IN THE MIND...
What can imagery tell us about success and failure in arithmetic?

Demetra Pitta and Eddie Gray
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

ABSTRACT

In this paper we consider the quality and use of imagery as described by eight to twelve-year-old children at extremes of mathematical achievement. The analysis considers the relationship between children's interpretation and description of images triggered by words, symbols and icons and imagery associated with mental processing of number combinations. Children of different achievement tend to concentrate on different objects or different aspects of an object. This leads to qualitative differences in imagery: that of 'low achiever's' tends to be episodic and active whilst that of the 'high achievers' tends to be semantic and generative. This contrast is reflected in the different use of imagery. For the former it appears essential to thought, to the latter it is a thought generator. We suggest these distinctions contribute towards the success and failure in simple arithmetic seen in the qualitatively different thinking associated with the proceptual divide.

INTRODUCTION

The focus of this paper is to seek further insight into children's success or failure within elementary arithmetic. By considering imagery and its use we seek to add a further dimension to the discussion on qualitative difference in thinking (Gray, 1991; Gray & Tall, 1994).

Children use imagery in their thinking more than adults (Kosslyn, 1980) and through their role in the child's thought processes, the properties of images will have far reaching consequences on children's concepts and reasoning (Bruner, Oliver and Greenfield, 1966; Piaget and Inhelder, 1971) and therefore place major constraints on cognitive processes. Gonzalez and Kolers (1982 argue that in the context of arithmetic mental representations of the objects will effect mental operations .

In our consideration of children's use of imagery in elementary arithmetic we develop our argument within a framework which adds a further dimension to the suggested divergence in thinking that arises from qualitatively different interpretations of arithmetical symbolism (Gray and Tall, 1994). Our discussion evolves from the Piagetian notion of 'interiorised action' (Beth and Piaget, 1966) and focuses on children's success and failure in making the cognitive shift associated with interiorisation. Gray (1991) observed that such differences may arise because of qualitatively different thinking. The 'less successful' focus on the use of procedures and the development of competence in utilising them, whilst the 'more able' demonstrate flexibility in the use of efficient procedures and/or the use of encapsulated processes. It is our intention to take the debate further by associating the notion of achievement and 'qualitative difference' with the part that imagery may play before the quality is communicated. At issue is the notion that given the opportunity to carry out actions on objects all children will abstract the actions underscoring an arithmetical process and encapsulate these actions as a concept. Our evidence suggests that of considerable influence in making such a cognitive shift is what the child 'chooses' to conceptualise and the use they make of such knowledge. We suggest that by focusing on different actions and/or different aspects of the action, individual children may 'encapsulate' different things. This may promote images which are not only of a different quality but, dependant on this quality, are also subject to different use. On one hand it provides an aid to thought which supports flexible thinking. On the other it is essential to thought confirming a procedural dimension which causes great difficulty and eventually leads to failure.

ACTIONS AND SYMBOLS

Our attempt to seek insight into the role of imagery in success and failure within elementary arithmetic cannot be divorced from the modalities that generate the image. These are largely visual and tactile but we cannot ignore the auditory since qualities of the other two may be sharpened by the child's interpretation of the focus of attention directed by the pedagogue.

> "I find it easier **not** to do it [simple addition]with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. It can take longer to work out the sum than it does to work out the sum in my head. "If we don't [use our fingers] the teacher is going to think, 'why aren't they using their fingers.....they are just sitting there thinking'...we are meant to be using our fingers because it is easier....which it is not". (Amanda, age 9)

What insight from a child who is herself experiencing difficulties; she is attempting to decrease her dependence on perceptual items and concentrate on figural representations of items (see Steffe, von Glaserfeld, Richards and Cobb, 1983). Of course such a transition may be fraught with difficulty, particularly if associated with an interpretation of thinking unrelated to cognitive reality. Such an instance is provided by James, a nine-year old, whose effort to use "*fingers in my head*" caused evident tension and was unsuccessful. For James this appeared to be a natural transition, it was his perception of what others were doing. He "*wanted to be like the clever children* [who] *did things in their heads*". He didn't realise that they were doing things quite differently. In his response to number words and symbols James was trying and *do* something in his head by actually representing a counting activity which involved using fingers. He didn't realise that the 'clever' children were using number facts that they '*knew*'.

Within our context for our discussion is the way in which actions on concrete objects become mathematical abstractions. The notion of numerical concepts being formed from actions with physical objects forms the background for the conceived cognitive development of simple arithmetic (see, for example, Piaget, 1965; Steffe, *et al*, 1983; Kamii, 1985; Gray and Tall, 1994). However, though we know such a shift takes place we do not know how it takes place. Notions such as "interiorisation" (Beth & Piaget, 1966) and "encapsulation" (Dubinsky, 1991) draw attention to the result of the process whereby dynamic actions become conceptual entities but there is little to give us a sense of how these notions may be developed. Sfard (1991) indicates that there is a three phase process. She sees acquaintance with the process, 'interiorisation', and the eventual squeezing of lengthy sequences of operations, 'condensation', as quantitative changes preceding the qualitative change denoting an ontological shift. This shift is described as reification, a process through which a new entity may become detached from the process which produced it. In a sense such notions indicate how growing procedural competence may provide a platform for procedural encapsulation. However, though Sfard does suggest that "some kinds of inner representation [verbal or visual] fit one kind of [mode of thinking] better than another" (p. 7), there are few examples of the sorts of imagery that may associated with these phases, how such imagery is used, and the possible consequences of any qualitative differences in it.

