NOUNS, ADJECTIVES AND IMAGES IN ELEMENTARY MATHEMATICS

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This paper considers the quality of images described by children at extremes of mathematical achievement. Two groups were presented with auditory and visual stimuli and asked to consider the images prompted by them. Using the grammatical notions of noun and adjective, we consider the qualitative differences in the properties and relationships identified. The similarity and the differences identified between images of concrete nouns and images of numerical nouns are illustrated. High achievers concentrate on relationships and abstract qualities of concrete, numerical nouns, icons and symbols. Low achievers highlight surface details and emphasise concrete qualities of concrete nouns and icons and see numbers as adjectives associated with concrete nouns.

INTRODUCTION

The encapsulation of arithmetical processes is regarded to be fundamental to the development of numerical concepts. Such encapsulation may be seen as the “re-concretising” of what are essentially abstract aspects of mathematics; the concept of five, abstracted from the process of counting five things, is identified through the label ‘five’ and the symbol 5. An action becomes an “object of thought” (Piaget, 1985, p. 49), an abstract noun associated with a numerical symbol which ambiguously represents process and concept: a procept carrying concrete and abstract ideas.

Concept formation in number development is seen to involve generalisation and abstraction from actions on physical objects. An underlying assumption in cognitive development is that eventually pedagogue and learner share common ground based upon their shared perceptual experiences. These experiences may be shared through actions on objects, for example counting, and the iconic and symbolic representations that consolidate and represent a compression of such actions. However, it is hypothesised within this paper that an understanding of the objects of action, may strongly influence the quality of encapsulation.

The paper considers children’s imagery as it is exposed through verbal description. To gain insight into what children choose to communicate when asked “What comes into your mind when you hear the word... or see the icon... or symbol...”, we take the view that an image is mediated by description (Kosslyn, 1980; Pylyshyn, 1973). Thus we rely extensively on language, but we realise that no precise claims can be made about the exact nature of the images. It would appear that differences between high and low achievers are not solely due to their ability to interiorise actions. We will suggest that the encapsulation of different objects–including mathematical ones–and/or the encapsulation of different components of objects may have a large part to play in mathematical achievement. Though children may look at the same thing with their “minds eye” they may see, use and manipulate things so differently. This may have consequences for their mathematical achievement (Gray & Pitta, submitted).

Identifying Mathematical Objects

We suggest that levels of abstraction are dependant upon whether or not mathematical objects are perceived to be real, and thus named as nouns, or whether they are associated with other objects and may be more adjectival in quality. Pimm (1987) suggests that such complex shifts are indications of the specialised use of mathematical ideas.

Contemporary theories of cognitive development in number may be associated with the Piagetian view that numerical concepts arise from internalised actions; processes become encapsulated as concepts. Such a notion may allow us to refer to numbers as mathematical objects and talk about them as if they were real things. Dörfler (1993) suggests that the existence of such an object may not be needed; in reviewing our own thought patterns we may not so much focus on an object but on the many relationships associated with it. However, “it” implies that such relationships are associated with a subject or object of discussion. Gray & Tall (1994) indicate that the flexibility associated with numerical relationships is crucial to the notion of proceptual thinking. To develop such thinking there is a need to recognise what the objects of mathematics are, to name them and to organise our knowledge about the relationships between them.

RESEARCH METHOD

In the belief that images created for mathematical items bear similarities with those images created for non-mathematical items, twenty four children were selected from within in a “typical” school of the English Midlands to describe images associated with mathematical and non-mathematical nouns and a range of symbols and icons. The children represented the age span 8+ to 12+, thus providing a sample size of six from each year, three ‘low achievers’ and three ‘high achievers’. Achievement was measured by children’s levels in the Standard Assessment Tasks of England and Wales (SCAA, 1994) or scores obtained within the Mathematical Concepts and Skills components of the Richmond Attainment Tests (1974). Children were interviewed individually for half an hour on at least four separate occasions over a period of eight months.

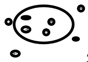
Children were presented with a range of auditory and visual items prior to mentally solving a bank of elementary numerical expressions. Here we consider the results of the auditory and visual items those within the numerical component being reported within Gray & Pitta (submitted). Responses were obtained using semi-structured interviews recorded through field notes and audio and video tapes. At each interview children were asked to talk freely about their imagery and what came to mind with each item. The auditory ones included common nouns such as “ball” and “car”, and abstract nouns such as “number”, “fraction” and “five”. On presentation of with each item children were asked to talk about their first image. Then, at a later date, they were asked to provide a “explanation” that would help a martian understand it. The visual components, presented on a separate occasion, included symbols such as "5", $3 \div 4$, and , $\frac{3}{4}$ and icons such as  (two quarters),  (dancing man),  ,(marbles) and  (honeycomb).

