The Nature of the Object as an Integral Component of Numerical Processes[1]

Eddie Gray, Demetra Pitta and David Tall

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL
UK

This paper presents an outline of research studies indicating the existence of qualitatively different thinking in elementary number development, drawing on empirical evidence obtained over the last ten years. Evidence first signalled qualitative differences in numerical processing (Gray, 1991) which was seminal in the development of the notion of procept (Gray & Tall, 1994). More recent studies of the role of imagery in elementary number processing (Pitta & Gray, 1997) indicate that qualitatively different outcomes may arise in the abstraction of numerical concepts from numerical processes because children concentrate on different objects or different aspects of objects which are integral components of numerical processes.

INTRODUCTION

The notion that numerical concepts are formed from actions with physical objects underpins the conceived cognitive development of simple arithmetic (see, for example, Piaget, 1965; Steffe, et al., 1983; Kamii, 1985; Gray & Tall, 1994). These conceptions share common ground. The properties by which the physical objects are described and classified need to be ignored; and attention is focused on the *actions* on the objects which have the potential to create an ‘object of the mind’, which has new properties associated with new classifications and new relationships. For some there may be a cognitive shift from concrete to abstract in which the concept of number becomes conceived as a construct that can be manipulated in the mind. For others, however, meaning remains at an enactive level; elementary arithmetic remains a matter of performing or representing an action.

The focus of our work has been to consider this difference in thinking and its consequences. What is it that children are doing differently and why? Using rich empirical evidence we develop a cognitive theory which may account for these differences. Our research paradigm has focused on extremes of mathematical achievement and examined children’s interpretations of arithmetical symbolism and the associated imagery. Our conclusions highlight the influence of different cognitive styles influencing the nature the object which is an integral component of children’s numerical processing. We suggest that such differences effect the cognitive shift from concrete to abstract thought and will have consequences on children’s numerical development.

A THEORETICAL FRAMEWORK

Piaget (1973, p. 80) believed that the growth of numerical knowledge in the child stemmed, “not from the physical properties of particular objects but from the actual actions carried out by the child on the objects”. He wrote of how the coordination of actions became mental operations—“actions which could be internalised” (Piaget, 1971, p. 21)—and suggested that “actions or operations become thematised objects of thought or assimilation” (Piaget, 1985, p. 49). The formation and meaning of knowledge, within the context of learning as well as in mathematics, stemmed from active thinking and operating on the environment.

Substantial interest in the cognitive development of mathematics has focused on the relationship between dynamic actions and conceptual entities. For some, grammatical metaphors sharpened the subtle changes that form the basis for numerical constructs. Dienes (1960) described how a predicate (or action) becomes the subject of a further predicate which may in turn become the subject of another and so on. The qualitative benefit from making predicates the servant rather than the master of thought were clear:

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians (Dienes, 1960, p. 21)

Using a similar analogy Davis (1984) signalled the qualitative changes associated with actions becoming objects of thought.

> The procedure, formerly only a thing to be done–a verb–has now become an object of scrutiny and analysis; it is now, in this sense, a noun. (Davis, 1984, p. 30)

These distinctions, together with theories accounting for the transformation of processes into concepts have helped to shift attention from *doing* mathematics to *knowing* mathematics. The way in which dynamic actions become conceptual entities has been variously described as “interiorisation” (Beth & Piaget, 1966), “encapsulation” (Dubinsky, 1991), or “reification” (Sfard, 1991). Dubinsky and his colleagues (Cottrill et al., 1996) formulate the encapsulation as part of the APOS theory (action-process-object-schema), in which actions become repeatable as processes which are then encapsulated into objects to later become part of a mental schema. Sfard also indicates the cognitive shift as a three phase process: *interiorisation* of the process, then *condensation* as a squeezing of the sequence of operations into a whole, then *reification*—a qualitative change manifested by the ontological shift from *operational* thinking (focusing on mathematical processes) to *structural* thinking (focusing on properties of, and relationships between, mathematical objects).

Gray & Tall (1994) focused on the role of mathematical symbolism representing either a **pro**cess to *do* or a con**cept** to *know*. To emphasise this dual meaning the term **procept** was introduced. Procepts start as simple structures and grow in interiority with the cognitive growth of the child. The word "concept" rather than "object" was used because terms such as "number concept" or "fraction concept" are more common in ordinary language than "number object" or "fraction object". Furthermore, the term is used in a manner related to the "concept image" consisting of "all of the mental pictures and associated properties and processes" related to the concept in the mind of the individual (Tall & Vinner, 1981, p. 152). In this sense there is no claim that there is a "thing" called "a mental object" in the mind.

