Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic

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In this paper we consider the ***duality between process and concept*** *in mathematics, in particular* ***using the same symbolism*** *to represent both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The* ***ambiguity of notation*** *allows the successful thinker the* ***flexibility in thought*** *to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema. We hypothesise that the successful mathematical thinker uses a mental structure which is an amalgam of process and concept which we call a* **procept**. *We give empirical evidence from simple arithmetic to show that this leads to a qualitatively different kind of mathematical thought between the more able and the less able, in which the less able are actually doing a more difficult form of mathematics, causing a divergence in performance between success and failure.*

Introduction

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation.
>
> William P. Thurston, Fields Medallist, 1990.

Mathematics has been notorious over the centuries for the fact that so many of the population fail to understand what a small minority regard as being almost trivially simple. In this article we look at the way in which mathematical ideas are developed by learners and come to the conclusion that the reason why some succeed and a great many fail lies in the fact that the more able are doing qualitatively different mathematics from the less able. The mathematics of the more able is conceived in such a way as to be, for them, relatively simple, whilst the less able are doing a different kind of mathematics which is often intolerably hard. "A small variation in notation" will be seen to hide a huge gulf in thinking between those who succeed and those who eventually fail.

Process and Procedure

It will prove fruitful in our discussion to distinguish between our use of the terms "process" and "procedure". The term "process" will be used in

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a general sense, as in the “process of addition”, the “process of multiplication”, the “process of solving an equation”, to mean a cognitive or mathematical process (or both). A “procedure” will be used in the sense of Davis (1983, p.257) to refer to a specific algorithm for implementing a process, for instance the “count-all” or “count-on” procedures to carry out the process of addition, or an individual’s idiosyncratic procedure using mental or physical props, such as counters or imagined fingers to carry out a computation.

The Perceived Dichotomy between Procedure and Concept

Hardly a decade passes without concern being expressed over the general level of children’s attainment in mathematics, the nature of the mathematics curriculum or the quality of children's learning of the subject. In the USA the NCTM Standards reflect the perceived need to improve children’s performance. Within the United Kingdom the imposition of a National Curriculum (1989) is aimed at “raising standards” of performance in all subjects, including mathematics. The requirements of this curriculum distinguish between the skills or procedures that an individual needs to have acquired in order that they can *do* things, and the concepts or basic facts which they are expected to know on which they operate with their skills. This suggests a fundamental dichotomy between procedures and concepts, between things to *do* and things to *know*. However, in mathematics, we shall see that the truth is somewhat different.

Procedural aspects of mathematics focus on routine manipulation of objects which are represented either by concrete material, spoken words, written symbols, or mental images. It is relatively easy to see if such procedures are carried out adequately, and performance in similar tasks is often taken as a measure of attainment in these skills.

Conceptual knowledge on the other hand is harder to assess. It is knowledge that is rich in relationships. Hiebert & Lefevre (1986) describe conceptual knowledge as:

> a connected web … a network in which the linking relationships are as prominent as the discrete pieces of information … a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognises its relationship to other pieces of information.
>
> (Hiebert and Lefevre 1986, pp. 3–4)

Flexible thinking using conceptual knowledge is likely to be very different from thinking based on inflexible procedures. Yet procedures still form a basic part of mathematical development. Indeed, there is distinct evidence that such procedures can play a subtle role in concept formation in that the learner’s interiorization of procedures can lead to their crystallization as mental objects that can form the focus of higher conceptual thought.

Process becoming conceived as Concept

Piaget speaks of the *encapsulation* of a process as a mental object when

> ... a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower.
> (Beth & Piaget 1966, p. 247).

Dienes uses a grammatical metaphor to describe how a predicate (or action) becomes the subject of a further predicate, which may in turn becomes the subject of another. He claims that

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians. (Dienes, 1960, p.21)

In an analogous way, Greeno (1983) defines a "conceptual entity" as a cognitive object which can be manipulated as the input to a mental procedure. The cognitive process of forming a (static) conceptual entity from a (dynamic) process has variously been called "encapsulation" (after Piaget), "entification" (Kaput, 1982), and "reification" (Sfard, 1989). We shall use these three terms interchangeably in the remainder of the article, favouring the original word "encapsulation".

This encapsulation is seen as operating on successively higher levels so that:

> ... the whole of mathematics may therefore be thought of in terms of the construction of structures,... mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget 1972, p. 70)

From the viewpoint of a professional mathematician:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston 1990, p. 847)

Sfard (1989) expresses the way in which the stratification occurs by talking about *operational* mathematics, in which the operations at one level become reified as objects to become basic units of a higher level theory.

At the foundation of arithmetic is the concept of number. This has its origins in the process of counting. The sequence of number words become part of a procedure to point at successive elements calling out each number word in turn until the last word is identified as the number of elements in the collection. The process of counting is encapsulated in the concept of number. (The relation between process and object in the growth of number knowledge in the domain of counting is a focus of attention of Gelman & Meck (1986)).

There is thus widespread evidence for the encapsulation of processes into objects in mathematics. The apparent dichotomy between procedural and conceptual knowledge needs closer analysis to see how this encapsulation features in the divergence between inflexible procedures and flexible concepts.

The Role of Symbols

We believe that the role of the symbols is of paramount importance. Cockcroft (1982) noted that mathematical symbolism is both the strength and weakness of mathematical communication. We would like to take this fundamental paradox a stage further; mathematical symbolism is a major source of both success and distress in mathematics learning. But what is it about mathematical symbols that can be so problematic?

