PUPILS' IMAGES OF TEACHERS' REPRESENTATIONS

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This paper suggests that learner's procedural and conceptual knowledge of mathematics is constructed, in part, from their mental representations of teachers' external representations. These mental representations reproduce the learner's perceptual experiences and are thus referred to as images. In the first instance individual images may be quasi-sensory or language like in format. Some pupils form a general image of the structure inherent in the external representations whilst others have specific images which embody the surface characteristics of those representations teachers use to communicate the mathematics. For such pupils the medium becomes the message.

INTRODUCTION

Teachers use a variety of representations, involving verbal, written, pictorial and concrete material presentations, to communicate their mathematics to pupils. Pupils are asked to replicate the mathematical activity through the use of these ‘external’ representations. Their talk, writing, drawing or actions indicates the nature of their knowledge construction. This paper considers the mental representations pupils form from these varied stimuli and seeks to answer the question "What kinds of mental representations do pupils form from the representations given and how do they make use of them?" A preliminary study in an English school for 5 to 11 year olds suggests that pupil’s internal representations are initially 'image-like' in that they reproduce, in part, their previous perceptual experiences. These images may not be visual or tactile yet, when they are evoked for mental calculation, pupils use language associated with spatial or motor aspects of the external representation. The study also suggests that the pupils' construction of mathematical meaning appears to be based on their internal representations of the teacher's external representations. It is conjectured that the way in which these internal representations 'stand-in' for the original experiences varies between pupils of different abilities. It is also conjectured that individuals' mathematical knowledge construction may be transformed from a quasi-sensory mental representation (visual, auditory, tactile) to a more general, language like, representation over different time periods.

The paper provides a brief review of some of the literature on representations and images which has influenced and informed the study and gives some detail of the phenomenographic methodology. The focus for the initial classroom observations and semi-structured pupil interviews has been whole number place value and addition and subtraction of two digit numbers. The implications of these findings and other recent research at Warwick are considered to conclude that a longitudinal study of teachers' and pupils' representations is needed to investigate the influence of the one upon the other.

REPRESENTATIONS AND IMAGES

Lesh, Post, & Behr (1987) suggest that five types of representation are available to mathematics teachers: world contexts, manipulatable models, pictures and diagrams, spoken language and written symbols. Such representations are not *the mathematics* but transformations of the mathematics into communicable form, a process that Kang & Kilpatrick (1992) have termed 'didactic transposition'.

The way in which pupils form mental representations from their mathematical experiences is open to debate. The Piagetian view is that mental representations are constructed by the individual. This is, in a sense, a compromise between theorists such as Fodor, who assumes an *innate* representational language of thought, and others, such as Vygotsky, who insist on the priority of public external representations that are *copied* to become the internal representations (For a fuller discussion see Olson & Campbell, 1993).

The study reported in this paper starts with the same premise as Lesh & Kelly (1997), that the learner's internal conceptual structures cannot simply be received from others but are developed from, and expressed using, external systems of representation. The assumption is also made that aspects of the internal representation can be inferred from the children's use of external representations, in this case through their language and the physical movements that they use (see also Thomas, Mulligan, & Goldin, 1996). However, no precise claims can be made about the nature of these internal representations (Kaput, 1992)

The pupils' encounter with a teacher's representations is a multi-sensory experience and the first interest of this study is the image formed from the child's perception of this experience. An image "reproduces in part some previous perceptual experience in the absence of the original sensations" (Russell, 1956) or, more generally, it is a quasi-sensory experience that is a "concrete re-presentation of sensory, perceptual, affective or other experiential states" (Richardson, 1969). In this sense an 'image' of Dienes blocks might be the almost tangible sense of feeling them or moving them, a recalled vision of them on a desk, the recall of a teacher talking about them, a memory of thinking how to add tens and ones, or remembered pleasure in piling them higher than anyone else.

In the field of visual imagery Kosslyn (1980, 1996) has demonstrated that images can have depictive, picture like, qualities that could not be explained by purely propositional, language like, mental representations. His model suggests that images are short-term memory representations generated from long-term memory representations that may have a depictive or propositional form. He also proposed a "Representational-development hypothesis" which has strong parallels with the development from procedural to proceptual thinking in some individuals (see for example Gray & Tall, 1994). Kosslyn suggested that the type of internal representation that is predominantly used changes with age and that later types of internal representation, being more powerful, supplement and eventually overshadow older ones.

