Knowledge Construction and Diverging Thinking in Elementary & Advanced Mathematics

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ABSTRACT: This paper begins by considering the cognitive mechanisms available to individuals which enable them to operate successfully in different parts of the mathematics curriculum. We base our theoretical development on fundamental cognitive activities, namely, ***perception*** *of the world,* ***action*** *upon it and* ***reflection*** *on both perception and action. We see an emphasis on one or more of these activities leading not only to different kinds of mathematics, but also to a spectrum of success and failure depending on the nature of the focus in the individual activity. For instance, geometry builds from the fundamental* ***perception*** *of figures and their shape, supported by action and reflection to move from practical measurement to theoretical deduction and euclidean proof. Arithmetic, on the other hand, initially focuses on the* ***action*** *of counting and later changes focus to the use of symbols for both the process of counting and the concept of number. The evidence that we draw together from a number of studies on children's arithmetic shows a divergence in performance. The less successful seem to focus more on perceptions of their physical activities than on the flexible use of symbol as process and concept appropriate for a conceptual development in arithmetic and algebra.*

Advanced mathematical thinking introduces a new feature in which concept definitions are formulated and formal concepts are constructed by deduction. We show how students cope with the transition to advanced mathematical thinking in different ways leading once more to a diverging spectrum of success.

1. CONSTRUCTING MATHEMATICAL KNOWLEDGE

Mathematical development occurs in a biological brain. To enable a structure with complex simultaneous activity to pursue sequential thought in a coherent way requires a special mechanism. Crick suggests:

> The basic idea is that early processing is largely parallel: a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) "object(s)" can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attention system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e. attending to one object after another) not highly parallel (as it would be if the system attended to many things at once). (Crick, 1994, p. 61)

The powerful thinking that develops in mathematics takes advantage of this biological phenomenon. The filtering out of most activity to focus on a few elements requires that these elements be distilled to their essence so that they are "small enough" to be considered at one time. It also requires that each of these elements be appropriately linked to other relevant structures in the huge memory store to allow it speedily to become a new focus of attention as required.

One method to cope with the complexity of a sequence of activities is repetition and practice until it becomes routine and can be performed with little conscious thought. This frees the conscious memory to focus on other items (Skemp, 1979). For instance, in using tools, the techniques become part of unconscious activity whilst the individual can focus on more utilitarian or aesthetic issues. Although such repetition and interiorisation of procedures has been seen as an essential part of mathematics learning, for decades it has been known that it has made no improvement in the understanding of relationships (see for example, Thorndike, 1922; Brownell, 1935). More importantly, if used exclusively, it may lead to a form of procedural thinking that lacks the flexibility necessary to solve novel problems (see for example, Schoenfeld, 1992).

A more powerful method of dealing with complexity is through the human use of language. Here a single word can stand not only for a highly complex structure of concepts and/or processes but also for various levels in a conceptual hierarchy. Perception of figures is at the foundation of geometry, but it takes the power of language to make hierarchical classifications. Figures are initially perceived as gestalts but then may be described and classified through verbalising their properties, to give the notions of points, lines, planes, triangles, squares, rectangles, circles, spheres, etc. Initially these words may operate at a single generic level, so that a square (with four *equal* sides and every angle a right angle) is not considered as a rectangle (with only *opposite* sides equal). Again, through verbal discussion, instruction and construction, the child may begin to see hierarchies with one idea classified within another, so that "a square is a rectangle is a quadrilateral", or "a square is a rhombus is a parallelogram is a quadrilateral". The physical and mental pictures supported by linguistic descriptions may become conceived in a more pure, imaginative way. Points have "position but no size", straight lines are truly straight, with "no thickness and arbitrary length", a circle is the locus of a point a fixed distance from the centre and so on. Such a development leads to platonic *mental* constructions of objects and the development of Euclidean geometry and Euclidean proof. Thus, a focus on perceived objects leads naturally through the use of language to platonic mental images and a form of mathematical proof (as in Van Hiele, 1959, 1986).

On the other hand, the idea of counting begins with the repetition of number words, with the child's remembered list of numbers steadily growing in length and correctness of sequence. The act of counting involves pointing at successive objects in a collection in turn and saying the number words, "there are one, two, *three* things here." This may be compressed, for instance, by carrying out the count silently, saying just the last word, "there are [one, two,] three", heard as "there are …three." It is thus natural to use the word "three" not just as a counting word, but also as a number concept. By this simple device, the counting process "there are one, two, three," is compressed into the concept "there are three." (Gray and Tall, 1994).

This compression is powerful in quite a different way from the compression in geometric thinking. In geometry, a word represents a generic concept (say "square") in a hierarchy of concepts. In arithmetic the number word is also part of a hierarchy (a counting number is a fraction is a rational number is a real number). However, the major biological advantage of numbers arises not from this hierarchy but from the way in which the number words can be used to switch between *processes* (such as counting or measuring) and *concepts* (such as numbers). Not only are number symbols "small enough" to be held in the focus of attention as concepts, they also give immediate access to action schemas (such as counting) to carry out appropriate computations. In the biological design of the brain, they act not only as economical units to hold in the focus of attention, they also provide direct links to action schemas.

When numbers have become conceived as mental entities, they may themselves be operated upon. For instance, two numbers may be added to give their sum through a development that again involves a process of compression. The addition of two numbers begins as "count-all", involving three counting stages: "count one set, count another, put them together and count them all". This is compressed through various stages including "count-on", where the first number is taken as the starting value and the second is used to count-on to give the result. Some of these results are committed to memory to give "known facts". They may then be used in a conceptual way to "derive facts", for instance, knowing that 5+5 is 10, to deduce that 5+4 is one less, namely, 9.

