Difficulties teaching Mathematical Analysis to Non-Specialists

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This paper reports the effects of teaching mathematical analysis to students who are to be teachers of elementary school children, yet who take analysis as the final summit of their mathematical studies at university. The students concerned divided into three groups. A tiny minority understood the formalities of the subject and the need for logical proof, the majority attempted to learn definitions by rote but in the main failed to understand the underlying concepts, and the remainder used inappropriate concept images from earlier mathematics. This paper questions the rationale of teaching formal analysis at degree level for those who are not specialist mathematicians.

Introduction

This paper considers the almost insignificant effect that a course in analysis had in changing the quality of mathematical thinking of a group of students who, training to be elementary and secondary school teachers, follow the course as a high point of their university degree programme. Evidence from written assessment and individual interview shows that only a tiny minority of the students are moving in a direction that would eventually enable them to utilise the formal aspects of mathematics. The majority did not recognise the need for formality. It was a surprise to find some students, even at this level, attempting to generalise from the particular; despite their extensive work with real numbers, their concept image had not expand to take in the notion of the concept definition. Knowing the concept definition by heart did not guarantee that they understood the concept (Vinner, 1992). Their experience prior to meeting the formal definitions not only affected the way in which they formed mental representations of the concepts (Tall, 1992), but frequently became manifest through their efforts to resolve problems with an inappropriately “evoked concept image” (Tall & Vinner, 1981).

A high proportion of pre-university mathematics teaching tends to emphasise calculation and manipulation of symbols to get “answers”. In such an atmosphere the acquisition of the concepts has an intuitive basis which is founded upon experience (Tall, 1992). Such a paradigm contrasts starkly with that utilised to develop advanced levels of mathematical thinking; formal definitions give rise to concepts whose properties are reconstructed through logical deductions.

The study of analysis may be seen as an attempt to introduce the student to the formality that is the hallmark of the working mathematician; the general thought patterns of the students are encouraged to change from a mode which relies extensively on the formation of concepts through the encapsulation of process as concept (Gray & Tall, 1994), to a mode which is structured within the realms of concept definition. However, the transition from one form of thinking to the other is a difficult one. Though mathematicians use definitions and formal language in a meaningful way to compress mathematical arguments, the learners method of thinking about mathematical concepts can depend on more than the form of words used in a definition.

Vinner (1992) has outlined students possible responses to cognitive tasks associated with the implied use of definition: the desirable one in which the student is not supposed to formulate a solution before consulting the concept definition, and a more usual model where the respondent is unaware of the need to consult the formal definition but places emphasis on a concept image. In the instances considered in this paper we show there was little very little evidence of the former but a considerable emphasis on the latter. But perhaps it is impossible to avoid the mathematical tensions that arise between the mathematcs tutors' desire to introduce students to the rigour of mathematical proof and the student perceptions that may be dominated by other considerations:

> "When I got the piece of work back my main concern was with what I had got. Unfortunately being so preoccupied with other things I am doing....I am fully aware of the fact that the things I did last year and even last term are going to be out of my head unless I think about them again. What I said to you earlier about relating everything, well it just goes against that philosophy... basically I have a problem of relating..." (Third year undergraduate student)

The context

At the end of a first course in Analysis, 20 students, all following a four year course leading to a teaching degree with mathematics as their main subject, were given written tasks that required a demonstration of their understanding of the use of definitions introduced during the course. Though there were three items within the package of assessed work we will consider student responses to the first, a problem which focused on the their understanding of real functions and the definitions associated with differentiability and continuity. As a result of the analysis of the students efforts seven students were invited to take part in more detailed individual interviews.

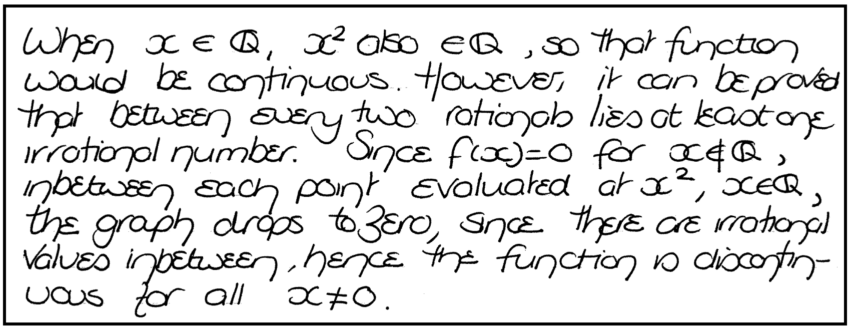
This first item invited students to:

Explain why the function $f(x)=\begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ is discontinuous for *all* $x \neq 0$,

The students written responses showed that the majority of them tried to avoid as much as possible the use of formal language; they worked mainly with an image and/or tried to use a dynamic or procedural version of the definition. In their responses it was possible to identify the coexistence of these characteristics with older images that had remained unchanged by the new theory. In other instances it was possible to identify "incorrect" images constructed on a misunderstanding of the theory.