Kosslyn further suggests that visual imagery will necessarily be used in response to a question about a concrete object where the information has not been stored as part of a propositional representation or can not be deduced from propositional representations. As propositional knowledge increases and deduction becomes easier then visual imagery may be used less. With increasing use of facts the image is more likely to take on a propositional format. If the child has few representations in other formats he has little choice but to use his visual image. Imagery generation may be preferred because it is quicker than propositional reasoning which requires more or different processing resources. A similar point is made by Intons-Peterson & McDaniel (1991) that the less familiar we are with a task the closer is our imagery to the original perception.

Paivio (1986) refers to "proto-typical" mental representations of conceptual categories that are either the best example or a composite of typical features of that category. Kosslyn (1996) suggests that proto-typical images tend to be stronger because they have been accessed more often but that a particular exemplar may be stronger if it has been refreshed frequently. In the context of this paper children may form proto-typical images from the variety of teacher's representations for example, a sense that 62 is 6 tens and 2 ones without reference to columns or blocks. Others may have an image based on the most frequently used exemplar (for example counting on for addition) or most recently used exemplar (for example Dienes Blocks for addition).

METHOD

The assumption made in this exploratory study is that the images formed by pupils can not be studied in isolation from the context of the classroom and the interaction between the pupils and teachers. It was therefore regarded as essential to observe the common experiences of the learners as a basis for the analysis of their different conceptualisations. The focus is on what the teachers and pupils say and do in the lessons and how the pupils make reference to their experiences when questioned in interviews.

The research approach adopted is a naturalistic qualitative one which can be termed "phenomenographic" in that it is an investigation of people's understanding of phenomena and it seeks to categorise and explain the qualitatively different ways in which people think about the phenomena. The initial discovery of previously unspecified categories of thinking may be peculiar to the researcher and context but the test of their validity is in their applicability for other researchers and as a source of explanation of differences in learning outcomes (Marton,1988).

The study was conducted in a school for 5 to 11 year olds in a large middle-income village near Birmingham, UK in the period October 1997 to July 1998. One lesson per week was observed with follow up interviews with individual children. Audio transcriptions of children's interviews were supported by detailed field notes. The teacher observed is an experienced male and his 33 pupils were those judged by their previous teachers to be the most able of the 80 Y2 (6 to 7 years old) in the school. Samples of pupils were interviewed in October, March/April and July.

RESULTS

The discussion of the results will draw upon the several representations used by the class teacher (Mr. K.) to demonstrate 2-digit addition. These included a Number Track, ranging from 1 to 105, a Hundred Square, Dienes Base Ten Blocks, cards printed with single digits and the written algorithm. The pupils practised a representation-specific procedure with each of the materials. Though each representation is structure-oriented, in the sense used by Resnick & Ford (1981), the "transparency" of the correspondence between the material and the structure is variable (Meira, 1998). The validity of using such a variety of representations of the same mathematics has been questioned (Dufour-Janvier, Bednarz, & Belanger, 1987) but the intention of this study was to observe the effect of the variety not to criticise it. To trace the relationship between teacher's representation and pupils' internal representations over the 9-month period two themes are examined:

- The common experience of the pupils and their individual conceptualisations from the experience.
- The medium term proto-typical representations that are formed by the pupils.

Conceptualising from Experience

A typical lesson prior to the October interviews shows one of the teacher's representations involving Dienes Blocks used in demonstration mode:

> The teacher gave Mandy two tens (In response to the question "Another way of putting it?" Mandy said "twenty") and 4 ones. When asked "How many altogether?" Mandy said "Twenty-four". A similar demonstration was used with Nina. She was given 1 ten and 2 ones and responded correctly to similar questions. The teacher requested Mandy and Nina to "Now put them together in my hands". In response the two children put the *tens in one hand* and the *ones in the other*. The teacher then requested "How many altogether?" adding "Look how easy it is to add them instead of all individual cubes"**.**
>
> Pupils were then directed to work on two-digit addition questions presented in the textbook as pictures of Dienes Blocks with written numerals