This power of mathematical symbols to evoke either process or concept caused Gray & Tall (1994) to give the notion a formal name. The amalgam of a *process*, a *concept* output by that process, and a *symbol* that can evoke either process or concept is called a *procept*. In elementary arithmetic, procepts start as simple structures and grow in interiority with the cognitive growth of the child. Although other theorists (including Dubinsky, 1991 and Sfard, 1991) use the term "object", we prefer the word "concept" because terms such as "number concept" or "fraction concept" are more common in ordinary language than "number object" or "fraction object". Further, the term is used in a manner related to the "concept image" which consists of "all of the mental pictures and associated properties and processes" related to the concept in the mind of the individual (Tall & Vinner, 1981, p. 152). Procepts are generic and increase in richness with the growing sophistication of the learner. There is no claim that there is a "thing" called "a mental object" in the mind. Instead, a symbol is used which can be *spoken*, *heard*, *written* and *seen*. It has the distilled essence that can be held in the mind as a single entity, it can act as a link to internal action schemas to carry out computations, and it can be communicated to others.

1.1 Piaget's three forms of abstraction

Piaget spoke of three forms of abstraction. When acting on objects in the external world, he speaks first of *empirical abstraction*, where the focus is on the *objects* themselves and "derives its knowledge from the properties of objects" (Beth & Piaget, 1966, pp. 188-189). On the other hand, a focus on the *actions* leads to *pseudo-empirical abstraction* which "teases out properties that the action of the subjects have introduced into objects" (Piaget, 1985, pp. 18-19). Further constructions can then be accomplished by *reflective abstraction*, using existing structures to construct new ones by observing one's thoughts and abstracting from them. In this way:

> … the whole of mathematics may therefore be thought of in terms of the construction of structures,... mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures.
>
> (Piaget, 1972, p. 703)

Note here that reflective abstraction seems to be formulated as a mental version of "pseudo-empirical abstraction", in which an "operation" on a mental entity becomes in its turn an "object" at the next level. Some authors (for example, Dubinsky, 1991) have taken this to mean that reflective abstraction only occurs by processes becoming conceived as conceptual entities through a process of "encapsulation" or "reification". Given Piaget's two notions of abstraction from the physical world, the question naturally arises as to whether there are corresponding forms of reflective abstraction focusing on mental objects and on mental actions. Our analysis would support this position. In the cognitive development of geometry, there is a clear shift from the mental conception of a physical triangle to the mental construction of a perfect platonic triangle. The former is imagined drawn on paper, with lines having thickness joining points having size, the latter has perfectly straight edges with no thickness and vertices with position but no size. We therefore suggest that there are (at least) *two* forms of reflective abstraction, one focusing on *objects*, occurring, for instance, in Euclidean geometry, the other focusing on *actions on objects* (usually represented by symbols), for instance, in arithmetic, algebra and the calculus.

Our focus on perception, action and reflection is therefore consistent with Piaget's three notions of abstraction, with the additional observation that reflective abstraction has a form which focuses on objects and their properties, as well as one which focuses on actions and their encapsulation as objects.

1.2 Theories of process-object transformation

The notion of (dynamic) processes becoming conceived as (static) objects has played a central role in various theories of concept development (see, for example, Dienes, 1960; Piaget, 1972; Greeno, 1983; Davis, 1984; Dubinsky, 1991; Sfard, 1991; Harel & Kaput, 1991; Gray & Tall, 1994).

Dubinsky and his colleagues (e.g. Cottrill *et al.* 1996) formulate a theory which they give the acronym APOS, in which *actions* are physical or mental transformations on objects. When these actions become intentional, they are characterised as *processes* that may be later encapsulated to form a new *object*. A coherent collection of these actions, processes and objects, is identified as a *schema.* In more sophisticated contexts, empirical evidence also intimates that a schema may be reflected upon and acted on, resulting in the schema becoming a new object through the encapsulation of cognitive processes (Cottrill *et al*, 1996, p.l72).

Sfard (1991, p. 10) suggests that "in order to speak about mathematical *objects*, we must be able to deal with the products of some *processes* without bothering about the processes themselves". Thus we begin with "a process performed on familiar objects" (Sfard and Linchevski, 1994, p 64). This is then "condensed" by being seen purely in terms of "input/output without necessarily considering its component steps" and then "reified" by converting "the already condensed process into an object-like entity." Sfard postulates her notion of "reification" within a wider theory of *operational* and *structural* conceptions, the first focusing on processes, the second on objects (Sfard, 1989, 1991, 1994). In several papers she emphasises that the operational approach—constructing new objects through carrying out processes on known objects usually precedes a structural approach to the new objects themselves.

Such theories, which see the construction of new mental objects through actions on familiar objects, have a potential flaw. If objects can only be constructed from cognitive actions on already established objects, where do the initial objects come from?

Piaget's theory solves this problem by having the child's preliminary activities involving perception and action of the physical world. Once the child has taken initial steps in empirical or pseudo-empirical abstraction to construct mental entities, then these become available to act upon to give a theoretical hierarchy of mental constructions.

Sfard's theory concentrates on later developments in older individuals who will already have constructed a variety of cognitive objects. Dubinsky also concentrates on undergraduate mathematicians. However, the APOS theory is formulated to apply to all forms of object formation. Dubinsky, Elterman & Gong (1988, p. 45), suggest that a "permanent object" is constructed through "encapsulating the process of performing transformations in space which do not destroy the physical object". This theory therefore follows Piaget by starting from initial *physical* objects that are not part of the child's cognitive structure and theorises about the construction of a *cognitive* object in the mind of the child. It formulates empirical abstraction as another form of process-object encapsulation.

