CHAPTER 5

The Place of Counting in Number Development

Arithmetical Beginnings

In most infant classrooms during a mathematics lesson we can see some form of counting. Children count counters, pencils, shapes, marbles, acorns, toys–anything. If it is countable it is counted. Counting is one of the actions young children participate in using everyday objects from the real world. Objects from this world can be seen. On the one hand they are manipulated, named and described, providing the initial phases in the development of geometry, on the other they can be quantified, giving the initial phases in the development of arithmetic.

Geometric growth stems from handling real world objects. They are seen and manipulated, conceived of visually and holistically as they appear to the senses, and named. It takes time for the regularities of these shapes to be described in a more subtle verbal form, for their properties to be identified and relationships based on those properties to be established. Arithmetic also has a physical counterpart originating in the real world, so it has visual elements but it is possible for these elements to change. Actions on objects of the real world form objects that are part of the arithmetical world. However, whereas perception and manipulation lead to the gradual accommodation of geometrical concepts the formation of numerical concepts is far more subtle. It involves a shift in attention from the objects of the real world to the objects of the arithmetical world– numbers and their symbols.

The part that counting has to play in the shift of attention is the focus of this chapter. We will see how actions upon objects of the real world may be steadily compressed to form objects of the arithmetical world. Children who utilise the underlying strength of these new objects have a source of flexibility and power which provides them with a stepping stone towards further mathematical growth. Those who do not become trapped within the complexity of the actions and left bewildered as arithmetic and mathematics generally becomes ever more complex.

Counting plays a sophisticated and central role in the development of number concepts. But how is this done? If it is so fundamental why is it that well into Key Stage 2 we may see some children still relying extensively on it to add and subtract –even to establish multiplication facts? Perhaps it seems natural that if so much energy and time is expended upon the development of sound counting skills within Key Stage 1, some children within Key Stage 2 appear to be reluctant to use alternative approaches. The more we work at remembering how to do something the more we are likely to use the remembered approach; it is perhaps the case that the more we remember how to *do,* paradoxically, the less we may *know*.

"It will help you if you count".

It will prove fruitful to distinguish between the terms "process" and "procedure" which will be used extensively in the chapter. The word "process" is used in a general sense, as in the "process of counting", the "process of addition", or the "process of subtraction". It need not be something that is currently being carried out in thought or by action, for example we may speak of the process of addition without actually performing it. Nor is there any implication that the process must be carried out in a unique manner. For instance, the process of addition may be carried out by counting or by some other method. The term "procedure" is used to describe a specific algorithm for implementing a process. Flexibility in carrying out a process will play a fundamental role in our story. Within simple arithmetic counting may stimulate the growth of such flexibility but it may also inhibit it. How may such a contraction arise? What is it about counting that may cause such a paradox?

Try to visualise James. He is a small five year old sitting amongst a group of similarly aged children carrying out counting activities. His voice carries above the others within the group "... five, six, **seven**". There is silence as he then quietly writes. A closer examination of what he is doing reveals that he was adding 3 + 4 using his fingers to count-on from four. Now think of Joseph who is eight. He is also trying to add 4+3. He is sitting motionless but his lips are moving and, looking closely, we can see his eyes moving slowly from left to right. The lip movements stop and then start again. His eyes repeat their movement. This happens several times with always a little more tension evident in the deep frown on Joseph's face. Eventually

Joseph's teacher interrupts his concentration. "*Use counters or your fingers Joseph. It will be easier.*" There is a some relief on Joseph's face as he goes for the second option without making it too obvious–he uses his fingers under his desk. Why? He explained later that he "*...wants to do things like the clever children. They do it in their heads*". Counting on his fingers under the desk helps him continue his subterfuge: it still looks as if he is doing things like the "clever children". The teacher shares his secret. It is almost as if they have entered into a conspiracy–the current difficulty has been sorted out because he has been told how to *do* the sum using an easier approach. On the next combination however we begin to see the lips move and the eyes roll. Again his teacher very quietly intervenes and suggests that he should use his fingers "*...it will help you understand what you are doing*".

