Tackling the Problems: An Explanation for Success and Failure

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Abstract

Arithmetical word problems are believed to provide a rich source of experience from which we may confirm children's understanding of arithmetical concepts. By drawing together the relationship between the different approaches children use to solve elementary arithmetical problems this paper considers the importance of procedural compression on the development of understanding. Children's interpretations of arithmetical symbolism forms the basis for a theory of compression from which the qualitatively different thinking that leads to long term success or failure may be considered.

Introduction

It has been known for some considerable time that when faced with similar problems whether contextual or context free, children use a variety of approaches. Some indicate misconceptions, some provide the correct solution for the wrong reason, others provide considerable insight into the child's mathematical thinking By considering a theory drawn from a model of the relationship between the strategies children use to solve elementary problems in arithmetic, this paper considers the pivotal role that the interpretation of mathematical symbolism plays in the development of qualitatively different thinking about mathematics. Such a theory suggests that the arithmetic of the successful is conceived in such a way as to be, for them, relatively simple, whilst the less successful are doing a different kind of arithmetic which is often intolerably hard.

The context of word problems within mathematics, and within arithmetic in particular, is an interesting one. It arises from the belief that a student's mathematical knowledge and understanding is assumed to comprise problem-solving skills as well as the facts and procedures that they have studied. The development of such skill has frequently been associated with the use of word problems (Kantoski, 1981). However, the contemporary view, certainly at a more advanced level, is that problem-solving means solving other problem types such as non routine mathematics problems and real (application) problems. Through such means we may encourage the development of active mathematical thinking; a process that not only expands our understanding but also enables us to increase the complexity of the ideas we can handle (Mason, Burton & Stacey;1982)

It is suggested that word problems presented in the elementary mathematics curriculum do not fit such a paradigm. More frequently, the basic underlying pedagogical and epistemological assumption is that they provide useful practice. Additionally, particularly in early arithmetic, they allow pedagogues to introduce general conceptual patterns framed within a ‘real life’ context. However, solutions may not reflect “appropriate understanding” nor provide any notions of children’s misconceptions. Some insight may only be gained if we talk to children.

Consider the following examples:

First, Lauren, aged nine, had indicated that she could not explain what a seventh was. However, she was able to answer the question “find one seventh of fourteen?” and give the correct answer “two”.

When we look closely at the quality of this “two” we see that the procedure she used did not help her to find one seventh but one half. This was not because of a conscious attempt to find one half and neither was it because she recognised the relationship between a half and a seventh. She had recalled the relationships between the number triple 14, 7, 2 and quickly established that 14 partitioned into sevens gives two. Her justification for the answer implied that she already knew the solution and only invented a procedure to explain it. Her approach, counting zero to seven and then seven to fourteen, was indicative of her misconception of seventh. Equally this procedure, in different circumstances, may have been too difficult to use since it implied an iterative approach.

In the second incident, Ricky, was asked : “How long is half of a 30cm piece of string?” There was some surprise when Ricky answered, “that’s one that you can’t find exactly–its between 15 and 20”. When he explained his approach the reason became very clear.

> “Look at my fingers. We can count in five’s. Five, ten, fifteen, twenty, twenty-five, thirty. There’s six fingers”. Six fingers were extended.
>
> “If we want to find a half we can start again. “Five, ten, *fifteen*”. The first three counted fingers were grouped together. Ricky continued to count the remaining three extended fingers, “twenty, twenty-five, *thirty*”. These fingers were then grouped.

Ricky went on to explain that a half of the thirty was the gap between the fingers extended to show fifteen and twenty.

From these two incidents we see Lauren using a mental representation of a number line with a questionable procedure to justify a correct solution. Ricky, using a visual representation with a suitable procedure but misinterprets the representation and consequently has difficulty obtaining a solution.

Framework for a theory.

A sequence of the developmental levels of conceptual structures and solution procedures for word problems within the field of elementary arithmetic was given by Fuson (1992). Within this sequence it is possible to see elements of the courser analysis presented by Carpenter, Hiebert and Moser (1981). Their general classifications that have now become prominent in discussions of children's strategies in this field. They have proven to be of considerable use to classroom teacher's and, be taken with Sowder's (1988) analysis, could profitably provide a support to teacher's diagnosis of a child's quality of achievement when solving word problems.

Gray (1991), identified that the strategy classification associated with elementary word problems may also be associated with related context free problems. Thus for a word problem such as:

> *John had 4 pencils. His mother gave him 3 more. How many pencils does John have now?*

or the associated context free problem 4+3, the addition process which involves the two single digit numbers may be implemented through:

- a procedure which involves counting both numbers and then counting the combined total: **count-all**
- a procedure through which the value of one set is conceptualised and the addition process completed by counting using: **count-on** (or count-up/ count-back)
- the use any other known number fact: **derived fact**
- knowing directly: **known fact**.

When children's preferred approaches over a range of problems were considered it was revealed that there was a divergence in qualitative thinking between children who used a combination of different strategies. On one hand there were those who wished to remain at a procedural level and made use of a variety of representations to support counting procedures–dots, eyes, fingers etc. On the other, there were children who provided no evidence of the use of external representations. Although some of these used mental counting strategies and others knew the solutions, yet others were making use of knowledge that was known and operated at a conceptual level which was very flexible:

"Four and four is eight and one loose is seven"

It is not easy, as we shall see, to fit "knowing" into any developing theory but, overall, what the children were revealing, without articulating, was a divergence based upon their procedural and conceptual understanding. The spectrum of performance revealed that, for those who relied extensively on procedural methods, mathematics was an activity to be carried out in accordance with a sequence of actions. In contrast, other children saw mathematics as actions on numbers as arithmetical objects; the numbers had a quality which was both concrete and available for manipulation.