Image using

Though all of the students had been taught the concept definition, only one student used it to solve the problem. By far the greater majority of students provided evidence of attempts to reconstruct a proof through a concept image, without reference to the concept definition; in some cases with verbal reconstruction (Figure 1):



When $x \in \mathbb{Q}$, x^2 also $\in \mathbb{Q}$, so that function would be continuous. However, it can be proved that between every two rationals lies at least one irrational number. Since $f(x)=0$ for $x \notin \mathbb{Q}$, inbetween each point evaluated at x^2, $x \in \mathbb{Q}$, the graph drops to zero, since there are irrational values inbetween, hence the function is discontinuous for all $x \neq 0$.

Figure 1: Verbal reconstruction of the proof for continuity

One attempted to be more explicit about the image (Figure 2):

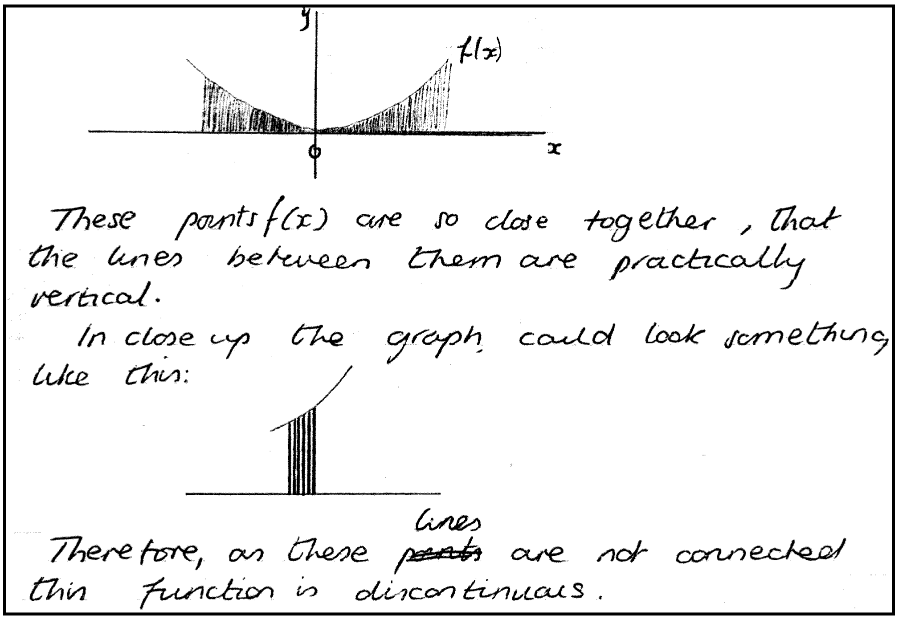


Figure 2: An evoked concept image of continuity

Though this student was interviewed later, no further insight into his image was forthcoming. However, some was gained from the interview with another student. Asked to draw pictures of functions that could not be differentiated this student drew the graphs shown in Figures 3 and 4.

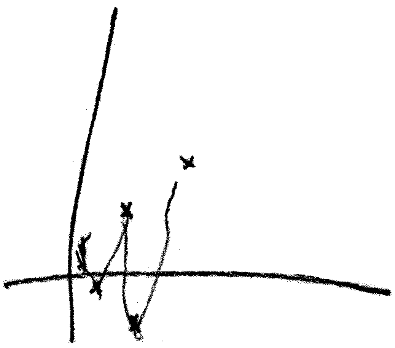


Figure 3. Student image of a non-differentiable and discontinuous function

As she drew Figure 3 the student commented:

> "I still think if you could differentiate at a point [pointing to a cross]... if you joined those together [joining up the crosses] like that you could still find the gradient at a certain point....you can have the gradient between two of those points, that would be the gradient if it was a straight line."

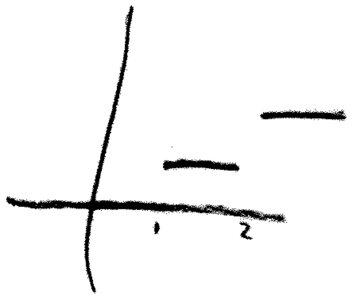


Figure 4. Student image of a non-differentiable and discontinuous function

As she drew the second graph the student was asked if it was possible to join the points:

> "No... because I've seen one similar to that [Figure 3] on a graphical calculator, and I've seen that one as well [Figure 4]."

The student was asked if she could provide an example of the formula for the function she had seen drawn in the calculator, but she replied:

> "I can't remember. It wasn't exactly that [Figure 3] it was similar. It had lots of little bits there [in the calculator] and then got wider."

