Beyond the Physical Actions:
A Perspective on Cognitive Development in Arithmetic.

Eddie Gray & Demetra Pitta

ABSTRACT

This chapter focuses on qualitative differences in children's thinking about arithmetic. In doing so it places a perspective on cognitive development which integrates the role of successive process to concept encapsulation, the representation of these by numerical symbols and the way in which imagery may mediate between the two. The evidence suggests that the different ways in which individuals process information using these notions at a given time can be either beneficial or severely compromising for their current and future development. The chapter concludes by briefly considering the potential of a graphic calculator as a mechanism that supports a focus on the process of evaluation and the meaning of the symbolism. Such an activity may offer a way into arithmetic that helps those children who are experiencing difficulty.

INTRODUCTION

The developmental increase in children's mathematical thinking in primary school years is self-evident, not just in terms of knowledge they possess, but in the style and capacity of their reasoning and thought. However, while the development itself is apparent, the psychological mechanisms and the components that cause it are not fully identified or understood. In an attempt to gain insight into this issue the same question seems to recur "*What are children really doing in their heads?*" We believe that the answer indicates qualitative differences that in a numerical context are a source of flexibility and power for one child but a hindrance to long-term cognitive development to another.

It is the intention of this chapter to consider the way in which different forms of visual imagery are associated with numerical achievement. To develop the chapter we focus first upon the evolution of numerical concepts with particular attention to the role of symbols. We then consider the notion of mental representation and in particular the relationship between visual image and numerical achievement. Our conclusion suggests that there are qualitative differences in the kinds of visual image that may occur through a child's interpretation of numerical symbols. It is suggested that children who tend to rely on active and episodic images may have difficulty recognising the advantages of the numeration system that is the key to more sophisticated levels of understanding in arithmetic.

The development of numerical thinking

Tall (1995) has suggested that the construction of mathematical knowledge may be seen as a two-pronged affair grounded in the learner’s interaction with the environment. One is linked to perception and manipulation and is hypothesised to be a visio-spatial to verbal deductive transformation, the other builds upon successive process-to-concept encapsulations using manipulable symbols. Though Piaget (1973) indicated that the ability to think mathematically consists of gathering information from actual actions, very different types of mathematics follow from manipulation and the use of visual clues on the one hand and the representation of processes by symbols on the other.

The initial stages in the development of geometry and of number sense have physical counterparts that originate in the real world that possess visual elements. Whereas in geometry the identification of properties and relationships may lead to the classification of the objects of the environment, within number the actions on physical objects lead to the development of procedures, through which processes are named, conceptualised and symbolically represented. Counting may be seen as an example of such a process. Though the learner’s initial focus may be countable objects and the action of counting, this focus needs to change to take account of the symbols which represent both the process of counting and the concept of number (Gray & Tall, 1994).

Counting begins with the repetition of number words, with the child’s remembered list of numbers steadily growing in length and correctness of sequence. The act of counting involves pointing at successive objects in a collection in turn and saying the number words, “there are one, two, *three* things here.” This may be compressed, for instance, by carrying out the count silently, saying just the last word, “there are [one, two,] three”, heard as “there are …three.” It is thus natural to use the word “three” not just as a counting word, but also as a number concept. By this simple device, the counting process “there are one, two, three,” is compressed into the concept “there are three.”

Hypothesised cognitive structures associated with this development provide clear examples of the way in which physical objects play a fundamental role in the construction of cognitive objects. For example, Steffe, Richards, Cobb and von Glasersfeld (1983) provide a detailed analysis of the way in which actions with physical objects become metamorphosed through a variety of increasingly abstract representations. Such a change in the nature of the object upon which the action focuses makes it possible to act upon the results of carrying out the processes without bothering about the processes themselves.

This form of compression is powerful in quite a different way to the compression associated with compression in geometric thinking. In arithmetic the number word is part of a hierarchy (a counting number is a fraction is a rational number is a real number). However, the major biological advantage of numbers arises not from this hierarchy but from the way in which the number words can be used to switch between *processes* (such as counting or measuring) and *concepts* (such as numbers). Not only are number symbols “small enough” to be held in the focus of attention as concepts, they also give immediate access to action schemas (such as counting) to carry out appropriate computations. They act not only as economical units that may be held in the focus of attention, but they also provide direct links to action schemas.

Arithmetical Symbols–A Cognitive Perspective.

