COUNTING: THE CONTINUING STORY

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In her interesting synthesis of the significance of counting Maclellan[1] presents a story that should be extended to gain a full perspective of its importance. This extended story is a very important one because it can set the scene for the child's broader mathematical development. Though it starts by linking the act of counting with the concept of number, it is one which may have two different conclusions. One ends with the acquisition of flexibility which can be the source of great strength and power to the learner, whilst the other, though it may provide a basis for short term success, leaves the learner with the prospect of attempting to deal with a much harder form of mathematics which will eventually leading to failure.

The development of numerical concepts and skills within young children has received such attention over the past two decades that the importance of meaningful counting as a basis for arithmetical development would now appear to be beyond question. However, it is too simplistic to believe that many children count because of the intrinsic satisfaction they derive from the activity. Many children count because they cannot do otherwise. What is acceptable in the development of early number concepts is not so acceptable when children are attempting combinations in the teens and may be the source of extreme stress and difficulty when children are acquiring skill with basic addition and subtraction algorithms. An excessive reliance on counting consigns children to a mathematical cul de sac.

But I race ahead of myself. Let us take up the story from the point where Maclellan leaves it – counting seen as the earliest way in which children can begin to construct the concept of number.

Recently David Tall and I were looking at the concept of number and the way in which we may describe it. At the University of Warwick we stopped some students and lecturers as they walked between lectures and asked them two simple questions:
"What does the word triangle mean to you?"
"What does the word five mean to you".

There was very little hesitation amongst the students as they attached meaning to the word "triangle". "It's a two dimensional shape with three sides", "...with three angles", "...it can have equal sides and equal angles." Within the space of a few minutes different students gave a broad spectrum of meaning with relative ease. However, when attempting to provide some sense to the word "five" these same students hesitated and searched for a response. The student who replied that it was "The number that comes after four and before six" was not let off the hook. She was asked to tell us what the word "four" meant.

The explanations given by most of our interviewees gave the sense of five in terms of countable items; "It's five apples I suppose". But even as they responded in a similar vain some students laughed nervously. They seemed to guess that there was something more to it and thought it ridiculous that they couldn't give a verbal answer to satisfy their own intuitive understanding. A mathematics students response became very complex and quite beyond the scope of this article – but in any case he hadn't read the book properly.

Here is a strange dilemma. We are all familiar with numbers. We use them everyday of our lives and yet here we find undergraduates who find it difficult to explain precisely what is meant by the word "five". There doesn't appear to be the same difficulty with the word "triangle"...most of us have seen one so we can describe it. Perhaps it is a difficult task because the number five is not a concrete object like a triangle; it is a word in a sequence of words we use for counting.

Through their experience of the counting process children learn to fuse a counting action with a sequence of number words; they learn that the last number word tells them how many things have been counted. Familiarity with counting enables the child to use the number name to stand for countable items, but an important shift in thinking is made when the focus is expanded to include the number words used in the process of counting and these same words being conceived of as number concepts in their own right. On the one hand it is possible to recreate the process that gave rise to the named quantity and on the other it is possible to talk of that quantity as if it were a "thing". By giving the "thing" a name we can speak about it and we can hear about it. We create a symbolism that can be written and can be read. These forms of communication allow the symbol to be shared in such a way that it has, or seems to have, its own shared reality. "Five" is an abstract concept, but through using it in communication and acting upon it with the operations of arithmetic, it takes on a role as real as any physical object. The "thing" can become a tool which can be taken out of its original context and assume the characteristic of an object. It is as if the created object had substance in the same way as a triangle has substance.

The concept of number is formed from the compression of counting processes, it is an action encapsulated as an object. To make sense of the word "triangle" we do not encapsulate processes. We increase our understanding by looking at shapes that posses the qualities which are characteristic of things defined as triangles. Our understanding grows through examining shapes that possess similar underlying qualities albeit there may be some differences in these, for example, sides of the same length, sides of different length, possession of a right angle, etc. These additional qualities help us to refine our understanding of triangle. It is this difference between numerical concepts which arise from the compression of processes and elementary geometric concepts that arise from our experiences of handling, assessing and comparing that our interviewees found difficult to articulate.

