Subsidising Definitions in a Linear Algebra Course

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Abstract

Within Advanced Mathematical Thinking notions are introduced through definitions that have no related process. Using an undergraduate Linear Algebra course as a frame for discussion, this paper examines the way in which, in response to a perceived need, some definitions may be subsidised by procedural use of properties. Such an approach enables students to achieve success in the academic components of the course. However, for some, this approach restricts opportunities to display mathematical creativity.

Linear Algebra is an introductory course of advanced mathematics at University level. During their pre-university courses mathematics students will have met some components of the course, such as matrix arithmetic and solution of simultaneous linear equations. This will have two consequences:

(i) The fact that the course builds upon already familiar concepts supports the development of a positive attitude by the students.

(ii) The ideas may have been introduced in a procedural way which will encourage students to continue seeking such an approach when reintroduced to these ideas.

This paper presents some of the teaching techniques employed in a first-year Linear Algebra course and how these techniques affect students' learning. Our findings are based on observation of the lectures, analysis of the course notes, detailed consultation with the lecturer and videotaped group discussions with a group of 8 students.

Overview

Most of the contemporary theories concerning the nature of mathematical understanding suggest that there exist qualitative differences in the way that people understand mathematics. Skemp (1976), for example, made the distinction between relational and instrumental understanding, while Hiebert & Lefevre (1986) distinguished knowledge into conceptual and procedural.

To understand a new notion in elementary mathematics students have to undergo a cognitive shift incorporating lengthy procedures in mathematical concepts. This conversion of actions or operations into what Piaget (1945) described as "thematised objects of thought or assimilation" (p. 49) was described by the term *encapsulation* (Dubinsky, 1991).

Cottrill, Dubinsky, Nicholls, Schwingendorf, Thomas & Vidakovic (1996) formulated the APOS theory, from the acronym of the words action, process, object and schema. *Actions* are physical or mental transformations of objects to obtain other objects. When these actions become intentional they are characterised as *processes* which may be encapsulated to form a new *object*. A coherent collection of these actions, processes and objects, linked in some way, is identified as a *schema*. A schema can be reflected upon and transformed and thus result in the formation of a new object.

In elementary mathematics each mathematical notion is almost always introduced through some kind of action or operation on a physical or a mental object. In advanced mathematics though, we usually introduce abstract concepts by presenting their definitions; we have an object upon which we shall build our further operations without having a previous process to sustain our object.

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. … It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions. (Tall, 1991, p. 20)

Mathematical definitions are supposed to be minimal and where possible elegant (Vinner, 1991). This means that a definition does not contain parts which can be inferred by other parts of it, a fact which makes most definitions extremely difficult for students to understand. The result is that most students are unable to build any processes based on these objects and even when that happens they cannot proceed for more than two steps—that is to the next object.

Teaching the Linear Algebra Course

There seem to be two ways of sequencing the contents of Linear Algebra; the computation-to-abstraction approach and the abstraction-to-computation approach (Harel, 1987). The first approach suggests that matrix arithmetic and linear systems should precede vector spaces and linear transformations, in order to enable the students to develop the language and reasoning needed for understanding the more abstract material. The second approach starts with vector spaces and linear maps and then matrices and simultaneous linear equations are treated as applications of the former.

In this particular Linear Algebra course the approach chosen was the computation-to-abstraction one, because the lecturer felt it would be more beneficial to start with already familiar concepts and use them as building blocks for the development of the more abstract notions of vector spaces and linear transformations. Various introductory strategies were used in order to present the new material, such as abstraction –introducing abstract ideas by initially illustrating them by specific examples (Harel, 1987)– and embodiment (Dienes, 1960) –translating definitions and theorems in terms of given situations. The difference between these two processes lies in the timing of the presentation of the particular situation; in abstraction it comes before the concept is defined, whereas in embodiment, it follows the formal definition.

Another strategy used for the introduction of vector spaces, in particular, was consistent with the APOS theory (Cotrill *et al.*, 1996). Starting with vectors in $\Re^n$, objects already familiar to the students, addition and scalar multiplication were defined, initially as actions on these vectors. When these actions were interiorised, along with their properties (the 10 vector space axioms), they became processes, which then were used to form the new object 'vector space'. These notions were later extended to include vector spaces other than $\Re^n$, resulting in the schema of a general vector space.

Discussion

Teaching Linear Algebra in this way has some indisputable benefits. Firstly, students acquire knowledge in a way familiar to them from their pre-university experience; starting from actions and processes and then proceeding to objects. Secondly, they manage to tackle successfully their weekly assignments, which means that they meet the academic standards of the course. And thirdly, and perhaps more importantly, they feel that Linear Algebra is a subject which they understand without having to put in a tremendous amount of effort; a fact that discourages them in most of their other courses.

On the other hand, such an approach does not encourage students to make the essential shift form school to university mathematics, since their focus remains on processes instead of objects. In other words, they are satisfied by being able to 'do' things and they refer to the formal definitions only if they get stuck with the manipulation of the processes. There are also some students who feel that this approach makes Linear Algebra seem too easy for them and they do not find it challenging enough as a subject.

As a final note, we should say that this research is taking place in the Mathematics Department of one of the most demanding British Universities and the students involved seem to have a solid mathematical background, a fact confirmed by their results on some initial examination tests. This particular course though, was designed not only for the Mathematics students but also for students on combined degrees, such as Mathematics and Statistics and Mathematics and Physics, who are considerably weaker. More detailed analysis will reveal whether this method of teaching Linear Algebra is even more beneficial on such less able students.

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