Three isues would appear to arise from the students efforts to compensate for their inability to providere the appropriate concept definition:

- The image associated with the "linked points" of a graph prevents any formal association between the concept definition and an appropriately formed concept image.
- Such an image may be reinforced by misconceptions that arise from an automatic use of graphic calculators and computer programs; initially students may not associate the relationship between the graph, the defined function and the associated procedure (see also Hunter, Monaghan & Roper, 1992).
- Students learn images and intuitive ideas by rote; some seem not to worry about basic foundations upon which to relate knowledge meaningfully.

Reconstructing the definition

The images some of the students constructed differed significantly from the ideals that mathematicians would wish to be constructed from the definition. The statement "Between two rationals there exists an irrational; between two irrationals there exists an irrational" was translated by some students to mean:

> "since there is always an interval around each rational p/q where x is an irrational..."

> "there is always an interval around some rational x where x is irrational and..."

Such representations provide an example of student's imprecision in the use of mathematical language and their difficulty in dealing with quantifiers which may arise from interpreting theorems in such a way.

Helped by a "redefined" model:

The $\mathbb{R}$ must ∴ be ordered...rational... irrational...
rational ... irrational.... etc.

one student simplified his arguments to prove that the function given in the first question is continuous at x=0. This student could conclude:

> let α be the $\mathbb{R}$ either side of $x = 0$ (ie $(\pm\alpha)$
> from earlier explanation 0 will lie between two irrational numbers (ie $\pm\alpha$ = irrational numbers)
>
> $$f(0+\alpha) = f(\alpha) = f(\text{irrational}) = 0$$
> $$f(0-\alpha) = f(-\alpha) = f(\text{irrational}) = 0$$
> $$\therefore\ f(0+\alpha) \to 0 \quad \text{and } f(0-\alpha) \to 0$$
>
> $\therefore$ at $x = 0$ $f(x)$ is left and right continuous

During the interview this was placed into a context by the student:

> "We had worked out in class that between any two rationals you always find an irrational, between any two irrationals you'll always find a rational, so from that I deduced that if you took two rationals you'll always be able to find an irrational in between, so I put down on my assignment that it was alternating between rationals and irrationals, which is wrong I think...Why do I think it is wrong? To be absolutely honest with you I haven't really looked at it properly to work it out which I know I should, but all I remember is thinking that I was right when I did the question."

Such a student would require some considerable time to synchronise his model with the proposed theoretical model. These students will not have this time.

The individual interviews confirmed the evidence received from assessment. Each interview started with a series of common questions to establish students understanding of the formalities and central concepts arising from the analysis course. The students selected for interview (N=7) were drawn on one hand from those whose written work had shown evidence of the interplay between personal description and a concept image and those who, on the other, displayed the inappropriate use of a concept image.

Space precludes presentation of the "formal" questions but the following synthesis will allude to them and highlight the most important issues that arose from the interviews.

- None of the students gave the formal definition of continuity and neither could they state how to calculate correctly "the derivative of f at a point in the domain D where $f: D \to IR$.".

- Since the student's examples of differentiable and non differentiable functions were the same as those given for continuous and discontinuous functions, it is hypothesised that their concept images of these notions were the same (Vinner & Tall, 1981). Their confusion over these two ideas could be seen even when they attempted to provide a formal definition:

> " A function is continuous if it can be differentiated at every point within a range"

"A continuous [function], you can differentiate that.....if you have two points on it, it is continuous between the two points then you can differentiate that"

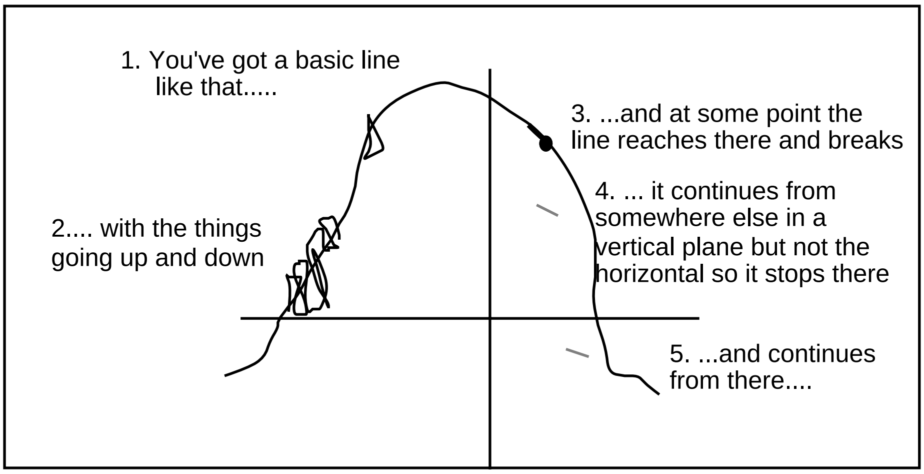
- Whether or not a function could be defined at a point determined whether or not it was continuous for some students

"...where you pick two points and a point between can be defined as well. You've got a curve which continues because whichever point you pick there's always another point on the line, there's no gaps in the curve."