The encapsulation of arithmetical processes into an arithmetical concept may be seen much more clearly if we consider the process of addition becoming the concept of sum. Children's growing sophistication in handling the addition of numbers can be seen as a steady compression from counting processes to handling number concepts. We can see this by considering the sum 4+3. The most elementary method used by children is to count 4 objects, then count 3 objects and then to put all the objects together and recount the total starting from one. This succession of three counting processes is called "count all", which, for some children it can be a fairly lengthy procedure.

The next stage occurs with the realisation that it is not necessary to count the first set in this way. Now the number 4 is seen as a number object and the child simply counts on a further 3 numbers in the number sequence. The sum of 4+3 becomes, $\underline{4}$, 5, 6, 7. This process is called "count on".

A further stage of compression occurs when both numbers are seen as objects and the child remembers the sum as a known fact so that the sum of 4 and 3 is 7. Experience usually leads on to the encapsulation of 4 + 3 as a known fact but such facts can be learnt in two distinct way: by rote or in a meaningful way. If they are learned in a meaningful way it may later prove possible to use them flexibly to derive new knowledge, for instance we may use the fact that 4+3 is 7 to establish the fact that 14+3 is 17. This is called a "derived fact".

It can be seen that the two interpretations based upon counting evoke different *processes* to accomplish the addition of 4 + 5 whilst the third may evoke a *concept*–the *sum* of 4 + 5. The fourth interpretation uses higher order thinking to decompose and recompose concepts such as those of the third component. We now begin to see more clearly what underlies the above experience. The notation represents a process to do, which can be progressively compressed to be manipulated as a mental object.

The move from "count all" to "count on" is not as simple as it seems. "Count on" is a sophisticated double counting process. To calculate 4+3 by "count on" requires not only counting on beyond 4 in the number sequence but also to keep a check that precisely three numbers are being counted. Amongst young children we may see different units being used to aid such a counting process – counters, fingers, even rhythmic movements of the head. Indeed, we may see children use any number of idiosyncratic aids to support their counting procedure, particularly if they continue to use count-all and the combination involves a number from the teens.

It is possible to see children up to the age of eight or nine solving simple addition or subtraction combinations using counting procedures but the compression of such procedures into known and derived facts gives a child a much more powerful method of doing arithmetic. As they encounter problems with larger numbers children are able to use the knowledge they posses to solve the new problems:

9 + 6 = 15 because “9 + 1 = 10; 10 + 5 = 15”
8 + 6 = 14 because “6 + 4 = 10; 10 + 4 = 14”

Children who know facts and use them flexibly find arithmetic far easier than those who have to rely on counting. Such flexibility can be seen in stark contrast to the difficulties experienced by children whose purpose appears to be satisfying the need to gain competence in counting. Whilst counting procedures may be successful for simple combinations they become extremely difficult with larger numbers.

When subtraction problems are encountered they prove much easier for the flexible child who may see a subtraction problem as being equivalent to a known addition fact:

9 – 7 = 2 because 2 + 7 = 9
14 – 6 = 8 because 4 + 2 = 6: 8 + 2 = 10: 10 + 4 = 14

The child who relies on the addition procedure of “count-on” is much more likely to see subtraction as the inverse procedure of “count-back”. Count-back can become very difficult as the combinations become progressively harder. Christopher, who was seven, had been very successful using counting to obtain solutions to addition and subtraction combinations to 10. He was so successful, and covert, that as a result of the Key Stage 1 Standard Assessment Task he had been attributed with achievement at Level 2 in knowing and using number combinations to 10. However, the greater proportion of solutions he gave for the addition combinations were obtained through count-on, the subtraction ones through count-back. When he attempted subtraction combinations to 20 in an attempt to achieve level 3 the difficulties emerged. When faced with 19–13 he tried count back. In the first instance he had to establish a procedure which would support the counting process identified as count-back. At the same time he had to keep keeping track of how many were counted. Identifying the procedure took several seconds but eventually he decided to use his fingers. He then started the count back procedure uttering out loud the backward number sequence from 19:

<u>19</u>, 18, 17, 16, 15, 14......15......14, 13, 12, 11, 10, 9, 8, 7, 6.