Instead a symbol is used which can be *spoken, heard, written* and *seen,* which is capable of evoking appropriate processes to carry out necessary manipulations in the mind of the individual and which can be communicated to share with others.

Theories which refer to the cognitive shift from process to object are process driven, but they form an important backdrop for the theory of procepts. Indeed Anna Sfard's notion of duality (Sfard, 1991) and discussions with her in 1989 were important in its early development. Procepts are dynamic and generative—"things" that are the source of great flexibility and power. The problem in the cognitive context is to identify why some children implicitly seem to recognise this fact but others do not.

A FOCUS ON ELEMENTARY ARITHMETIC

'Encapsulation' theories—and here the one word is used as a matter of convenience—have intrinsic differences but also share common ground in attempting to account for process/object links. Notions such as 'interiorisation' or 'repeatable actions' may lead to quantifiable differences in procedure but not qualitative differences in thinking. Such a distinction is implicit in the finely-grained analysis of counting units of Steffe et al. (1983). Decreasing dependence on *perceptual* material permits children to eventually count *figural* representations of perceptual material; the counting process continues in the absence of the actual items. *Motor acts*, such as pointing, nodding and grasping, that accompany the counting process, can be taken as further substitute units for perceptual items. Dependence on these three forms of unit is further reduced by the realisation that the utterance of a number word, the verbal unit, can be taken as a substitute for countable items that could have been coordinated with the uttered number sequence. However, these changes though quantifiably different, are qualitatively similar—each procedure is an analogue of a fundamental counting process. The concept of unit becomes wholly abstract when the child no longer needs any material to create countable items nor is it necessary to use any counting process.

THE EMPIRICAL EVIDENCE

Theories of encapsulation focus on the manner in which processes are encapsulated as objects. However, the individual's perception of the original objects plays a vital role. Counting starts with objects perceived in the external world which have properties of their own; they may be round or square, red or green or both round and red. These properties need to be ignored if the counting process is to be encapsulated into a new entity—a *number* which is *named* and given a *symbol*. It is our contention that *different perceptions of these objects*, whether mental or physical, *are at the heart of different cognitive styles that lead to success and failure in elementary arithmetic*.

Three themes dominate the empirical studies used in building the resulting theory:

- differing cognitive styles reflected by children's approaches to elementary number combinations when they could not recall solutions,
- process/concept links as represented by the tactics used to carry out elementary computations,
- the nature of any imagery associated with these tactics.

Differing Cognitive Styles

Gray (1991) built on the classification of children's solution strategies for solving addition and subtraction problems (Carpenter et al., 1981; Carpenter & Moser, 1982) which was, in turn built on the classification including count-all (CA), count-on (CO), derived fact (DF) and known fact (KF) (Groen & Parkman, 1972; Groen & Resnick, 1977). This research had two aims: to consider whether the classification in contextual situations could be transferred to context-free ones and to identify fall-back strategies chosen by children when they failed to know a fact—in short, to identify a cognitive hierarchy. Herscovics & Bergeron (1983) had emphasised that any such cognitive hierarchy need not apply to specific individual children. Gray found that the sequence of fall-back strategies revealed a divergence in thinking between different individuals.

The assumption was that if the child preferred to solve numerical problems by remembering the answer (known fact), if this is not known, the most efficient alternative is to use another known fact to derive a solution. Should both these strategies fail, it was assumed that the child will resort to the next preference by counting. Logic, supported by evidence from other work (e.g. Fuson, 1982; Secada et al., 1983; Steffe, et al., 1983) seemed to indicate that the descending order of preference, theoretically available to all children, could be considered as a direction of regression.

Figure 1 represents a model of this regression for addition and subtraction in which count-back (CB) and count-up (CU) are indicated as alternatives (Woods, Resnick & Groen, 1975). It is implicit in this range of strategies that the child's use of counting methods could reveal something more about understanding of counting than the mere use of a procedure to solve the problems. For example, children who use count-all or count-on as dominant strategies for addition often see subtraction as the inverse of these operations. Whilst such strategies are necessary pre-conditions for a child to relate addition and subtraction (Steffe et al., 1983), it was hypothesised that children using either display qualitatively different thinking from those only using one or the other.

