TRYING TO GET IT RIGHT:
A PERSPECTIVE ON COUNTING

Eddie Gray
Mathematics Education Research Centre
University of Warwick

Introduction

Recently, at the University of Warwick, David Tall and I asked some students and lecturers two questions: "What *does the word triangle mean to you? What does the word five mean to you"*.

There was very little hesitation amongst the students as they placed some meaning on the word "triangle". "It's a two dimensional shape with three sides", "...with three angles", "...it can have equal sides and equal angles." Within the space of a few minutes different students gave a broad spectrum of meaning with relative ease. However, when attempting to provide some sense to the word "five" these same students hesitated and searched for a response. The student who replied that it was "The number that comes after four and before six" was not let off the hook. She was asked to tell us what the word "four" meant.

The explanations given by most of our interviewees gave the sense of five in terms of countable items; "It's five apples I suppose". But, even as they responded in a similar vain some students laughed nervously. They seemed to guess that there was something more to it and thought it ridiculous that their verbal response couldn't satisfy their own intuitive understanding. An undergraduate mathematician tried to give a very complex response but, as David remarked, "he hadn't read the book properly".

Here is a strange dilemma. We are all familiar with numbers. We use them every day of our lives and yet we may find it difficult to explain precisely what they are. There doesn't appear to be the same difficulty with the word "triangle"... most of us have seen one so we can describe it. Perhaps it is a difficult task because the number five is not a concrete object like a triangle; it is a word in a sequence of words we use for counting.

Counting and the Number Concept

If we look closer at counting we may gain some insight into the foundation of the dilemma. Although counting procedures do not involve formal mathematics, efficient and meaningful counting forms the basis for early number development. Experience of the counting process provides the child with the opportunity to fuse a counting action with a sequence of number words and they learn that the last number word tells them how many things have been counted.

An important shift in thinking is made when the child's focus is expanded to include the number words used in the process of counting and the use of these same words as number concepts in their own right. On the one hand it is possible to recreate the process that gave rise to the named quantity and on the other it is possible to talk of that quantity as if it were a "thing". By giving the "thing" a name we can speak about it and we can hear about it; we can write it and we can read it. These forms of communication allow the symbol to be shared in such a way that it has, or seems to have, its own shared reality. "Five" is an abstract concept, but through using it in communication and acting upon it with the operations of arithmetic, it takes on a role as real as any physical object. The "thing" can become a tool which can be taken out of its original context and assume the characteristic of an object; the counting process becomes encapsulated as a numerical object and it is as if the created object has substance in the same way as a triangle has substance.

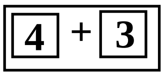
To make sense of the word "triangle" we do not encapsulate processes. We increase our understanding by looking at shapes that posses the qualities which are characteristic of things defined as triangles. It is the difference between numerical concepts which arise from the compression of processes and elementary geometric concepts that arise from our experiences of handling, assessing and comparing that our interviewees found difficult to articulate.

Interpreting Arithmetic Symbolism

We see then that early arithmetic involves the process of counting which becomes compressed into the concept of number with symbols playing a pivotal role. Children's growing sophistication in handling the addition of numbers can be seen as a steady compression from counting processes to handling number concepts.

As an example let us consider the expression "4 + 3". When faced with this expression children at different stages of development might respond to it in a number of different ways. "4+3" can mean either the process "add four and three together” (which can be performed by various counting procedures) or the concept “the sum of four and three”. We can see the concept of 4+3 evolving from the compression of a series of counting processes:

Count-all

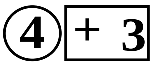


(count four, count three, count-all)

The first, count-all involves three separate processes; the process of counting four, the process of counting three and then the process of counting-all.

Count-on

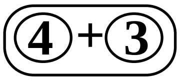
...a compression of count-all into a shorter procedure.



(number 4, count-on 3)

It is realised that there is no longer a requirement to count out both inputs. One may be held in the mind as the numerical object “four” and the second amount counted on to it.
Four plus three becomes “four, ... five ,six, seven”.

Know that 4+3 is 7:



(number 4, number 3, result 7)

Count-on may provide the basis for further compression. The process of counting becomes consolidated as a known fact—it has led to the concept of sum. “Four plus three is seven.”

Derived Fact

“I know 4 + 4 is eight so 4 + 3 is one less”

Known facts may provide flexibility when dealing with harder combinations. We use the fact that 4 + 3 is 7 to establish the fact that 14 + 3 is 17.

We now begin to see more clearly what previous experience is telling children: that notation represents a *process* to do, which can be progressively compressed to be manipulated as a mental *object*. A child may have one or more of these interpretations at a given time and of course some facts, mainly the doubles are easier to remember than others; they may be acquired as part of a rhythmic pattern “two and two is four, three and three is six...”. Those who count-all may know a few facts but they are unlikely to derive facts (Gray & Tall, 1994). Some children may use a variety of approaches whilst other rely extensively on counting. For those that remain procedural counting can be a source of security in some aspects of arithmetic but it can prove to be horrendously difficult in others. More successful children leave count-all behind and compress knowledge with great flexibility and fluency using a combination of count-on, known facts and derived facts.

It can be seen that the two interpretations based upon counting evoke different *processes* to accomplish the addition of 4 + 3 whilst the third may evoke a *concept*–the *sum* of 4 + 3. The fourth interpretation uses higher order thinking to decompose and recompose concepts such as those of the third component. We now begin to see more clearly what underlies the above experience. The notation represents a process to do, which can be progressively compressed to be manipulated as a mental object.