It is a surprise that in this case he was successful. But see the difficulty that emerged. He was a bit unsure of the backward counting sequence—and how many of us actually think of giving some reinforcement to this sequence without counting—that he momentarily forgot whether he was going up the sequence or coming down it. Christopher also has some difficulty keeping a check beyond ten on his fingers. If we allow children to use their fingers for counting, and indeed such use may be a necessary stage in the compression process, we should be aware that all too frequently such an approach is self taught. It may provide relative success for simpler combinations but procedure used may not generalise to larger numbers. Children may have to invent new methods to extend their earlier counting procedures to a more complex situation.

Jane (7) was very successful using her fingers to subtract from numbers less than ten. 9 – 2 caused her no problems. Without counting she extended nine fingers. Again without counting, she curled down two of them and, yet again without counting, she looked at the remaining extended fingers and said “Seven!”. Her earlier experiences of using her fingers for counting had been assimilated to the point were she could subitise quantities on her fingers very easily but such a procedure could not easily be used for 11–2. She had to attempt counting but she only had ten fingers. It took her several seconds to devise a method which involved counting her left thumb twice and then subitising the remainder. On this occasion she was successful but an attempt to modify of the same method caused horrendous difficulties when she attempted 20 – 5.

The important thing about the children mentioned above is that the counting they used for combinations to ten was so efficient, and frequently very covert, that at the age of seven they were each regarded as “average” ability in their number work. At one level their mastery and confidence in their personal counting techniques provided an automatic response to particular combinations but it is suggested that these same qualities were reasons why underlying number relationships were obscured. Even young children who are constructing the concept of number begin to look for relationships between the “things” if they do not have to rely exclusively on counting. For example Thomas, who was just five, volunteered information about the numbers that make five “four and one make five....two and three make five....five and nothing make five....”. When asked what 1 + 4 made he had some difficulty—he couldn’t remember. It wasn’t an insoluble problem. He counted on his fingers to find it; a more difficult relationship was established through the use of counting.

If we look closely at any of the seven and eight or nine-year-old year old children that we teach we are able to contrast the difference in the quality of thinking between those who extensively rely on counting procedures and those who use flexibility brought about by the compression of these procedures. However, up to this point, the examples presented do not consider the even greater difficulties experienced by “less able” children. Thomas who was nine provides us with an example of their difficulties. He had complete confidence in the use of his fingers. Indeed he had previously stated that “I use my fingers for counting. That is what they are for”. He was very successful with numbers to ten but where numbers beyond ten were used he did not have a general counting procedure which he could use for all combinations but each one presented a new situation accompanied by the search for an appropriate procedure. For 6 + 8 he used count-all so that by the time he reached ten he had exhausted his supply of fingers. Very, very slowly he continued the count imagining fingers to the right of his right hand. The real problem was that 6 + 8 was only part of two digit combination, 36 + 48. His counting procedures, even though lengthy, gave correct solutions to what he regarded as two separate combinations 6 + 8 and 3 + 4. The solution to the two digit combination was given as 74.

It may be argued that the differences in methods of approach to basic number combinations outlined above occur because children are at a different points in a spectrum of mathematics learning. Whilst this may well be true in some cases Gray & Tall [2, 3] indicate that this does not hold true for *all* children. Their evidence shows that *all* children *do not* progress through the ever more complex stages of mathematics reflecting, even at different rates, similar styles of mathematical thinking. Their analysis suggests that there are points of bifurcation that lead to different qualities of thinking and it is hypothesised that these in turn have implications for long term mathematical development.