Two children, both counting, but there is a tremendous difference in the quality of this counting over such a spectrum of age. The five year old is experiencing counting as part of a programme of conceptual development which may eventually give him choices. Joseph and children like him are counting because they are unable to do anything else–they have no choice. Faced with the problem such as 4+3 they translate it into a counting action. They have had plenty of experience doing this: an addition and subtraction signs means count, albeit a different sort of counting. It can take so long to do the counting and get an answer that such children may not remember the numbers they started with. Once again they have practised a counting procedure. Simple arithmetic is about counting. Joseph didn't realise that when the other children were doing things in their heads they were using methods that are far easier than his. Even when he has difficulty, he is advised to try an approach that is harder than theirs. That this is so was aptly explained by Amanda, who is nine,

> "I find it easier **not** to do it [simple addition]with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. It can take longer to work out the sum than it does to work out the sum in my head."

What insight from a child who herself is having difficulty with arithmetic. She focuses on the two features emerging from elementary arithmetic that could create a dichotomy: being able to *do* and *think* at the same time. Doing the first may get in the way of the second but within the classroom environment there can be clear difficulties in making the choice:

> "If we don't [use our fingers] the teacher is going to think, 'why aren't they using their fingers.....they are just sitting there thinking'...we are meant to be using our fingers because it is easier....which it is not". (Amanda, age 9)

The dual face of numbers

Through their experience of a counting procedure young children learn to associate a counting action with a sequence of number words. The last number word tells them how many things have been counted and, with familiarity, the child can use this number word to stand for countable items (see Maclellan, Chapter 4, this volume). So, when counting five objects the child points to each object in turn and repeats the sequence of number words "one, two, three, four, five". The final five tells them that "five" have been counted. Later when they hear the word "five" or see the symbol "5" they may associate these things with five countable items.

One of the interesting things about numbers and, no doubt one that we all know but do not make explicit to the children we teach, is that we treat them as if they were real 'things'– "five is half of ten", "three and four make seven". Though there is no need to associate these things with other, real life, objects to make sense of them, we can do so if we wish.

Early counting experiences associate numbers with things. By doing so it is possible that for the learner the notion of *seven* lines may have qualitatively little difference to the notion of *straight* lines. Of course there is one difference, the first is the result of an action and is arithmetical, the second the result of perception and is geometrical. However, the word *seven* and the word *straight* may be associated with the objects to which they refer, they may be properties of the set of lines. To continue associating numbers with other objects is very limiting. To say "my sister is *seven*" may have a similar quality to saying "my sister is *tall*". The seven is concretised by being associated with real items: the numbers are used as adjectives and associated with other nouns. Real power in arithmetic derives from not only being able to see this but to also see the numbers as nouns – as ‘things’. When we do this we can establish relationships between the ‘things’, something even young children after their initial experience with counting can do. They will tell us that the things can be seen differently. Nicky (5) looked at the two expressions 2 + 1 and 1 + 2 and said “*See those two, they are both the same, they’re both three*”. Paul, his friend, thought he could give all of the numbers that make five. He started by saying “*Two things together make a number*” and quickly he reeled of examples that make five, “*...two and three, four and one, five and nothing*”. He had some trouble with one and four but decided he could do it if he counted.

Within these two examples we see two different but complementary aspects of numerical symbols whether spoken or written. They refer to countable items and they compress ideas that allow us to see other relationships. Those that recognise symbolic ambiguity which either triggers the re-creation of the number through a counting process or its use as a thing have a very powerful tool at their finger tips. Those that don’t, like Joseph, are more unfortunate–they only have half of the key available to them. Unfortunately, this is the more difficult half.

Compressing counting procedures

Children’s growing sophistication in handling counting procedures may be seen as an example of a steady compression which can eventually permit choice between these and the use of number concepts. We can see this by considering the relationship between the addition process and the concept of sum as seen in the addition of 4 + 3.

The most elementary method used to carry out the addition process is to count 4 objects, (1, 2, 3, 4,) then to count 3 objects, (1, 2, 3,) then to put all the objects together and count the total, (1, 2, 3, 4, 5, 6, <u>7</u>). This succession of three separate counting procedures is called “count-all”.

The next stage occurs when it is realised that it is not necessary to count a set of four followed by a set of three. One of the numbers, 4 for example, may be seen as a number object and the child can simply *count-on* a further 3 numbers in the number sequence. The sum of 4+3 becomes, <u>4,</u> 5, 6, 7. We may see “count-on” as another procedure used to carry out the process of addition. It may be spontaneously constructed and “invented” by children (Baroody & Ginsburg, 1986), “personalised” (Gray, 1991), or “taught” (Fuson & Fuson, 1992).