At the undergraduate level, Dubinsky (1991) extends APOS theory to include the construction of axiomatic theories from formal definitions. APOS theory is therefore designed to formulate a theory of encapsulation covering all possible cases of mental construction of cognitive objects.

Our analysis has different emphases. We see the differences between various types of mathematical concept formation being as least as striking as the similarities. For instance, the construction of number concepts (beginning with pseudo-empirical abstraction) follows a very different cognitive development from that of geometric concepts (beginning with empirical abstraction) (Tall, 1995). In elementary mathematics, we see two different kinds of cognitive development. One is the van Hiele development of geometric objects and their properties from physical perceptions to platonic geometric objects. The other is the development of symbols as process and concept in arithmetic, algebra and symbolic calculus. It begins with actions on objects in the physical world, and requires the focus of attention to shift from the action of counting to the manipulation of number symbols. From here the number symbols take on a life of their own as cognitive concepts, moving on to the extension and generalisations into more sophisticated symbol manipulation in algebra and calculus. Each shift to a new conceptual domain involves its own subtle changes and cognitive reconstructions, however, what characterises these areas of elementary mathematics is the use of symbols as concepts and processes to calculate and to manipulate.

1.3 A new focus in advanced mathematical thinking

When formal proof is introduced in advanced mathematical thinking, a new focus of attention and cognitive activity occurs. Instead of a focus on symbols and computation to give answers, the emphasis changes to selecting certain properties as definitions and axioms and building up the other properties of the defined concepts by logical deduction. The student is often presented with a context where a formal concept (such as a mathematical group) is encountered both by examples and by a definition. Each of the examples satisfies the definition, but each has additional qualities, which may, or may not, be shared between individual examples. The properties of the formal concept are deduced as theorems, thus constructing meaning for an overall umbrella concept from the concept definition. This didactic reversa1—constructing a mental object from "known" properties, instead of constructing properties from "known" objects causes new kinds of cognitive difficulty.

The new formal context—in which objects are created from properties (axioms) instead of properties deduced from (manipulating) objects—not only distinguishes advanced mathematical thinking from elementary mathematical thinking, it also suggests that different kinds of "structure" occur in the structural-operational formulation of Sfard. In elementary mathematics, for example, a "graph" is described as a structural object (Sfard, 1991). In advanced mathematics, the Peano postulates are said to be structural (Sfard, 1989). Thus, a structural perspective may refer to visual objects in elementary mathematics and Bourbaki-style formal structure in advanced mathematics.

1.4 A theoretical perspective

The preceding discussion leads to a theory of cognitive development in mathematics with two fundamental focuses of attention—*object* and *action*—together with the internal process of *reflection*. In line with Piaget we note the different forms of *abstraction* which arise from these three: *empirical abstraction*, *pseudo-empirical abstraction* and *reflective abstraction*. However, we note that reflective abstraction itself has aspects that focus on object or on action.

We see abstraction from physical objects as being different from abstraction from actions on objects. In the latter case, action-process-concept development is aided by the use of symbol as a pivot linking the symbol either to process or to concept. Procepts occur throughout arithmetic, algebra and calculus, and continue to appear in advanced mathematical thinking. However, the introduction of axioms and proofs leads to a new kind of cognitive concept—one which is *defined* by a concept definition and its properties *deduced* from the definition. We regard the development of formal concepts as being better formulated in terms of the definition-concept construction. This focuses not only on the complexity of the definition, often with multiple quantifiers, but also on the internal conflict between a concept image, which "has" properties, and a formal concept, whose properties must be "proved" from the definitions.

We therefore see elementary mathematics having two distinct methods of development, one focusing on the properties of *objects* leading to geometry, the other on the properties of *processes* represented symbolically as procepts. Advanced mathematics takes the notion of *property* as fundamental, using properties in concept definitions from which a systematic formal theory is constructed.

2. DIVERGING COGNITIVE DEVELOPMENT IN ELEMENTARY MATHEMATICS

2.1 Divergence in performance

The observation that some individuals are more successful than others in mathematics has been evident for generations. Piaget provided a novel method of interpreting empirical evidence by hypothesising that all individuals pass through the same cognitive stages but at different paces. Such a foundation underlies the English National Curriculum with its sequence of levels through which all children pass at an appropriate pace, some progressing further than others during the period of compulsory education.

Krutetskii (1976, p. 178) offers a different conception with a spectrum of performance between various individuals depending on how they process information. He studied 192 children selected by their teachers as 'very capable' (or 'mathematically gifted'), 'capable', 'average' and 'incapable'. He found that gifted children remembered general strategies rather than detail, curtailed their solutions to focus on essentials, and were able to provide alternative solutions. Average children remembered specific detail, shortened their solutions only after practice involving several of the same type, and generally offered only a single solution to a problem. Incapable children remembered only incidental, often irrelevant detail, had lengthy solutions, often with errors, repetitions and redundancies, and were unable to begin to think of alternatives.