The evidence is based on responses to a range of elementary context-free addition and subtraction problems given by 72 children from two schools. Identified by age (7+ to 11+) and achievement ('above average', 'average' and 'below average'), the children's responses demonstrate that the above classifications are suitable for context-free items. The results indicate that some children wish to remain at a procedural level which, in terms of information processing, make things very difficult for them, whilst others operate at a conceptual level which is more flexible. The notion of different cognitive styles leading to a diverging outcomes came from the observation that the less able, who relied extensively on counting procedures, were "*making things more difficult for themselves and as a consequence become less able*" (Gray, 1991, p. 570) whilst in contrast, the ability to "*compress the long sequences [of procedures] appeared to be almost intuitive to the above-average child" (ibid.).*

Process Concept Links and the Proceptual Divide

Drawing upon the children's interpretations of symbolism, the differing cognitive styles evident in this first study were later placed within the context of a proceptual divide between those children who processed information in a flexible way and those who invoked the use of procedures. Those doing the former have a cognitive advantage. They link procedures to perform arithmetic operations with number concepts through cognitive links relating process and concept. Two pieces of evidence seem to support the notion that these differences are manifestations of qualitatively different thinking. The first considers the cumulative responses made by children in the above study to subtraction combinations for which they could not recall a solution. Figure 2 shows the response rate to various number combinations. (The 'known fact' responses are omitted, but implicit, since the responses total 100% in each case.) Though distinct age groups are not identified within this figure the general distinctions are clear to see. The left hand side shows how the high achievers use almost all derived facts and a few examples of counting, whilst the right hand side shows few derived facts and a large percentage of counting. The proceptual divide is clearly shown.

It may be argued that discrete "snapshots" of children fail to take account of the stage theory proposed by Piaget. This suggests that given time all children go on from pre-operational to concrete operational and finally to formal operational thinking. This theory implies that all children should be able to encapsulate counting procedures into numerical objects. Observation within any classroom shows that this is not the case.

The ability to simply recall facts may muddy the theoretical waters. Far more significant is the way in which the children may use the facts they already know to establish those as yet unknown. Recognition of this difference may allow us to distinguish between those facts that are isolated pieces of knowledge and those that are usefully connected to others. Trends pointing to longer term differences were confirmed by a small scale longitudinal study (Gray, 1993).

A group of 29 children (from a different school) were grouped according to their level of achievement in the numerical components of the Standard Assessment Tasks (SEAC, 1992) given to all children within the UK at the end of Key Stage 1 (7+). These tests identify levels of competence normally expected of the "average" seven year old and may also be used to identify children at each extreme of the spectrum of achievement. The children were interviewed individually after the test on a range of context-free elementary number combinations that formed the basis for the test. The same children were interviewed one year later on the same items and on items which reflected their mental approach to two digit addition and subtraction.

The results of the elementary components (figure 3) show how over the two interviews counting procedures, frequently very inefficient, dominated the strategies used by children who did not reach the average standard. In the second phase of interviews the children's counting approaches were sufficiently robust to cope with all combinations to ten but they remained unreliable for combinations to twenty. There is an extensive use of derived facts by those who achieved an above average level of achievement.

Although as teachers we often ask children what, to us, appears to be the same question, various children may interpret it very differently. The expression 4+3 actually signals children to do different things. To some it is a concept to know. To others it is a process to do. It is conjectured that such differences may be manifestations of different stages of cognitive development in various children:

> When a procedure is first being learned, one experiences it almost one step at a time; the overall pattern and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity–it becomes a thing. It, itself is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. (Davis, 1984, p 29–30)

However their ‘permanency’ may also a reflection of different cognitive styles reflected in, for example, the cognitive shift associated with encapsulating the process of addition as the concept of sum. Within figure 4 we see this as the result of the qualitative compression of the lengthy count-all procedure into the shorter one that is count-on. The evidence seems to suggest that different cognitive styles may lead to the bifurcation in thinking that is evident in the proceptual divide.

The common pedagogical approach to numerical processes builds on the belief that number development should commence with enactive approaches and that, given sufficient time, all children will encapsulate arithmetical processes into numerical concepts. The existence of a proceptual divide would seem to indicate that this is *not* the case and even when teaching programmes have been designed to shift the lower achievers focus from processes to thinking strategies (see, for example, Thornton, 1978) lower achievers resist a change from the security offered by their well known counting procedures. Further, we conjecture that positive efforts to make the relationships implicit in proceptual thinking explicit to those that do not have the associated flexibility run the danger of being seen by some as a new set of procedural rules.