Several studies (for example, Dubinsky, 1991) suggest that such differences may occur because some children go through a process of 'procedural encapsulation' whilst others do not. Gray & Tall (1994) suggest a more supple underlying theory that relates not just to conceptual differences but to the versatility that may be achieved through the flexible use of processes and concepts.

The notion of procedural encapsulation is somewhat complicated within mathematical contexts. A procedure such as counting has the potential to lead to conceptual understanding but its repetition through practice can result in the acquisition of procedural competence. This, because of growing automacity, may impinge upon encapsulation or lead to the acquisition of knowledge through rote learning and sheer memory work. Knowledge acquired this way may muddy the theoretical waters. In simple arithmetic known facts may be either rote or meaningful.

A theory of compression.

The term *procept* was born to describe the amalgam of *pro*-cess and con-*cept* represented by the same symbolism. It arose from the observation that symbolism played a pivotal role by being seen sometimes as concept sometimes as process. Such a theoretical notion eases the description of the innate flexibility that may be obtained from the composition and recomposition of encapsulated mathematical objects. Thinking that projects the ability to use this flexibility through the use of a procedure, an encapsulated procedure or the flexible compression and decompression of encapsulated procedures we defined as *proceptual thinking*. In contrast, thinking that projected the ability to apply a specific algorithm to carry out a process is seen as *procedural thinking*. We suggest that major differences in learners' attempts to become "mathematical" may be considered within the framework of the theory of procepts.

The theory of procepts is in essence a theory of compression. If we reconsider the approaches children use to deal with the elementary contextual and context free combinations we now begin to see more clearly what previous experience is telling children: notation represents a *process* to do, which can be progressively compressed to be manipulated as a mental *object*. As children's general conceptual structures progress, lengthy procedure such as count-all may first be compressed into count-on–often making use of "count-on from largest". Compressing such a procedure lays open the possibility of a further compression into a known fact; an addition procedure is encapsulated into the concept of sum.

Quality of thinking

> To overcome the peculiarities of the human brain with its massive long term memory and small short-term working memory, some method of compression must be pursued to allow the working memory to cope with complex situations. (Gray & Tall (1994, p.2)

Short term memory plays a crucial part in the actions carried out by individuals. Only through being processed in working memory can information from the sensory part of the system enter a person's long term memory. It is only when information is called out of the person's long term memory can stored information be used in the course of thinking. We see how Lauren used her long term memory to find one seventh of fourteen. However, her response suggests that we should not always use practice to confirm understanding but sometimes to disconfirm it.

Although it does not appear possible to extend the capacity of short term memory it does seem possible to aid retention so that working memory is able to cope with complex situations. One method, almost routine in many schools, is to practice a procedure until it becomes routinised to the point that it may be applied in particular situations with a degree of automacity which does not require much conscious thought. We may place many word problems into this realm of practice. However, though practice may make imperfect (Foster, 1993), it does seem reasonable to believe that if the learner expends a considerable amount of time in developing efficiency with a procedure its use may dominate particular situations.

Procedural compression provides the potential for the emergence of styles of thinking triggered by different interpretations of mathematical symbolism. On the one hand we see procedural thinking based on routine manipulation of procedures, on the other, the flexibility of proceptual thinking. Such a distinction leads to the notion of a proceptual divide between those who think procedurally and those who think proceptually.

Standard tests of attainment may fail to discriminate between the two though the consequences may go a long way towards explaining long-term success and failure in simple arithmetic (Gray, 1993,1994)

Concluding remarks

In our efforts to answer the question "What is it that makes mathematics so trivially easy for some and yet so horrendously difficult for others?" we see the theory of procepts providing one way of considering the differences that arise. However, it was quickly realised that there were mental objects in mathematics which are *not* procepts. The concept of number is formed from the compression of counting processes, it is an action encapsulated as an object. To make sense of the word "triangle" we do not encapsulate processes that arise from arithmetical actions. We increase our understanding by looking at shapes that posses the qualities which are characteristic of things defined as triangles.

One of the problems with the formulation of a mathematics curriculum is that it is conceived by people who are successful. These are people who are able to think in sophisticated ways because of their ability to compress knowledge. Their mathematics is seen from a mature viewpoint in which the structures have great richness and interiority and they therefore have perceptions of simplicity in which this structural richness plays an implicit fundamental role. Learners do not yet have this conceptual richness, but the belief is that as long as teachers explain to them how to manipulate sophisticated thoughts they are giving them power and strength. However, short term need in the form of procedural growth, tends to take precedence over long term development. Failure becomes a more distinct possibility than long term success.

Frequently, to confirm understanding, we provide practice, but the more children work at remembering procedures the more they are likely to use them. If all of their effort goes into this solution to their problems they may remember procedures but, paradoxically, they may possess less understanding.

Those who are successful see their mathematical landscape in its entity; they begin to see how things fit together. For many learners there is a temptation is to nip up the nearest hill and take a local view. As teachers we may subscribe to this temptation for the short term benefit of the learner. Unfortunately these local views may not fit together; they may well be separated by the chasm that is the proceptual divide.

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