It is interesting to note that Sinclair & Sinclair (1986) sense that with pre-school children – for whom written symbolism is absent – the distinction between procedural and conceptual knowledge seems far less appropriate. Following Piaget, their discussion focuses once more on the theme of *action* (process) becoming the *object* of thinking, the process becoming the concept.

Sfard (1989) comments that the ability to conceive mathematical notions as processes and objects at the same time, although ostensibly incompatible, is in fact complementary. Yet she asks “How can anything be a process and an object at the same time?”. We suggest that the answer lies in the way that professional mathematicians cope with this problem.

The *ambiguity* of symbolism for process and concept

A clue to the manner in which process and concept are combined in a single notion can be found in the working practices of professional mathematicians and all those who are successful in mathematics. They employ the simple device of *using the same notation to represent both a process and the product of that process*. As Thurston’s father noted in the initial quotation, *a*/*b* and *a* divided by *b* are just synonyms … a small variation in notation. In practice there is rarely a variation – the same notation is used for either process or product.

Examples pervade the whole of mathematics:

- The symbol 5+4 represents both the process of adding through *counting all* or *counting on* and the concept of *sum* (5+4 is 9),
- The symbol 4×3 stands for the process of repeated addition “four multiplied by three” which must be carried out to produce the product of four and three which is the number 12.

- The symbol 3/4 stands for both the process of division and the concept of fraction,
- The symbol +4 stands for both the process of "add four" or shift four units along the number line, and the concept of the positive number +4,
- The symbol –7 stands for both the process of "subtract seven", or shift seven units in the opposite direction along the number line, and the concept of the negative number –7,
- The algebraic symbol 3*x*+2 stands both for the process "add three times *x* and two" and for the product of that process, the expression "3*x*+2",
- The trigonometric ratio sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ represents both the process for calculating the sine of an angle and its value,
- The function notation $f(x)=x^2-3$ simultaneously tells both how to calculate the value of the function for a *particular* value of *x* and encapsulates the complete concept of the function for a *general* value of *x*,
- An "infinite" decimal representation $\pi=3.14159\ldots$ is both a process of approximating π by calculating ever more decimal places and the specific numerical limit of that process,
- The notation $\lim_{x\to a} f(x)$ represents both the process of *tending to a limit* and the concept of the *value of the limit*, as does $\lim_{n\to\infty} s_n$, $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$, and $\lim_{\delta x\to 0} \sum_{x=a}^{b} f(x)\,\delta x$.

Mathematicians abhor ambiguity and so they rarely speak of such a device, yet it is widely used throughout mathematics. By using the notation ambiguously to represent either process or product, whichever is convenient at the time, the mathematician manages to encompass both – neatly side-stepping a possible object/process dichotomy. We believe that the ambiguity in interpreting symbolism in this flexible way is at the root of successful mathematical thinking. We further hypothesise that its absence leads to stultifying uses of procedures that need to be remembered as separate devices in their own context ("do multiplication before addition", "turn upside down and multiply", "two negatives make a positive", "add the same thing to both sides", "change sides, change signs", "cross-multiply" etc.).

We conjecture that the dual use of notation as process and concept enables the more able to tame the processes of mathematics into a state of subjection; instead of having to cope consciously with the duality of concept and process, the good mathematician thinks ambiguously about the symbolism for product and process. We contend that the mathematician simplifies matters by replacing the *complexity* of *process-concept duality* by the *convenience* of *process-product ambiguity*.

The Notion of Procept

We do not consider that the ambiguity of a symbolism expressing the flexible duality of process and concept can be fully utilised if the distinction between process and concept are maintained at all times. It is essential that we furnish the cognitive combination of process and concept with its own terminology. We therefore use the portmanteau word "procept" to refer to this amalgam of concept and process represented by the same symbol. However, we wish to do this in a way which reflects the cognitive reality. So, in the first place we say that:

An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.

This preliminary definition allows the symbolism to evoke either process or concept, so that a symbol such as 2+3 can be seen to evoke either the process of addition of the two numbers or the concept of sum.

The definition caused us a great deal of heart-searching because we wanted it to reflect the observed cognitive reality. In particular we wanted to encompass the growing compressibility of knowledge characteristic of successful mathematicians. Here, not only is a single symbol viewed in a flexible way, but when the same object can be represented symbolically in different ways, these different ways are often seen as different names for the same object. Thus a young child might see 4+5 as "one more than 8, which is 9", because the 5 is seen as 4+1 and the other 4 plus this 4 makes 8, so 1 more makes 9.

In order to reflect this growing flexibility of the notion and the versatility of the thinking processes we extend the definition as follows:

A *procept* consists of a collection of *elementary procepts which have the same object.*

In this sense we can talk about the *procept* 6. It includes the process of counting 6, and a collection of other representations such as 3+3, 4+2, 2+4, 2x3, 8–2, etc. All of these symbols are considered by the child to represent the same object, though obtained through different processes. But it can be decomposed and recomposed in a flexible manner.

We are well aware that mathematically we could put an equivalence relation on ekenentary procepts, to say that two are equivalent if they have the same object and then define a procept to be an equivalence class of elementary procepts. However we feel that this kind of mathematical precision overcomplicates the cognitive reality. The nature of the procept is dependent on the cognitive growth of the child. It starts out with a simple structure and grows in interiority in the semse of Skemp (1979). Indeed, we simply see an elementary procept as the first stage of a procept, rather than an element in an equivalence class, which would grossly overcomplicate matters.

We characterize *proceptual thinking* as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation.