Table 1 highlights the most powerful descriptive concepts and categories used for a discussion of the results.

	Auditory Items	**Symbolic /Visual Items**
1. Not Known	Unable to give meaning or any sense of recognition	Unable to give meaning or any sense of recognition
2. Associations and contextual	Child unable to pinpoint meaning–child conjectures and provides an associative theme or context.	Child unable to pinpoint meaning–child conjectures and provides an associative theme or context.
3.1 Single example	Single example that does not include symbol	
3.2 Multi examples	Several examples of the item	
3.3 Symbolic examples	Symbolic references with general characteristics: prototypical example	
4.1 Visual concrete examples.	Details of visual characteristics given. Descriptive.	Details of visual characteristics given. Descriptive.
4.2 Imaginative Extensions	Item forms basis for imaginative and/or concrete extensions.	Item forms basis for imaginative and/or concrete extensions.
4.3 Insight to abstract qualities.	Descriptions of non-visual characteristics. Insight into meaning and relationships–tend to resemble definitions	Description of non-visual characteristics. Insight into meaning and relationships–tend to resemble definitions.
5.0 Proceptual interpretation.	Emphasis on equivalence and interpretation.	Process and concept described. Examples of equivalence and interpretation

Table 1: Classification of responses to auditory and visual items

RESULTS

Because of space limitations we present just an outline of the results expanding where necessary to support later discussion.

1. Images associated with verbal items.

Figure 1 shows the classification of children's responses to auditory items and provides an indication of how they interpreted the nouns. The grouped responses, ▭ , indicate where over 25% of the total of either first image or martian explanations fall into particular classifications (12 responses for each item, number of responses in each sample 96)

- The first images of the 'low achievers' tended to be either 'association' or 'single example'. The former providing some indication of episodic memory, for example, "*people playing football*" (ball) or the recollection of personal events, for example, "*Reminds me of a friend. She was five and played with me, then she moved*" (five). Single examples included, "*my dads car, two seater*", "*a blue car*", "*figure 3*", "*figure 5*". Images classified as 'visual/description' were quite strongly linked to “five”. Several of these were related to the way the five was written.