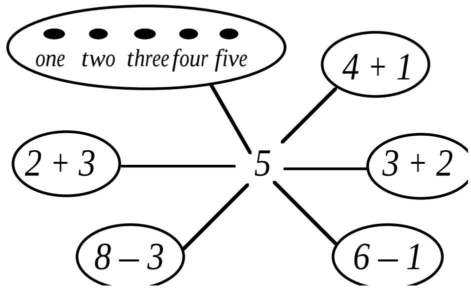
The move from “count all” to “count on” is not as simple as it seems because “count on” is a sophisticated double counting process. To calculate 4+3 by “count on” requires not only counting on beyond 4 in the number sequence but also to keep a check that precisely three numbers are being counted. Amongst young children we may see different units being used to aid such a counting process – counters, fingers, even rhythmic movements of the head. Indeed, we may see children use any number of idiosyncratic aids to support their counting procedure, particularly if they continue to use count-all and the combination involves a number from the teens. All to frequently mastery and confidence in a personal counting techniques can provide an automatic response to particular combinations but it is suggested that these same qualities are reasons why, for some children, underlying number relationships are obscured. Children can use known facts to build unknown ones; known facts used in a meaningful way help children to *derive* other facts. As combinations become more difficult those who know the facts and can use them flexibly find arithmetic far easier than those who have to carry out procedures.

A related form of compression may lead to known subtraction combinations but it is conjectured that the successful child is the one who blurs the distinction between addition and subtraction; subtraction is just another way of looking at addition. Such flexibility can be seen in stark contrast to the difficulties experienced by the children who use procedures.

The ambiguity of symbolism

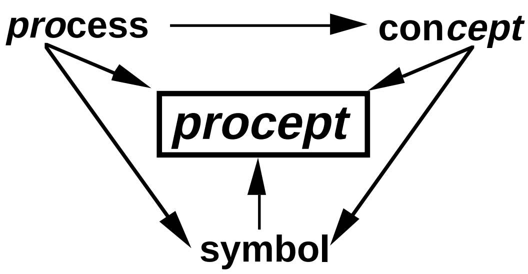
In simple arithmetic the counting process and the object formed from that process are fused in the representation given by the spoken or written symbolism—something even the undergraduate mathematicians didn’t manage to explain. As a simple example we may consider the symbol “5”. It can be spoken, it can be heard, it can be written it can be seen. Whenever we see the symbol or hear its name we can recreate the counting process that gave rise to the symbol or use the concept of five without any reference to countable items, as if it were a real thing The symbol five represents both a process of counting five and the object five.

But it doesn't end there. Many different processes give rise to the object five. Not only the process of counting... one, two, three, four, five, but also the process of adding four and one, the process of adding three and two... the process of taking three away from eight, of two away from seven...



The symbol "5" represents the compression of a considerable amount of information. It can ambiguously represent the outcome of a counting process or the concepts identified by "sum" and "difference". It is for this reason that Gray & tall (1991, 1994) introduce new terminology and call such a symbol a *procept*.; the portmanteau word procept refers to the combination of *pro*cess and con*cept* represented by the same symbol.

A procept is of course a special kind of concept. It is one which is first met as a process. Symbolism is then introduced for the product of that process and this takes on a dual meaning representing the process and the object created by the process.



Those fortunate enough to maintain sight of the ambiguity of symbolism can develop and utilise its power. It is conjectured that such ambiguity provides a means of expressing the cognitive development which gives power and flexibility to the more able whilst at the same time providing some explanation for the difficulties of the less able: it exhibits the proficiency with which the more able are diverted away from the procedural cul-de-sac which becomes the inevitable destination of the less successful. Flexible thinking that arises from the ability to view symbolism *either* as a trigger for carrying out a procedure *or* as the representation of a mental object which may be decomposed, recomposed and manipulated at a higher level is at the root of powerful mathematical thinking to overcome the limited capacity of short-term memory. At Warwick we call this form of thinking *proceptual thinking.*

However, many children see the object of the game remaining within the realm of *doing*,; of becoming successful in carrying out the procedure. Their thinking focuses on a specific algorithm for implementing a process – it is *procedural* in nature. Procedures require more short term memory than compressed objects and are therefore harder to handle. Count-on may well be the point of bifurcation in arithmetic (Gray, 1993) leading to a widening gulf between the successful proceptual thinkers and the less successful procedural thinkers which we have termed the *proceptual divide* (Gray & Tall, 1994).

Conclusion

We all appreciate the difficulties that young children experience in simple arithmetic and the consequences this may have for competence in more advanced skills. An analysis such as that presented above does suggest that the child's arithmetical competence will be improved if we concentrate on the development of flexibility. One of the problems with young children is that they spend so much of their time in early arithmetic counting to development of mastery and confidence in personal counting techniques. These may become automatic responses to particular combinations but it is suggested that this is one reason why underlying number relationships are obscured. Because of their lack of success in arithmetic it seems almost natural that some children turn to their well tried and tested procedures, even if they are difficult to generalise and may even be inappropriate for the task in hand. Extensive practice in simple arithmetic may cement the very counting procedures that will eventually lead to failure. We therefore need to combine the practice of those facts which are essential building blocks within the system with the flexible means whereby they can be manipulated most easily. Such flexibility is only achieved if the child is in possession of some number facts. The learning of basic number facts and number tables can promote the development of this flexibility if they are automated. But this does not mean that such knowledge is simply set within a rote learning context. Those who are more successful at arithmetic have more than this. They use the arithmetical symbol in a flexible way in which it is both a process which enables them to do mathematics and a mental concept which enables them to think about it.

References:

Gray, E. M. & Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education*, (Vol. 2, pp. 72–79). Italy: Assisi.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*. **25**, 2, 115–141.