An important aspect of the two counting procedures used to carry out the addition process is the procedural compression signifying a change from lengthy procedures associated with count-all, to the more contracted ones of count-on. However, the move from one to the other is not as simple as it seems. Count-on is a sophisticated double counting process. To calculate 4+3 requires not only counting on beyond 4 in the number sequence but also keeping a check that precisely three numbers are being counted. In the infant class we may see some children using counters and others using fingers. Sometimes the only evidence that the child is counting comes from the close observation which shows a nodding head, moving eyes or moving lips. Joseph tried this but he couldn’t easily keep two numbers in focus at once.

Of course we shouldn’t pretend that count-all and count-on are the only classifications which describe the steady compression of counting procedures. Steffe *et al* (1983) provide us with a clearer picture of the things children create when they count; they indicate a growing sophistication in the objects used. Children may even change the numbers around, start with the largest “*because it is nearer the answer*” and count-on the smallest. Baroody and Ginsburg (1986) see this distinction not only as an important step on the way to learning more formal arithmetic but, they suggest, it also provides an indication of the child’s efforts to reduce the number of steps and the time used to carry out the process. Quite clearly then it is possible to provide even finer gradations than those that form the focus of this chapter. Recognising them may provide insight into the way in which children’s thinking is developing. Fixation on any one may provide a longer term prognosis of the child’s achievement in arithmetic and in mathematics as a whole.

It is not our purpose to dwell on the finer details but to provide a coarser analysis which may be useful within the classroom and provide a sense of where such procedures may lead to. It is the notion of “compression” which helps us do this. Whichever form of classification we use, the coarse one or the finer gradations which give a more detailed picture, we see that with experience children compress lengthier procedures into shorter procedures. The procedure may not only be quicker but it is suggested that its operation also uses up less memory space and makes it more possible to directly link the inputs to the outputs–to *know* the solution.

Directly linked to 7, the sum of 4+3 becomes a *known fact*. In any isolated incident it is not easy to distinguish whether or not such facts are meaningful or rote learned. The difference may only become apparent when such facts are decomposed and recomposed to give "*derived facts*" (see Thompson, Chapter 6, this volume). For instance we may use the fact that 4+3 is 7 to *derive the fact* that 14+3 is 17. It would even be possible to use another known fact such as 4+4 to derive the sum of 4+3.

By considering the compression of counting procedures we can begin to see more clearly what the experiences we give children may be leading to and, more importantly, what this experience is telling them. Count-all and count-on evoke different processes of counting whilst known facts can evoke the concept of sum. Consequently when young children are presented with elementary number combinations such as 4+3 they can interpret the notation in two qualitatively different ways; as a *process* to do, which can be progressively compressed to be manipulated as a mental *object*.

The child who has compressed counting procedures into known and derived facts possesses a powerful tool with which to achieve success in arithmetic. If they encounter problems with larger numbers they are able to use the knowledge they already have. As combinations become more difficult those who **know** facts and **use** them flexibly find arithmetic far easier than those who have to carry out counting procedures. For such children subtraction may become just another way of looking at addition; it is relatively easy for the flexible child who can use a related addition fact.

Such flexibility can be seen in stark contrast to the difficulties experienced by the children who use counting procedures. These procedures may be successful for simple combinations but they may become extremely difficult for larger numbers. Concrete materials can be used to support (or rather, avoid) the double counting that is so frequently a feature of children's difficulties. This can give the semblance of progress when little progress has actually been achieved and the subtleties of the double-counting of the count-on algorithm have not been sufficiently well appreciated to be carried out without physical supports. Joseph's failure without concrete aids and his eventual success with them, albeit with small numbers, aptly illustrate this point.

The child who relies on count-on for addition is much more likely to use its inverse, count-back, for subtraction. This can be horrendously difficult even with some physical support. Consider Jenny who attempted to count-back 13 from 19 keeping a check on the double count by using her fingers. "*19, 18, 17, 16, 15, 13.....14, 15, 14, 13, 12…*". It is surprising that she arrived at the correct solution. Though she almost immediately recognised her miscount at 15 this caused her some added difficulty: was she counting-up or counting-back? To overcome such difficulti we often use a number line to help children count-back but this may have a fatal flaw. Counting back on a number line may be no more than an example of single-counting, hardly more sophisticated than count-all, and it may not generalise into a flexible form of subtraction.