Though such imagery forms the nucleus of our attention, the issue of retrieval is a contentious one that generates models focusing on either fact retrieval (for example. Ashcraft, 1982; Seigler and Shrager, 1984; Cambell and Graham, 1985) or on both individual fact retrieval and rule and procedure generated responses (Baroody, 1985; Baroody and Ginsburg, 1986). Differences in the use of the two were observed between different children and within individual children by Gray (1991) and these differences revealed a divergence in thinking which led to initial conclusions about the preferences children had when dealing with unknown combinations. Some wished to remain at a procedural level, which, in terms of information processing made things very difficult for them, whilst others operated at a conceptual level which was very flexible. The notion of divergence stemmed from the observation that the less able who relied extensively on procedure were "making things more difficult for themselves and as a consequence become less able" (Gray, 1991, p. 570). In contrast, the ability to "compress the long sequences [of procedures] appeared to be almost intuitive to the above-average child"(ibid).

This distinction was later placed within a theoretical context which emphasised the duality and ambiguity associated with interpretations of mathematical symbolism (Gray & Tall, 1994). Symbolism may represent a **pro**cess to *do* or a con**cept** to *know* and to emphasise this double meaning they coined the term **procept.** It is suggested that the ability to recognise and use mathematical procepts gives great flexibility to the learner. They have the choice between doing mathematics by using a procedure or drawing upon relationships inherent in understanding a concept. Thinking associated with such flexibility Gray & Tall termed **proceptual**. The manifestation of the divergence between procedural led to the notion of the **proceptual divide**, and proceptual thinking. Though it is hypothesised that such a divide can occur at numerous points in mathematical development, Gray (1993) placed particular emphasis on that which occurs during the compression of counting procedures into numerical concepts and, in particular he saw count-on as the point of bifurcation as is seen in Figure 1.

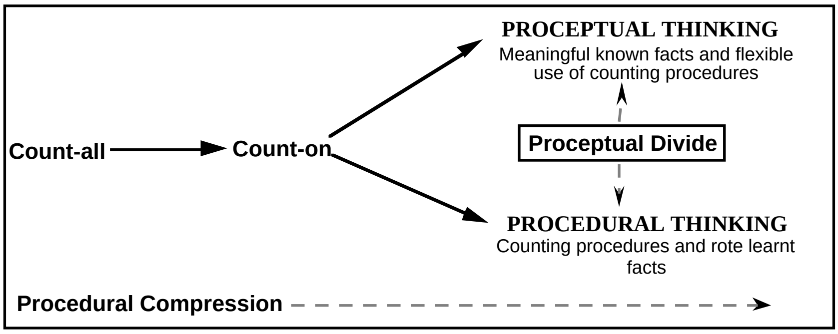

Figure1: A proceptual divide formed from qualitatively different compression's of the count-on procedure.

Procedural compression is associated with qualitative changes in conceptual entities and "involves the transition from enactive concepts to the construction of novel mathematical objects" (Cobb, 1987, p.3). Within Figure 1 we see this as a compression of the lengthy count-all procedure into the shorter one that is count-on. . It is conjectured that count-on may be a source of divergence in thinking. One route leads towards flexible proceptual thinking associated with the abstraction of the process of addition as the concept of sum, the other leads to the development of the procedural competence associated with applying a particular process. Evidence for the distinction has largely been obtained by considering perceptual and symbolic differences which were observed directly and/or reported by children. However, interpreting the distinctions has not taken into account the role of imagery in the qualitative differences that arise. It is to this that we now turn

IMAGERY AND NUMBER PROCESSING

Piaget suggested that new knowledge is constructed by the learner through the use of "active methods" which required that "every new truth to be learned be rediscovered or at least reconstructed by the student" (Piaget, 1976, p. 15). His interest lay in the way that the coordinations of the actions associated with active methods became mental operations. It was not the objects themselves that were important but actions on the objects. Whether or not *all* children who display competence in the procedural aspects of early number activities undergo this process of constructive abstraction–which Kamii (1985) suggests is a construction of the mind rather than something that exists in objects–or indeed whether or not they abstract the appropriate thing is a mute point. The abstraction of a basic counting unit may form a platform from which children may gradually replace slower count-based approaches with more efficient fact retrieval processes. However, through our insight into children's images we suggest that such procedural compression may not be so easily achieved by low achievers.

Images belong to an environment which is confined to the individual. They are restricted to a world no one else can enter and efforts to consider them are fraught with difficulties; our understanding relies upon words and/or pictures but because of their disguised nature it is only possible to make conjectures about them. Well wrapped possessions, they may be covered in many layers and sometimes found as discrete packages. We may believe it is possible to shake the package to find out what is inside, but by doing this we risk breaking it. The pitfalls, particularly in terms of operational definitions and interpretation, are clearly identified by Pylyshyn (1973).