We see number as an elementary procept. A symbol such as “3” evokes both the counting process “one, two, three” and the number itself. The word “three” and its accompanying symbol “3” can be *spoken*, it can be *heard*, it can be *written*, and it can be *read*. These forms of communication allow the symbol to be shared in such a way that it has, or seem to have, its own shared reality. “Three” is an abstract concept, but through using it in communication and acting upon it with the operations of arithmetic, it takes on a role as real as any physical object.

It is the construction of meaning for such symbols, the processes required to compute them, and the higher mental processes required to manipulate them, that constitute the abstraction of mathematics. Indeed the ambiguity of notation to describe either process or product, whichever is more convenient at the time, proves to be a valuable thinking device for the professional mathematician. It is therefore opportune to use this notion to see how it features in the development of successful mathematical thinking. As an example we consider the development of the notion of addition and the related (inverse) operation of subtraction.

Number awareness involves both the process of counting and the product of the process. Underscoring the counting process, which exploits and generates one-to-one correspondence, is a series of co-ordinated actions and ideas, which leads eventually to number invariance which might be termed the “threeness” of three. The symbol “3” inextricably links both procedural and conceptual understanding. But conceptual understanding implies that the relationships inherent in all of the different components that form 3 are also available (1 and 1 and 1; 2 and 1; 1 and 2; one less than 4 etc). The symbols 1+1+1, 2+1, 1+2, 4–1 all have output 3 and together form part of the procept 3. All these different proceptual structures allow the number 3 to be decomposed and recomposed in a variety of ways either as process or object. In this way the various different forms combine to give a rich conceptual structure in which the symbol 3 expresses all these links, the conceptual ones and the procedural ones, the processes and the product of those processes. This combination of conceptual and procedural thinking is what we term *proceptual thinking*.

The flexibility of the procept of number is fundamental to the development of arithmetic. The procedural and conceptual approaches that children use to form the sum of two or more amounts introduced through word problems have been well documented (for example Fuson, 1982; Carpenter et al, 1981, 1982; Baroody & Ginsburg, 1986). Translating some or all of these approaches into a conceptual hierarchy for addition formed part of the focus of these and other studies (Herscovics and Bergeron, 1983; Secada, Fuson and Hall, 1983; Gray, 1991; Fuson & Fuson, 1992).

Those for addition involve a number of different procedures, including "count-all" (count each set separately then count the two together), "count-on from first" (count on the number of elements in the second set, starting from the number in the first set), "count on from largest" (put the largest set first and count on the smaller number of elements in the second set), together with higher order strategies, such as "knowing the fact" or "deriving new facts from known facts". There are corresponding procedures for subtraction "take away" (count the big set, count the subset to be taken away then count the set which remains), "count-back" (start from the larger number and count back down the number sequence to find the number remaining) "count-up" (start from the number to be taken away and count-up to the number given), together with higher order strategies, either "knowing the facts" or deriving the facts from other known facts.

Even finer gradations of these categories have been proposed and can be helpful in distinguishing children's thinking processes. Here we wish to use the notion of procept to analyse the procedures in a more integrated manner, referring only to the growing facility for compression of ideas from the processes of counting to the procept of number.

The procedure of count-all consists of three separate sub-procedures: count the first set, count the second set, then combine the sets as a single set and count all the objects (figure 1).

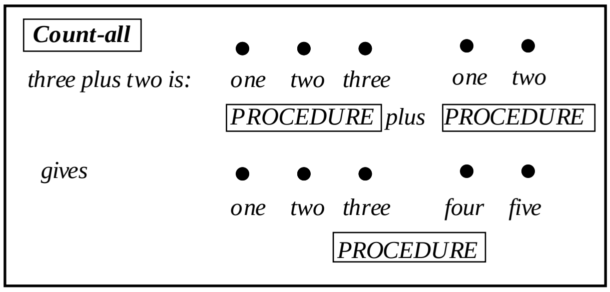


Figure 1 : count-all as a combination of procedures

The child records the process of counting-all as the product of that procedure, the number five. As the procedure occurs in time, the link between input (3 plus 2) and output (5) is more likely to be viewed as a counting procedure rather than to be encapsulated and remembered as a known fact (3+2=5). Count-all is a procedure extending the counting process rather than an encapsulated procept.

The count-on procedure is a more sophisticated strategy than count-all (see, for example, Secada *et al*, 1983; Carpenter 1986; Baroody and Ginsburg, 1986; Gray 1991). The first number is considered as a whole and the second number is interpreted as a counting procedure. (It is actually a sophisticated double-counting procedure where 3+2 involves saying “four, five”, whilst simultaneously keeping track that “two” extra numbers are being counted.) We therefore see count-on as “procept plus procedure”; the first number is a procept and the second number a procedure (figure 2).

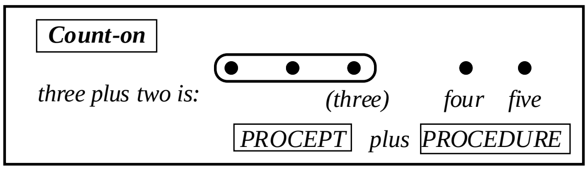


Figure 2 : counting-on as procept plus procedure

We believe that “count-on” as a procedure can have two qualitatively different outcomes, as a (counting) procedure of addition or as the procept of sum.

i. *Count-on as procedure* is essentially a compression of count-all into a shorter procedure. It remains a procedure that takes place in time so that the child is able to compute the result without necessarily linking input and output in a form that will be remembered as a new fact. Some children – often with a limited array of known facts – may become so efficient in counting, that they use it as a universal method that does not rely on the vagaries of remembered facts.

ii. *Count-on as procept* produces a result that is seen both as a counting process and a number concept. The notation 3+2 is

> seen to represent both the process of addition and the product of that process, the sum.

