A Perspective on Mental Arithmetic

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Introduction

Curriculum development within mathematics is often predicated on the idea that all individuals pass through the same learning sequence albeit at different paces. Allowing for individual differences there seems to be a general assumption that there is a universality in the learning process. A National Curriculum, for example, will specify the sequences of levels through which children shall grow. Wide demands that children learn the 'basics' implicitly support the notion that we provide practice to confirm "understanding. However, the more we work at remembering procedures the more we are likely to use them. If all of our effort goes into this solution to our problems it is perhaps the case that the more we will have to remember but, the less we may understand.

Physical objects, structural aids and other metaphors have increasingly dominated classrooms in an attempt to make arithmetic more accessible to the majority. Manipulables are often seen as panaceas through which children are led to "understand" what they are doing. However, our experience within mathematics classrooms is dominated by two observations. First, we can see children who appear to make sense of the subject matter no matter what pedagogy is used to stimulate learning. Secondly we see children who even though they may have made some considerable effort to acquire proficiency in the skills fail dismally. Why should this be so? We suggest that one reason for this difference is that children focus on qualitatively different aspects of arithmetical activity.

In the mind

To gain some sense of the way this may be seen in mental arithmetic let us go into a fictional classroom where eight year old children are spending some time doing mental arithmetic. The children we meet are real and at different levels of mathematical achievement. We will try to establish what children of different levels of achievement are doing in their minds–what sorts of imagery supports the child when dealing with mental arithmetic associated with elementary number combinations. To link the notions of achievement and ‘qualitative difference’ with the role of imagery we will make the assumptions that an image is mediated by a description (Kosslyn, 1980).

For a few moments we watch as the teacher asks for responses to elementary number combinations. Some of the children are excited, always trying to catch the teacher’s eye, keen to respond to every question. Some are more reticent responding more infrequently. Scattered around the room we see others who are sitting with deep frowns on their foreheads. They are concentrating on manipulating their fingers. Somewhere from within the room a child gives a correct answer. The finger counters slowly puts down their hands and rest for a moment. They look as if they hope this ‘mental arithmetic’ will soon end but another question follows. After a moments hesitation they once again return to fingers. Glancing around the room we see some who do not appear to be part of the proceedings. We walk across to one of these. It’s James, a quiet nine year old. He too is sitting with a deep frown but gradually the furrows are getting deeper. It is an ‘easy’ question, 4+3 but it is obvious that James is in difficulty.. He tells us that he is trying to count fingers in his head. He knows that others do things in their heads so he tries to do the same. Across the room Michael and Rebecca are eagerly trying to catch the teacher’s eye. They know the answer. “It’s two fours take away one–that’s seven” says Michael. I just know it says Rebecca. The furrows on James’ brow are deeper. He doesn’t seem to realise that the others who are doing things in their heads are doing it differently.

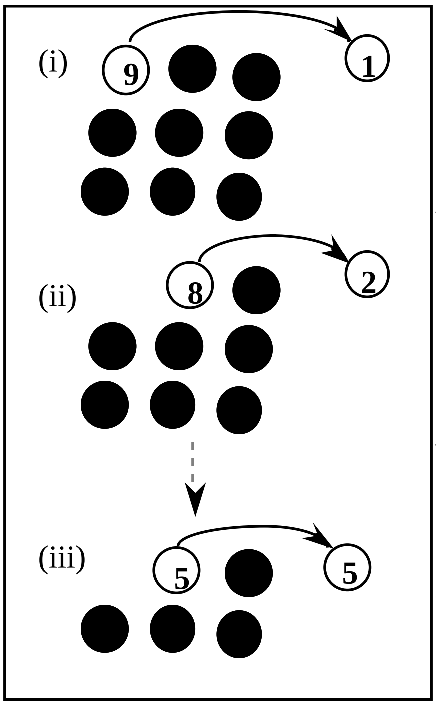

Figure 1

The current emphasis on mental calculation is laudable, but before we over-criticise quality of teaching and replace one pedagogic emphasis with another we should try to make sense of what ‘mental’ means to these children. We stop at John who in his teacher’s view is a ‘low achiever’. He tells us that when he was subtracting 4 from 5 he ‘saw’ a row of five white counters flash up in his mind. Almost instantly these were replaced by a single black counter. “The answer’s one”, he says. We stay with him as he tries 6+3. This time he sees six white counters arranged “like on a dice” and three black counters arranged in a line next to the six. “I have this pattern in my mind so I immediately know it is nine”.