The formation of numerical concepts from actions with physical objects is the core of the perceived cognitive development of simple arithmetic (see, for example, Piaget, 1952; Steffe *et al*, 1983; Kamii, 1985; Gray & Tall, 1994). These perceptions share common ground; the nature of the object that is the basis of the action becomes more and more abstract. This permits a steady growth in procedural sophistication — somewhat lengthy procedures with real objects are steadily compressed to form the basis for the encapsulation of numerical concepts. Though it is not clear how this actually takes place, we know it does take place. Steffe *et al* suggest that it is manifest through the progressive abstraction of the nature of the unit (or object) that is integral to the counting process. They hypothesise that decreasing dependence on *perceptual* material permits children to eventually count *figural* representations of perceptual material; the counting process continues in the absence of the actual items. *Motor acts*, such as pointing, nodding and grasping, that accompany the counting process, can be taken as further substitute units for perceptual items. Dependence on these three forms of unit is further reduced by the realisation that the utterance of a number word, the verbal unit, can be taken as a substitute for countable items that could have been co-ordinated with the uttered number sequence. Gray & Tall (1994), concentrating on the process of counting, suggest that increasing sophistication may be seen in the compression of lengthy count-all procedures to shorter ones associated with the process of count-on. The latter can provide a basis for the acquisition of known facts. Thus procedural compression is associated with qualitative changes in conceptual entities and "involves the transition from enactive concepts to the construction of novel mathematical objects" (Cobb, 1987, p.3). In such a way a mental entity, behaving as if it were a real thing, is named and symbolised. However, "we may not have anything in our mind which is like a physical object we have symbols which we can manipulate *as if* they were mental objects" (Tall, 1995, p. 65).

When numbers have become conceived as mental entities, they may themselves be operated upon. However we suggest that the manner through which children's understanding of concepts is mediated through their mental representations of verbal and written symbols is crucial to the development of elementary numerical concepts and to more complex mathematical ideas. Symbols are the essential means through which concepts are communicated but their full development depends upon the corresponding evolution of concepts to a degree when they can be used independently of any exemplar or embodiment (Skemp, 1979). Within mathematics they are an efficient means of storing and conveying information, not least because they allow the compression of a considerable amount of information into a small place. The paradox is that symbolism is both the reasons for the power of mathematics and of the complexity for many trying to learn it (Cockroft, 1982)

What is happening in the mind?

For some elementary arithmetic remains a matter of performing or representing actions. The more overt acts and strategies associated with counting are well documented through the work of Steffe *et al* (1983), Carpenter, Hiebert & Moser (1981), Siegler and Jenkins (1989), Gray (1991), Gray and Tall (1994). Less well-documented are more covert processes, particularly those associated with "*what is happening in the mind*". To provide some insight into this we turn to mental representation and in particular imagery. In doing this we are cognisant of Piaget and Inhelder's (1967) belief that the object which is the core of the mental image should be given equal attention to that given to the real object.

> Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived...Now in all probability the image, an internalised imitation, is consequently derived from motor activity, even though its final form is that of a figural pattern traced on the sensory data. (Piaget & Inhelder, 1967, p. 17)

Historically mental representations have been interpreted by analogy with physical representations (Paivio, 1986). Some physical representations may be picture-like while others are language-like. Picture-like representations may include photographs, drawings, maps and diagrams and have been variously described as having analogue, iconic, continuous and referentially isomorphic properties which imply that they map onto the objects or events in a non-arbitrary way. Language-like representations include natural human language as well as more formal systems such as the symbolism of mathematics. Characterised as being non-analogue, non-iconic, digital or discrete, the relationship between the language-like representations and the object is completely arbitrary. Thus representations may vary in their level of concrete-abstractness. At one extreme we have highly concrete, iconic, modality-specific representations whilst at the other we have completely abstract, amodal representations that are only arbitrarily related to real world objects. Such a variation implies that the nature of the 'object' that dominates the mental representation may not only differ by its degree of abstraction but it is conjectured that different mental representation may be put to different use.

It would be too much for this chapter to focus on both 'picture-like' and 'language-like' mental representations. Consequently, our discussion will concentrate on the former in the context of elementary arithmetic. To aid our purpose we will use the notion of image to identify 'picture-like' mental representations. We do so recognising that we are excluding a valuable category in our understanding of children's thought processes. However, we believe that even discussion of the qualitative differences in children's 'picture-like' mental representations provides insight into differences in children's arithmetical behaviour.

The observation that some individuals are more successful than others in mathematics has been evident for generations. Piaget (1952), for example, provided a novel method of interpreting empirical evidence by hypothesising that all individuals pass through the same cognitive stages but at different paces. Such a view is usually implicit in the construction of many standard curricula (See for example DES 1995; NCTM, 1989) but often these fail to take into the different ways individuals do mathematics. A different conception was offered by Krutetskii (1976) who identified a spectrum of performance between various individuals that depended upon how they process information. In is within this frame that we consider the role of imagery.

The importance of imagery in cognitive development has been identified by psychological research. Kosslyn (1980) suggested that children use it in their thinking more than adults do. However, the properties of images can place major constraints on cognitive processes thus having far reaching consequences on children's concept development and reasoning (Bruner, Oliver and Greenfield, 1966; Piaget and Inhelder, 1971).