When input numbers and their sum can be held in the mind simultaneously then the result is a meaningful known fact which may be envisioned as a flexible combination of procept and procept to give a procept (figure 3).

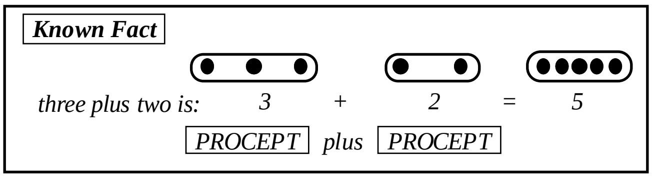


Figure 3 : (Meaningful) known fact as procept plus procept

It is important here to distinguish between a meaningful “known fact” that is generated by this flexible form of thinking and a fact that is remembered by rote. In any isolated incident such a distinction may be hard to make. The difference is more apparent in a wider context when it can be seen to be linked to other known facts in a flexible manner.

The manipulation of meaningful known facts leads to “derived facts”. For instance, faced with “four and five”, one may know that “four and four makes eight”, so the response is that it is “one more”, which is “nine”. The language used by children in such situations shows that they freely decompose and recompose the component parts in a proceptual way. We shall see later that merely “knowing facts” does not necessarily lead to “deriving facts”. The derivation depends on the flexibility of the known facts. Indeed, some facts, such as those for, say “16+3 is 19” based on 6+3 is 9 can be so fast as to be virtually instantaneous. Thus it is not always possible to distinguish between a “known fact” and a quickly constructed “derived fact”.

Furthermore, the existence of flexible proceptual knowledge means that the number 5 can be seen as 3+2 or 2+3 and if 3 and something makes 5, the “something” must be 2. A proceptual view of addition is so intertwined with subtraction that subtraction facts are easily related to those for addition.

The need for flexibility in arithmetic is a regular feature in the literature. For instance, Steffe, Richards & von Glaserfeld (1981) and Fuson, Richards & Briars (1982) both suggest that the use of the sequence of number words for the solution of addition and subtraction leads to the understanding that addition and subtraction are inverse operations, and this contributes to the flexibility of solving addition and subtraction problems. However, proceptual flexibility gives new insight. At the *derived fact* level, addition and subtraction are so closely linked that subtraction is simply a flexible reorganisation of addition facts. At the *procedural* level, addition as “count-on” is considered to have subtraction as inverse through "count-back" or "count-up". We shall see that less successful children often favour "count-back" as the natural reverse process. The cognitive complexity of counting back is enormous. The child must count the number sequence in reverse starting from the larger number and keep track simultaneously of how many numbers have been counted. A sum such as 16–13 by count-back requires the recitation of 13 numbers in reverse order from 16 down. Such procedures, especially when carried out by less successful children, are highly prone to error. The more able proceptual thinker has a simpler task than the less able procedural thinker, so that the likely divergence between success and failure is widened.

The fundamentally different ways of thinking exhibited by children performing arithmetic usually represented by the terms procedural and conceptual, may be described more incisively as *procedural* and *proceptual*. Proceptual thinking *includes* the use of procedures. However, it also includes the flexible facility to view symbolism either as a trigger for carrying out a procedure or as a mental object to be decomposed, recomposed and manipulated at a higher level. This ambiguous use of symbolism is at the root of powerful mathematical thinking to overcome the limited capacity of short-term memory. It enables a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mathematical process. It includes both concepts to *know* and processes to *do*.

Qualitatively Different Approaches to Simple Arithmetic

Gray (1991) interviewed a cross-section of children aged 7 to 12 from two mixed ability English schools to discern their methods of carrying out simple arithmetic exercises. Towards the end of the school year, when the teachers had intimate knowledge of the children for over six months, he asked the teacher of each class to divide their children into three groups – "above average", "average" and "below average" according to their performance of arithmetic – and to select two children from each group who were "representative" of that group. The two schools each provided 6 children from each of 6 year groups, making 72 children in all, 12 from each year divided into 3 groups of 4 children according to their teachers' perceptions of their performance on arithmetic. In what follows we shall refer to the year groups by age, so that, for instance, 9+ refers to children who would be nine during the school year. They were interviewed over a two month period starting six months after the beginning of the year, so at the time of interview a child designated as 9+ would be in the range 8 years 6 months to 9 years 8 months.

Figures 4 and 5 consider the types of response made by the above average and the below average children to a range of addition and subtraction problems subdivided into three levels.

The three categories of addition problems considered:

A single digit addition with a sum of ten or less (e.g. 6+3, 3+5)

B addition of a single digit number to a teen number the sum being twenty or less than twenty (e.g. 18+2, 13+5)

C addition of two single digit number with a sum between 11 and 20 (e.g. 4+7, 9+8)