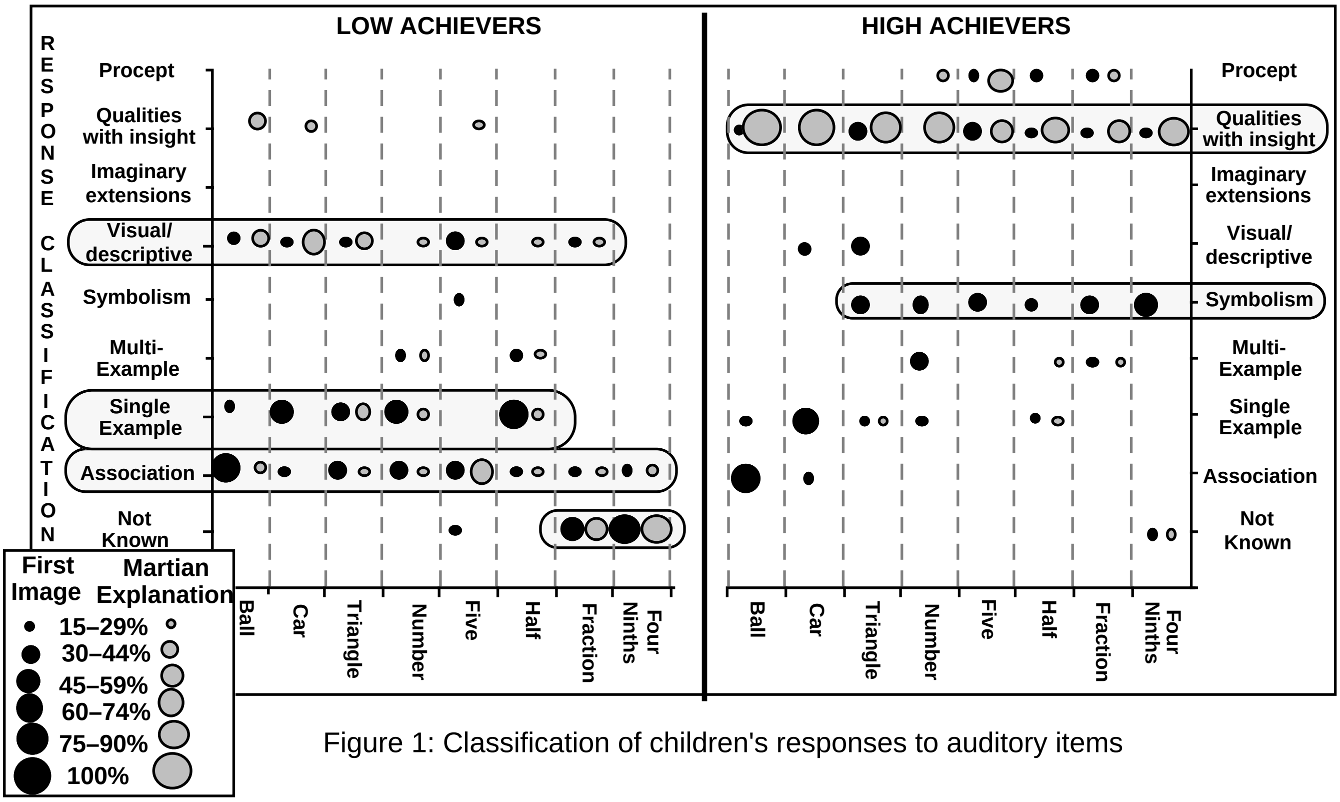


Figure 1: Classification of children's responses to auditory items

- The low achievers explanations for the martian included an extensive number of surface details for the common nouns. Phrases such as “*squares with patterns*”, and “*different colours*”, helped to describe ball, whilst surface details such as “*windows*”, “*seats*”, and “*boot*” were extensively used for ‘car’. 50% of those whose initial images for the word ‘ball’ were associated with “*football*”, provided explanations based upon the visual characteristics of a football. The abstract nouns evoked a range of responses. The dominant description for the word ‘five’ was “*number*” and it was strongly associated with other objects, for example, “*my sister is five*”. It consistently had a concrete context placed upon it, more-so than the other items. ‘Number’ was either associated with mathematics or described by the visual properties of individual symbols “*some are bent and lines, some are circles*”. ‘Fraction’ was described as “*a number on another number*”, “*half a number–numbers with a line between”*.
- Higher achievers also gave ‘association’ or ‘single example’ in their first responses to the common nouns but martian explanations involved a 100% shift towards descriptions based upon abstraction and non-visual qualities. Other items were extensively described in terms of ‘symbolic’ image or ‘qualities with insight’, the latter dominating the children’s descriptions to the martian.

It may well be that the lower achievers provided better descriptions to the martian since the greater proportion of their information was related to visual attributes or the use of items. Frequently, particularly with the common nouns, they indicated that they would show a picture or a model. High achievers, by providing ‘qualities with insight’, frequently ignored the concrete and more fundamental characteristics through which the item may be recognised and focused immediately on deeper qualities.

2. Images associated with iconic representations and symbols

Figure 2 shows the classification of children's responses to visual items. The grouped responses, ☐ , show where over 25% of the total of explanations fall into particular classifications (12 responses for each item, number of responses in each sample 120).

