SPATIAL STRATEGIES AND VISUALISATION:

Eddie Gray
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

Abstract

This paper is an invited response to the Imagery and Visualisation Research Forum of PME XX111. It takes as its starting point the keynote paper *The Role of Visualisation in Young Students Learning* (Owens,1999). The review attempts to consider the paper in the wider context of mathematical thinking. It addresses difficulties associated with the investigation of imagery and draws the conclusion that a reappraisal of the data could suggest the possible existence of a divergence in spatio-visual thinking similar to that in arithmetical/algebraic thinking.

Introducing a Perspective

Writing in 1996, Owens suggested that:

> Student's responsiveness during active engagement in problem-solving activities is precipitated by their own thinking and feelings … responsiveness implies a degree of understanding as well as involvement and interest in the activity.
>
> (Owens, 1996, p. 101)

With this notion in mind, I present a personal review of the paper *The Role of Visualisation in Young Students Learning* (Owens, 1999). In doing so, I am also aware that each comes to the task of considering the paper with different personal repertoires of knowledge. Inevitably, we see the paper in our own personal way.

Within her paper, Owens considers two features designed to inform teachers about young children's early spatio-mathematical development. The first is a 'framework' that provides a basis for teachers to assess children's thinking and build a teaching programme. The second is a mechanism for assessing the children against the framework. It is claimed that an important aspect of the two is the relationship between spatial understanding and visualisation. Indeed, some of the tasks are 'specifically designed to encourage visualisation' and the framework itself is associated with a 'hierarchical' list of imagery strategies.

My personal reading of Owens has been influenced by two questions:

- first, how should we interpret the fact that some pupils seem to find the study of mathematics relatively easy whilst others find it virtually impossible?

- second, what can it add to our efforts to establish some kind of underlying theoretical structure that may help us further our understanding of mathematical thinking?

Perception, action and reflection

Geometry builds from the fundamental perception of figures and their shape, supported by action and reflection to move from practical measurement to theoretical deduction and Euclidean proof. This was a theme which Tall (1995) alluded to in his plenary address to PME in Brazil. He spoke of the way in which elementary mathematics begins with '*perceptions of*' and '*actions on*' objects in the external world.

> The perceived objects are first seen as visio-spatial gestalts, but then, as they are analysed and their properties are teased out, they are described verbally, leading in turn to classification (first into collections and then into hierarchy's… Tall, 1995, p 61.

To establish the distinction between the perceptive and manipulative aspects of early spatial development and the verbal/symbolic development of arithmetic and algebra Tall suggested that a different kind of development stemmed from *actions on* objects. Here the process of counting is developed using number words that become conceptualised as number concepts.

Early mathematical concepts are strongly associated with preliminary activities involving perception and action within the physical world and reflection on both perception and action (Gray, Pitta, Tall & Pinto, in press). Such a development requires the ability to concentrate the mind and give careful thought to an act or idea and then to filter out irrelevancies and separate notions from their context. It involves the construction of relationships between and amongst objects and of the inter-relationships of the actions on them. It may be that such a process works to the advantage of the more successful. An emphasis on one or more of the activities of perception, action or reflection leads not only to different kinds of mathematics, but also to a spectrum of success and failure depending on the nature of the focus in the individual activity.

In any context involving an action with objects, the individual has the possibility of attending to different aspects of the situation. Indeed, this is an issue that Cobb, Yackel and Wood (1992) see as one of the great problems in learning mathematics, particularly if learning and teaching are approached in a representational context. In their search for substance and meaning, some children may be distinctly disadvantaged right at the start of their mathematical development, but it is a disadvantage that may not make itself apparent in the earlier stages of cognitive development.

Any considered attempt to establish the way children think as opposed to simply measuring their level of achievement has to be welcomed. I see the Owens paper in this context. But there is more — it hints at a divergence in spatial thinking that may match the one identified in arithmetic/algebraic thinking. This is not to say that I applaud the paper in all of its aspects. As a research paper it has many technical weaknesses, one of the major ones being the extensive inclusion of the mechanisms of assessment (Table 1) at the expense of empirical evidence from which the reader may make judgements. What we see is the researcher's

interpretation of the evidence. In a sense, we have a description of the researcher's image of the child's image.

The framework and possible links with van Hiele

The earlier work of Owens (see for example 1992, 1996, 1998) led to the development of a framework to inform teachers of young students' early spatial mathematical development. The 1992 test was developed at the recognition level of spatial development (Owens, 1992,) and was therefore associated with Van Hiele Level 1 (van Hiele 1986).