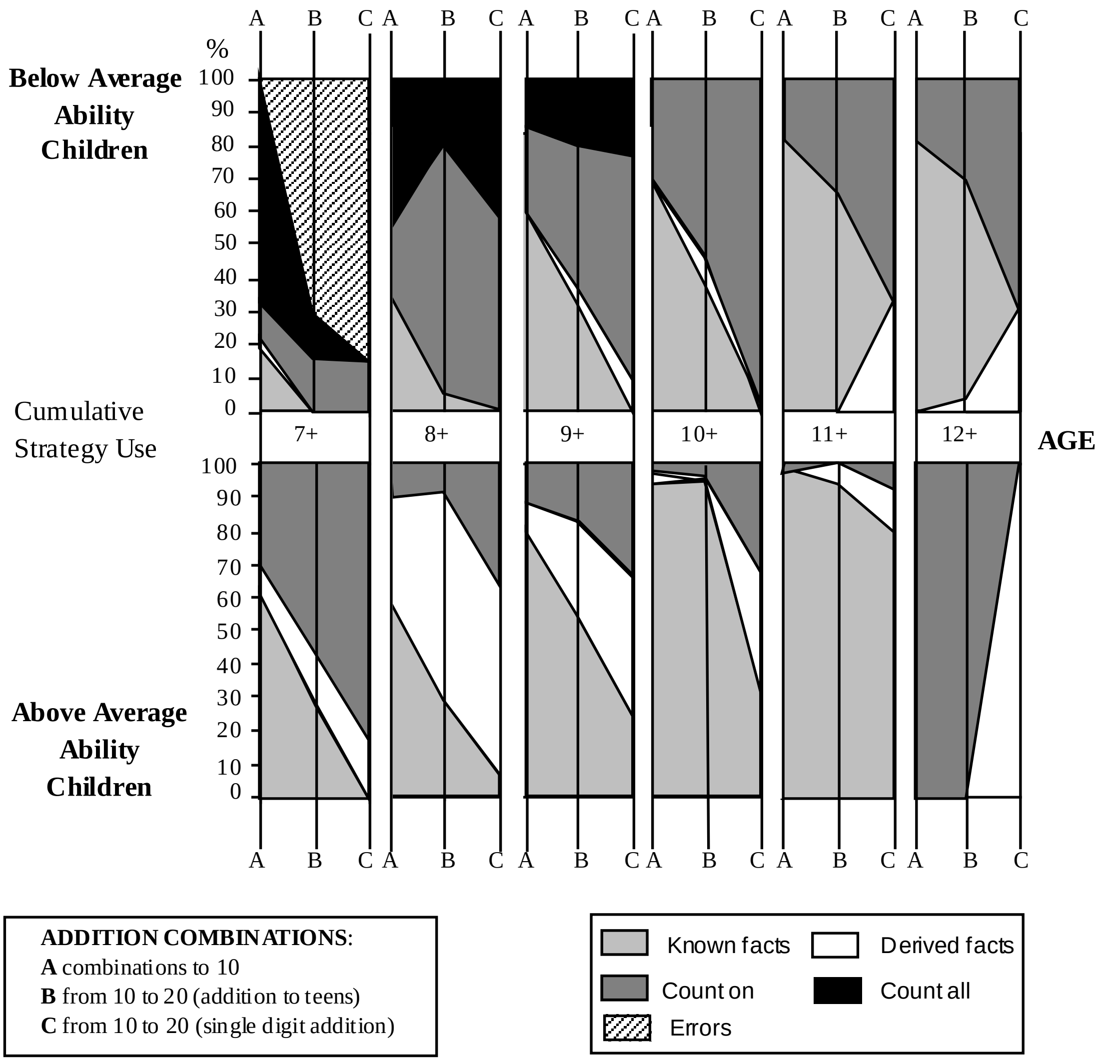


Figure 4 : Strategies used to obtain solutions to simple addition combinations by groups of below average and above average children of different ages.

The striking difference between the two groups is seen by comparison in the use of procedural methods (counting) and the use of derived facts. In most cases the 7+ below average children were unable use an appropriate method to obtain correct solutions for the category B and C problems. In contrast to the above average ability children, the below average ability children, apart from one exception, do not provide any evidence of the use of derived facts to obtain solutions to the number combinations in category A.

Even though the number combinations used in category B used units combinations previously met in category A, the contrast between the two ability groups, particularly those aged 9+ and below, could hardly be more evident.

The above average ability children either:

(i) made use of their known category A combinations to to obtain a solution to the units component of the category B combinations and then adding on ten, or

(ii) derived the solution to the units component of category B and then added on ten.

Thus for one above average ability child 15+4 may be solved through knowing 5+4 *is* 9 and then adding on 10, whilst for another 5x2 is ten so 5+4 is 9, add 10 is 19. Even though they may have known the relevant category A solution the equivalent aged below average ability children used a procedural method to obtain the solution to category B combinations.

The category C combinations proved to be more difficult for the below average children to remember. although those within the 11+ and 12+ age groups extensively solved 9+8 as a result of knowing 9+9.

Figure 5 (adapted from Gray & Tall, 1991) concentrates on three categories of subtraction:

A single digit subtraction (e.g. 8–2),

B subtraction of a single digit number from one between 10 & 20 (e.g. 16–3, 15–9),

C subtraction of a two digit number between 10 and 20 from another two digit number. (e.g. 16–10, 19–17).

Note the low incidence of known facts in the 8+ below average children and the related absence of derived facts; meanwhile the 8+ above average children have over 50% known facts in category A (up to 10) and a high incidence of derived facts in all three categories. With a good knowledge of subtraction facts to 10, these more successful children are able to derive almost everything they do not know, and only then resort to counting.

If this 8+ above average group is compared with the 10+ below average group with a similar profile of known facts, it can be seen that the 10+ below average do not use their known facts to derive any new facts in category C, and only occasional facts in category A and B. Even though these two groups have similar attainments in known facts, they have radically different methods of obtaining what is not immediately known.