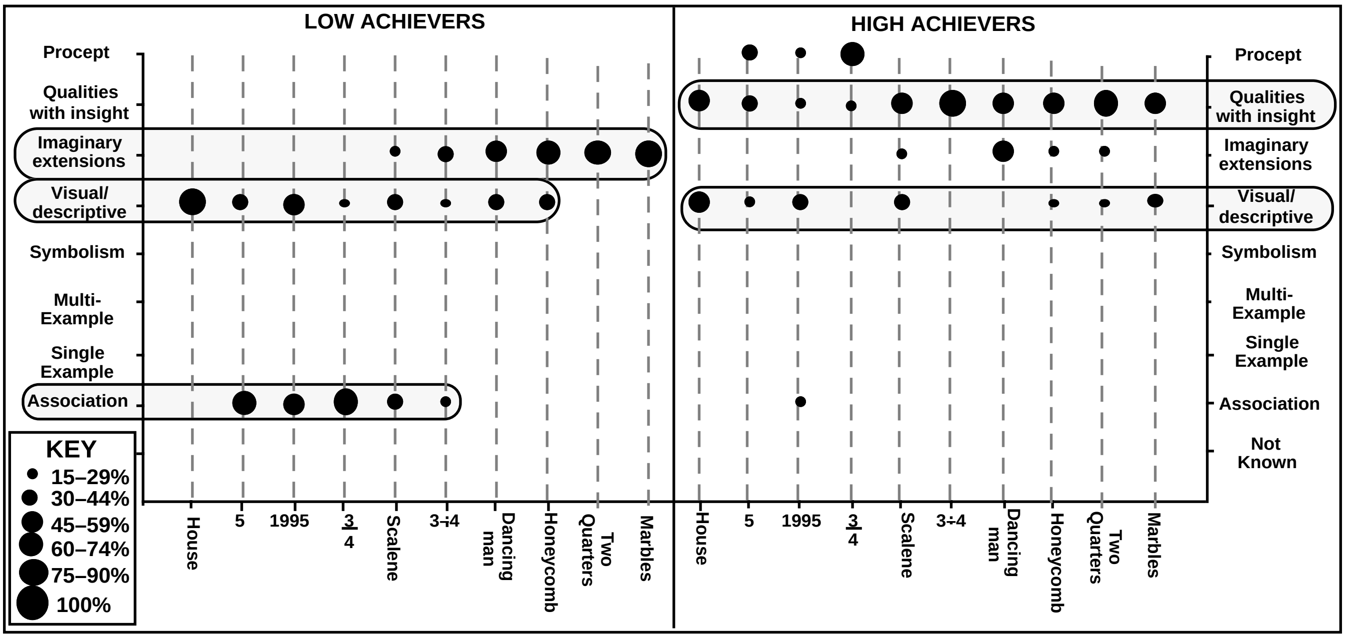


Figure 2: Classification of children's responses to symbolic and iconic presentations (Percentages)

- Low achievers had more uncontrolled reaction than high achievers to the visual items. The four iconic representations, 'marbles', 'two quarters', 'honeycomb' and 'dancing man', were strongly associated with 'imaginative description'. They were generally considered in an informal and isolated way. Detail was added as if they were "pictures out of focus". The children's efforts were directed towards inventing a story about them and giving each "picture" colour through surface details. Though the descriptions used a variety of contexts they were divorced from formal abstract vocabulary or notions such as number or shape.
- High achievers tended to look behind the icons and discuss their qualities with an insight that, where appropriate, had mathematical overtones. They recognised that icons could represent an idea. Those that could be associated with mathematics, for example, 'marbles', 'honeycomb' and 'two quarters' were described using formal mathematical language and the children provided extensive evidence of their ability to give a description of the non-visual characteristics. (Of course we are aware that these children may have anticipated the nature of "favourable" responses but as is indicated in the discussion the development of their responses changed with item difficulty). However, irrespective of age, the low achievers provided no indication that they were attempting to provide favourable responses.

- As they had shown with the numerical nouns, the low achievers extensively associated the mathematical symbols with personal detail and episodic memory. For example, the symbol **5** was strongly associated with somebody who was five, "*someone is five years old*", "*the five on a birthday card*". The 'imaginary extensions' given for $3 \div 4$ indicated the inclusion of concrete examples such as "*apples*", "*sweets*" ,"*chocolate*". In essence these children need to make reference to other concrete nouns when using numerical nouns
- The high achievers consistently considered the underlying qualities of the symbols and in several instances provided examples of proceptual thinking, for example $3 \div 4$ was identified as "*three over four, 3/4's, 75%, three out of four*". The children appeared confident in communicating their knowledge of abstract numerical nouns.

DISCUSSION

When directed towards a particular word, icon or symbol there are an indefinite range of conclusions that the child could have drawn from the event. The children's mental representations of any concept appear to possess similar characteristics. Either the children direct attention towards its core aspects –the essential features or definition–or they direct us towards identification features through which we may recognise instances of the concept. However, the properties of the former may be different from those associated with the latter.

The similarities in the children's descriptions of imagery are remarkable both for their consistency across the range of items presented, and for the differences they display between the high achievers and the low achievers. We consider these by looking at each group of items–words, icons and symbols–separately and presenting examples which highlight the qualitative differences between the children.

Looking at the Words

First we look at the word 'five'. It tells us how old or how many of something. The low achieving children provided comments such as "*my sister is five*". In such a context "five" indicates a property of "sister" and it has conceptual characteristics analogous to the concepts of "tall" or "big" used in a adjectival way. The auditory responses of the low achievers were strongly associated with such aspects, children talked freely about "five books" or "five fingers"–the concrete objects being the books or the fingers–but did not direct attention at the core of the concept, either the counting process or the numerical concept which was the noun.

