NUMBER PROCESSING:
Qualitative differences in thinking and the role of imagery

Eddie Gray

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

Demetra Pitta

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

This paper considers imagery associated with children's mental processing of basic number combinations. Children's verbal and written descriptions are used as a means of accessing their imagery and we see how the tendency to concentrate on different objects leads to qualitative differences in imagery and its uses. Children described as 'high achievers' provide evidence of an implicit appreciation of the information compressed into mathematical symbolism.. In contrast, 'low achievers' create images strongly associated with visual stimuli suggesting that these children, far from encapsulating arithmetical processes, are mentally imitating them.

INTRODUCTION

> "I find it easier **not** to do it [simple addition]with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. It can take longer to work out the sum than it does to work out the sum in my head." (Emily, age 9)

Although not explicit in Emily's comment, the meaning associated with her notion of 'concentrate' was related to the mental manipulation of a collection of dots. She was describing the difficulty associated with the simultaneous engagement of external referents–fingers–and the mental scan of a different series of referents–dots. The latter was preferred but the former was used because:

> "If we don't [use our fingers] the teacher is going to think, 'why aren't they using their fingers.....they are just sitting there thinking'...we are meant to be using our fingers because it is easier....which it is not". (Emily, age 9)

There is no doubt that Emily is only one of many children who prefer to do things 'mentally', or as has been described so frequently by children "in my brain". Many do so because they know things and engage in a form of automatic processing. Others have to make a conscious effort and do so, not because they realise that such effort, with practice, may gradually become automatic, but because of the social environment of the class; "*We are not allowed to use fingers*", "*I am too old for counters" and* perhaps the saddest from a boy of 10 who "*wanted to do things like the clever children*".

Recognising that others do things mentally does not give such children an insight into how things are done by others. This is the focus of this paper. It considers the relationship between procedures, concepts and images in simple arithmetic. To establish the latter it assumes that an image is mediated by a description (Kosslyn, 1980; Pylyshyn, 1973). It builds upon the notion that the language and concrete items associated with objects of thought possess different connotations. These have implications for the quality of children's imagery (Pitta & Gray, submitted) and their processing ability.

The evidence suggests that whilst proceptual thinkers focus on the flexibility of the symbolism and hold symbols as “objects of thought”, procedural thinkers may construct and utilise mental images which support their procedural interpretations of symbolism. If it is appropriate, they quickly translate the symbol into another object of thought, finger images, a number track or marbles. It is suggested that mental manipulation with these objects places such strain on the limits of the child’s working memory that it impinge against the continuing compression required for “constructive abstraction” (Kamii, 1985) and the development of proceptual thinking.

IMAGERY IN NUMBER PROCESSING

The means through which the co-ordination of actions may become mental operations was of interest to Piaget who believed that new knowledge is constructed by the learner through the use of “active methods” which required that “every new truth to be learned be rediscovered or at least reconstructed by the student” (Piaget, 1976, p. 15). Whether or not *all* children who display competence in the procedural aspects of early number activities undergo this process of constructive abstraction –which Kamii suggests is a construction of the mind rather than something that exists in objects–or indeed whether or not they abstract the appropriate thing is a mute point. The abstraction of a basic counting unit may form a platform from which children may gradually replace slower count-based approaches with more efficient fact retrieval processes. However, such procedural compression may not be so easily achieved by low achievers.

These observations lead us to consider imagery, though, because of the disguised nature of mental images it is only possible to make conjectures about them. They may appear to be well wrapped possessions, covered in many fine layers and sometimes even hidden in discrete packages. We may believe it is possible to shake the package to find out what is inside, but by doing this we run the risk of breaking it. The pitfalls, particularly in terms of operational definitions and interpretation are clearly identified by Pylyshyn (1973).

In cognitive psychology, it has been traditional to characterise mental representations as symbolic: a pattern stored in long term memory which denotes or refers to something outside itself (Vera & Simon, 1994). Such a characterisation is based on the assumption that the knowledge structures possessed by humans are symbolic representations of the world. Images exist, are used and may influence thinking.