It has been traditional to characterise mental representations as symbolic: a pattern stored in long term memory which denotes or refers to something outside itself (Vera & Simon, 1994). Such a characterisation is based on the assumption that the knowledge structures possessed by humans are symbolic representations of the world. It is suggested, though controversially so, that these representations divide into analogical and propositional aspects–essentially sensory dependent and language like representations. The classical analogical representation is the visual image–though images can be formed from other modalities–which appears to have all of the attributes of actual objects or icons. Seron, Pesenti, Noel, Deloch and Cornet (1992) suggest that quantity directly represented by "patterns of dots or other things such as the alignment of apples or a bar of chocolate" (p. 168) may be deemed to be analogical. Such imagery is 'characterised by its appearance in the absence of the objects to which it refers' (Mead, 1938, p. 224). They take up some form of mental space in the same way that physical objects take up physical space and they can be mentally moved or rotated (see Boden, 1988). Propositions, as mental representations, may represent conceptual objects and relations through, for example, mathematical symbols or spoken words.

The relationship between different forms of representations may be seen through the presentation and solution of arithmetic facts (Deahenne & Cohen, 1994). Symbolic, verbal and analogical representations support the transcoding of numbers into whatever internal code is required for the task in hand. Symbolism promotes direct verbal routines and flexible transformations by proceptual thinkers. Amongst procedural children, where symbolism is more iconic (static) we see the occurrence of analogical forms of imagery which may inhibit the potential for flexible interpretation. To consider such a relationship we assume that an image is mediated by a description (Kosslyn, 1980; Pylyshyn, 1973).

PRELIMINARY CONSIDERATIONS: A FRAMEWORK FROM WORDS, ICONS, PICTURES AND SYMBOLS

The focus of the work within this paper arises from the notion that an individuals imagery of non-arithmetical objects may carry some similarities or relationships to imagery in an arithmetical context. It is fundamental to the thesis of this paper that different qualities of mathematical abstraction are dependant upon the perceived reality of mathematical objects. In the arithmetical context such an interest leads to four research questions:

(i) Which object and which aspects of the object are dominant in children's imagery when dealing with elementary number combinations?
(ii) Which action and what particular aspects of the action do children take into their minds when solving elementary number combinations?
(iii) What is the quality of the image generated as a result of the above considerations.
(iv) To what sort of use is this imagery put?

To gain some insight into the answers to these questions a series of snap shots of 24 children representing the chronological ages from 8+ to 12+ were considered. The children were selected from the extremes of mathematical achievement. This provided a sample consisting of three children within each of two achievement bands, 'high achievers' and 'low achievers', across the fours years of a middle school in the English Midlands, 24 children in all,. To establish a relationship between children's qualitatively different approaches to simple arithmetic and the quality of their imagery we examined children's responses to a range of auditory and visual stimuli. Achievement was measured by children's levels in the Standard Assessment Tasks of England and Wales (SCAA, 1994) or scores obtained in the Mathematical Concepts and Skills components of the Richmond Attainment Tests (France, *et al,* 1974). Children were interviewed individually for half an hour on at least four separate occasions over a period of eight months. Though it is imagery associated with obtaining solutions to elementary number combinations that is the main focus of this paper we start our analysis by presenting a context for later discussion.

In the first phases of the interview sessions the children were presented with a range of auditory and visual items and were asked to talk freely about their imagery and what came to mind with each item (see Pitta and Gray, 1996). Though there are an indefinite number of conclusions that may be drawn from each item, the analysis of the results of this component indicates that similarities in the children's descriptions of imagery are remarkable both for their consistency across the range of items, and for the differences they display between the 'high achievers' and the 'low achievers'.

When responding to the auditory items 'low achievers' tended to highlight the descriptive qualities of items which were strongly personalised, qualities also evident when the children responded to the visual items. However, there was a tendency to associate these items with a story in the sense that they were seen as pictures that required colour, detail and a realistic content. In contrast, 'high achievers' concentrated on the more abstract qualities within both series of items. Though they initially focused on core concepts, they could traverse at will a vertical network of knowledge through which they abstracted these notions or representational features. An overall summary of the analysis of the children's responses to the auditory and visual items is given in Table 1 (adapted from Pitta & Gray 1996).

Table 1: A comparison of children's interpretations of words and icons

	Low Achievers	High Achievers
Words	• Concretised • Unable to reject information • 'Horizontal' thinking – directed towards surface features. • Imitation	• Focus on abstract qualities • Able to reject information • 'Vertical' thinking. Direct attention towards core features or definitions. • Thought generator
Icons	• Interpreted as a "picture out of focus" an incomplete concrete reality which needs focusing. • Given colour, detail and realism (with imagination). • Display "horizontal" thinking– 'imaginary' extensions' similar in quality. • Imitation	• Concentrate on abstract qualities. • Able to ignore detail – concentrate on interpretation • "Vertical" thinking associated with free movement between abstract and descriptive aspects. • Thought generator