"Continuity is every single point has another value"

Some others had a confused image that they could not synthesise in words:

"I don't know the definition but I know that it's where all the points if you drew them in a graph all points.....well they are not up and down all over the place".



Unable to write the definition of continuity one student indicated that the images of continuous function she possessed were from graphical work (Fig. 5):

"I can vaguely describe what a continuous function is on a graph".

Figure 5: A students evoked concept image of a continuous function

This student's attempt to describe such a function with her graph were almost indecipherable. However, she did indicate that

"I am just remembering a few things but it is not coherent at all".

Discussion

It seems that a great problem in dealing with mathematics lies in the fact that the theory was constructed upon aims that students do not achieve. Partially, this is because the composite theory is not made explicit but hidden behind the formal language and apparently clear hierarchies which mathematicians use to present the subject matter. Students have difficulty linking the language and the sequential steps of the hierarchies to form an overall theory encompassed within an understanding of the reasons for its formation. Many, destined to acquire definitions by rote learning, attempt to support these through intuitive ideas and the reproduction of procedural aspects of the theory. Even though they may be given intuitive experiences to support the formal aspects, being unable to understand the relationship, they evoke previously established concept images which are not good enough to build upon. They acquire definitions with no supporting content; they evoke images from within the school mathematics curriculum.

Additional evidence for such an hypothesis could be found in the students' efforts to classify numbers as rational or irrational. One student who had no understanding of the difference and who made little effort to obtain any, stated that:

> "I always look these up when I need to know what they are. I've got a list of all the different symbols and what things mean and I usually refer to that when I need to know, but it hasn't stuck yet."

Another, though aware of the definition, preferred to use a concept image when analysing $0.\dot{9}$:

> "If you rounded that up it would be a rational number."

His explanation of this comment indicated that he did not understand what one means:

> "I don't know, it's just like .999... is too close to 1 but I don't know whether that makes any difference to a rational or an irrational number being so tiny. I'm just guessing."

A third had difficulty classifying zero as rational or irrational but even though he attempted to work with the formal definition he failed because the latter was misunderstood.

> "...zero isn't it ? I don't know...Maybe it's an irrational. I'm not really sure whether you can have division by zero....Zero divided by zero, normally you can't have zero on the bottom of a division line because it's undefined, so therefore it can't be defined as *p* over *q* so it must be irrational."

This evidence of students rote learning, both of the definition and the concept image, must be placed alongside additional evidence which illustrates that students knowledge of mathematical concepts may take on a variety of identities (Duffin & Simpson, 1993). We suggest that though such variety may be strongly associated with students conception of real numbers, the real numbers may still not be natural in the sense used by Duffin & Simpson even for students at this level.

Conclusion

This paper presents some evidence that arises from the mismatch that can occur when students who are not candidates for advanced mathematics are faced with the rigours of advanced mathematical thinking. The vignettes serve to support the evidence provided by Vinner (1992) but we would wish to look more closely at the longer term prognosis for the mathematical development of the students considered. Although only one student provided evidence of a reasonable understanding of the place of concept definition in analysis, all of the students described within this paper achieved at least pass grades in their assessed work–largely through a kindly interpretation of the marks.

From the evidence of the assessment and the individual interviews the students may be seen to fall into one of three groups:

- A very small group (N=2) which seemed to be moving towards a formal understanding of the subject matter using the formal definitions meaningfully or recognising the need for formal language and logical proof.
- A second, much larger group, (N=10) who, though they evoked the use of a concept image to support personal description, did not effectively use formal definitions. The majority of these students revealed that they had initial difficulties interpreting problems in the context of the theory. Such difficulties could be manifest through the limited considerations they gave to crucial aspects of the problems, for example, considering rational cases but not irrational ones, or arguments augmented with superfluous–in the sense that they provided more than the necessary–repetitive considerations.
- A third group of students (N=8) used inappropriate concepts images formed from earlier mathematical conceptions which remained largely unchanged as a result of the course in analysis. Such students attempted to establish a formal result by generalising from specific cases or they displayed an inability to link procedural and conceptual images of function and graphical representation.

The laudable desire to lead these students towards the formality of mathematics was thwarted for two reasons. Not only do they not appear to be ready to start the course–and thus the assumptions underlying the move to formality were not met–but, more importantly, they will have no opportunity to consolidate their knowledge to the point where concept definition and concept image have appropriate associations. When faced with formal aspects of a theory which they do not construct for themselves, students can ignore not only its convenience but also the arbitrary and respective reasons for each theoretical construction and each definition; important links can be missed and such deficiency will give way to a collection of fragments which bear little relationship to each other.

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