It is suggested, though controversially so, that symbolic mental representations divide into analogical and propositional representations–essentially sensory dependent and language like representations. The classical analogical representation is the visual image–though images can be formed from other modalities–which appears to have all of the attributes of actual objects or icons. They take up some form of mental space in the same way that physical objects take up physical space and they can be mentally moved or rotated (see Boden, 1988). Propositions, as mental representations, may represent conceptual objects and relations through, for example, mathematical symbols or spoken words. Gray & Tall (1994) suggest that the symbols of elementary arithmetic serve the ambiguous purpose of representing processes *and* concepts.

Deahenne & Cohen, (1994) suggest that the relationship between different forms of representations may be seen through the presentation and solution of arithmetic facts. Symbolic, verbal and the analogical representations support the transcoding of numbers into whatever internal code is required for the task in hand. It is transcoding approaches which require the use of working memory in the absence of external representations that we are particularly interested in this paper. Symbolism promotes direct verbal routines and flexible transformations by proceptual thinkers. Amongst procedural children, where symbolism is more iconic (static) we see the occurrence of analogical forms of imagery which we suggest may inhibit the potential for flexible interpretation.

METHOD

Twenty four children were selected within in a "typical" school of the English Midlands to represent the chronological ages 8+ to 12+. This provided a sample of six children from each year, three 'low achievers' and three 'high achievers'. Achievement was measured levels obtained in the Standard Assessment Tasks of England and Wales ((SCAA, 1994)) or scores obtained from the Mathematical components of the Richmond Attainment Tests (1974). Children were interviewed individually for half an hour on at least four separate occasions over a period of eight months.

Following the presentation of range of auditory and visual items (Pitta & Gray, submitted) the children were presented with a series of one and two digit addition and subtraction combinations, for example, 6+3, 9–5, 13+5, 15–9. Children's responses were obtained using semi-structured interviews recorded through field notes, audio and video tapes. Children were asked to talk freely about their imagery and what came to mind during the solution processes for each item. Solution approaches were classified similarly to that of Gray & Tall (1994). Whilst external representations were partially identified through children's sensory motor activity, evidence of images relied extensively on verbal and written description by the children. Though no precise claims can be made about the nature of their imagery it is evident that a pattern does emerge.

RESULTS

First and very briefly, because of space limitations, we draw together the general solution strategies and associate these with the type of representations used.