The special feature of mathematics is its symbolism consequently, the auditory and visual items that formed the focus of the first phases of the study included numerical items presented both orally and symbolically. These items, which included 'five', '5', 'half' and '$\frac{1}{2}$' evoked qualitatively different responses from children of the two achievement groups. These bore striking similarities to the responses for the non mathematical items and they reflected the degree to within which the children were involved with the abstract qualities of the numerical objects. The higher the involvement the more the child was able to talk about the items at an impersonal level. When hearing the word ‘five’ or ‘half’ ‘high achievers’ frequently referred to the symbol, when seeing the symbol they frequently used phrases such as “it is” to illustrate the semantic aspects of the object. For example, the word ‘five’ drew responses such as “*it is two plus three, one hundred take away ninety five*”, or “*it is prime because it is only divisible by one and five*”. This does not mean to say that they did not attach qualities arising from episodic memory, such as “*I had five candles on my cake for my fifth birthday*”, to these items; “high achievers’ were able to do both. On the other hand ‘low achievers’ almost always displayed examples of episodic memory, concretised the item, “*I have five fingers*”, or associated its use with some arithmetical action such as counting. The qualitative differences between the responses of the ‘high achievers’ and the ‘low achievers’ may be summarised in Table 2 (adapted from Pitta and Gray 1996)

Table 2: A comparison of children’s interpretations of symbols

	Low Achievers	High Achievers
Symbols	• Order to carry out an action • Concretised by either: (a) associating with a concrete item or (b) identifing as an icon. • “Horizontal thinking” demonstrated through procedural association. • Imitation	• Recognised as both the holder of an idea and an action. • Detached from concrete qualities, associated with abstraction • “Vertical thinking” demonstrated by proceptual flexibility. • Thought generator

A comparison of Table 2 with Table 1 indicates how the imagery associated with non-arithmetical objects may carry similarities with imagery of named arithmetical objects and symbols. Such similarities may be summed up by concluding that images of the low achievers are episodic and active, whilst those of the high achievers are semantic, and generative. We use the terms ‘episodic’ and ‘semantic’ to draw a distinction between images that arise from memory associated with the recollection of personal happenings and events, and images associated with organised knowledge associated with meaning and relationships. The former is based upon access to former experience, the latter does not depend upon the learning episodes that provided the basis for knowledge (see, Tulving, 1985).

IMAGES IN SIMPLE ARITHMETIC.

After the auditory and visual items the children were presented with a series of one and two digit addition and subtraction combinations, for example, 6+3, 9–5, 13+5, 15–9. Responses were obtained through semi-structured interviews recorded by field notes, and audio and video tapes. Children were asked to talk freely about their imagery and what came to mind before and during the solution processes for each item. The solution strategies were classified as in Gray & Tall (1994). Whilst external representations were partially identified through children’s sensory motor activity, evidence of images relied extensively on verbal and written description by the children.

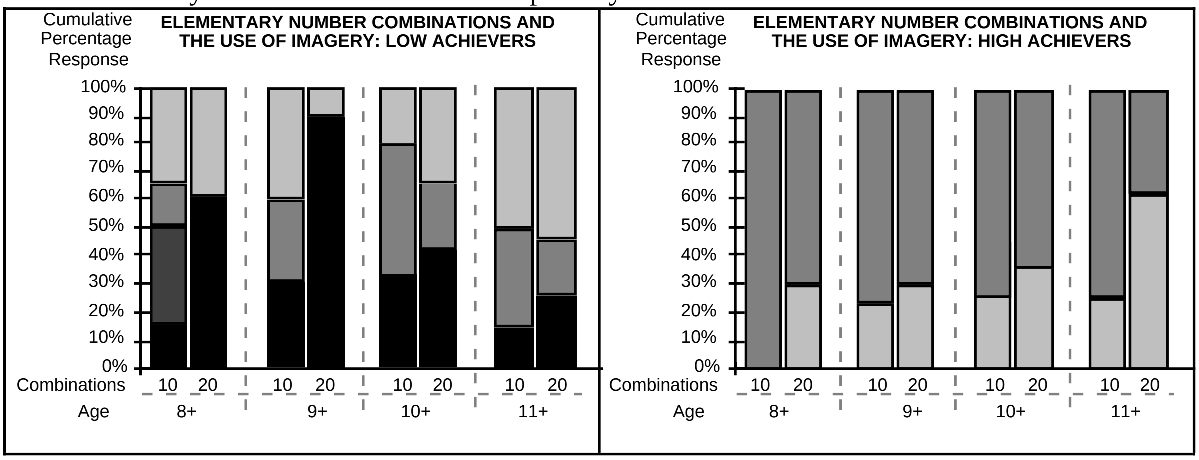

Figure 2: The use of imagery and other approaches when solving elementary number combinations by children at extremes of achievement

Approaches to the numerical element of study indicate that there is a clear distinction between the variety of strategies used by the ‘low achievers’ and the ‘high achievers’(See Figure 2).

KEY
- Perceptual Objects
- Simultaneous engagement perceptual objects and imagery
- Imagery
- Unclassified

Unclassified responses (see adjacent key) were those where it was difficult to obtain a clear notion of what the child was doing. These responses did not include the use of perceptual items and were frequently obtained when children gave automatic responses. Though use of perceptual items dominated overall, there was evidence of the use of imagery across the spectrum of age for addition and subtraction combinations to ten but the combinations to twenty only evoked descriptions of imagery from amongst the eleven and twelve year olds. In contrast, no high achievers used perceptual items to solve the combinations but the use of imagery was reported across the spectrum of the age range and the range of combinations. Within each group the number of possible responses for combinations to ten is 54 whilst for twenty it is 60.

