THE INTRICATE BALANCE BETWEEN ABSTRACT AND CONCRETE IN LINEAR ALGEBRA

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Abstract

This paper focuses upon the strengths and weaknesses of disparate approaches to Linear Algebra. By identifying the theoretical distinctions between the abstraction-to-computation approach and the computation-to-abstraction approach, it presents examples of how one lecturer, recognising the cognitive obstacles associated with the nature of Linear Algebra, used the latter in the development of a first-year University course. Student reaction suggests that this laudable effort may be addressing procedural difficulties, but compounding conceptual ones, since the delivery of advanced mathematics material in a 'concrete' manner, can militate against the use of concept definitions and this may have broader implications for further mathematical development.

Introduction

Linear Algebra is one of the first courses of advanced mathematics at University level. Along with Analysis, it is intended to shift the students' way of thinking from school mathematics towards advanced mathematical thinking. It is probably the first 'real' mathematics course that students have to encounter, since it requires limited mathematical prerequisites, yet its theory is systematically built from the ground up (Tucker, 1993; Hillel & Sierpinska, 1994). In addition, Linear Algebra brings together methods and insights of geometry and algebra, and its wide range of applications in modern mathematics make it an essential component of all scientific courses (Tucker, 1993). Most importantly though, students have to become familiar with its main themes, such as vector spaces and linear maps, since they are central in the further development of pure mathematical theory (Tucker, 1993).

This paper intends to summarise the existing literature concerning the cognitive obstacles within Linear Algebra and some of the teaching methods employed to overcome these difficulties. Also, it considers how a particular Linear Algebra course was delivered, having those difficulties in mind, and what the impact was on a small group of high achieving students. It concludes by suggesting that there should be a balance between concrete and abstract approaches in Linear Algebra, however difficult to achieve, since, in some instances, in our effort to solve a problem we may create a new one.

Difficulties within Linear Algebra

During their pre-university courses, mathematics students will have met some components of the course, such as matrix arithmetic and solution of simultaneous linear equations (AEB GCE Syllabuses, 1999; NEAB GCE A/AS Syllabuses for 1998) . This, unfortunately, does not guarantee a smooth transition to the stark (?) Linear Algebra. On the contrary, as Hillel and Sierpinska (1994) argue, "both the teaching and learning of linear algebra at the university level is almost universally regarded as a frustrating experience" (p. 65).

Some of the reasons for the difficulties faced by the students are not confined to the content of Linear Algebra, but are a result of the transition from elementary to advanced mathematics.

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students; initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions.
>
> (Tall, 1991, p. 20)

Linear Algebra, though, has certain particularities which can also impede students' learning and understanding. The heart of these is that Linear Algebra was developed to unify, simplify and model already existing problem solutions, rather than to solve new problems (Harel &Trgalová, 1996). Students can solve many problems within a linear algebra course without using the relevant theory but by mere implication of direct manipulation techniques (Hillel, & Sierpinska, 1994).

Harel & Tall (1991) distinguish between three types of generalisations within advanced mathematics; *expansive, reconstructive* and *disjunctive generalisation*, and argue that the successive generalisations of vector sum and scalar multiples from $\mathbf{R}^2$ to $\mathbf{R}^3$ to $\mathbf{R}^n$ are an expansive generalisation for the students, since it involves "applying the same techniques to each coordinate in successively broader systems" (p. 39). The passage from $\mathbf{R}^n$ to the abstract concept of a vector space, on the other hand, requires a re-constructive generalisation.

> The learner is presented with a name for the concept ("the vector space *V*") and some of its properties (the axioms) and –usually guided by an expert–must follow a subtle and difficult process of construction and meaning of *V* and its properties by deduction from the axioms. This is further complicated in the learner's mind by the fact that the properties *to be deduced in V are known* to hold $\mathbf{R}^n$, causing the problem for the student that, although these properties are "obvious" in the (only) examples (s)he understands, judgement must be suspended on their truth in *V* until they are shown to follow by deduction from the axioms. (Harel & Tall, 1991; p. 39)

Even though the theory of Linear Algebra is universally applicable, when it comes to solving problems, there is a wide variety of algorithms for any certain task, with the restriction that different algorithms work in different settings. Thus, students are faced with the further difficulty of having to decide which is the most appropriate algorithm to tackle their problem. Carlson (1993), for example, notes that the procedure needed to find a basis for a vector space of row vectors is different than that to find a basis for a vector space of functions.

An additional disadvantage of the unifying character of Linear Algebra is the variety of representations that students have to get accustomed to. The word 'vector', for example, firstly introduced in the context of concrete $\mathbf{R}^n$ subspaces, can mean different things depending on the corresponding vector space. Hillel & Sierpinska (1994), argue that the initial representation of a vector as a string of numbers, becomes shaken when students realise that the representation of a vector depends on the choice of basis.

As linear algebra is one of the first undergraduate mathematical courses, students are required –probably for the first time– to deal with abstract concepts instead of numeric manipulations (Carlson, 1993). They have to start "thinking about the objects and operations of algebra not in terms of relations between particular matrices, vectors and operators but in terms of whole structures of such things: vector spaces over fields, algebra's, classes of linear operators, which can be transformed, represented in different ways, considered as isomorphic or not, etc." (p. 65).