Our research also shows a divergence in performance. We do not use the evidence collected to imply that some children are doomed forever to erroneous procedural methods whilst others are guaranteed to blossom into a rich mathematical conceptualisation. We consider it vital not to place an artificial ceiling on the ultimate performance of any individual, or to predict that some who have greater success today will continue to have greater success tomorrow. However, the evidence we have suggests that the different ways in which individuals process information at a given time can be either beneficial or severely compromising for their current and future development. A child with a fragmented knowledge structure who lacks powerful compressed referents to link to efficient action schemas will be more likely to have greater difficulty in relating ideas. The expert may see distilled concepts which can each be grasped and connected within the focus of attention. The learner may have diffuse knowledge of these conceptual structures which is not sufficiently compressed into a form that can be brought into the focus of attention at a single time for consideration.

Far from not working hard enough, the unsuccessful learner may be working very hard indeed but focusing on less powerful strategies that try to cope with too much uncompressed information. The only strategy that may help them is to rote-learn procedures to perform as sequential action schemas. Such knowledge can be used to solve routine problems requiring that particular technique, but it occurs *in time* and may not be in a form suitable for thinking about as a whole entity.

2.2 Focus on objects and/or actions in elementary mathematics

The observation that a divergence in performance exists in the success and failure of various students does not of itself explain *how* that divergence occurs. To gain an initial insight into aspects of this divergence, we return to our initial notions of perception, action and abstraction. We earlier discussed global differences between geometry (based on perception of figures, supported by action and extended through reflection), and arithmetic (based on actions of counting objects that are initially perceived and reflected upon). Now, within arithmetic we consider the effect of different emphases on action, perception and reflection.

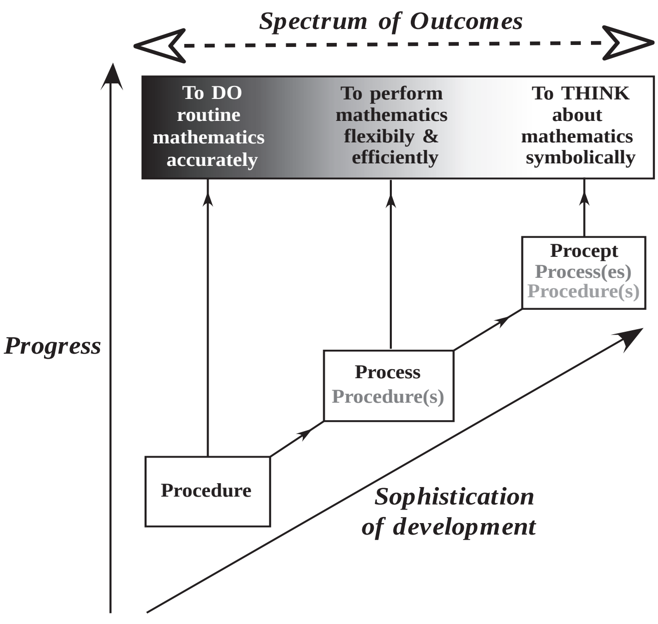
Whenever there is an activity involving actions on objects, the complexity of the activity may cause the individual to focus only on part of the activity. For instance, it is possible to focus on the objects, on the actions or a combination of the two. Cobb, Yackel and Wood (1992) see this attention to objects or actions as one of the great problems in learning mathematics, particularly if learning and teaching are approached from a representational context. Pitta and Gray (1997) showed that certain observed differences in children's arithmetic performance could be linked to the learner's focus either on objects, on actions, or on a combination of both.

To investigate the way in which children may focus on different aspects of a situation, Pitta (1998) placed five red unifix cubes before some seven-year-old children at the extremes of mathematical ability. She asked the children to indicate what they thought about when they saw the cubes and what they thought would be worth remembering about them. The four more able children all had something to say about the cubes using the notion of 'five'. They all thought that 'five cubes' was worth remembering. In contrast, the four lower ability children talked about the pattern, the colour, or the possible rearrangements of the cubes and considered these to be worth remembering.

Different contexts require a focus of attention upon different things. Within an art lesson it may be important to filter out those things that may not immediately be seen to be part of an aesthetic context. Number may be one of these. In the mathematical context it is important to filter out those things that may not be seen to be mathematical. Yet, in the activity just considered, low achievers seemed less able to do this, continuing to focus on their concrete experience. High achievers, on the other hand, were able to separate the inherent mathematical qualities from the actual physical context. They could also, if required, expand their discussions to include other aspects of the activity, revealing cognitive links to a wider array of experience. Such differences may become manifest in the way in the activity is remembered. It is hypothesised that low achievers focus upon the physical aspects of the activity, which are assimilated in an episodic way. High achievers appear to focus upon the semantic mathematical aspects, which are accommodated in a generic way (Pitta and Gray, 1997).

2.3 The Proceptual Divide

The divergence in success between extremes of success and failure can be usefully be related to the development of the notion of procept. Gray and Tall (1994) suggest that interpretations of mathematical symbolism as process or procepts leads to a *proceptual divide* between the less successful and the more successful. On the one hand, we see a cognitive style strongly associated with invoking the use of procedures, on the other a style more in tune with the flexible notion of procept. Those using the latter have a cognitive advantage; they derive considerable mathematical flexibility from the cognitive links relating process and concept. In practice, there is a broad spectrum of performance between different individuals in different contexts (figure 1).



In a given routine context, a specific procedure may be used for a specific purpose. This allows the individual to *do* mathematics in a limited way, provided that it involves using the learned procedure. Some individuals may develop greater sophistication by being able to use alternative procedures for the same process and to select a more efficient procedure to carry out the given task speedily and accurately. For instance, the procedure of "count-on from largest" is a quicker way of solving 2+7 (counting on 2 after 7 rather than counting-on 7 after 2). Baroody & Ginsburg (1986) suggest that growing sophistication arises from the recognition that a single mathematical process may be associated with several procedures. Woods, Resnick & Groen (1975) note that this element of "choice" can be indicative of increased sophistication. However, it is only when the symbols used to represent the process are seen to represent manipulable concepts that the individual has the proceptual flexibility both to *do* mathematics and also to mentally manipulate the concepts at a more sophisticated level (Gray & Tall, 1994).