Clearly such responses give rise to two issues:

(i) the place of verbal imagery. Such imagery was not clearly identified within the study,

(ii) the quality of automatic responses. There were clear differences in the way children of different achievement talked about automatics responses. ‘Low achievers’ tended to use the phrase “thought it”, high achievers the words “knew it”. In such instances ‘low achievers’ would report seeing no imagery whereas there was a strong likelihood that high achievers did see imagery in symbolic form.

Although the range of strategies used to obtain solution are of interest, the proportional use of strategies associated with procedural and proceptual interpretations of symbolism differ very little from those reported by Gray and Tall (1994)

From Figures 2 the following appears to be clear:

- Amongst low achievers within each age group there was an increase in the use of perceptual items as the combinations became more difficult, Such an increase was associated with a decline in the reported use of imagery.
- Amongst high achievers there was an increase in the use of imagery as combinations became more difficult.

Such evidence could of course be pointing towards two different things. Taking the achievement groups as distinct groups it supports the notion of qualitatively different thinking. Taking the children as a whole we could conclude that it increases our awareness that imagery is a mediator between the use of perceptual objects and automatic symbolic manipulation (see also Steffe *et al* 1983). However, we will leave that debate until another time and focus on the qualitative differences associated with the imagery of the two groups.

Whereas imagery identified by ‘high achievers’ was always symbolic in form, imagery reported by ‘low achievers’ was dominated by the analogical form. In some instances where it was symbolic the common feature emerged–there were strong similarities in the way in which low achievers saw imagery in the arithmetical items and in the way in which they saw imagery in the non-arithmetical items. Their analogical images reflected the need to concretise mathematical symbols and were active and indispensable to solution procedures. Frequently highly detailed,, these images, like those associated with symbols, indicated the children’s inability to filter out information.

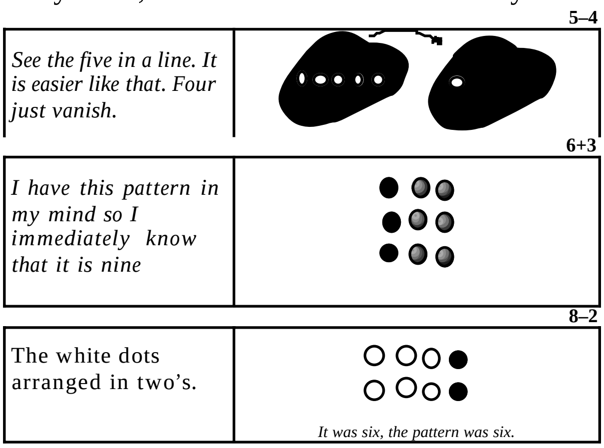

Figure 3: Analogical images based upon discrete objects

Analogical images were essentially of two different forms; those based on discrete objects such as counters or marbles, and those which possessed qualities analogous to the number track or number line.

Figure 4, established from diagrammatic representations given by children, illustrates some of the images within which counters or marbles appeared to play a central role. They are not presented as a hierarchy but as a demonstration of the way in which the mathematical symbols were concretised. The instances within the figure show how patterns associated with numbers are utilised in the image. In the first example, 5–4, a ten year old boy described how five “*white counters*” in a row flashed up only to be almost instantly replaced by the single counter. The other two instances were variously described as marbles or counters which possessed colour. It is conjectured that such images could support the later development of known facts and derived facts. It may well be that the examples given may support a view that the children knew the responses but were not confident enough to rely on the knowledge. Several of the ‘low achievers indicated that “*…even if you know something it could be wrong so it is better to work it out*.” Such a comment adds supports to the view of Krutetskii (1976) “One method of solution may be an obstacle to another” (p. 338).

Though the examples given above did not involve counting, most of those given by ‘low achiever’s did. In some instances marbles or counters were used in a dynamic way to reflect a count–on procedure. Figure 5 and 6 indicates diagrammatic copies of representation given by a nine-year-old and an eleven-year-old. These are associated with the solutions to 9–5 and 7+4 .

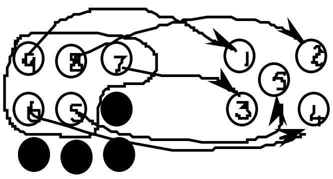

Figure 4

In the first we see the dynamic image that grows from a pattern of nine. The procedure used was count-back and as each counter was counted it was moved and assigned a new numerical value. When the count back of five had been completed the child knew from the pattern "*that 3 and one makes four*".

Within Figure 5 we see how each phase of the solution procedure evolved from the previous one. First the "black" seven appeared with "*four white balls*". One of the balls had an eight written above it and the eight moved to take the place of the seven which disappeared, There were now three white balls the one nearest the eight having a nine written over it. This now moved to take the place of the eight, and so on.