In an arithmetical context it has been argued that imagery of numbers is often highly imaginative, unconventional and built up over time (Thomas, Mulligan & Goldin, 1995). Pirie & Kieran (1994) see the notion of 'image having' replacing the need for actions or the specific instances of image making but suggest that quality of the image can influence the quality of later understanding:

> "Image having is the level at which the learners actually have some images for concept and thus they no longer need to rely on the actions that occasioned the understanding and can carry and use the ideas they have constructed. This does not imply, however, that their images are complete, appropriate or even sufficient for the work in hand. Many learners develop strong early attachments to particular dominant images and this can seriously hamper later growth of

understanding”

(Pirie & Kieran, 1994, p. 247)

The relationship between different forms of mental representation associated with number sense (symbolic, verbal and analogical representations) may be seen through the presentation and solution of arithmetical facts (Dehaene & Cohen, 1994). In this context the visual image is seen to be the classical analogical representation (though it may also be supported by modalities from other senses such as touch, smell, orientation and so on). “Patterns of dots or other things such as the alignment of apples or a bar of chocolate may be deemed to be analogical” (Seron, Pesenti, Noel, Deloch and Cornet 1992 p. 168). “Characterised by its appearance in the absence of the object to which it refers’ (Mead, 1938, p. 224), such images take up some form of mental space in the same way that physical objects take up physical space and they can be mentally moved or rotated (Boden, 1988).

Imagery and Elementary Arithmetic

In the field of elementary arithmetic we suggest that as ideas become more subtle and the connections to the physical world become more extended, different forms of arithmetical development may be determined from an analysis of children’s imagery. By drawing on the assumption that an image is mediated by description (Kosslyn, 1980; Pylyshyn, 1973) we suggest that arithmetical actions with real life objects may have consequences for the quality of the image which is created. This in turn may well have an influence on the way in which the image is used.

It is suggested that qualitative differences in the forms and the use of imagery may be seen amongst children who are at the extremes of numerical achievement. Gray and Pitta (1997) suggest that the images of ‘low achiever’s’ appear to be episodic and active whilst those of the ‘high achievers’ are semantic and generative. The notions of ‘episodic’ and ‘semantic’ are used to make a distinction between those images that are associated with the recollection of personal happenings and events and those images which are associated with knowledge linked to more abstract meaning and relationships. The former is based upon access to former experience, the latter does not depend upon the learning episodes that provided the basis for knowledge (see, Tulving, 1990).

In their investigations (Gray & Pitta, 1996; Gray, Pitta & Tall, 1997) children aged 7 to 11 were invited to respond to verbally and visually presented elementary addition and subtraction combinations using written and spoken symbols in the absence of any other form of representation. The children were asked to find solutions to number combinations to ten and then to twenty for each combination explain what was happening in their heads.

Those identified as ‘low achievers’ translated the symbols into numerical processes supported by the use of either perceptual objects or figural representations of perceptual objects. These were then used to carry out a particular procedure to obtain a solution. Though the nature of the object may have varied, for example counters, marbles, fingers and even symbols, these variations were masked by a qualitative similarity — each different object served as a counting unit. The difficulty of the combination appeared to dictate whether or not the children used an overt procedure, like counting on fingers, or a covert procedure that was only clarified by responding to the request to indicate what was happening in the mind.

In the latter instance it was frequently noted that the figural representations were analogues of the real life object and often these possessed shape and colour and each was associated with a numerical label and counted in a sequential way. Of course this caused extreme difficulty, the analogical image associated with discrete objects invoked concentration on the action of mental counting. We suggest that such image was essential to action because it maintained a focus and even though difficult and limited in scope some children tried to maintain this sort of image when dealing with combinations to twenty. However, in all of those instances where children used analogical images the procedural difficulties soon ensured that they fell back to the use of perceptual items.

In contrast, ‘high achievers’ focused on symbols and those abstractions that enable them to make choices. Reported images associated with elementary arithmetic were always symbolic but more importantly most references to seeing symbols in the mind were associated with notions of “flash”. For combinations that were known either the input, for example 3 + 4 or the result ‘7’ were reported to “flash” through the mind. In instances where children reported the use of derived facts it was frequently the numerical transformation that 'flashed'. For instance when given 9 + 7 one eleven year old produced the answer 16 accompanied by the statement. "10 and 6 flashed through my mind.

The overall evidence obtained from the study suggests that children who are 'low achievers' in mathematics are unable to detach themselves from the search for substance and meaning–no information is rejected, no surface feature filtered out. Mental procedures associated with the use of the counting sequence may be seen as an analogue a physical experience. The procedure with which this sequential episode is re-enacted (count all, count-on, count-back) reflects the child's familiarity and competence with it, their personal preference (Gray 1991) and their knowledge of specific number combinations. For these children its conjectured that for any arithmetical combination where mental imagery is used the general number sequence is specified in two ways:

(a) the number that counting should start and end at, (this could be one, it could be the largest quantity), and
(b) the objects that are to be manipulated.