One week later, after some further practice with the representations, 11 pupils were asked to work out one question, 24 + 53 (presented on paper), in their heads. They were then asked how they had worked it out. Though none of the pupils spontaneously mentioned visualising Dienes Blocks, Elspeth's response has a clear trace of the teacher's representation:

> Elspeth: Well you add the tens together then you add the units because its like *in one hand* you have the tens and *in the other* you have the units

Elspeth was one of only 2 pupils who gave the correct answer to this problem. The other pupil counted on from 53 using fingers to help.

Though the above example lesson involves the teachers use of Dienes Blocks a dominant representation previously presented to the children had been the Hundred Square used for two-digit addition. When the children were prompted to think about either a Hundred Square or Dienes Blocks (their choice), none could mentally manipulate them without considerable assistance. Max's responses, however, suggest that he had formed an image related to the Hundred Square:

Max If it was 24 add 11 it would be 35 ... Because it was diagonal like that. (moves head to the right and down)

Asked what 24 add 10 was he replied 34 and explained

Max Because you just go down and it would just go back under there .. 'cos that would go in tens and then that would stay in the same place and the tens would just go back under there.

In each instance it sounds as if he is describing a visual image yet when requested to move down 5 squares from 24 he could not do so. A possible explanation is that he has a "global" image (Kosslyn, 1996) of a Hundred Square that lacks the visual resolution for him to scan very far over it.

Another instance where the internal representation is a consequence of what has been attended to and extracted (see Kosslyn, 1980) is given by Neal:

Neal You get 24 in your head then add on the 5 and the 3... 32

Here he has attended to the separate adding of tens and ones.

A second indication of a relationship between the teacher's representation and the child's conceptualisation is provided from an example obtained in March. During a lesson that focused upon adding one to a three digit number a temporary teacher summarised previous teaching in the following way:

> "You have been throwing dice and all sorts of things. You also looked at rolls of raffle tickets like this… | | 186 | | (Drawn on board). What comes next? Why wasn't the 1 or the 8 changed?" A pupil replies "Because you are not adding tens or hundreds."
>
> The teacher next wrote 199 in the middle position and again asked "What comes before? What comes after?" Going on to comment: "But that means I'm altering the tens and hundreds. That's because I can't have more than 9 in any column."
>
> The teacher continued to talk to the group of pupils who had experienced difficulty. To illustrate "*going to*" the next number she held up 3 individual-digit cards and then *changed* the units digit card for a different one. She indicated that only the units digit changed except when the 9 is changed for a 0 and then the tens are changed as well. The children were invited to use these similar cards to help them with additional exercises using raffle tickets. In the exercise which followed half of them made mistakes by altering the wrong digit.

The usual class teacher also makes use of these individual-digit cards to illustrate changing digits to increase a number by one, ten or a hundred. Interviews conducted during March and April (gap caused by Easter holiday) included a question focusing upon children's conceptualisation of adding one. Fifteen children were asked: "What comes next after three hundred and seventy nine?" and then "How did you decide that?"

The results of these interviews suggested that three quarters of the children were influenced by these experiences. The separate digit reasoning of the pupils has traces of the single-digit representation involving changing individual digits of a number. Two thirds of this 'influenced' group (9 children) obtained the correct solution:

Hazel(30/3) Because um three hundred and seventy nine we have to *change* the um ten, we have to change the 7 to an 8 and we take 9 to an oh.

Elspeth(30/3) Well when you add, you said that it was seventy nine, and add one on and it equals to seven, so, eighty because when you add one on to 9 it has a nought at the end.

Brian(20/4) well you just um well um you don't think about the first number, well, you don't *change* the first number but you um you have to change the second two numbers because it's *gone on to* a nine and it's like *going on* to a ten

However, one third of those who made use of separate digit reasoning attempted to add one to the wrong digits:

John (30/3) … 389 … I *changed* the ten to the next

Ann(1/4) 4 hundred . 480 'cause if you just keep it on 3 it wouldn't be right

Christine(1/4) ... 4, … 5 hundred, and, … fifty. Well I was thinking about, I added one more on to, the units makes ten and then I, added it on so it was 4 made 5, the hundreds, and 7 would … uh… becomes 5 little dots

The use of words such as 'change', 'changed' and 'gone on to' suggests that these particular pupils' representations are image-like in that there are spatial allusions to the teacher's representation and an echo of the teacher's words. It is conjectured that these representation-specific words are not those that the children would use if their mental representation were simply based on counting.