So what causes the proceptual divide? We may conjecture that pedagogy may account for it in some degree. There does exist a certain ‘conspiracy’ between pedagogue and learner which is manifest in the belief that being shown how to do something solves current difficulties (see, for example, Skemp, 1977). We conjecture that one cause of the proceptual divide is the qualitatively different focus of attention which, on the one hand places the emphasis upon concrete objects and actions upon these objects, and on the other on abstraction and the flexibility intrinsic within the encapsulated object. Why is it that some children seem to implicitly recognise this power but others do not?

IMAGERY AND ELEMENTARY ARITHMETIC

To gain a partial answer to this question our attention turned to imagery. Our fundamental thesis was that different qualities of mathematical abstraction were influenced by the child’s cognitive style and that the relationship between achievement and qualitative difference

may be determined by considering:

- the nature of the object that was dominant in children’s imagery
- the way imagery is used within elementary arithmetic.

Psychological research has identified the importance of imagery in cognitive development and children use it more in their thinking than adults (Kosslyn, 1980). Its role in the child’s thought processes, cause it to have far-reaching consequences on children’s concepts and reasoning (Bruner, Oliver & Greenfield, 1966; Piaget & Inhelder, 1971) and therefore images place major constraints on cognitive processes.

The relationship between different forms of representation may be seen through the presentation and solution of arithmetic facts (Deahenne & Cohen, 1994) and in the context of arithmetic mental representations of the objects will effect mental operations. (Gonzalez and Kolers, 1982). Children's internal representation of numbers are often highly imaginative and unconventional and built up over time (Thomas, Mulligan & Goldin, 1995) but the possession of an image of a mathematical idea implies that the individual does not need actions or the specific instances of image making (Pirie & Kieran, 1994). However, they may be eidetic in the sense they can be visual representations of previously scanned material (Leask, Haber & Haber, 1969) and fully formed from something presented (Mason, 1992) though classification of this phenomena is a problem (Gregg, 1990).

To associate the notions of achievement and 'qualitative difference' with the role of imagery we make the assumption that an image is mediated by a description (Kosslyn, 1980; Pylyshyn, 1973). Following Pylyshyn we make the assumption that the representation conforming to an image is more like a description than a picture. The classical notion is usually of a visual image–though images can be formed from other modalities—which appears to have all of the attributes of actual objects or icons. In the context of numerical development Seron et al. (1992) suggest that images of quantities directly represented by "patterns of dots or other things such as the alignment of apples or a bar of chocolate" (p. 168) may be deemed as analogical.

Methodology

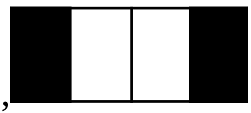
Paivio (1991) suggested that the generation of an image promotes the development of a trace in the brain that integrates the separate components of the item in question. Accessing a part of the information encoded in memory prompts the retrieval of all other pieces of information contained in the image (Woloshyn, Wood & Pressley, 1990). To gain a sense of the nature of children's imagery associated with both concrete and abstract objects and the relationship this may have with mentally processing elementary number combinations, 24 children, selected to represent the extremes of ability, 'low achievers' and 'high achievers', across four age groups, 8+ to 12+, were first asked to respond to auditory and visual items and then asked to provide mental solutions to a series of elementary arithmetic combination in addition and subtraction.

The research methodology used semi-structured clinical interviews (see also Gray & Pitta, 1996; Pitta & Gray, 1996). Items which prompted discussion were presented in a way that gave the interviewees the freedom to follow their own inclinations. Data from each individual was collected in a variety of ways including records of achievement and teacher assessment. The initial selection of children was made from full class records. Each individual interview was audio and video taped and subsequent transcriptions formed the basis for response classification. When responding to each item within the auditory and visual sections, children were asked to provide a first notion of ‘what came to mind’ when they first heard or saw the item. They were then given 30 seconds to talk aloud about the item in question. For auditory items, children were also asked to explain the item so that an extra-terrestrial may understand what it was.

The items within the auditory section contained words such as ‘ball’, ‘car’, ‘triangle’, ‘five’, ‘fraction’ and ‘number’; in the visual section were icons representing ‘two quarters’, a ‘dancing man’, geometric shapes forming a ‘house’, and so on, and symbols such as ‘5’, and ‘3÷4’.

Qualitative differences in interpretation

Though there are a wide variety of conclusions that may be drawn from each item, the analysis of the results indicates that similarities in the children’s descriptions of imagery are remarkable both for their consistency across the range of items, and for the differences they displayed between the ‘high achievers’ and the ‘low achievers’.