1. Strategies and Representations: Combinations to Ten

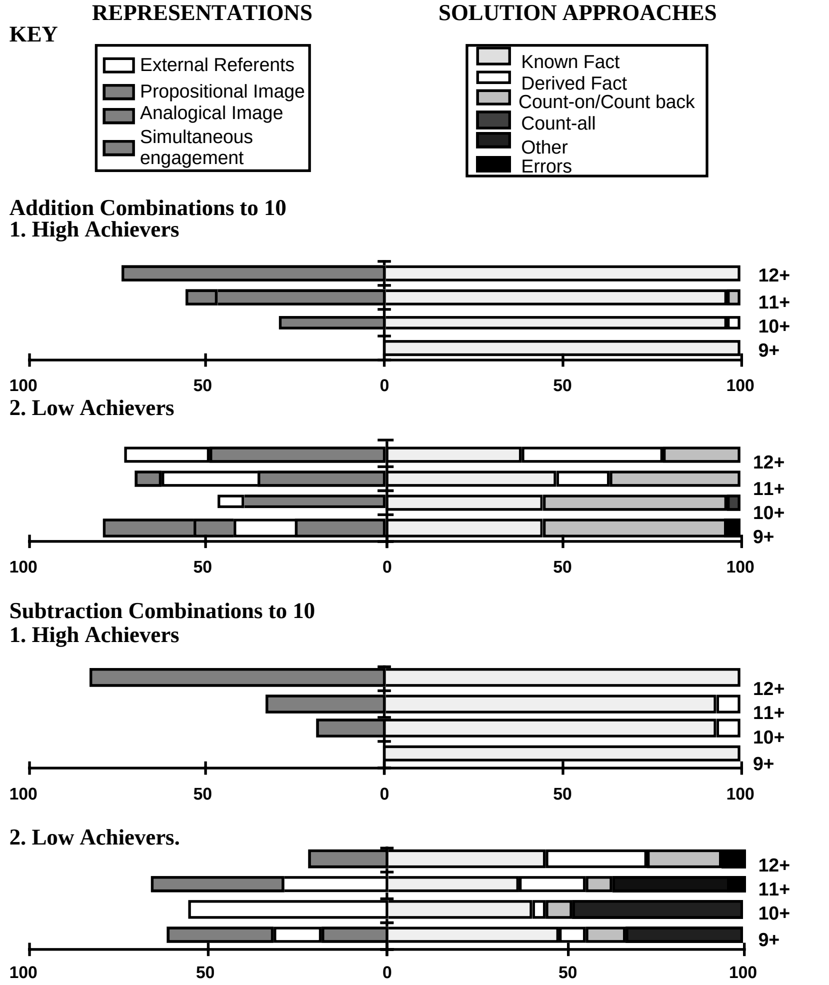


Figure 1: Strategy combinations and representations used to solve addition and subtraction combinations to ten (percentages)

Figure 1 (p.4) shows the strategies and associated representations used by the low and high achievers to obtain solutions to the number combinations to ten. The representations are subdivided to illustrate percentages which indicate:

- the use of external referents such as fingers.
- where children's verbal description may be associated with conceptual objects represented by numerical symbols. This notion is loosely tied in with that of propositional images.
- mental imagery associated by analogy with external referents–analogical images.
- the simultaneous engagement of external referents with an analogical representation

Several features emerge from the analysis of Figure 1 (p.4):

- Amongst the high achievers there is the almost complete absence of procedural methods associated with counting and there is no evidence of the use of external representations–verbal enunciation was associated with images of numerical symbols, either the expressions themselves or the final solutions.
- Amongst low achievers we note:
  - the imagery of 11+ and 12+ children when solving addition combinations is dominated by symbolism supported by analogical representations.
  - the absence of symbolic representation amongst these two year groups when dealing with subtraction was associated with the fairly extensive use of external referents by the 11+ group.
  - the increasing use of external referents amongst the younger children and, in some instances, we note that these are simultaneously engaged with analogical representations.

At this point, the use of only immediate recall and counting methods amongst the 9+ and 10+ “low achievers” indicates qualities which would enable them to be identified as procedural. The 11+ and 12+, since they collectively display the integrated mixture of known facts, the use of known facts and some evidence of counting procedures may be seen to display proceptual qualities when dealing with addition and subtraction combinations to ten.