The numerical procept

The evidence suggests that there are two main interpretations of arithmetical expressions such as 4+3. One makes use of numerical concepts and relationships whilst the other triggers the use of counting procedures. These lead us to an important feature of arithmetical symbolism. Not only does it provide a sense of what to *do* but also what to *know*.

Now we begin to see what is so special about arithmetical symbolism. It is really so very simple. Numerical symbols don't represent either a process or an object; they represent both at the same time. Consider, as an example, the symbol "5". It can be written....it can be seen. It can be spoken ....it can be heard. "5" represents the fusion of a number name with a counting process. We can recreate the counting process whenever we see the symbol or hear its name. But we can also use the concept of "five" without any reference to countable items. Many different processes give rise to the object five. Not only the process of counting...one...two...three...four...five... but also the process of adding four and one, of adding three and two, two and three, of taking three away from eight, or two away from seven, of halving ten, and so on. All of these processes give rise to the same object. The symbol “5” represents a considerable amount of information not least the counting *pro*cess by which it is named and con*cept* or idea by which it is used. Gray and Tall (1994) believed such a fundamental ambiguity deserved its own terminology. This is embraced within the notion of *procept*: a symbol which ambiguously represents both *pro*cess and con*cept*. There are many numerical symbols that evoke either process or concept:

- 3+2 is either the process of *addition* of 2 and 3 or the concept of sum, 7
- 3/4 can mean (amongst other interpretations) the process of *division* of 3 by 4 or the concept of fraction ¾
- 3×4 represented the process of *repeated addition* and the concept of product, 12.

Not all mathematical symbols are procepts but they do occur widely, particularly in arithmetic, algebra and aspects of higher mathematics. We may consider number as a procept–process and a concept represented by the same symbol. Children who use count-all recreate the process embedded within each symbol. Children who use count-on may use either the process or the concept: they may treat one number as an object and use the process embedded in the other to increment in ones. Though usually shorter than count-all, count-on remains a counting process which takes place in time. By using it a child may be able to compute the result without necessarily linking input and output in a form that will be remembered as a new fact. Some children – often with a limited array of known facts – may become so efficient in counting, that they use it as a universal method that does not involve them in the risk of attempting to use a limited number of known facts. However, count-on may lead to development of a procept. It can produces a result that is seen both as a counting procedure and a number concept.

The proceptual divide

In the early stages, number is widely seen as a counting process. It is only when the child realises that the number of elements is independent of the way in which the elements are arranged and of the order in which they are counted that number can begin to take on its own stable existence as a mental object. During Key Stage 1 most children count at least some of the time, and some children count all the time. Those who count quickly can succeed in the number facts to 10 almost as well, and sometimes better, than those who know or can manipulate number facts. But those who achieve higher levels do so because they begin to see numbers as mental objects to be manipulated (Gray, 1994). The more successful may still count, but they do so less and less, and when they do count they use the technique sparingly in subtle ways which are more likely to succeed than those that continue to count on a regular basis. The latter may develop intricate counting techniques using imaginary fingers, parts of the body, selected objects in the room, and so on, to cope with the number facts to twenty. But in doing so they give themselves a harder job to do than those who use number facts in a flexible way.

The divergence between those who interpret processes only as procedures and therefore make mathematics harder for themselves, and those that see them as flexible procepts is called the *proceptual divide* (Gray & Tall, 1994). It is hypothesised that the difference between success and failure lies in the difference between the use of procepts and procedures. Those who use procedures where appropriate and symbols as manipulable objects where appropriate are said to be proceptual thinkers. It is further hypothesised that count-on is one procedure that causes a bifurcation between those who display the ability to think proceptually and those who think in terms of procedures (Gray, 1993).

This divide between success and failure is found throughout the mathematics curriculum. At any stage, if the cognitive demands on the individual grow too great, it may be that someone, previously successful, founders. Like Joseph, they may ask “tell me how to do it”, anxiously seeking the security of a procedure rather than the flexibility of procept. From this point on failure is almost inevitable. It is for this reason that mathematics is known chiefly as a subject in which people fail, fail badly, and fail often.