When responding to the auditory items, the ‘low achievers’ tended to highlight the descriptive qualities of items in strongly personalised terms, qualities also evident when the children responded to the visual items. However, there was a tendency to associate these items with a story in the sense that they were seen as pictures that required colour, detail and a realistic content. In contrast, ‘high achievers’ concentrated on the more abstract qualities within both series of items. Though they initially focused on core concepts, they could traverse at will a hierarchical network of knowledge from which they abstracted these notions or representational features. An overall summary of the analysis of the children’s responses to the auditory and visual items is given in table 1. Each item triggered ‘low achievers’ to provide descriptions which were qualitatively similar whereas ‘high achievers’ used each comment to trigger a qualitatively different comment. For example, amongst the ‘low achievers’ the notion of ‘ball’ consistently evoked images from boys of ‘scoring a goal’ or playing in the school team whilst the visual item ‘two quarters’, , was frequently identified as curtains on a window and the detail was added by describing the curtains as “flowery”, “rose coloured” or “green”.

Responses to symbolic items (table 2) bore striking similarities to those with words or icons and reflected the degree within which the children were involved with the abstract qualities of the objects. The higher the involvement, the more the child was able to talk about the items at an impersonal level. On hearing the word ‘five’ or ‘half’, ‘high achievers’ often referred to the symbol using phrases such as “it is” to illustrate semantic aspects of the object. For example, the word ‘five’ drew responses such as “it is two plus three, one hundred take away ninety five”, or “it is prime because it is only divisible by one and five”. This does not mean to say that they did not attach qualities arising from episodic memory, such as “I had five candles on my cake for my fifth birthday”; high achievers’ were able to do both. On the other hand ‘low achievers’ almost always displayed examples of episodic memory, concretised the item, “I have five fingers”, or associated its use with some arithmetical action such as counting.

Comparing table 2 with table 1 indicates how imagery associated with non-arithmetical objects may carry similarities with imagery of named arithmetical objects and symbols. Such similarities may be summed up by concluding that images of the low achievers are episodic and active, whilst those of the high achievers are semantic, and generative. We use the terms ‘episodic’ and ‘semantic’ to draw a distinction between images arising from memory associated with the recollection of personal happenings and events and images associated with organised knowledge having meaning and relationships. The former is based upon access to previous experience, the latter no longer depend on learning episodes that provided the basis for knowledge (see Tulving, 1985).

The qualitatively different responses to the words, icons and symbols indicates that the ‘low achievers’ were reluctant to reject information and, if there was little to describe, they created it by establishing stories around the items using images from their known physical world, often as participants in the image, elaborating the detail whenever it seemed that such embellishment was required. In some instances they drew upon one image which acted as a symbol, for example, “my football”, “my mother’s car”. The objects referred to were invariably real, quantifiably different, but qualitatively the same. In contrast, ‘high achievers’ filtered out the superficial to concentrate on the more abstract qualities of the items. Though they focused on real world concepts, they were also able to relate to a hierarchy of ideas which allowed them to refer to objects in the abstract by using qualitatively different notions or representational features.

Images in Elementary Arithmetic

Such differences became marked when images associated with children’s responses to the range of elementary number problems were considered. Again ‘low achievers’ tended to concretise and focus on all of the information. Symbols were translated into numerical processes supported by the use of figural objects that possess shape and in many instances colour. Transformation of numerical symbols into mental analogues of physical objects took two general forms. The first was as ‘dots’ or ‘marbles’ (see figure 5, adapted from Pitta & Gray, 1997). The use of such objects frequently involved mental processes akin to subitising. As may be seen from the diagrammatic representations arrays of dots supported the mental activity but in all instances where they were described such images were limited and there was no evidence of their use on numbers in excess of ten.

Frequently ‘low achievers’ reported imagery strongly associated with the notion of number track although the common object which formed the basis of each ‘unit’ of the track was derived from fingers. In some instances children report seeing full pictures images of fingers, in others it was ‘finger like’. The essential thing is that the object of thought was ‘finger’ and these invoked a double counting procedure. Other images used to support double counting were dynamic invoking actions. Figures 6 and 7 indicates diagrammatic copies of representation given by a nine-year-old and an eleven-year-old. These are associated with the solutions to 9–5 and 7+4 .

In the first we see the dynamic image that grows from a pattern of nine. The procedure used was count-back and as each counter was counted it was moved and assigned a new numerical value. When the count back of five had been completed the child knew from the pattern “that 3 and one makes four”.