Count-on and its complimentary procedures of count-up and count-back may be seen to play a pivotal role in the early development of this thinking as it relates to arithmetic. These procedures can be important steps towards the encapsulation of arithmetical concepts. They can also be the reasons why the underlying relationships are obscured and this can have serious consequences for developments which are considered to be algebraic.

As part of their programmes of study within the National Curriculum young children are required to recognise the use of a symbol as an unknown. Notions of “unknowns” are frequently developed through missing addend sentences, of the form $3 + \square = 10$. There are of course several ways of doing this; recalling the number combination, knowing something about the relationships that link the number triple 3, $\square$, 10, where $\square$ can only be 7, or employing a counting procedure. To children who do the two former the problem is trivial. To those who attempt the latter it can be very difficult. Children who operate procedurally using count-all don’t know how many are in the second count —“What has to be counted?” . Those who operate with count-on need count up to 10 from 3 keeping a check on the amount counted. But what about $\square + 3 = 10$ – “What do I start with?” Think of the horrendous difficulties such children may have when faced with $\square + \Delta = 10$. This form of missing addend problem abounds in commercial texts. To establish the framework for the symbolism their should be the presupposition that children possess some knowledge of number relationships which may include the understanding that addition and subtraction are complimentary. Without this platform, the spectrum of performance that may be observed in classrooms will highlight that children having the greatest difficulty are those attempting to do the hardest mathematics. Far from giving them insight into early stages of algebraic thinking the focus of attention is on searching for an appropriate arithmetic procedure.

As teachers we want to help our children do mathematics but behind this desire lies a potentially dangerous conspiracy between teacher and child. We offer support to child “invented” methods to deal with mathematics even when the eventual outcome of such a method may lead to tremendous difficulty. To resolve the difficulty, which all too frequently arises amidst the other pressures within the classroom, we respond to the cry “show me how to do it” by doing just that. For a time everybody is happy. My colleague David Tall has put this very succinctly; the child receives gratification, the teacher is pleased that the child can do something and parents and politicians are satisfied that progress is being made. But if we continue to show children the procedures of mathematics we may end up by confining them to a cul-de-sac of mathematical short-sightedness which ends up in terminal failure.

The ability to count is at the foundation of the number concept and arithmetic. Its extended application can also be seen as the parting of the ways leading to a spectrum of performance in simple arithmetic. At first it is necessary to focus on the specific and more evident action and then only later focus on the subtle and generative deeper concepts. But, as we have seen, interpretations of simple arithmetic can all too frequently lead too extensive reliance on the former without taking the leap of insight that gives the latter. We may perpetuate this by continuing to present the thousands of examples that children do day in and day out from their mathematics texts. Some children begin to see the conceptual simplicity fairly easily. They have a new route within which to apply their flexibility. We also know that others, far from gaining insight into this simplicity, continue to only gain practice in procedures which may in fact inhibit their insight.

If, alongside their experiences which develop procedural competence, we could only provide these children with the opportunities which allow them to concentrate on the relationships involving the objects produced by the counting procedures. If only, in some way, such parallel experiences removed the need for lengthy sequences of action so that the child could concentrate on the inputs and the outputs without the procedural clutter.

There are possible ways ahead using new technology. In arithmetic this means allowing children to use a calculator to help focus on the meaningful relationships. The facility on modern graphic calculators which enable several calculations to be seen on screen at the same time is particularly helpful in this respect. With such a tool children who may become focused on the procedural aspects of the arithmetic can be refocused on the concepts without the strain of carrying out the procedure. As teachers, we may choose the emphasis. We need to recognise that the more successful child is the one who ambiguously uses either process or concept, which ever is most appropriate at the time. In such a way we may safeguard the importance of counting within a framework which ensures that we are attempting to lessen the gap that occurs between the thinking that gives short term results and that which gives the flexible thought processes characteristic of the more successful.

References:

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