Strategies and Representations: Combinations to Twenty

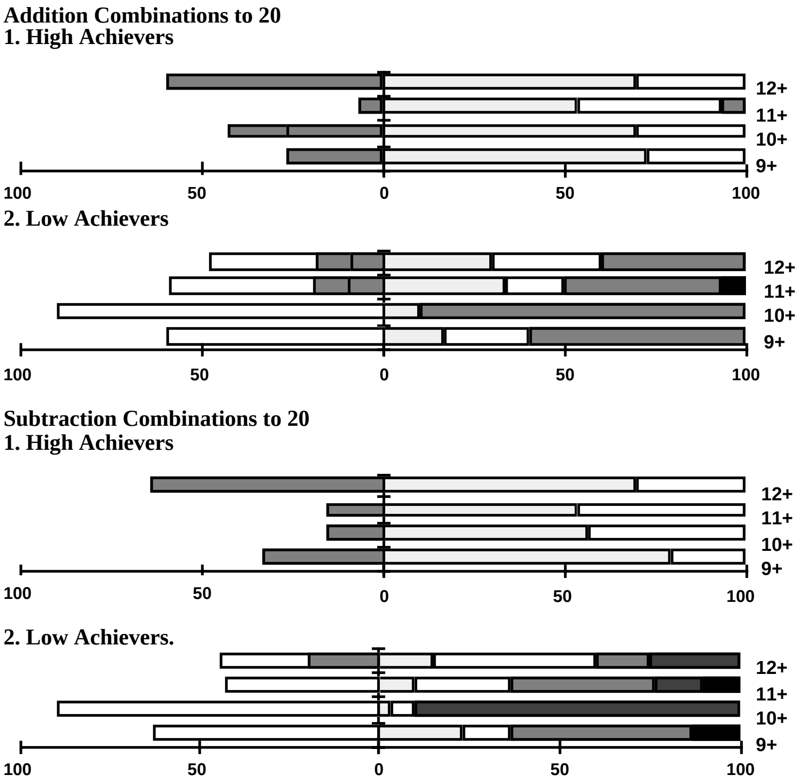


Figure 2: Strategy combinations and representations used to solve addition and subtraction combinations to twenty (percentages)

The classifications identified for Figure 2 are as those specified for Figure 1. We note immediately the greater proportion of derived facts used by the high achievers and the more extensive use of counting, particularly with external referents, by the low achievers.

- The proceptual thinking of the high achievers may be identified through enunciation that refers to images of symbols associated with the initial expressions, semantic transformations of the expressions or from the solutions.
- Amongst Low achievers we see that reference to symbolic images is far less evident. The fairly extensive use of derived facts amongst the 11+ and 12+ children is no surprise. Their strategies generally serve to support the evidence given from different samples cited in Gray and Tall (1994).

In general, the evidence shows that the high achievers did not use external representations to solve any of the problems. They either recalled solutions or provide extensive evidence of semantic elaboration, both approaches being associated with “images of arithmetical symbols”. Amongst the low-achievers, only the 11+ and 12+ indicate any reference to imagery without the simultaneous engagement of external referents.

Amongst low achievers, we detect a decline in symbolic related imagery and a “regression” from internal to external representations that is both age related and associated with problem difficulty. On the whole their imagery is associated with analogical representations which support counting procedures. We suggest this soon forces them to reach the limits of working memory and makes life so extremely difficult for them that they recognise the "safety" in using external referent. Gear *et al* (1991) have suggested that a component of developmental difficulties in mathematics is a working memory deficit. In our next section we provide an alternative reason which suggests that on the contrary these low achievers may show an extraordinary use of working memory. Their problem is one associated with its use as well as its capacity.

DISCUSSION

The verbal reports of the proceptual children provides evidence of the important role that symbolism plays:

1. In those instances where we are able to identify known fact responses, these symbols have skeletal qualities, they carry the ideas and offer the potential for process/concept ambiguity. They require no detail to make them operational. Placed on the minds scratch pad they are interpretations of input data or precursors to verbal output but they are associated with retrieval of simple facts without regard to quantities involved. However, our evidence to date does not allow us to contribute to the controversy that may surround notions of verbal coding (see Dehaene & Cohen, 1994)

2. The different degrees of complexity associated with the use of derived facts, particularly with number combinations to 20, provided a variety of examples where expressions were decomposed into simpler known facts, for example, 9+8=8+8+1, 15–9=15–10+1. Perhaps one of the points of interest was the tendency of the 11+ children to indicate that they "did not see anything" although notions of "thought it" were strongly in evidence. This is an issue that we feel needs further clarification. We suggest that nothing was written on the scratch pad and verbal coding could have taken place.

Finding solutions to the expressions through derived facts requires two features not necessarily apparent when using known facts. The first is the possession of a good understanding of the quantities involved in the original problem, for example noticing that 9 is close to 10, and the second involves the use of working memory. However, we suggest that use of the latter is minimised because the children almost intuitively recognise cognitive referents associated with the inputs–disregarding perceptual properties they focus on the relationships associated with the objects of thought–the procept.

It was this ability to recognise the proceptual characteristics of the expressions and their associated symbolism that highlighted the difference between the low achievers and the high achievers. The former had proceptual options available to them but we are not in a position to indicate whether or not their images at this point were functionally significant. The evidence from the low achievers appears to be quite different; no matter what numbers they were dealing with, each individual, on failing to recall a fact, generally they evoked a procedure which they saw common to all combinations. Usually this involved counting, particularly if external referents were used, but this was not always the case when imagery was reported. Usually images given by the low achievers appeared to be functionally significant–they appeared to have a direct role in the processing procedure.