Within figure 7 we see how each phase of the solution procedure evolved from the previous one. First the “black” seven appeared with “four white balls”. One of the balls had an eight written above it and the eight moved to take the place of the seven which disappeared, There were now three white balls the one nearest the eight having a nine written over it. This now moved to take the place of the eight, and so on.

Such images were essential to the action; they maintained the focus of attention. The objects of thought of the ‘low achievers’ were analogues of perceptual items that seemed to force them to carry out procedures in the mind, as if they were carrying out the procedures with perceptual items on the desk in front of them. When the image failed they used the real items. For these children mathematics involved action and to carry out the action they used real things.

Symbolism enables us to utilise short term memory to better effect but the differences between the ‘low achievers’ imagery associated with symbolism and that described by the ‘high achievers’, was stark. It is here that we may see clearly the ‘low achiever’s’ inability to filter out information thus providing the contrast between their uneconomical use of memory and the ‘high achievers’ economic use. Here, we should explain that we use the word ‘economic’ not simply to illustrate differences in the detail but also in arrangement as well as quality.

Symbolic images played considerably less part in processing for ‘low achievers' than they did for 'high achievers'. It was also reported far less than analogical images.

Figure 8 shows an eight-year-old child's diagrammatic representation of imagery associated with 3+6. The child described all the numbers going around in his head in circles. "The number I want moves out and I count them. Then they go back and new numbers go out." In this case it was first the '3' and the '6'. These became "blacker" than the other numbers. The three moved back and became four and the six moved back and became 7. For this child such imagery was only associated with number combinations to ten. For the other numerical items perceptual units were used.

The notion of "spinning" seemed to be a common feature of the 'low achievers' descriptions, implying that images remained for some time and possessed movement. Even when adding 2+1 a nine year old reported seeing all of the operation symbols "spinning around on one side and a big black 3 on the other". In some instances images were associated with approximation. When adding 6+3 another nine-year-old reported seeing "a jumble of numbers with 8 and 9 standing out because they are near the answer." This was a similar response to that given by a twelve-year-old who, when doing the same combination reported an image that consisted of 3,6,9,12,15, and 18. "All the numbers were in the three times table". Whilst the "three and the six stayed there because they were part of the nine, the twelve, fifteen and the eighteen just fall away."

The use of symbolic imagery amongst 'high achievers' was far more economical. The word "flashing" dominated their descriptions instead of "spinning". Images came and went very quickly. "I saw '3+4' flash through my mind and I told you the answer", "I saw a flash of answer and told you." It was not unusual for the children to note that they saw both question and answer "in a flash", sometimes the numerical symbol denoting the answer "rising out of" the symbols representing the question. In instances where children reported the use of derived facts it was frequently the numerical transformation that 'flashed'. For instance when given 9 + 7 one eleven year old produced the answer 16 accompanied by the statement. "10 and 6 flashed through my mind."

Discussion

Clearly the quality of imagery generated differs considerably. On the one hand we see the dominant objects being either physical, such as fingers and counters, or figural representations of physical items. On the other we see it as an object of thought. 'Low achievers' concentrate on analogues of physical actions, and where they use symbolism they continue to carry out actions associated with such analogues. Their images are not so much associated with "knowing" mathematics but with "doing" mathematics. In contrast the symbolic images of ‘high achievers’, appear to act as though generators. They appear to flash as memory reminder’s, momentarily coming to the fore so that new actions or transformations may take place.

The evidence that comes from children’s imagery associated with elementary arithmetic combinations is given in table 3. Comparison with table 1 and 2 clearly shows the similarities and differences between the two groups of children over the range of items that formed the basis for comparison. We once again see the tendency of ‘low achievers’ to concretise and focus on all of the information. Imagery in the numerical context is strongly associated with procedural aspects of numerical processes. The children carry out procedures in the mind as if they were carrying out procedures with perceptual items on the desk in front of them. ‘High achievers’ appear to focus on those abstractions that enable them to make choices. Their ability to reject information is again apparent. We suggest that such differences have overriding consequences for children’s mathematical achievement. The one conclusion that may be drawn for the use of analogical images is that it would seem to place a tremendous strain on working memory. Gear et al. (1991) have suggested that a component of developmental difficulties in mathematics is a working memory deficit. We would suggest that on the contrary these low achievers show an extraordinary use of working memory. Their problem is one associated with its use and not its capacity. Not only is the child focusing on the representation but also on discrete numbers in that representation.

The ability to filter out information and see the strength of such a simple device as a mathematical symbol appears to be confined to the high achievers. The evidence suggests that children who are ‘low achievers’ in mathematics appear unable to detach themselves from the search for substance and meaning—no information is rejected, no surface feature filtered out.