Though the evidence suggests that the object to be manipulated may vary, the clarity of the image associated with these objects depends upon how heavily the child leans towards perceptual representation. The more a child depends on perceptual objects the more concrete their mental objects.

A different picture emerged from the high achievers. Not only did they use knowledge they knew to build knowledge they did not know but if identified it was clear that their mental representations were symbolic. These seemed to come and go very quickly. It is hypothesized that they acted as thought generators and reminders that momentarily come to the fore as new transformations or precursors of verbal comment.

Such differences have serious consequences, which may be contributory factors to the formation of the proceptual divide (Gray & Tall, 1994). We do not suggest that the notion of a divergence in performance means that some children are doomed forever to erroneous procedural methods whilst others are guaranteed to blossom into a rich mathematical conceptualisation. It is vital not to place an artificial ceiling on the ultimate performance of any individual, or to predict that some who have greater success today will continue to have greater success tomorrow. However, the evidence suggests that the different ways in which individuals process information at a given time can be either beneficial or severely compromising for their current and future development (see for example Gray & Tall, 1994; Gray & Pitta, 1997; Pitta, 1998)

A child with a fragmented knowledge structure who lacks powerful compressed referents to link to efficient action schemas will be more likely to have greater difficulty in relating ideas. The expert may see distilled concepts, which can each be grasped and connected within the focus of attention. The learner may have diffuse knowledge of these conceptual structures which is not sufficiently compressed into a form that can be brought into the focus of attention at a single time for consideration. Far from not working hard enough, the unsuccessful learner may be working very hard indeed but focusing on less powerful strategies that try to cope with too much uncompressed information. The only strategy that may help them is to rote-learn procedures to be performed as sequential action schemas. However, though such knowledge can be used to solve routine problems requiring that particular technique, it occurs *in time* and therefore it may not be in a form suitable for thinking about as a whole entity.

Understanding the notation system

The actions in simple arithmetic are meant to be the platforms from which children can give meaning to symbols. However this is only part of the story associated with the development of numerical understanding. There is another story to tell, that of understanding the numeration system. Dehaene (1997) presents evidence to indicate that the time required to name a number of dots grows slowly from 1 to 3. It increases sharply beyond this limit. At the very same point the number of errors increases sharply. He argues that it was possibly in recognition of these limitations that different human societies converged to denote the first three or four numbers by an identical number of marks, I, II, III, and the following numbers by what are essentially arbitrary symbols. For the Western world it was the later recognition of the strength behind the relative simplicity of Arabic symbols, in association with the invention of a symbol for “zero” and a unique base number, that gave complete transparency to the magnitude relations between 5, 50,500 and 5000. We now use the far-reaching consequences of such a discovery — our measurement and monetary systems are based upon the same underlying relationships. Those able to retrieve the meaning of the arbitrary shapes that are numerical symbols have a source of such power that they fail to understand why others do not possess it. However, though such a system may appear to be straight forward for those who understand it, it has taken almost four hundred years to become common in the Western world after being brought to its attention by Simon Stevin (Sarton, 1935). Its acquisition remains a source of difficulty for many. We suggest that if the child's focus of attention is consistently directed towards the active aspects of numerical symbolism, the transformations to be derived from the numeration system will be overwhelmed by the need to count (see Gray, 1994)

Of course now our attention should turn to possible action. One approach at encouraging more flexible thinking used a graphic calculator with a multi-line display retaining several successive calculations for a child to use in a learning experiment (Gray & Pitta, 1997). The experience was found to have a beneficial effect in changing the mental imagery of a child who previously experienced severe conceptual difficulty. Before using the calculator, the child's arithmetic focused on counting using perceptual objects or their mental analogues. After a period of approximately six months use with the graphic calculator, it was becoming clear that she was associating a different range of meanings with numbers and numerical symbolism. She was beginning to build new images, symbolic ones that could stand on their own to provide options that gave her greater flexibility. In contrast to interaction with concrete objects which requires the individual to interpret what is going on, interaction with the graphic calculator offers a system in which the individual could build and test concepts first by observing and then by predicting and testing what happens. The child could directly control the form of presentation. The evidence suggests that if practical activities focus on the process of evaluation and the meaning of the symbolism they may offer a way into arithmetic that helps those children who are experiencing difficulty. At a more sophisticated level it can provide a source to the insights of pattern and relationship which are the characteristics of the notation system. It may be doing an injustice to the learner if the benefits of this are left because actions, external and/or internal, dominate their thinking about arithmetic.

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Eddie Gray
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

Demetra Pitta
Psevdas Primary School
Larnaca
CYPRUS