual worlds. As students grow older, links with the embodied are reduced and reliance on the proceptual is increased. ‘Proof’ is introduced via numeric checking or algebraic manipulation or through verbal descriptions of visual figures in geometry.

Pure mathematics retains implicit use of the embodied to varying degrees but focuses more on the proceptual and formal worlds. A statement is ‘true’ if it has a formal proof.

Individuals in pure mathematics reach the formal level in different ways. Some think in a ‘natural’ manner that builds embodied imagery into formal definitions and deductions. Others are ‘formal’ thinkers who reject embodiment and attempt to work only from the formal definitions using formal deduction. In practice, however, *all* mathematicians use a biological brain that depends, consciously or unconsciously, on embodied conceptions.

Applied Mathematics, Engineering, etc focuses more on increasingly sophisticated symbolic activities. It is embodied and proceptual rather than formal.



TECHNOLOGICAL TOOLS

– *complement* human activity.

- The brain performs many activities simultaneously and is prone to error, concept image as support.
- the computer carries out individual algorithms accurately and with great speed. No conceptual baggage.

Arithmetic – algorithms for four rules +, −, \square , \div ($\sqrt{\quad}$ etc)

Algebra – rules for substitution, eg $X*(-Y)$ may be replaced by $-(X*Y)$ where X and Y are expressions.

Derive (an early version) simplified

$$\frac{(x+h)^n - x^n}{h} \quad \text{to give} \quad \frac{(x+h)^n}{h} - \frac{x^n}{h} .$$

the limit option as h tends to 0, gave not nx^{n-1} , but

$$\frac{n \ln(x) - \ln(x/n)}{h} .$$

The current version of *Derive* has added a new rule to give:

$$\frac{n-1}{n} x^{n-1} .$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad \text{calculated mentally using graphic image.}$$

Computer algebra system may use:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{D(\sin x)}{D(x)} = \frac{\cos 0}{1} = 1 .$$

RELATIONSHIPS BETWEEN BRAIN, MIND AND TECHNOLOGY

Given the constraints and support in the biological brain, the concept imagery in the mathematical mind can be very different from the working of the computational computer. A professional mathematician with mathematical cognitive units may use the computer in a very different way from the student who is meeting new ideas in a computer context.

Students using *Derive* on hand-held computers to draw graphs of functions were asked:

‘What can you say about u if $u=v+3$, and $v=1$?’

None of seventeen students improved from pre-test to post-test and six successful on the pre-test failed on the post-test.

Hunter, Monaghan & Roper (1993)

A symbol manipulator replaces the mathematical procedures of differentiation by the selection of a sequence of procedures in the software. For instance *Derive* requires the user to take the following sequence of decisions carried out by touching the appropriate keys:

- select **Author** and type in the expression,
- select **Calculus**, then **Derivative**,
- specify the variable (e.g. **x**),
- **Simplify** the result.

A comparison of two schools in the UK, one following a standard course, one using *Derive* is as follows:

Please explain the meaning of $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

... All the students in school A gave satisfactory theoretical explanations of the expression but none gave any examples. However, none of the *Derive* group gave theoretical explanations and only two students [out of seven] mentioned the words 'gradient' or 'differentiate'. Four of the *Derive* group gave examples. They replaced $f(x)$ with a polynomial and performed or described the sequence of key strokes to calculate the limit. (Monaghan, Sun & Tall, 1994.)

Students using the symbol manipulator saw differentiation as a sequence of keystrokes in a specific symbolic example rather than a conceptual idea of 'rate of change'.

WHERE DOES THE TI-92 FIT IN?

The TI-92 has

- a numeric-symbolic interface which carries out proceptual operations on behalf of the user.
- a version of Cabri-geomètre that allows flexible exploration.

THE EMBODIED WORLD

It can be used to visualise graphical and geometric aspects of mathematics. The geometric interface has an enactive element moving the figures around (in rather a clumsy way!). The visual interface is limited but has great potential for exploring visual ideas of differentiability, continuity etc which we will study later.

THE PROCEPTUAL WORLD

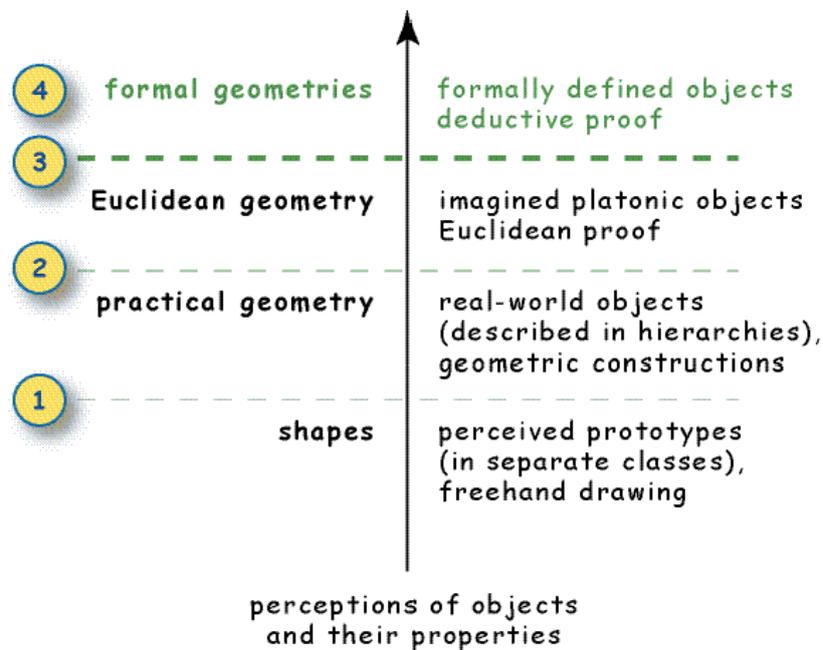
It is an *implicit* proceptual tool that does the work without the user necessarily having insight into what its happening conceptually. Thus it needs complementary activities to give insight into the use of the symbolism.

THE FORMAL WORLD

It is not a formal system in the mathematical sense of definition-proof-theorem. However, some communities (eg applied mathematics, engineering, etc) require an embodied-proceptual world rather than a formal world and it has considerable value here.

I shall also show how aspects of the embodied use can lead to the need for, and the development of, mathematical proof.

Preparation for Workshop 1: Cabri



Cabri is a piece of software that can be used at many levels.

Ideas:

1. An investigation to look at shapes with properties and their names. Eg design a trapezium and investigate whether it can be a square. Consider what it is that students may be thinking in terms of the underlying properties. Try some other figures.

2. Moving from practical geometry to Euclidean geometry, design a figure that has certain properties (e.g. 2 line segments meeting at their midpoints) and see how one can deduce that it has certain other properties (eg the lines are the diagonals of a parallelogram). Invent other examples.

3. There are many investigations at this level. See, for example, the TI-92 booklet or consult the web.

4. For those who want a challenge, read up about the Poincaré construction of hyperbolic geometry in Cabri and do it! (Go to Google and look for **hyperbolic geometry, cabri**, to find

<http://mcs.open.ac.uk/tcl2/nonE/nonE.html>

5. *Challenge*: I have heard it said that it is possible to trisect an angle using Cabri. *I do not know if it is true or false*. Either explain why it is not possible or find a proof and show it to me.