In a particular case, all three levels (procedure, process, procept) might be used to solve a given routine problem. It might therefore be possible for individuals at different levels of sophistication to answer certain questions in a test at a certain level. However, this may be no indication of success at a later level because the procept in its distilled manipulable form is more ready for building into more sophisticated theories than step-by-step procedures. On the other hand, all too frequently, children are seen using procedures even when they are inappropriate, inefficient and unsuccessful (see for example Gray, 1993). Those who operate successfully at the procedural level are faced with much greater complexity than their proceptual colleagues when the next level of difficulty is encountered.

2.4 Mental representations and elementary mathematics

The notion of a proceptual divide illustrates the extreme outcomes of different cognitive styles. We now turn to asking *why* such a difference occurs. To gain a partial answer to this question we now consider mental representations, particularly those in imaginistic form.

Pitta & Gray (1997) describe the way in which two groups of children, 'low achievers' and 'high achievers', report their mental representations when solving elementary number combinations. Differences that emerged showed the tendency of low achievers to concretise numbers and focus on detail. Their mental representations were strongly associated with the procedural aspects of numerical processing—action was the dominant level of operating (see also Steffe, Von Glasersfeld, Richards and Cobb, 1983). In contrast, high achievers appeared to focus on those abstractions that enable them to make choices.

The general impression was that children of different levels of arithmetical achievement were using qualitatively different objects to support their mathematical thinking. Low achievers translated symbols into numerical processes supported by the use of imaginistic objects that possessed shape and in many instances colour. Frequently they reported mental representations strongly associated with the notion of number track although the common object that formed the basis of each 'unit' of the track was derived from fingers. In some instances children reported seeing full picture images of fingers, in others it was 'finger like'. The essential thing is that the object of thought was 'finger' and the mental use of finger invoked a double counting procedure. The objects of thought of the low achievers were analogues of perceptual items that seemed to force them to carry out procedures in the mind, almost as if they were carrying out the procedures with perceptual items on the desk in front of them. Pitta and Gray suggest that their mental representations were essential to the action; and they maintained the focus of attention. When items became more difficult, the children reverted to the use of real items.

In contrast, when high achievers indicated that they had "seen something", that "something" was usually a numerical symbol. More frequently these children either responded automatically or reported that they talked things over in their heads. However, when they did describe mental representations the word "flashing" often dominated their description. Representations came and went very quickly. "I saw '3+4' flash through my mind and I told you the answer", "I saw a flash of answer and told you." It was not unusual for the children to note that they saw both question and answer "in a flash", sometimes the numerical symbol denoting the answer "rising out of" the symbols representing the question. In instances where children reported the use of derived facts it was frequently the numerical transformation that "flashed". For instance when given 9 + 7 one eleven year old produced the answer 16 accompanied by the statement. "10 and 6 flashed through my mind". Here we have vivid evidence of powerful mental connections moving from one focus of attention to another. Such a child evidently has flexible mental links between distilled concepts that allow quick and efficient solutions to arithmetic problems.

This ability to encapsulate arithmetical processes as numerical concepts provides the source of flexibility that becomes available through the proceptual nature of numerical symbolism. Recognising that a considerable amount of information is compressed into a simple representation, the symbol, is a source of mathematical power. This strength derives from two abilities; first an ability to filter out information and operate with the symbol as an object and secondly the ability to connect with an action schema to perform any required computation. We suggest that qualitative differences in the way in which children handle elementary arithmetic may be associated with their relative success. Different cognitive styles seem to indicate that differing perceptions of tasks encountered lead to different consequences, one associated with *performing* mathematical computations, the other associated with *knowing* mathematical concepts.

Mental representations associated with the former appear to be products of reflection upon the actions and the objects of the physical environment. One consequence of mathematical activity focusing upon procedural activity is that it would seem to place a tremendous strain on working memory. It does not offer support to the limited space available within short-term memory.

3. THE TRANSITION TO ADVANCED MATHEMATICAL THINKING

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. … It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions. Tall, 1991, p. 20

The cognitive study of "advanced mathematical thinking" developed in the mathematics education community in the mid-eighties (see, for example, Tall, 1991). Euclidean proof and the beginnings of calculus are usually considered "advanced" at school level. However, the term "advanced mathematical thinking" has come to focus more on the thinking of creative professional mathematicians imagining, conjecturing and proving theorems. It is also applied to the thinking of students presented with the axioms and definitions created by others. The cognitive activities involved can differ greatly from one individual to another, including those who build from images and intuitions in the manner of a Poincaré and those more logically oriented to symbolic deduction such as Hermite.

Piaget's notion of "formal operations" indicates the ability to reason in a logical manner:

> Formal thought reaches its fruition during adolescence … from the age of 11–12 years … when the subject becomes capable of reasoning in a hypothetico-deductive manner, i.e., on the basis of simple assumptions which have no necessary relation to reality or to the subject's beliefs, and … when he relies on the necessary validity of an inference, as opposed to agreement of the conclusions with experience. Piaget, 1950, p. 148.

In a similar manner, the SOLO taxonomy identifies the formal mode of thinking where:

> "The elements are abstract concepts and propositions, and the operational aspect is concerned with determining the actual and deduced relationships between them; neither the elements nor the operations need a real-world referent" Collis & Romberg, 1991, p. 90.