Such interpretations would seem to indicate that, in a sense, the children's understanding is on a plateau. The given property does not provide a cue to the presentation of a property at a higher level of thought. Thinking which generates the description is essentially "horizontal"–it moves from one surface feature to the next and so on. In the numerical sense it moves from one series of countable objects to the next, for example, from books, to fingers.

Essentially such "horizontal thinking" was also experienced when children considered the ball. Children who describe the colour of a ball, or who went to great lengths to describe the visual characteristics of a football were also describing features not essential to the conceptual core.

Children who describe a ball as a spherical object, a round or ovoid object used in games etc. did not focus on horizontal, frequently discrete, properties. They provided a sense that their descriptive qualities could move in a vertical plain which they traversed to provide notions of the core concept or representational features almost at will.

Looking at the Icons

Similar patterns of behaviour were identified with the iconic representations. Again, though there are an indefinite number of conclusions that may be drawn from each item, the low achievers focused extensively on visual and imaginative characteristics. These were concrete, realistic and of a similar qualityin the sense that they were seen as pictures that required colour, detail and a realistic content. High achievers concentrated on the more abstract qualities.

Again we see notions of horizontal thinking arising when the children discussed the imagery of ‘two quarters’, ‘honeycomb’ etc. They provided imaginary extensions which were essentially the same in quality i. e. “*window with curtains, window with shutters, lift doors*.”

Amongst the high achievers their was evidence that item difficulty influenced their vertical movement. The icon ‘two quarters’ provided evidence of such movement being in a “top down” form. Initially it triggered ‘higher level’ mathematical interpretations that gradually moved towards imaginative description–it was a “*shape in four quarters, half shaded....picture of a window*”, or “*two out of four, half, cupboard, windows where they don’t use shutters*”. The more complex “honeycomb” presented evidence of a different analysis–a “bottom up” interpretation. Initially descriptive, each quality seemed to provide cues for the next level of processing –“*Hexagons, symmetrical, four light, four dark, one quarter and three quarters*”.

Looking at the Symbols

The special feature of mathematics is its symbolism. The qualitative evidence indicates that children’s interpretations of both linguistic and iconic stimuli have strong similarities with their interpretations of mathematical symbolism. ‘Association’ and ‘visual/description’ again dominates low achievers responses and these are accompanied by the need to concretise the symbols. This was achieved in two ways:

(i) by associating concrete items with the symbols, for example “*Three can’t be divide by four because there would be a remainder. You can do it with apples though*”, and,

(ii) allowing the symbols themselves to become the concrete items and then to describe the lines and curves which were their external features. This happened with several numbers to once again provide evidence of the horizontal thinking.

Amongst high achievers the abstract nature of the symbolism tended to draw upon either a bottom up analysis or, depending on familiarity, a top down one–vertical thinking. The "unfamiliar" $3 \div 4$ prompted responses of the former kind such as "*Fraction of some sort, not a whole number. Ah! Yes, three quarters*". 3/4 promoted both, "*Three quarters, four pieces, 3 of one sort, one of another, three quarters, one quarter, ratio 3:1....*", "*three quarters, point 75, 3 over four, three out of four, equivalence, four squares, three shaded*". This also shows that these children could describe the notions without the need to concretise them.

High achievers seem to be consistent in using the mathematical symbol as a procept and applying vertical processing characteristics in their elementary arithmetic. Similarly low achievers remain consistent in their need for concrete referents (external or internal) and horizontal thinking. For them:

- The mathematical symbol appears to be quickly translated into a concrete item, either as external referent or mental image.
- This concrete item can be changed–fingers can become sticks etc. but the quality of the item remains the same–a further example of horizontal processing.

Such transformation leads to counting processes repeated without reflection on the input/output link which, we suggest, inhibits the abstraction of, for example, the process of addition into the concept of sum.

CONCLUSION

The notion of action encapsulation lies at the heart of many contemporary theories of cognitive development in mathematics. However, actions on objects possess connotations that, we suggest, have strong implications for the quality of children's imagery. Children with different understanding of the nouns, icons and symbols associated with mathematics concentrate on different aspects of these to the point where they may attempt to encapsulate different kinds of action. The mathematics of the low achievers remains abstract; its symbols need concretising and its pictures focusing. By not understanding the nature of the abstract nouns or the symbolic nature of icons and numerical symbols we suggest they may not form the generalisations and relationships that are the hallmarks of proceptual thinking.

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