Dominated by perception, van Hiele Level 1 suggests that appearance becomes the mechanism through which learners operate on shape and other geometric configurations. However, though perception is the foundation of geometry, it takes the power of language to make hierarchical classifications. Figures are initially perceived as gestalts but then may be described and classified through verbalising their properties, to give the notions of points, lines, planes, triangles, squares, rectangles, circles, spheres, etc. Initially these words may operate at a single generic level, so that a square (with four *equal* sides and every angle a right angle) is not considered as a rectangle (with only *opposite* sides equal). Again, through verbal discussion, instruction and construction, the child may begin to see hierarchies with one idea classified within another, so that "a square is a rectangle is a quadrilateral", or "a square is a rhombus is a parallelogram is a quadrilateral".

Owens' contribution to this development has been to tease out the coarser framework and provide a finer grained analysis that may be beneficial to teachers. Focussing on orientation and motion, part whole recognition and the use of language and classification, she presents what may be a useable set of criteria to establish children's thinking. However, I wonder how these criteria differ from the van Hiele level 1, that of establishing a visual gestalt, and van Hiele level 2, that of being able to characterize shapes by their properties. It seems self evident though that these two need embedding in an associated language context in recognition that:

> The criterion for having a concepts is not that of being able to say its name, but that of behaving in a way indicative of classifying new data according to the similarities which go to form this concept. (Skemp, 1986. P. 26)

Evidence of divergent thinking

Of course, it is important to place these notions into a context which has meaning for the teachers of young children. However, it is equally important not to over-generalise from the outcomes — a factor that does seem to dominate this part of the paper, particularly since we are considering a "theoretical framework". We may accept that students 'need to recognise shapes in different orientations' but it is far less easy to accept (in the absence of any empirical data from which to make a personal analysis) that:

- movement is imagined by students as they make associations between shapes
- that the need to complete shapes using imagery are necessary skills for students.

Equally, it is difficult to accept that *all* students will "realise" and/or "develop relationships" or "associate particular words" or use "words to represent their imagery". These expressions project the hopes of the pedagogue not the realism of the learning experience.

In the cognitive context, we see the framework presented as a model which informs us about children's thinking. In the pedagogic context, we see the model as a framework from which teaching could be developed. Such a suggestion reminds me of efforts to turn Piagetian theories into a mathematics curriculum. Indeed, we see similarities with the Piagetian notion of a stage theory and Owens claims that the 'imagery strategies' through which the different aspects of spatial knowledge become evident 'are more or less likely to emerge and be used by children in an order'. Any divergence from this hierarchy of development is simply explained in terms of 'intuition or incidental learning'. No evidence in the paper supports either of these arguments. Most appears to be based upon teaching experiments (for example Owens, 1992, 1996) and/or through the analysis of children's spatial problem solving abilities (see for example Owens & Clements, 1998). Any claims about a hierarchical process of growing strategy sophistication would seem to be best addressed by a longer-term developmental study with individual students.

The notion that there are different strategies associated with children's spatial thinking is important and perhaps we are seeing the first stages in an attempt to mirror for spatial development the research in elementary arithmetic which has proven to be so beneficial. The evidence presented within typical examples suggests that there are differences in the way that children think. Though the study refers to a sample of 50 children, 'typical responses' presented for the reader to make judgements are only drawn from 6. For four of these children it is possible to consider their responses over the 6 items which stimulated responses. An analysis of the responses of these four suggests distinct behaviours not an ordered development. The data within Table 3 may be re-interpreted; not to simply provide 'typical responses' but to indicate qualitatively different thinking representing a spectrum of varying between perceptual and figurative extremes. We see, for example, that "*" always uses "emergent" strategies, "perceptual" strategies dominate the thinking of "◊", pictorial imagery that of "⇒" and the more sophisticated strategies are used by "●" and "■". Responses associated with the emergent strategy, suggest that the child identified as "*" is at a stage of "pre-recognition" (Clements and Battista, 1992) based upon a deficiency in perceptual activity whilst child "◊" appears to be firmly embedded in perceptual strategy use. My conclusion from the evidence would be that some operate on a perceptual level, whilst others operate at a figurative level which itself may reflect differing degrees of sophistication.

A focus on imagery and visualisation

Piaget and Inhelder drew our attention to the relationship between imagery and perception:

> Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived...
>
> (Piaget & Inhelder, 1967, p. 17)

As more progress is made on the research front the concept of image has become less clear (Cooper, 1995). In a sense, Owens adds to this lack of clarity. Through her implicit recognition that human cognition requires different representational constructs, she adds another dimension to our perceptions of image and visualisation. Images are significant components of cognition and their interpretation has particular relevance to the study of mathematics. However, our interpretation of the notions, and the evidence we use to describe and classify them, is often somewhat speculative and open to interpretation.