However, often these ideas are applied by Piaget to imagined *real-world* events and in the SOLO taxonomy to logical arguments in traditional algebra, involving relationships between symbols that no longer need have a perceptual referent.

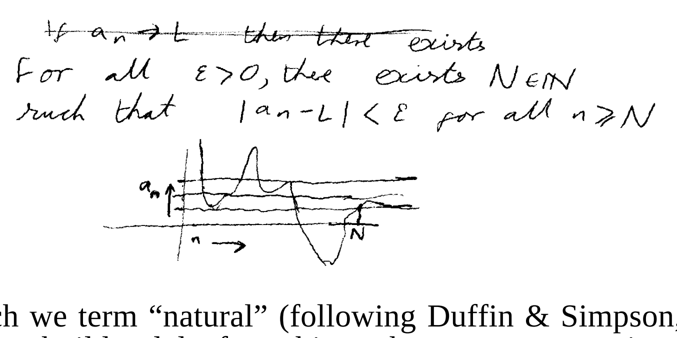
The notion of advanced mathematical thinking is more subtle than this. It involves the creation of new mental worlds in the mind of the thinker which may be entirely hypothetical. Mathematicians do this by reflecting on their visual and symbolic intuitions to suggest useful situations to study, then to specify *criteria* that are necessary for the required situation to hold. This is done by formulating *definitions* for mathematical concepts as a list of axioms for a given structure, then developing other properties of this structure by deduction from the definitions. A considerable part of research effort is expended in getting these criteria precise so that they give rise to the required deduced properties. What is then produced is more than a verbal/symbolic list of definitions and theorems. Each individual theoretician develops a personal world of concept images and relationships related to the theory. These may include ideas that suggest what *ought* to be true in the given theory before necessarily being able to formulate a proof of what *must* follow from the definitions.

Definitions of structures—such as “group”, “vector space”, “topological space”, “infinite cardinal”—face in two ways. They face *back* to previous experiences which suggest what ideas are worth studying and *forward* to the construction of theorems which are true for any structure that satisfies the given criteria. They can cause great cognitive problems for a learner who must distinguish between those things in the mind which *suggest* theorems and other things that have already been *proved* from the criteria. The learner must maintain a distinction between the broad concept images formed from previous experience and new constructions—the *formal concept image*—which consists only of those concepts and properties that have been constructed formally from the definitions.

In practice, this often proves extremely difficult. Whereas mathematics researchers may have had experience at *making* new structures by constructing their own definitions, students are more likely to only be initially involved in *using* definitions which have been provided by others. Through their earlier life experiences they will have developed an image in which objects are “described” in words in terms of collecting together enough information to identify the object in question for another individual. The idea of giving a verbal definition as a list of criteria and *constructing* the concept from the definition is a reversal of most of the development in elementary mathematics where mathematical objects are thought to *have* properties which can be discovered by studying the objects and related processes. The move from the **object→definition** construction to **definition→object** construction is considered an essential part of the transition from elementary to advanced mathematical thinking.

This definition→object construction involves selecting and using criteria for the definitions of objects. This may reverse previous experiences of relationships. For instance, the child may learn of subtraction as an operation before meeting negative numbers and inverse operations. In formal mathematics the axioms for an additive operation in a group may specify the inverse $-a$ of an element a and define subtraction $b-a$ as the sum of b and $-a$. In this way the presentation of axiom systems as criteria for theoretical mathematical systems can strike foreign chords in the cognitive structure of the learner. Instead of proving results of which they are unsure by starting from something they know, they find they are trying to prove something they know starting from axioms which make them feel insecure.

Our experience of this learning process in mathematical analysis (Pinto & Gray, 1995; Pinto & Tall, 1996; Pinto, 1996, 1998) shows a spectrum of student performances signalling success and failure through following two complementary approaches.



One approach, which we term “natural” (following Duffin & Simpson, 1993) involves the student attempting to build solely from his or her own perspective, attempting to give meaning to the mathematics from current cognitive structure. Successful natural learners can build powerful formal structures supported by a variety of visual, kinaesthetic and other imagery, as in the case of student Chris (Pinto, 1998). He made sense of the definition of convergence by drawing a picture and interpreting it as a sequence of actions:

> *"I think of it graphically ... you got a graph there and the function there, and I think that it's got the limit there ... and then ε, once like that, and you can draw along and then all the ... points after N are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that's the n going across there and that's a_n. ... Err, this shouldn't really be a graph, it should be points."* (Chris, first interview)

As he drew the picture, he gestured with his hands to show that first he imagined how close he required the values to be (either side of the limit), then how far he would need to go along to get all successive values of the sequence inside the required range. He also explained:

> *"I don't memorise that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down."* (Chris, first interview)

However, his building of the concept involved him in a constant state of reconstruction as he refined his notion of convergence, allowing it to be increasing, decreasing, up and down by varying amounts, or constant in whole or part, always linking to the definition which gave a single unifying image to the notion. During his reconstructions, he toyed with the idea of an increase in N causing a resultant reduction in the size of ε, before settling on the preference for specifying ε, then finding an appropriate N.