This particular difficulty is not restricted to the context of Linear Algebra, but is common in almost all areas of mathematics. To understand a new notion in elementary mathematics students have to undergo a cognitive shift incorporating lengthy procedures in mathematical concepts. This conversion of actions or operations into what Piaget (1945) described as "thematised objects of thought or assimilation" (p. 49) was described by the term *encapsulation* (Dubinsky, 1991).

Cottrill, Dubinsky, Nicholls, Schwingendorf, Thomas & Vidakovic (1996) formulated the APOS theory, from the acronym of the words action, process, object and schema. *Actions* are physical or mental transformations of objects to obtain other objects. When these actions become intentional they are characterised as *processes* which may be encapsulated to form a new *object*. A coherent collection of these actions, processes and objects, linked in some way, is identified as a *schema*. A schema can be reflected upon and transformed and thus result in the formation of a new object.

The disadvantage of such an approach in Advanced Mathematics is that the students who are taught in this manner are not given a formal definition of the new object until the end –if then– of this whole learning process. Vinner (1991) argues that "it is hard to train a cognitive system to act against its nature and to force it to consult definitions either when forming a concept image or when working on a cognitive task” (p. 72). This situation can only get worse if the students’ concept image has been built through actions and processes, without considering the concept definition.

The course

The Linear Algebra course under consideration took place in the Mathematics Department of a very demanding British University. The duration of the course was 30 hours, split into hourly sessions three times a week, for ten weeks in the second term. The lecturer provided the students with complete and explicit notes, so that they could concentrate on understanding the material, instead of keeping their own notes. There was also a recommended textbook, which was Anton’s (1994) *Elementary Linear Algebra*. We should also note that the course was designed not only for Pure Mathematics students but also for students following combined degrees (Mathematics & Physics, Mathematics & Statistics, Mathematics, Operational Research, Statistics & Economics (MORSE)), a fact which explains the process-oriented nature of the course.

There seem to be two ways of sequencing the contents of Linear Algebra; the computation-to-abstraction approach and the abstraction-to-computation approach (Harel, 1987). The first approach suggests that matrix arithmetic and linear systems should precede vector spaces and linear transformations, in order to enable the students to develop the language and reasoning needed for understanding the more abstract material. The second approach starts with vector spaces and linear maps and then matrices and simultaneous linear equations are treated as applications of the former.

In this particular Linear Algebra course the approach chosen was the computation-to-abstraction one, because the lecturer felt it would be more beneficial to start with already familiar concepts and use them as building blocks for the development of the more abstract notions of vector spaces and linear transformations. Various introductory strategies were used in order to present the new material, such as ‘abstraction’ –introducing abstract ideas by initially illustrating them by specific examples (Harel, 1987)– and ‘embodiment’ (Dienes, 1960) –translating definitions and theorems in terms of given situations. The difference between these two processes lies in the timing of the presentation of the particular situation; in abstraction it comes before the concept is defined, whereas in embodiment, it follows the formal definition.

In order to demonstrate how these teaching techniques were employed, we have two extracts from the lecture notes; the first extract is a case of abstraction and the second of embodiment. These examples were chosen so as to reflect the nature of the delivery of the whole course.

<u>Abstraction</u>

Preparing for Eigenvalues and Eigenvectors

1. Draw a set of axes on a piece of paper.
2. Choose a vector from $\mathbf{R}^2$ and draw it on the axes.

3. Now multiply it be the matrix $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ to get a new vector. Draw the new vector you get.
4. Repeat these three steps 3 or 4 times, choosing a new vector each time. Notice that A sends vectors all over the place.
5. Now draw a new set of axes and plot the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.
6. Multiply each of these vectors by A and draw the result. Notice that these vectors stay on the line they started on. They have just been "stretched" in a positive or negative sense.
7. In these two cases,
$\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 2 \\ -3 \end{bmatrix}$
8. So, for certain vectors, multiplication be A just results in a "stretching" (which can include a change of direction) of the vector; in other words a scalar multiplication of the vector. Such a vector is called an **eigenvector** of A. The "amount of stretch" undergone by such a vector is called an **eigenvalue** of A.
9. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are eigenvectors of A, and 4 and -1 are corresponding eigenvalues of A.
10. In general, an eigenvector of A and its corresponding eigenvalue are related by the matrix equation

$$A\mathbf{x}=\lambda\mathbf{x}$$

where **x** is the eigenvector and λ is the eigenvalue.