We believe that this has serious consequences which contribute to the formation of the proceptual divide. The notion of procedural compression and the interiorisation of mathematical processes is strongly embedded in the literature. Interpretations of Piagetian notions that enactive approaches will form a foundation for procedural encapsulation may be associated with Bruner’s (1968) view that past experience may be conserved through such enactive approaches. Of course, whilst the latter must also be seen within the context of iconic and symbolic conservation, it would seem that far from ‘encapsulating’ enactive interpretations of arithmetical processes, the ‘low achievers’ are mentally imitating them.

The quality of image formed from enactive approaches is dependant upon what it is the child chooses to create an image of. This will influence the use to which the image is put. It is conjectured that this will not only have consequences for the quality of the action that is taken into consideration but it will also affect the quality of the object which dominates the child’s imagery.

It would seem reasonable that if some children concentrate on actions with physical objects and work hard to develop competence with these actions the more they are likely to use them.

Such considerations add a new quality to the notion of proceptual divide, one that is so strongly associated with image formation that it is possible that children's interpretations of mathematical actions may be strongly influences by their interpretations of their real world. In early mathematics children are faced with not one but *two* interpretations of their interaction with externally perceived objects. On the one hand it is the identification of the qualities of objects that arise from manipulation and perception which lead eventually to the development of geometrical concepts. On the other, though perception and manipulation are the dominant actions, it is the cognitive shift associated with the result of these actions that brings about the development of numerical concepts. The objects that are the catalysts for both strands of development are the same but the conceptual development is different. We believe that this has serious implications for pedagogy. Early years within school are dominated by enactive methods in the belief that given the appropriate experience all children will "encapsulate" arithmetical processes to form arithmetical concepts. *Observation within any classroom shows that this is not the case.* Children may be focusing on *different* aspects of their experience. For some the dominant focus is on objects and the actions *on those objects*, others are able to focus more flexibly on the results of those actions expressed as number concepts. The former may seek the security of counting procedures on objects rather than the longer-term development of flexible arithmetic. We need to determine which, so that we may provide the necessary support both to those who develop flexibly and also to those who, at the very start of their mathematical development, appear to be travelling a cognitively different route.

REFERENCES

Beth, E. W. & Piaget, J., (1966). *Mathematical Epistemology and Psychology*. (W. Mays, trans.). Dordrecht: Reidel.

Bruner, J. S., (1968). *Toward a theory of instruction*. New York: W. W. Norton & Co.

Bruner, J. S., Oliver, R. O., & Greenfield, P. M., (1966). *Studies in cognitive growth*. New York: Wiley.

Carpenter, T. P., Hiebert, J. & Moser. J. M., (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education* 12, 27-39.

Carpenter, T. P. & Moser. J. M., (1982). The development of addition and subtraction problem solving skills. In T. Carpenter et al. (Eds.) *Addition and Subtraction: A Cognitive Perspective,* (pp. 9-24). Laurence Erlbaum: New Jersey.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: beginning with a coordinated process schema, *Journal of Mathematical Behavior*, 15. 167–192.

Davis, R. B. (1984). *Learning mathematics: the cognitive science approach to mathematics education*. Norwood, NJ: Ablex.

Dehaene, S., & Cohen, L., (1994). Towards an Anatomical and Functional Model of Number Processing, *Mathematical Cognition,* 1, 1, 83–120.