Children using this form of imagery carried out mental actions which imitated a procedure they could use with real objects on the desk in front of them. There are difficulties however. They are not only using a mental double counting procedure but incrementing mathematical objects seen as symbols and decrementing analogues of physical objects. Indeed, though an active mental image of counters was used by the child dealing with 9–5, fingers were also utilised to support the counting procedure. Here is an example of the *simultaneous engagement* of a mental image with perceptual items. The later were not a focus of attention in the visual sense; the operation was purely tactile and used to support concentration which centred upon seeing the "*counters in my head*".

Images of discrete objects like counters provided some children with a degree of flexibility not associated with more static imagery like a analogical number line, or fingers in the head. We can leave the explanation to a nine-year-old

> "[with] the dots....it's....it's easier because you don't have to keep on thinking, 'No its that one I need to move, no its that one or that one', because it doesn't really matter which one you move" (Amanda, 9)

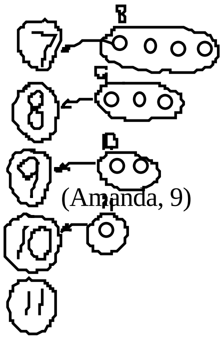

Figure 5

As children became older analogical images appeared to be more static. They reflected the children's growing awareness of number order as seen in the number track or a number line. Symbols could replace figurative items but counting along the 'line' was the dominant procedure used. External manifestation of analogical images may have been motor acts such as nodding or moving an eye but the distinction in the image displayed the qualitative difference between the imitation of counting using images of perceptual units and counting that now used the number word associated with the symbol as the counting unit.

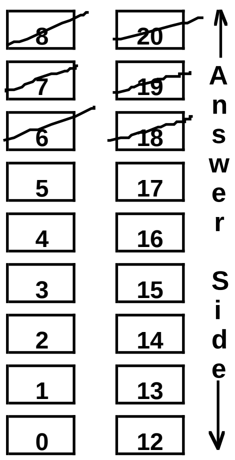

Figure 6

An eleven-year-old gave Figure 6 as a diagrammatic representation of what he saw in his head when subtracting 20–8. He described the two 'number tracks' as "*two calculators going around in opposite ways in my head*". The figure '8' and '20', were the first to appear, and then these were "crossed out" to be replaced by '7' and '19'. This process continued until '0' and '12' were reached.

An interesting feature of the children's use of analogical images was the difference in the quality of detail given by the children. The more a child's use of imagery oscillated with the use of perceptual items the greater was the level of detail associated with that imagery. It is suggested that children who do interchange between the two may be at a transition phase between the use of perceptual items and of an increasing awareness of motor acts (see Steffe, *et al*, 1983). The level of difficulty of the combinations also gave evidence of this qualitative change and in some instances we were able to see this happening with individual children. A twelve-year-old, described how for easy combinations, such as addition combinations to ten, for example 3+5, she "*saw two sets of lines*", one having five in the other having three. She placed the symbol '3' above the three lines and then counted-on. As combinations became more difficult, and in this case it was subtraction combinations to ten, these lines became joined together to form the diagrammatic representation of a hand. If the subtraction was "*very difficult*", for example, 9–5, the child saw a complete image of her hand and used the fingers to count back. For all number combinations to twenty she used her actual fingers.

Symbolism enables us to utilise short term memory to better effect but the differences between the 'low achievers' imagery associated with symbolism and that described by the 'high achievers', was stark. It is here that we may see clearly the 'low achiever's' inability to filter out information thus providing the contrast between their uneconomical use of memory and the 'high achievers' economic use. Here, we should explain that we use the word 'economic' not simply to illustrate differences in the detail but also in arrangement as well as quality.

Symbolic images played considerably less part in processing for 'low achievers' than they did for ‘high achievers’. It was also reported far less than analogical images.

Figure 7

Figure 7 shows an eight-year-old child’s diagrammatic representation of imagery associated with 3+5. The child described how all of the numbers were going around in his head like in circles. “*The number I want moves out and I count them. Then they go back and new numbers go out*.” In this case it was first the ‘3’ and the ‘6’. These became “*blacker*” than the other numbers. The three moved back and became four and the six moved back and became 7. For this child such imagery was only associated with number combinations to ten. For the other numerical items perceptual units were used.

The notion of “*spinning*” seemed to be a common feature of the ‘low achievers’ descriptions, implying that images remained for some time and possessed movement. Even when adding 2+1 a nine year old reported seeing all of the operation symbols “*spinning around on one side and a big black 3 on the other*”. In some instances images of symbols were associated with approximation. When adding 6+3 another nine-year-old reported seeing “*a jumble of numbers with 8 and 9 standing out because they are near the answer*.” This was a similar response to that given by a twelve-year-old who, when doing the same combination reported an image that consisted of 3,6,9,12,15, and 18. “*All the numbers were in the three times table*”. Whilst the “*three and the six stayed there because they were part of the nine, the twelve, fifteen and the eighteen just fall away*.”

The use of symbolic imagery amongst ‘high achievers’ was far more economical. It wasn’t the word “spinning” that dominated their descriptions but the word “*flashing*”. Images came and went very quickly. “*I saw ‘3+4’ flash through my mind and I told you the answer*”, “*I saw a flash of answer and told you*” were the most frequent comments though it was not unusual for the children to note that they saw both question and answer “*in a flash*”, sometimes the numerical symbol denoting the answer “rising out of” the symbols representing the question.