Embodiment

Definition We define the **adjoint matrix** of A, denoted adj A, to be the *transpose* of the cofactor matrix

Example $A=\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$

We replace each element of the matrix with its cofactor, to get the *cofactor* matrix, C.

$$C=\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} \\ \begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & \begin{vmatrix} & \\ & \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -6 & 7 \\ -9 & 3 & 9 \\ 5 & 0 & -10 \end{bmatrix}$$

We transpose the cofactor matrix to get the *adjoint* matrix, adj A.

$$\text{adj } A = C^T = \begin{matrix} -2 & -9 & 5 \\ -6 & 3 & 0 \\ 7 & 9 & -10 \end{matrix}$$

Another strategy used for the introduction of vector spaces, in particular, was consistent with the APOS theory (Cotrill *et al.*, 1996). Starting with vectors in $\mathbf{R}^n$, objects already familiar to the students, addition and scalar multiplication were defined, initially as actions on these vectors. When these actions were interiorised, along with their properties (the 10 vector space axioms), they became processes, which then were used to form the new object ‘vector space’. These notions were later extended to include vector spaces other than $\mathbf{R}^n$, resulting in the schema of a general vector space.

The students

The fact that this research is taking place in a highly regarded British Universities, as we noted earlier, means that the students who take part have a solid mathematical background, as indicated by the results in their A-level exams (University entrance requirements in Mathematics is 3 A’s) and their initial ‘mathematics techniques diagnostics test’. Our sample consisted of 8 high achieving Pure Mathematics students, seven of whom achieved first class grades in the course assessment (90% written examination, 10% awarded through weekly coursework).

The research was carried out by fortnightly videotaped group discussions (with two interviewers taking part), where the issues brought forward were associated both with the delivery of the course and the students’ understanding of the new notions. Some issues were brought up by the students themselves, initiated by certain difficulties they faced when doing their coursework.

In the following extracts from the transcripts of the discussions, the talk revolves around the notions of vector space, linear dependence and independence, bases and spanning sets. Having under consideration that the definition of vector space was introduced in the APOS pattern (as described above), we can clearly see that the students’ concept image of a vector space remains restricted to $\mathbf{R}^n$ spaces.

Int: Why are you always using the examples of $\mathbf{R}^2$, $\mathbf{R}^3$, ..., $\mathbf{R}^n$?

J: Because these are the easiest examples of vector spaces.

Int: But why are you associating the word ‘vector’ with vectors in space? We defined vector space as a set that satisfies those 10 axioms and any element of this vector space is a vector. So it does not have to be a vector in $\mathbf{R}^n$.

L: Because when we were first taught about vectors they were vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. And that’s what is still in my mind about what a vector is. I mean we have managed to extend it to $\mathbf{R}^n$ but not to other sets.

Int2: Does the previous image of vectors, the 2 or 3-dimensional image actually get in the way?

A: Yeah, actually it does.

When asked what the basis of a vector space is, no student gives the formal definition but they instead give an answer that applies only in a very specific context, in the same way that Carlson (1993) reports that “basis of a vector space” is for some students “the result of one specific algorithm applied in one specific context” (p. 29).

Int: What is the basis of a vector space?

J: Any two linear independent vectors.

Int: Can you have two linear independent vectors being a basis for $\mathbf{R}^4$?

J: Why not?

I: For a vector space of order *n* you need *n* linear independent vectors to span it.

A: Should the elements of a basis be all linear independent?

J: Yeah.

On the issue of the course delivery, the students seemed to have divergent views. Some appreciated the clarity and completeness of the setting out of the material, as well as the provision of algorithms for carrying out tasks, but for others the essence of mathematics lies in the challenge of having to struggle with theorems and proofs.

J: In specific cases there are tricks or rules you can use and it makes things easier than if you take the general approach to it. You have the general rule to cover yourself if you have any doubts.

Int: But why do you prefer to use these tricks?

J: Because it makes life easier. Would you use the definition of derivative in order to find the derivative every time?

Int: But is it not essential to know how you came up to this derivative, to know how to prove it?

I: It's not that essential to always carry in your head how to prove it, but to know where the results comes from and how it was derived. I think you'd be better off with knowing why it's true; that's what matters to me. If you just want to know the results you might as well study Physics!

L: Yeah, in applied maths we just had to use the tools, we didn't have to prove them.

Int: Do you think that the shift from school to University would be made easier if they gradually tried to change your way of thinking from the A-level to the more abstract?

J: There's something to be said about the 'shock treatment'. I think the sooner you get into the new way of thinking the better.

A: I think I'm learning the Analysis better because the lecturer threw us more or less straight in the deep end at the beginning of the course and we had to cope with it.

J: If you do the gradual thing then you might get the wrong impression. A year or something in the course you might realise "this isn't really for me". It's more honest to have the 'shock treatment'; this is what it's really like.

Conclusion

The position of Linear Algebra, as an initial course of advanced mathematics, implies that a bias towards either concrete or abstract approaches may cause difficulties. In this context, developing a Linear Algebra course based on the computation-to-abstraction approach does not imply that the students will not be introduced to both the concrete and the abstract aspects of the course. However, putting the emphasis on the 'concrete', may sway the students towards overlooking the 'abstract'. As a result, their concept images becomes enriched in processes but not in objects.

It is widely accepted that undergraduate students have to make a huge cognitive shift from school mathematics to advanced mathematical thinking. To this end, it is imperative that the balance between concrete and abstract is not only maintained but safeguarded in every possible manner.

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