Dienes, Z. P. (1960). *Building up mathematics*. Hutchinson Educational: London.
Dubinsky E., (1991). Reflective Abstraction. In D. O. Tall (Ed), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers. (1991).
Fuson, K. C., (1982). An analysis of the counting-on solution procedure in addition. In T. Carpenter et al. (Eds.), *Addition and Subtraction: A Cognitive Perspective*. Laurence Erlbaum: New Jersey.
Gonzalez, E.G. & Kolers, P.A., (1982). Mental manipulation of arithmetic symbols. *Journal of Experimental Psychology. Learning, memory and cognition*. 8, 308-319.
Gray, E. M., (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational studies in mathematics*, 22: 551–574.
Gray, E. M.& Tall, D. O., (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 25, 2, 115–141, (1994)
Gray, E M. & Pitta, D., (1996). Number Processing: Qualitative differences in thinking and the role of imagery. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol. 4, pp 155–162. Valencia: Spain.
Gregg, V., (1990). Eidectic Imagery. In M. W. Eysenck, (Ed.), *The Blackwell Dictionary of Cognitive Psychology*. Blackwell: Oxford.
Groen, G. J. & Parkman, J. M., (1972). A chronometric analysis of simple additio', *Psychological Review*, 79, 329-343.
Groen, G. J. & Resnick, L. B., (1977). Can preschool children invent addition algorithms?, *Journal of Educational Psychology*, 69, 645-652.
Herscovics, N. & Bergeron, J. C., (1983). Models of Understanding, *Zentralblatt für Didaktik der Mathematik*, 83, 75-83.
Kamii, C., (1985). *Young children reinvent arithmetic*. New York: Teachers College Press.
Kosslyn, S. M., (1980). *Image and mind*. Cambridge, Mass: Harvard University Press.
Leask, J., Haber, R. N., & Haber, R. B. (1969). Eidetic imagery in children: II Longitudinal and experimental results. *Psychonomic Monograph Supplements*, 3, 25–48.
Mason, J. (1992). Doing and construing mathematics in screen space. In B. Southwell, B. Perry and K. Owens (Eds.), *Proceedings of the 15th Annual Conference of the Mathematics Education Research Group of Australasia*, 1–17. University of Western Sydney: Nepean, Australia
Paivio, A. (1991). Dual coding theory: retrospect and current status. *Canadian Journal of Psychology*, 45, 255–287.
Piaget, J. & Inhelder, B., (1971). *Mental imagery in the child*. New York: Basic.
Piaget, J., (1965). The *child's conception of number,* (C. Gattengo & F. M. Hodgson, trans.), London: Routledge & Kegan Paul.
Piaget, J., (1971). *Genetic Epistemology*, (E. Duckworth, trans). New York: Columbia University Press.
Piaget, J., (1973). Comments on mathematical education, In A. G. Howson (Ed.), *Developments in Mathematical Education: Proceedings on the Second International Conference on Mathematics Education*. Cambridge, Cambridge University Press.
Piaget, J., (1985). *The Equilibrium of Cognitive Structures*. Cambridge MA: Harvard University Press.
Pirie, S., & Kieran, T., (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational studies in mathematics*, 26, 165–190.
Pitta, D. & Gray, E. M., (1997). In the Mind … What can imagery tell us about success and failure in arithmetic? In G. A. Makrides (Ed.), *Proceedings of the First Mediterranean Conference on Mathematics*, pp 29–41. Nicosia: Cyprus.
Pitta, D. & Gray. E. M., (1996). Nouns, Adjectives and Images in Elementary Mathematics. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol. 3, pp 35–42. Valencia: Spain.
Pylyshyn, Z. W., (1973). What the mind's eye tells the mind's brain. *Psychological Bulletin*, 80, 1–24.
Schools Examination and Assessment Council, (1992). *Standard Assessment Task, Pupil Sheet Booklet, Key Stage 1*. Lincoln: NFER-Nelson.

Secada, G. W., Fuson, C. K. & Hall, J. W., (1983). The transition from counting-all to counting-on in addition, *Journal for Research in Mathematics Education*, 14, 47-57.
Seron, X., Pesenti, M., Noel, M-P., Deloche, G., & Cornet, J-A., (1992). Images of numbers, or “When 98 is upper left and 6 sky blue”. *Cognition*, 44, 159-196.
Sfard, A., (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, 1-36.
Skemp, R. S., (1976). Relational understanding and instrumental understanding, *Mathematics Teaching*, 77, 20-26.
Steffe, L., von Glaserfeld, E., Richards, J., Cobb, P., (1983). *Children’s Counting Types: Philosophy, Theory and Applications*. Preagar: New York.
Tall, D. O. & Vinner, S., (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.
Thomas, N., Mulligan, J., & Goldin, G. A., (1996). Children’s representation of the counting sequence 1–100: Cognitive structural development. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol. 4, pp 155-162. Valencia: Spain.
Thornton, C. A., (1978). Emphasizing thinking strategies in basic fact instruction, *Journal for Research in Mathematics Education*, 9, 214-227.
Tulving, E., (1985). How many memory systems are there? *American Psychologist*, 40, 385–398.
Woloshyn, V., Willoughby, T., Wood, E. & Pressley, M. (1990). Elaborative interrogation and representational imagery facilitate adult learning of facts presented in paragraphs. *Educational Psychology*, 82, 513–524.
Woods, S. S., Resnick, L. B. & Groen, G. J., (1975). An experimental test of five process models for subtraction, *Journal of Educational Psychology*, 67, 17-21.