DISCUSSION

The ability to encapsulate arithmetical processes into numerical concepts provides the source of flexibility which utilises the proceptual nature of numerical symbolism. The ability to recognise that a considerable amount of information is compressed into a simple representation, the symbol, is a source of mathematical power. This strength derives from two abilities; first an ability to filter out information and operate with the symbol as an object and secondly the ability to draw upon the filtered out information whenever it is appropriate. We suggest that qualitative differences in the way in which children handle elementary arithmetic may be associated with the their success in being able to do these.

In the same way as we have summarised other evidence we may summarise the evidence that comes from children’s imagery associated with elementary arithmetic combinations (Table 3).

Table 3: Comparison of children’s use of imagery in an arithmetical context

	Low Achievers	High Achievers
Arithmetical context	• Concretised	• Focus on abstract qualities
	• Unable to reject information	• Able to reject information
	• ‘Horizontal’ thinking– directed towards procedural asociations with variations of the figural/imaginary items	• ‘Vertical’ thinking– attention directed towards known facts or transformations.
	• Imitation	• Thought generator
	• *Excessive memory, overload of WM*	• *Economic memory good use of LTM–use of symbols*

Comparison with Table 1 and 2 clearly shows the similarities and differences between the two groups of children over the range of items that formed the basis for comparison. We once again see the tendency of ‘low achievers’ to concretise and focus on all of the information. Imagery in the numerical context is strongly associated with procedural aspects of numerical processes. The children carry out procedures in the mind as if they were carrying out procedures with perceptual items on the desk in front of them. ‘High achievers’ appear to focus on those abstractions that enable them to make choices. Their ability to reject information is again apparent.

Clearly the general impression is that children of different levels of arithmetical achievement are using qualitatively different objects to support their mathematical thinking. On the one hand we see the dominant object being an object of the environment, a countable object, on the other hand we see it as an “object of thought” (Piaget, 1985, p. 49) formed through the cognitive shift associated with procedural compression. With the former imagery is frequently associated descriptive qualities of the object whilst with the later these properties are ignored until required.

Relying extensively, as they do, on perceptual objectives or figural representations of these objects, ‘low achievers’ focus on a particular procedure for carrying out the process. The mental actions are clear imitations of actions that may take place with perceptual items. They are not so much associated with “*knowing*” mathematics but with “*doing*” mathematics. If combinations are not known ‘high achievers’ transform them into something that is known.

The result is that the quality of imagery generated differs considerably. ‘Low achievers’ concentrate on analogues of physical actions, and where they use symbolism they continue to carry out actions associated with such analogues. Such activity appears to dominate thought. In contrast the symbolic images of 'high achievers', appear to act as though generators. They appear to flash as memory reminder's, momentarily coming to the fore so that new actions or transformations may take place.

We suggest that such differences have over-riding consequences for children's mathematical achievement. The one conclusion that may be drawn for the use of analogical images is that it would seem to place a tremendous strain on working memory. Gear *et al* (1991) have suggested that a component of developmental difficulties in mathematics is a working memory deficit. We would suggest that on the contrary these low achievers show an extraordinary use of working memory. Their problem is one associated with its use and not so much its capacity. Not only is the child maintaining sight of the analogical representation but also focusing on discrete numbers in that representation. Children such as James and Amanda, seen earlier, frequently display external evidence of the strain that is involved in mental calculation. They both use imagery but their imagery does not offer support to the limited space available within short term memory.

CONCLUSION

The ability to filter out information and see the strength of such a simple device as a mathematical symbol is something, we suggest, that is not part of the cognitive reality of the low achievers. The evidence obtained from the series of items within the current study indicates that children who are 'low achievers' in mathematics appear unable to detach themselves from the search for substance and meaning–no information is rejected, no surface feature filtered out. When creating imagery they seem to focus on **visual** characteristics and **parts** of objects. The similarities, between images associated with non-mathematical and mathematical items are striking. The children did not talk about an image as a **skeleton** upon which they may pin core ideas.

We believe that this has serious consequences which contribute to the formation of the proceptual divide. The notion of procedural compression and the interiorisation of mathematical processes is strongly embedded in the literature. Interpretations of Piagetian notions that enactive approaches will form a foundation for procedural encapsulation are associated with Bruner's view that past experience is conserved through such enactive approaches. This can provide a theoretical basis for mathematical metaphors to act as mediators in the development of mathematical concepts. The assumption is that all children will extract from this experience that which will enable them to become "experts" in particular aspect of mathematics. The evidence from this study suggests that children focus on qualitatively different aspects of the metaphor. It seems to point to such strong similarities between the mathematical and the non-mathematical that for some the notion of a proceptual divide may be pre-ordained.

The quality of image formed from enactive approaches is dependant what it is the child chooses to create an image of. This will have consequences for the quality of the object which dominates the child's imagery and the quality and features of the action that is taken into consideration. We suggest that these features will then influence the use to which this image is put. Such considerations add a new quality to the notion of proceptual divide, one that is so strongly associated with imagery that deeper research considering a longitudinal study of early number development associated with other curricular development is called for.