It may be correct to make the assumption that there is a functional equivalence between images and processes formed on images, and the corresponding external objects and the perceptual operations that the images and the imagined operations were thought to stimulate. To do so, however, suggests that images, like visual perceptions, have depictive or picture-like qualities. Here we are faced with an elementary problem — whether or not the notion of imagery can be synonymous with the notion of visualisation.

Guttierez (1996), in his excellent discussion of the various ways visualisation, image and mental image have been used in mathematics, suggests that a

> "mental image is a mental representation of a mathematical concept or property containing information based upon pictorial, graphical or diagrammatic elements [whilst] visualisation or visual thinking is a kind of reasoning based on the use of mental images".
>
> (Gutierrez, 1996, p. 6)

He considered that mental images were a basic element in visualization. Using the notions of concept image (the student's mental picture of a geometrical figure) and concept definition (the student's verbal definition to define a geometrical figure) Matsuo (1993) suggested that the more these converge. the more likely it is that the student moves from level 1 to level 2. Such a hypothesis would seem to be consistent with Tall's (1995) view and Skemp's (1986) criteria for the possession of a concept. Equally, the evidence suggested by Owens' responses from "●" and "■" would seem to support this view. Interestingly, Matsuo then suggested that there might be a difference between a state of understanding that is specified by the visual mode and a state of understanding which is clarified by the visual mode. I take this to mean a difference between the visual mode being necessary for understanding and the visual mode being a generator of

understanding. In an entirely different study that looks at imagery in the context of elementary arithmetic, Gray & Pitta (1997) draw a distinction between the use of imagery that is essential to thought and the use of imagery that generates thought.

It is possible that such a distinction may be interpreted from the evidence presented by Owens, where we see notions of perception and imagery being jointly classified to interpret imagery strategies. The fact that we see distinctions between children who rely extensively on perception and others who evoke imagery of a visual form may not be the whole of the story.

Though it would seem that the ability to visualise provides strength in the context of developing spatial awareness, the labels 'visualiser' and 'non-visualiser' may not be indicative of level of mathematical achievement. Pitta (1998) has shown that children at extremes of achievement may be strong visualisers. Equally, children at the extremes could be identified as non-visualisers. To focus solely on the incidence of visual imagery may not provide an insight into the qualitative differences between children, whereas a focus on imagery in both visual and non-visual form may.

The framework devised by Owens may lead us some way towards our understanding of children's understanding. It may also be suggesting that there is a diverging approach to elementary geometry which matches diverging approaches to elementary arithmetic (see Gray and Tall, 1994). Such a divergence, based upon qualitatively different interpretations that children place on spatial activity, may be manifested in the formation of qualitatively different forms of imagery and/or qualitatively different kinds of imagery. Visual imagery may be one of these forms but a broader view of the notion of image would not disregard the product of imaging in any modality.

I started this paper by asking two questions, one associated with the way in which children deal with the study of mathematics; the other associated with the formation of an underlying theoretical structure. The evidence from the framework developed by Owens would suggest that more sophisticated thinking reflects the ability to disregard details and focus on the generalizations that support choice. This was not to say that the detail was unavailable to them — it was and could be incorporated and used if needed.

The ability to create mental imagery is inherent in us all and it would seem that a visual stimulus would evoke visual imagery to a greater extent that verbal stimulus. Pitta (1998) has suggested that it is quite possible for children to posses the same visual image but to then attach different meanings to it. Therefore, whereas the visual image of the more successful child may be used to refresh memory or as a skeleton that ideas, equivalencies and relationships may be attached to, that same visual image may be used in an active mental episode for the less successful.

It is implicit from the Piagetian and the constructivist perspective that the knowledge and the beliefs that learners bring to a situation can influence the meanings they construct from that situation. Kay Owens uses the word responsiveness to mean a similar thing. The responsiveness of the sample in the survey illustrates that the differences in strategy use may not only be apparent when children's responses to the problems are considered but, also a manifestation of the use of memory. A tendency to focus on surface characteristics of the stimuli may be more strongly related to the use of short-term memory. On the other hand, emphasis on long term memory use may mean retrieval of mental imagery which could be both meaning related information or carry surface characteristics. The truth of the matter is we really do not know. The study of imagery in any context is fraught with difficulty. We make an assumption that report, description and external representation in the form of words, drawings and actions provide an indication of the nature of the mental image. Any efforts to explain or provide a description that relates to reality can be very different from the truth. However, I would suggest that the strength of this paper lies in the possible insights it provides to another divergence in mathematical thinking.

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