Implications for teaching and learning

If number is seen as a flexible procept, evoking a mental object, or a counting process, whichever is the more fruitful at the time, then children are likely to build up known facts in a meaningful way. Thus the "fact" that 4+3 is 7, becomes a flexible way of interchanging the notation 4+3 for the number 7. If 4 is taken from 7, then this number triple tells us that the number 3 remains. In this way, seeing addition as a flexible procept leads to subtraction being viewed as another way of formulating addition. Successful children learn how to derive new facts from old in a flexible way.

It helps us to realise that what we need to do is to help all children achieve the flexible form of thinking developed through compressing number processes into concepts. But at its highest level such flexibility is only achieved if children also know number facts and number tables. Too frequently, for some children, counting becomes the focus of attention. This can make it very difficult to compress the number processes into an object. We not only need to provide all children with opportunities to think about the power and flexibility of the symbolism but for some we need to give opportunities which may help them to make necessary links between combination and output without the use of a lengthy procedure. One method may be to give the child a calculator at the appropriate time. Of course this in itself will not help children learn number facts but it would help them to relate similar problems and observe the patterns. With a 'supercalculator', a calculator which has the potential for a graphical display, a child may *see* on display *at the same time* not only the fact that 3 + 4 is equal to 7 but also that 4 + 3 is 7, 2 + 5 is 7, 9 – 2 is 7, and so on. Because several combinations can be built and seen in sequence a child can easily try 13 + 4, 23 + 4, they may begin to see a pattern. By providing the super-calculator we are not only give an alternative representation for the numbers but we also provide an alternative way to deal with them. We offer a 'button pressing' procedure which permits the child to see a representation which includes inputs and outputs without lengthy counting procedures mitigating against their connection. In such a way we may help the child appreciate the pattern and develop some flexibility to solve harder combinations.

Will such a strategy improve all children's ability at arithmetic? In some senses there may be a problem. After all if some children, because of their lack of success at arithmetic, turn to using procedures whilst we may want to make them flexible, it may be that the best we can do is to teach them procedures to make them flexible. In other words we don't make them really flexible at all. Simply giving procedural children more examples to practise may help them in one way; it may make the procedures that they use a little faster and perhaps a little more efficient, but in other ways it could be very damaging in that what they do without guidance is to develop their own idiosyncratic methods which in fact make the mathematics far harder. We therefore need to combine the practice of those facts which are essential building blocks in the system with the flexible means by which they can be manipulated most easily. This may mean practising number combinations so that they become automatic, but this must not set arithmetic in the context of something which *must* be learned by rote. Those who are successful at arithmetic have more than this. They use the facts they know to build the ones they dont't know. They see the arithmetical symbol is in a flexible way: it is both a process which enables them to do mathematics and a mental concept which enables them to think about it.

References

Baroody, A.J. & Ginsburg, H.P. (1986). 'The relationship between initial meaningful and mechanical knowledge of arithmetic'. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case for Mathematics*. Hillsdale, N.J.: Lawrence Erlbaum Associates.

Fuson, K. & Fuson, A. M. (1992). Instruction supporting children's counting-on for addition and counting-up for subtraction. *Journal for Research in Mathematics Education, 23*, 1, 52–78.

Gray, E. M., (1994a). Spectrums of performance in two digit addition and subtraction. In J. P. Ponte and J. F. Matos (Eds.), *Proceedings of XVIII International Conference for the Psychology of Mathematics Education*. (Vol. III pp. 25-32). Lisbon: Portugal.

Gray, E.M. (1991) 'An analysis of diverging approaches to simple arithmetic: preference and its consequences'. *Educational Studies in Mathematics*, 22, 551-574.

Gray, E.M., (1993). Count-on: The parting of the ways in simple Arithmetic. In I. Hirabayashi, N. Hohda, K Shigematsu and Fou-Lai Lin (Eds.), *Proceedings of XVII*

International Conference for the Psychology of Mathematics Education. (Vol. I pp. 204-211). Tsukuba, Japan.

Gray, E.M., Tall D.O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 25**,** 2, 115–141.

Steffe, L. P., von Glaserfeld, E., Richard, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory and application.* New York: Praeger Scientific.

Words: 4506 + Refs 220