As an alternative to the "natural" approach, there is a second approach which Pinto (1998) termed "formal". Here the student concentrates on the definition, using it and repeating it as necessary until it can be written down without effort. Ross, for example explained he learned the definition:

> *"Just memorising it, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it gets imprinted and then I remember it."* (Ross, first interview)

He wrote:

A sequence (a_n) tends to limit L if, $\forall\, \varepsilon > 0$, $\exists\, N(\varepsilon) \in \mathbb{N}$ s.t. $\forall\, n \geq N$;

$$|a_n - L| < \varepsilon.$$

(Ross, first interview)

The focus in this case is on the definition and the deductions. Visual and other images play a less prominent role. Used successfully, this approach can produce a formal concept image capable of using the definitions and proving theorems as required by the course. At its very best the student will also be in a position later on to reconstruct knowledge, comparing old with new and making new links. However, it is also possible to develop the knowledge in a new compartment, not linked to old knowledge.

Both formal and natural learners can be successful in advanced mathematical thinking. However, they face different sequences of cognitive reconstruction. The natural learner may be in continuous conflict as (s)he reconstructs informal imagery to give rich meaning to the formal theory. The formal learner may have fewer intuitions to guide the way, but follows a course involving more new construction rather than reconstruction. At the end of the formalisation process, if the new knowledge is linked to the old imagery, then reconstruction is likely to be required at this stage.

Less successful students also have difficulties in different ways. Some (such as those in Gray & Pinto, 1995) saw the new ideas only in terms of their old meanings and could not make the transition to the use of definition as criteria for determining the concept. These could be described as natural learners who fail to reconstruct their imagery to build the formalism. Their informal concept image intimates to them that the theorems are "true" and they see no need to support informal imagery with what they regard as alien to both their need and their understanding.

Less successful students attempting the formal route may be unable to grasp the definition as a whole and cope with only parts of it. They may be confused by the complexity of multiple quantifiers, perhaps failing to give them their true formal meaning, perhaps confusing their purpose, perhaps concentrating only on a part of the definition.

It seems that the only way out for unsuccessful students, be they natural or formal learners, is to attempt to rote-learn the definitions.

> *Maths education at university level, as it stands, is based like many subjects on the system of lectures. The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. The pressure of time seems to take away the essence of mathematics and does not create any true understanding of the subject. From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject.*
>
> *Third Year Mathematics Student,* (Tall, 1993a)

4. CONCLUSION

In this paper, we have considered the interplay of perception, action and reflection on cognitive development in mathematics. Geometry involves a major focus on perception of *objects*, which develops through reflective activity to the mental construction of perfect platonic objects. Arithmetic begins by focusing on *actions* on objects (counting) and develops using procepts (symbols acting as a pivot between processes and concepts) to build elementary arithmetic and algebra.

In elementary arithmetic we find that the less successful tend to remain longer focused on the nature of the objects, their layout and the procedures of counting. Our evidence suggests that less successful children focus on the specific and associate it to real and imagined experiences that often do not have generalisable, manipulable aspects. We theorise that this places greater strains on their overloaded short-term memory. A focus on the counting procedure itself can give limited success through procedural methods to solve simple problems. High achievers focus increasingly on flexible proceptual aspects of the symbolism allowing them to concentrate on mentally manipulable concepts that give greater conceptual power. The flexible link between mental concepts to think about and action schemas to do calculations utilise the facilities of the human brain to great advantage.

We see the transition to advanced mathematical thinking involving a transposition of knowledge structure. Elementary mathematical concepts *have* properties that can be determined by acting upon them. Advanced mathematical concepts are *given* properties as axiomatic definitions and the nature of the concept itself is built by deducing the properties by logical deduction. Students handle the use of concept definitions in various ways. Some natural learners reconstruct their understanding to lead to the formal theory whilst other, formal, learners build a separate understanding of the formalities by deduction from the concept definitions. However, many more can make little sense of the ideas, either as natural learners whose intuitions make the formalism seem entirely alien, or as formal learners who cannot cope with the complexity of the quantified definitions.

The theory we present here has serious implications in the teaching of elementary and advanced mathematics, in ways which have yet to be widely tested. The obvious question to ask is “how can we help students acquire more beneficial ways of processing information?”, in essence, “how can we help those using less successful methods of processing to become more successful?” Our instincts suggest that we should attempt to *teach* them more successful ways of thinking about mathematics. However, this strategy needs to be very carefully considered, for it may have the result that we teach procedural children flexible thinking *in a procedural way*. This scenario would have the effect of burdening the less successful child with even more procedures to cope with. It might tend to make their cognitive structure *more complex* rather than *more flexible and more efficient*.

One approach at encouraging more flexible thinking (Gray & Pitta, 1997a,b) used a graphic calculator with a multi-line display retaining several successive calculations for a child to use in a learning experiment. The experience was found to have a beneficial effect in changing the mental imagery of a child who previously experienced severe conceptual difficulty. Before using the calculator, the child’s arithmetic focused on counting using perceptual objects or their mental analogues. After a period of approximately six months use with the graphic calculator, it was becoming clear in our interactions with her that she was associating a different range of meanings with numbers and numerical symbolism. She was beginning to build new images, symbolic ones that could stand on their own to provide options that gave her greater flexibility. The evidence suggests that if practical activities focus on the process of evaluation and the meaning of the symbolism they may offer a way into arithmetic that helps those children who are experiencing difficulty.

In the teaching of algebra, Tall & Thomas (1991) found that the act of programming could allow students to give more coherent meaning to symbolism as both process and concept. A computer language will evaluate expressions, so that, for instance, the learner may explore the idea that 2+3**x* usually gives a different answer from (2+3)**x* for numerical values of *x*. This can provide a context for discussing the ways in which expressions are evaluated by the computer. The fact that 2*(*x* +3), 2**x* +2*3, 2**x* +6, always give the same output, can be explored to see how different procedures of evaluation may lead to the same underlying process, giving the notion of equivalent expressions and laying down an experiential basis for manipulating expressions. This leads through a procedure – process – procept sequence in which expressions are first procedures of evaluation, then processes which can have different expressions producing the same effect, then concepts which can themselves be manipulated by replacing one equivalent expression by another.