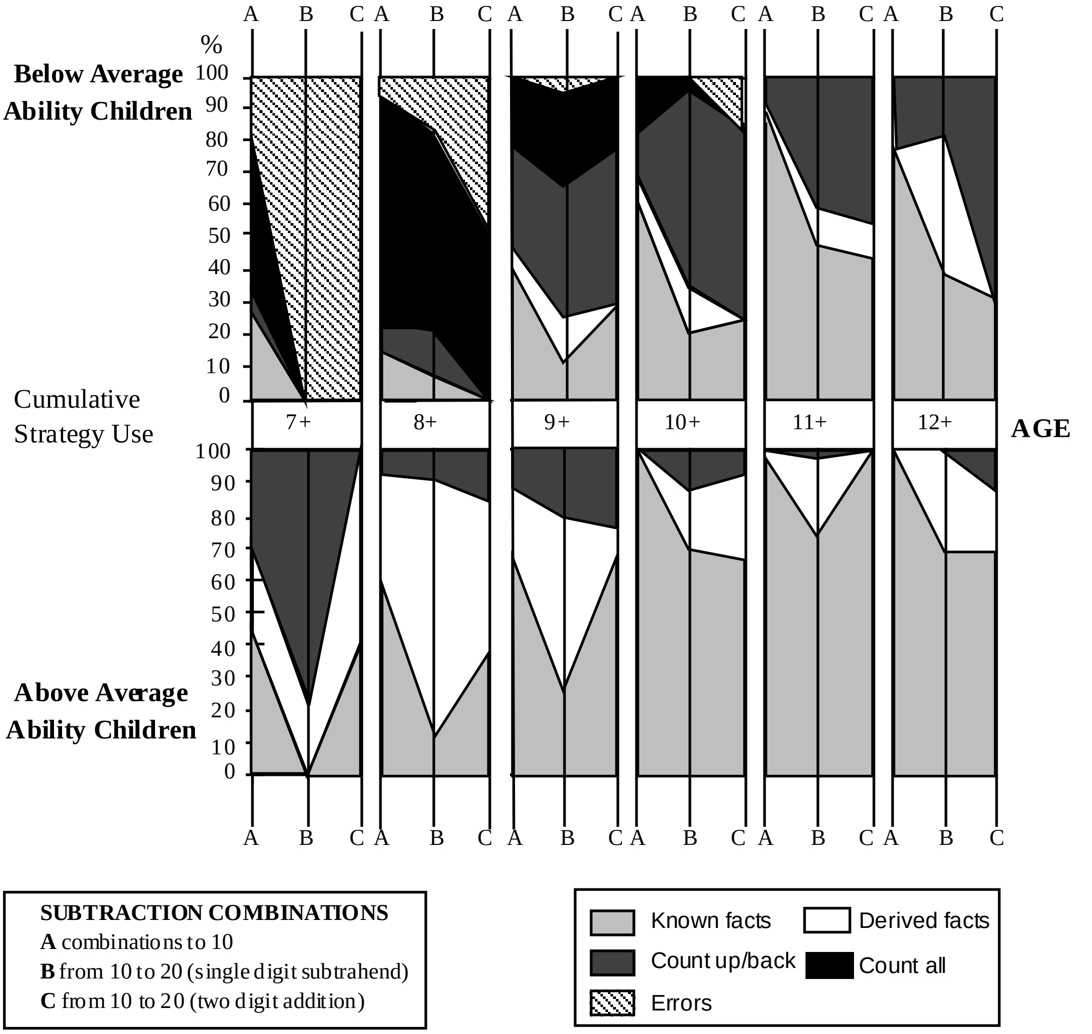


Figure 5: Strategies used to obtain solutions to simple subtraction combinations by groups of below average and above average children of different ages.

Throughout the age range there is a striking difference in what children do if they do not know the facts. The above average show a high incidence of derived facts and only in a small number of cases resort to counting. The below average rarely use derived facts, instead they almost always count.

A careful reconsideration of the individual data shows that there is even a difference in *how* they count. The below average nearly always select count-back as the natural process to take away, so that 16–13 is likely to be calculated by laboriously counting back 13 from 16. The above average are more likely to select the shorter counting strategy, so that 19–17 would more likely be a count-up, whilst 13–2 would be a count-back.

An apparently strange phenomenon occurs in the relative number of known facts in categories B and C. For below average children, at almost every stage fewer facts are known in category C than B, whilst the position is reversed in five out of six for the above average children.

The difference between these categories is that all the problems in category C are of the same kind: they involve the difference between two numbers between 10 and 20 which the more able can carry out by noting that the tens cancel, leaving a related subtraction problem with numbers between 0 and 9. Meanwhile category B has two distinct kinds of problem: those like 15–4 (which the more able can do by subtracting the units) and those like 12–8 which require an exchange. Thus, for students using derived facts, category C is easier than category B. But the less able, performing more often by count-back, find category B problems like 15–4 easier than category C problems like 17–13.

This phenomenon will have a consequent effect on the known facts. As the more able used derived facts to generate new known facts they are likely to know more facts where the derivations are easier. In practice easy derivations may be almost instantaneous and so appear to be "known". Thus the more able are likely to know more facts in category C than category B because of the greater difficulty of those in the latter category involving exchange. Meanwhile, the relative length of the count-back processes are likely to have the reverse effect on the fewer facts which become "known" to the less able.

We thus see that the more able tend to develop more flexible proceptual techniques whilst the less able rely on procedural methods of counting.

Individual examples make this more apparent. Michael (9+) is categorised as "below average" by his teacher, He chose to write 18–9 in the standard vertical layout as

$$\begin{array}{r} 18 \; - \\ \underline{\;9} \end{array}$$

and, as is usual in the decomposition process, put a 'little one' by the eight.