We listen as the teacher gives the next question

"9–5". Simon, another 'low achiever', is sitting quietly with his eyes raised to the ceiling but he does not see it. He is seeing something else; counters in his mind. He is manipulating the counters, eight black ones and one with a nine written on it. They are arranged as in Figure 1. The counter with nine on it moves and the 9 changes to a 1 (Figure 1 (i)). Now there are seven black counters and one with 8 written on it. This moves and the 8 becomes a 2 (Figure 1(ii)). This carries on until he is left with four black counters and one with a five written on it. The five moves but this time the number does not change (Figure 1(iii)). Simon "knew" the remaining pattern, "three and one make four".

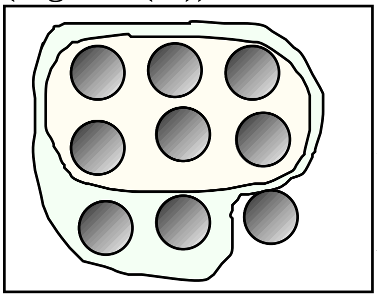

Figure 2

Simon used this nine pattern when he tried 9–8. In his mind he saw nine marbles arranged as before. He told us that he knew what a six pattern looked like "because that was two threes. Add two on to that and that was eight so there was one left". After explaining it he drew the pattern that he saw in his mind as in Figure 2.

By the time that Simon had completed this sum the teacher had moved on to 7+4. Martin was another child sitting quietly. He told us how he added this (Figure 3).

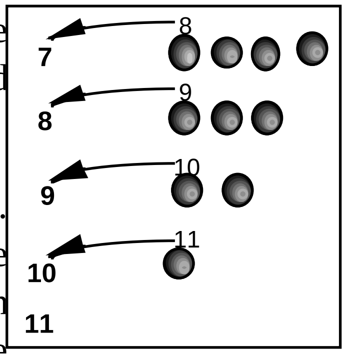

Figure 3

First a "*black*" seven appeared with "*four white balls*". One of the balls had an eight written above it and the eight moved to take the place of the seven which disappeared. There were now three white balls the one nearest the eight having a nine written over it. This now moved to take the place of the eight, and so on. Eventually Martin gave the answer 11.

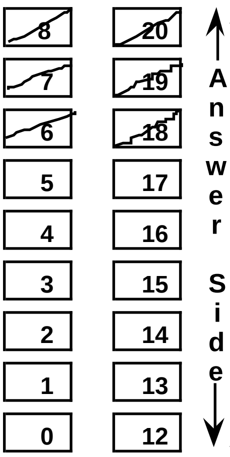

Figure 4

We stayed with Martin as he tried 20–8. He described two 'number tracks' as "*two calculators going around in opposite ways in my head*". The figure '8' and '20', were the first to appear, and then these were "crossed out" to be replaced by '7' and '19'. This process continued until '0' and '12' were reached (Figure 4).

It is interesting what we see amongst these children. All of their descriptions remind us of activities that they have performed with real objects within the classroom. Their descriptions emulate real activities but they are activities that are carried out in the mind.

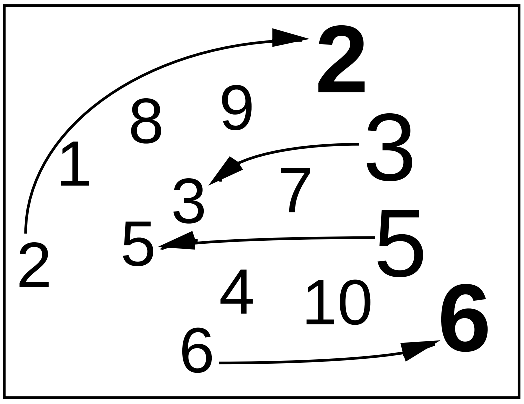

Figure 5

We stop at Emily. She is adding 3+5 (Figure 5)

She describes how all of the numbers are going around in head like in a circle. "*The number I want moves out and I count them. Then they go back and new numbers go out*." In this case it was first the '3' and the '5'. These became "*blacker*" than the other numbers. The three moved back and was replaced by two and the five moved back to be replaced by 6. The '2' then became a '1' and the '6' a '7' and then the '1' became '0' and the '7' an '8'. That is the answer.

Of course not all of the children had furrows across their brows, closed eyes or gazed into the air as they tried to deal with the combinations. John was bright eyed shooting his hand in the air every time a combination was given. He just knew the answer. His friend Aziz was equally excited but he described how numbers flashed across his mind. Adding 3+4 he described how *'"3+4' flash through my mind and I told you the answer"*. When the combination 7+6 was given Andrew, another who always wanted to give answers, told us how 7+7 and 14 flashed through his mind to be followed immediately by 14, 1 and 13 which again flashes away so that he was left with an image of 7+6 and 13.