A final group of children, approximately one quarter of the number interviewed, reflected a more powerful form of knowledge construction based upon a deeper conceptualisation of numeracy.

Jack(1/4) 'cause you know 79, eighty , just add a hundred to it, 380.

Clara(20/4) 'cause um I know, I know the hundred well what comes after 79 is 80

Steph(20/4) ... 380 I added one on .... I added another one on.

Representational Consistency

The pupils were interviewed again in early July. During the interim the children had had many lessons in which adding tens by increasing the tens digit has been demonstrated and practised with Dienes Blocks and individual-digit cards. Nine pupils had verbally presented questions requiring addition of ten in both interviews. There are striking similarities in their language used in response. For example

	March interview: 81 add on 10	July Interview: 38 add 10
Cora	*ninety* one. You just add one, I know nine comes after 8.	I just add one onto the tens
Jack	*ninety one*. If you just go 8, 9 then you just make it into a ten and you put the 1 on, *ninety one*	… forty-eight. I know my ten times table and then I just put the eight on

Christine	well I, … sort of ignore the units for a minute and just added like a ten on.	I would leave the un… units and just add on ten
Mandy	well cause in Mr K's we did adding on ten and I remembered cause 81 add on 10 is 91	Well we did it in Maths and he just said you add on ten so it would be the same
Brian	well I just change the tens and I just leave the units.	Well 'cause all you have to do is add the ten. You leave the units

In July questions on 2-digit addition and subtraction showed that half of the pupils used the same form of words for each, indicating reference to similar images or perhaps a single image of 2-digit manipulation. Some children's representations were very close to the teacher's written algorithm which he based on Dienes Blocks. There is evidence here of a proto-typical image being recalled and described to explain their calculation

DISCUSSION

Gray and Pitta (1996, 1997) have indicated that there are qualitative differences in the images formed and used by pupils of different abilities. They suggest that some children continue to mentally reconstruct the numerical procedure rather than encapsulate it. Lower achievers tend to use pictorial and iconic representations and attempt to mentally manipulate images of these. High achievers more often have symbolic images that appear to act as thought generators and memory aids. The divide between high and low achievers is also apparent in the images they projected of concrete nouns (ball, car) and abstract nouns (five, fraction). Pitta went on to conjecture that children may have a disposition toward different kinds of mental representations which transcends arithmetical and non-arithmetical boundaries (Pitta, 1998).

The image a learner uses when thinking about a piece of mathematics is an amalgam of verbal and non-verbal information derived directly from perceptual experience of the teacher's representation, a notion which Paivio (1986) terms "memory trace". This image is subsequently augmented by the experiences of using both external and internal representations but may no longer be evoked when recall of a known fact can replace its use. It would appear that some pupils seem more capable than others of constructing an efficient proto-typical image that embodies the structure inherent in the representations but disregards surface characteristics. Some pupils mentally manipulate digits to decide 380 follows 379, for others it has become a known fact, some continue to count on for 2-digit addition whilst others use a mental analogue of the written algorithm and yet others count on in tens and ones.

The preliminary studies in this paper provide evidence of the relationship between the teacher's representations and the pupils' images. It also suggests that the images can remain unchanged over a short time scale. Knowledge construction in arithmetic and algebra requires a cognitive shift to encapsulate active aspects of arithmetic into numerical concepts. The abstraction, which is the essence of this shift, requires that the surface characteristics of, and actions on, representations that are didactic transformations of the mathematics be eventually overshadowed. It remains to be seen whether or not Kosslyn's representational-development hypothesis is applicable to images used for mental calculation and place value and whether mental representations formed in mathematical and non-mathematical contexts follow similar development in individuals. A longitudinal study may resolve these issues.

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