"*This is the easy way of working it out. I can't take nine from eight but if I put a little one it makes it easier because now its nine from eighteen*".

He failed to realise that this is the same sum he started with and after a considerable time trying to cope with this problem he resorted to his more usual procedure for subtraction by placing eighteen marks from left to right on his paper, then starting from the left and counting from one to nine as he crossed out nine marks. He recounted the remaining marks from left to right to complete the correct solution by "take away".

The less able children are often placed in difficulties as they grow older because they feel a pressure to conform and not use "baby methods" of counting.

Jay (10+) solved the problem 5–4 by casually displaying five fingers on the edge of the desk and counted back, "*five, … four, three, two, one* ". At each count apart from the five he put slight pressure on each finger in sequence. The solution was provided by last count in the sequence. When attempting 15–4 he wanted to use a similar method but had a problem, declaring "*I'm too old for counters*!", but neither did he want to be seen using his fingers because "*My class don't use counters or fingers*." He felt he should operate in the same way as other children in his class (most of whom appeared to recall the basic facts from memory) yet he did not know the solutions and knew that he required a counting support. His use of fingers for obtaining solution for number combinations was almost always covert. When dealing with combinations to twenty he combined a casual display of ten splayed fingers on the edge of his desk with an imagined repetition of his fingers just off the desk. He spent a considerable time obtaining individual solutions and had a tendency to be very cautious in giving responses. He used his imaginary fingers to attempt to find a solution to 15–9 by counting back. Eventually he became confused and couldn't complete the problem.

Some of those deemed "below average" used some derived facts. However, the methods used differed from the "above average". The latter invariably performed all the calculations in their head. Some of the older below average children seemed to have a good knowledge of known facts yet seemed to need to display them on their fingers for visual support. This was categorised as "derived fact" because there was no visible evidence of actual counting. The counting process has been compressed to the stage where only the fingers need to be held up and the number facts recalled from the finger layout.

For instance, Karen (11+), the most successful of the "below average", made considerable use of her fingers in an idiosyncratic inventive manner. To perform the calculation 15–9, she held out five fingers on her left hand and closed it completely; she then held up four fingers on her right hand closed them and opened the right thumb, then redisplayed the five fingers of her left hand at the same time and responded "six". The whole procedure took about three seconds.

Her explanation showed a subtle understanding of number relationships (figure 6).

Nevertheless, the tortuous route that she followed showed that her inventiveness tended to relate to individual calculations and applied only to small numbers she could represent using her hands.

Other below average children who attempted to derive facts often had to do this based on a limited number of known facts that might not furnish the most efficient way to perform the calculation.

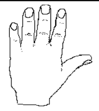
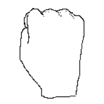
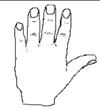
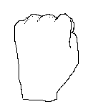
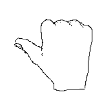
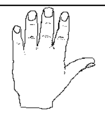
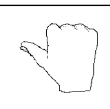
	DISPLAY		EXPLANATION TO CALCULATE 15 –9	
	Left Hand	Right Hand	Child's Explanation	Interviewer's Comment
STAGE 1			*Fifteen is ten and five. Forget the ten.*	Five fingers shown on left hand, (Other ten presumably held in mind.) Right hand closed.
STAGE 2				The child displays the nine to be taken away as a five and four.
STAGE 3				The left hand is closed, to cancel the displayed five leaving the previously displayed four.
STAGE 4			*Four from one of the fives making ten leaves one.*	The remaining four is taken from one of the fives held in the mind.
STAGE 5			*One and the other five from the ten make six.*	Remaining five in mind now displayed, giving a total of 5 and 1, which is 6.

Figure 6 : Subtracting nine from fifteen by an inventive route

Michelle (aged 10+), faced with "16–3", said "*ten from sixteen leaves six, three from ten leaves seven, three and seven makes ten and another three is thirteen*". Michelle seeks to find familiar number bonds to solve the problem. She sees 16 as 6 and 10, but takes the three from the 10 rather than from the 6 and ends up having to do the additional sum "six and seven".

Flexible strategies used by the more able produce new known facts from old, giving a built-in feedback loop which acts as an autonomous knowledge generator. Once they realise this, the more able are likely to sense that they need to remember less because they can generate more, reducing the cognitive strain even further. Meanwhile the less able who do attempt to derive facts may end up following an inventive but more tortuous route that succeeds only with the greatest effort, whilst the majority fall back on longer procedural calculations which cause greater cognitive strain or which have personal idiosyncrasies that fail to generalise.

> For unto everyone that hath shall be given and he shall have abundance: but from him that hath not shall be taken even that which he hath.
>
> (St Matthew, chapter 25, v.29)

The Proceptual Divide

We like to think of a procept conjuring up the adjective *plastic*: it is something flexible that can be re-moulded and reconstructed at will. As the learner progresses, the number procept grows in internal richness, in *interiority* in the terminology of Skemp (1979). It has more and more possibilities for flexible manipulation.

It is in the use of procepts that we consider lies a major difference between the performance of the more able and the less able in mathematics. The more able tend to display proceptual thinking whilst the less able are more procedural. The characteristics of these two form of thinking may be summarised as:

i. *Procedural thinking* is characterised by a focus on the procedure and the physical or quasi-physical aids which support it. The limiting aspect of such thinking is the more blinkered view that the child has of the symbolism: numbers are used only as concrete entities to be manipulated through a counting process. The emphasis on the procedure reduces the focus on the relationship between input and output, often leading to idiosyncratic extensions of the counting procedure which may not generalise.

ii. *Proceptual thinking* is characterised by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects which can be decomposed and recomposed in flexible ways.