In advanced mathematical thinking more research is required to test whether different methods of approach may support different personal ways to construct (and reconstruct) formal theory. Just as Skemp (1976) referred to the difficulty faced by a relational learner taught by instrumental methods (or vice versa), we hypothesises that there are analogous difficulties with natural learners being taught by formal methods (or vice versa). This suggests that more than one approach is required to teaching mathematical analysis. Some students may benefit from a study quite different from the traditional formal theory. For example, Tall (1993b) observed that a class of student teachers similar to those who failed to make any sense of the formalism (see Pinto & Gray, 1995) could construct natural insights into highly sophisticated ideas using computer visualisations even though this may not improve their ability to cope with the formal theory.

Success can be achieved for some students in various ways. These include giving meaning to the definitions by reconstructing previous experience, or by extracting meaning from the definition through using it, perhaps memorising it, and then building meaning within the deductive activity itself (Pinto 1998). However, not all succeed. Those who fail are often reduced, at best, to learning theorems by rote to pass examinations. How different this is from the advanced mathematical thinking of the creative mathematician, with its combination of intuition, visualisation and formalism combined in different proportions in different individuals to create powerful new worlds of mathematical theory.

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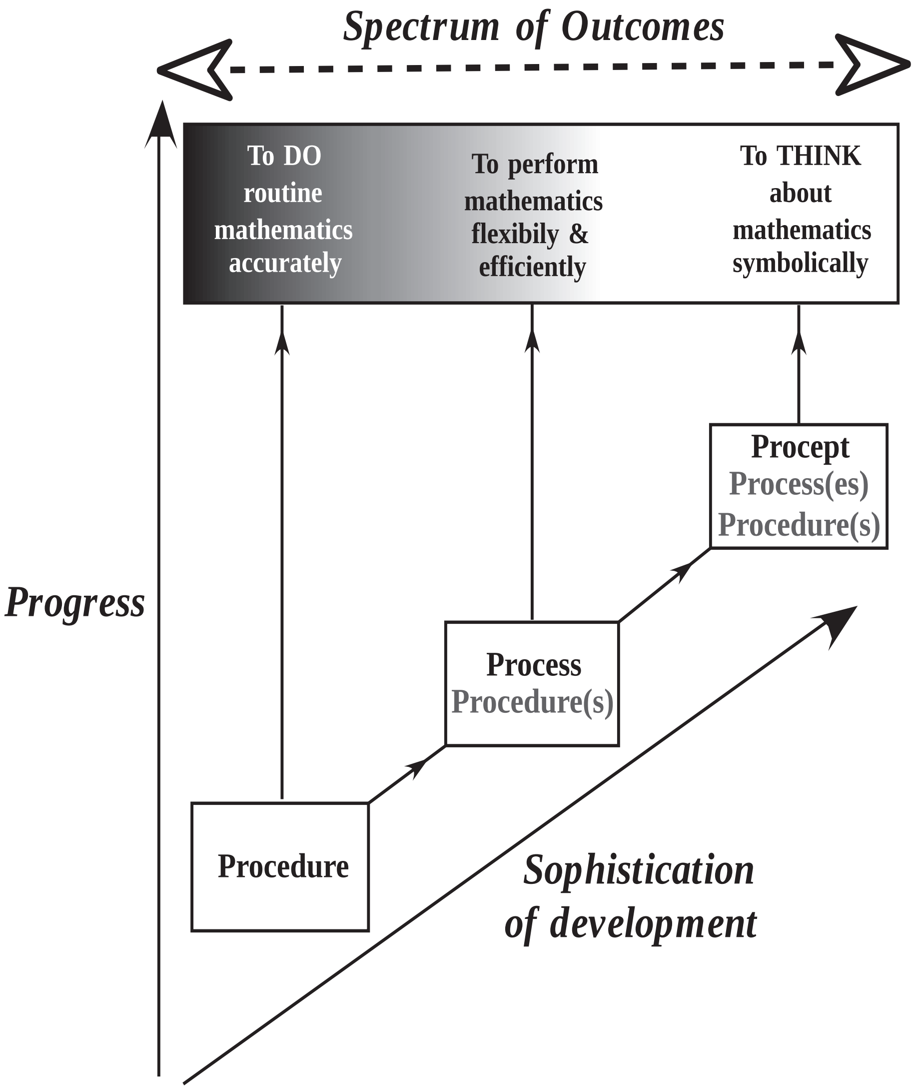
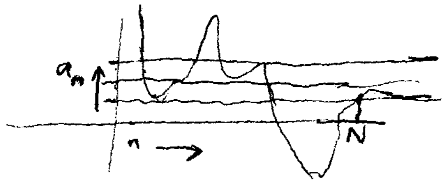


Figure 1: The Spectrum of Outcomes

Unnumbered figures integral to the text.

~~If $a_n \to L$ then there~~ exists
For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$
such that $|a_n - L| < \varepsilon$ for all $n \geq N$



A sequence (a_n) tends to limit L if, $\forall\ \varepsilon > 0$, $\exists\ N(\varepsilon) \in \mathbb{N}$
s.t. $\forall\ n \geq N$;

$$|a_n - L| < \varepsilon.$$