REFERENCES

Ashcraft, M.H., The development of mental arithmetic: A chronometric approach, *Developmental Review*, 2, 213-235, (1982).

Baroody, A. J., Mastery of basic number combinations: Internalisation of relationship or facts. *Journal for Research in Mathematics Education*, 16, 83-98, (1985).

Baroody, A. J., & Ginsburg, H. P., The Relationship between Initial Meaningful and Mechanical Knowledge of Arithmetic. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case for Mathematics*. (pp 75–112). Hillsdale, N.J.: Erlbaum, (1986).

Beth E.W. & Piaget J., *Mathematical Epistemology and Psychology*. (W. Mays, trans.). Dordrecht: Reidel, (1966).

Boden, M., *Computer models of mind. Cambridge*. Cambridge University Press, (1988).

Bruner, J. S., *Toward a theory of instruction.* New York: W.W. Norton & Co., (1968)

Bruner, J.S., Oliver, R.O., & Greenfield, P.M., *Students in cognitive growth*. New York: Wiley, (1966).

Cambell, J.I.D., & Graham, D.J., Mental multiplication skill: Structure, process and acquisition. *Canadian Journal of Psychology,* 39, 338–362, (1965).

Clark, J.M. & Cambell, J.I.D., Integrated versus modular theories of number skills and alcaculia. *Brain and Cognition*, 17, 204–239, (1991).

Cobb, P., Information-Processing Psychology and Mathematics Education–A Constructivist Perspective. *Journal of Mathematical behaviour*, 6, 3–40, (1986).

Dehaene, S., & Cohen, L., Towards an Anatomical and Functional Model of Number Processing, *Mathematical Cognition*, 1, 1, 83–120, (1994).

Dubinsky E., 'Reflective Abstraction'. In D.O.Tall (Ed), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers, (1991).

France, N., Hieronymus, A., & Lidquist, E.F., *Richmond Test of Basic Skills*. Windsor: NFER-Nelson, (1974).

Gonzalez, E.G. & Kolers, P.A., Mental manipulation of arithmetic symbols. *Journal of experimental psychology. Learning memory and cognition.* 8, 308-319, (1982).

Gray, E. M., An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational studies in mathematics*, 22: 551–574, (1991).

Gray, E.M., Count-on: The parting of the ways in simple Arithmetic. In I. Hirabayashi, N. Hohda, K Shigematsu and Fou-Lai Lin (Eds.), *Proceedings of XVII International Conference for the Psychology of Mathematics Education*. (Vol. I pp. 204-211). Tsukuba, Japan, (1993).

Gray. E. & Pitta, D., Number Processing: Qualitative differences in thinking and the role of imagery. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol.4, pp 155–162.Valencia: Spain, (1996).

Gray, E.M.& Tall, D.O., Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education,* 25, 2, 115–141, (1994).Kamii, C.,*Young children reinvent arithmetic.* New York: Teachers College Press, (1985).

Kosslyn, S.M., *Image and mind*. Cambridge, Mass: Harvard University Press, (1980).

Krutetskii, V. A., *The Psychology of Mathematical Abilities in Schoolchildren,* J. Teller, trans. (J. Kilpatrick & I. Wirszup, (Eds.). Chicago: The University of Chicago Press, (1976).

Mead, G.H., *The philosophy of the act*. Chicago: The University of Chicago Press, (1938).

Piaget, J., *The child's conception of number*, (C. Gattengo & F. M. Hodgson, trans.), London: Routledge & Kegan Paul, (1965).

Piaget, J., *A structural foundation for tomorrows education, To understand is to invent,* (G.A Roberts, trans.). Harmondsworth: Penguin, (1976).

Piaget, J., *The equilibrium of cognitive structures*. Cambridg MA: Harvard University Press, (1985).

Piaget, J. & Inhelder, B., *Mental imagery in the child*. New York: Basic, (1971).

Pitta, D. & Gray. E., Nouns, Adjectives and Images in Elementary Mathematics. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol.3, pp 35–42.Valencia: Spain, (1996).

Pylyshyn, Z.W., What the mind's eye tells the mind's brain. *Psychological Bulletin*, 80, 1–24, (1973).

SCAA, *Mathematics, Key Stage 2*. London: Schools Curriculum and Assessment Authority Publications, (1994).

Seron, X., Pesenti, M., Noel, M-P., Deloche, G., & Cornet, J-A., Images of numbers, or "When 98 is upper left and 6 sky blue. *Cognition*, 44, 159-196, (1992).

Sfard, A., 'On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics*, 22, 1-36, (1991).

Siegler, R.S., & Shrager, J., 'Strategy choices in addition: How do children know what to do?'. In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 229-293). Hillsdale, NJ: Lawrence Erlbaum Associates, (1984).

Steffe, L., von Glaserfeld, E., Richards, J., Cobb, P., *Children's Counting Types: Philosophy, Theory and Applications*. Preagar: New York, (1983).

Tulving, E., How many memory systems are there? *American Psychologist*, 40, 385–398, (1985).