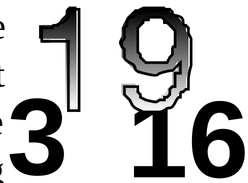

Figure 6

For many of the more able children the responses were almost automatic. They did not see things in their minds but they did seem to talk it over in their heads.. However some of the more able did see symbols. When Andrea was adding 3+16 she described how the 3 and the 16 appeared immediately in her mind as black numbers and then she saw 19 rising out of them (Figure 6).

It is interesting that the teacher thought she was giving children the same thing to do. However, in responding to the combinations children were doing qualitatively different things. Some translated the notions of 'add" and "minus" into counting procedures. Because of the absence of real manipulatives they tried to recreate these in their minds to support a counting procedure. This was of course very difficult and frequently involved double counting but it represented a tried and trusted method which the children have worked hard to learn as a result of their previous experience.

Comment

The mathematics curriculum may be signified by two qualitatively different components–things to *do* and things to *know*. The early emphasis is upon doing. The later emphasis is on knowing and mathematical strength comes from using what is known. There is a duality within this context which is reflected in the use of arithmetical symbols. Symbols signal a process to *do* and a concept to *know*. We see in our class of children that when asked to mentally resolve elementary number combinations this qualitative difference can remain.

Physical actions in simple arithmetic are meant to be the platforms from which children can give meaning to symbols. But to do this there needs to be a qualitative shift in thinking–the action needs to be encapsulated as a concept. Different kinds of mathematical behaviour and different levels of such behaviours are tied to understanding of these concepts. However, just because the concept exists and the child uses it does not mean that a child is fully aware of the relationship between ideas. Though most of the children within our class could eventually provide solutions to the combinations presented by their teacher we see that in the mind they are doing qualitatively different things.

When we reflect upon these differences two things strike us. First we see that children who are having some difficulty are focusing upon a mental representation of real objects and actions with them The objects in the mind are ‘real’ things that have ‘body’ and the actions are analogues of those that may be carried out with real things. The children seem unable to detach themselves from the search for substance—the objects and the actions associated with them are essential to thought. In contrast, many of those children who appeared to be giving ‘automatic’ responses saw ‘flashing’ images of symbols. These appeared to be used as ‘thought generators’. Such children have filtered out the detail associated with a particular procedures—their focus of attention was upon the abstract qualities of the symbols.

Within mathematics much of the curriculum and its associated pedagogy starts from the philosophical position of parallel stages of development in all individuals. Metaphors are seen to act as mediators in the development of concepts; ideas are fragmented into ‘easily’ manageable stages. The assumption is that all children will extract from these experiences that which will enable them to become “experts” in a particular area of mathematics. We suggest that children extract qualitatively different aspects of the metaphor and create within their minds representations which can hinder or support concept development. For some the focus is upon the physical objects and the actions with those objects. Others are able to focus upon the results of those actions expressed as number concepts. The former seem to seek the security of learned procedures, the latter on the development of a flexible arithmetic.

Clearly children are using qualitatively different objects to support their mathematical thinking. On the one hand we see the dominant object being an object of the environment, a countable object. On the other hand we see it as an "object of thought" (Piaget, 1985, p. 49) formed through the cognitive shift associated with procedural compression. With the former the mental representation may frequently be associated with descriptive qualities of the object whilst with the later these properties are ignored until required.

Even within our one class we may see the outcome of the differences. The use of 'real' objects tends to focus attention upon analogues of physical actions which dominate thought. The 'symbolic images' of 'high achievers', appear to act as thought generators. They appear to flash as memory reminder's, momentarily coming to the fore so that new actions or transformations may take place.

We suggest that such differences have over-riding consequences for children's mathematical achievement. The one conclusion that may be drawn for the use of analogical images is that though it would seem to place a tremendous strain on working memory, the children who are appearing to have difficulty with mental arithmetic, even at this elementary level, show an extraordinary use of working memory. We suggest that problem is one associated with memory use and not so much its capacity. Not only is the child maintaining sight of the analogical representation but also trying to focus on discrete numbers in that representation. Children such as James and Emily frequently display external evidence of the strain that is involved in mental calculation. They both use imagery but their imagery does not offer support to the limited space available within short term memory.

The quality of the mental representation formed through mathematical activity is dependant what it is the child chooses to create a representation of. We need to determine who is doing what. Only then may we provide the necessary support for those who at the very start of their mathematical development appear to traverse a cognitively different route.

Kosslyn, Stephen M. (1980). *Image and mind.* Cambridge, Mass: Harvard

University Press.
Piaget, J., *The equilibrium of cognitive structures*. Cambridg MA: Harvard University Press, (1985).