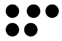
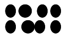
Pitta & Gray, indicate how low achievers interpretations of nouns, icons and symbols were strongly associated with the perceptive aspects of the stimulus. There appeared to be a need to concretise objects. It appears that such distinctive behaviour also guides these children's approach to basic number processing. In the mental world we may see an almost automatic representation of the stimuli as images of countable objects. These may be seen as analogues of, for example, fingers, tally's, number tracks or marbles, each providing an image of the quantity associated with particular numbers. On hearing the expression the children appear to disregard the semantic aspects and move immediately to analogical magnitude representations and use these as anchors for mental manipulation–numbers quickly become concrete objects.

The dominant representations identified amongst the low achievers were associated with a range of images from pictorial representations of a hand with fingers, through iconic representations of fingers and tally lines. The oldest children indicated how they labelled these tally lines and saw images of number tracks or number lines. The evidence was that children who developed such images used discrete objects with a double counting procedure. Two points emerge. First, the horrendous strain on working memory. Not only is the child maintaining sight of the analogical representation but also focusing on discrete numbers in that representation. This is associated with counting-up one set and counting back another. Indeed, one child described how two 'calculators', by description circular number tracks, operated in different ways, one keeping track of how many had been counted by decrementing in ones, the other keeping track of the answer which was incremented in ones. Every calculation, with slight modification, was the same–it always involved double counting. Indeed this was the case with all of the children who used such images–all involved double counting of linearly arranged objects, some labelled some not labelled. Such children seldom gave evidence of the use of derived facts. Indeed it is hypothesised that seeing images of discrete objects supports the counting process but does not lead to the realisation of the power and or compression associated with mathematical symbols. Instead of deriving facts and using what they know about numbers, a sort of vertical processing, the children display some element of creativity in changing their images of countable objects. They use different referents to carry out the same procedure, a form of horizontal processing (Pitta & Gray, submitted).

Such an interrelationship was developed by the few children who used dynamic images composed of marbles or dots. Images of pattern formation dominated their mental manipulation. Marbles can move position, fingers cannot. Fingers require sequential processing, marbles do not.

> "[with] the dots....it's....it's easier because you don't have to keep on thinking, "No its that

> one I need to move, no its that one or that one", because it doesn't really matter which one you move" (Emily, age 9)

But this was not the only advantage. because each item could move position independently of the others. A pattern of  may easily become  combining readily with  to make , or "two fours". In such a way derived facts may be developed and indeed this did lead to their use amongst two of the low achievers.

Amongst some of the younger low achievers the evidence of simultaneous engagement of mental imagery and external representation caused confusion until one representation dominated over the other. If we do two or more things mentally, for example, count-up, count-back and maintain a mental picture we gain some insight into the strains being placed on working memory.

CONCLUSION

There are limits to the size of working memory. Whether or not these limits are different for those children we identify as high achievers compared to those we see as low achievers is not resolved. Their implicit appreciation of the information compressed into numerical symbolism enables them to focus on the detail appropriate at the moment. However, this feature is not unique to their approach in mathematics. In the broader context symbolism, and the ability to focus on the many relationships asssociated with it, provides them with an economical means of utilising the power and space they have available. We would not like to give the impression that high achievers did not use and manipulate visual images. When dealing with more difficult two digit combinations all high achievers considered visual symbolic images in vertical form, even though they were given verbally,and made transformations which enabled them to process them more easily. Low achievers, giving more attention to different elements, found it even more difficult to mentally hold the initial inputs. They appear to place much greater reliance on a visual stimulus and create and manitpulate images associated with this. They have a much greater tendency to talk about things that may be captured by the senses and their imagery tends to be strongly associated with real concrete objects.

Notions of procedural encapsulation and the steady compression of lengthy counting procedures into numerical concepts imply that children recognise links between inputs and outputs. It would seem that far from encapsulating arithmetical processes some children reconstruct these processes mentally. Attempting to match their thoughts to given representations may only help them see things enactively, as with marbles, or iconically, as with the number line. It is those who realise that representations may be used to simplify ideas and are not intended to stand alone who will share in the construction of meaning.

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