It is our contention that whilst more able younger children evoke proceptual thinking to use the few combinations already known to establish more, less able children remain concerned with the procedures of counting and apply their efforts to developing competence with them. Procedural thinking in the context of developing competency with the number combinations can give guaranteed success and efficiency within a limited range of problems. But this efficiency with small numbers is unlikely to lead to success with more complex problems as the children grows older. Their persistence in emphasising procedures leads many children inexorably into a cul-de-sac from which there is little hope of future development.

This lack of a developing proceptual structure becomes a major tragedy for the less able which we call t*he proceptual divide*. We believe it to be a major contributory factor to widespread failure in mathematics. It is as though the less able are deceived by a conjuring trick that the more able have learned to use. Although all children are initially given procedures to carry out mathematical tasks, success eventually only comes not through being good at those procedures, but by moving on to the next stage of encapsulating them as part of a procept to solve the tasks in a more flexible way. The more able recognise that the process of addition first taught is not the main aim of the game and go on to succeed. The less able who try to do what they are asked: to master the counting procedure, seem cheated because when they finally do so, the game has moved on to a more advanced stage and left them behind.

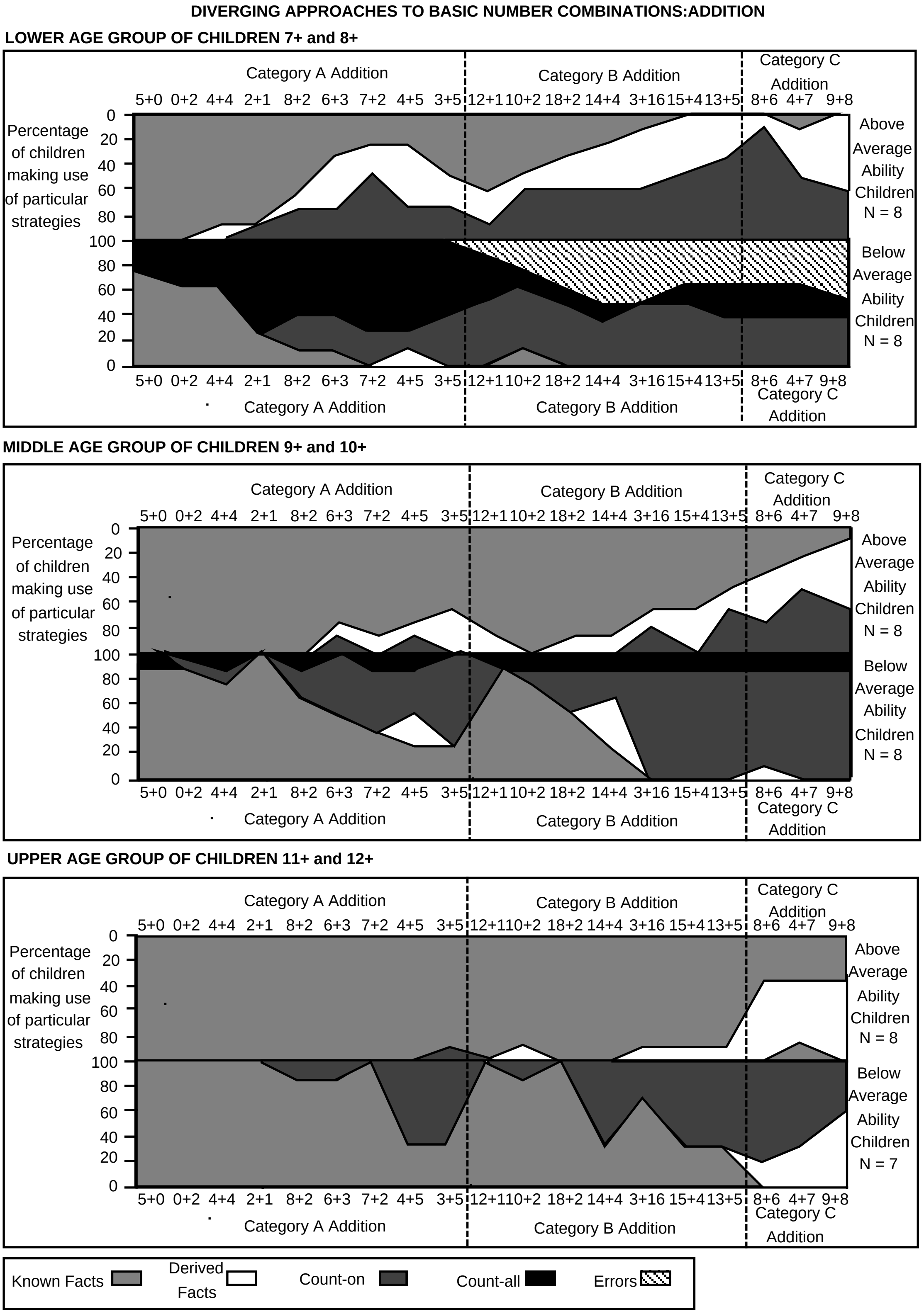


Figure 7: Diverging approaches to basic addition combinations. Age and ability comparisons

Figures 7 and 8 illustrate the divergence in strategy use between the less able in the study. They pair age groups together so that seven and eight year olds, nine and ten year olds, and eleven and twelve year olds are paired together in a more detailed analysis of the data in figures 4 and 5.

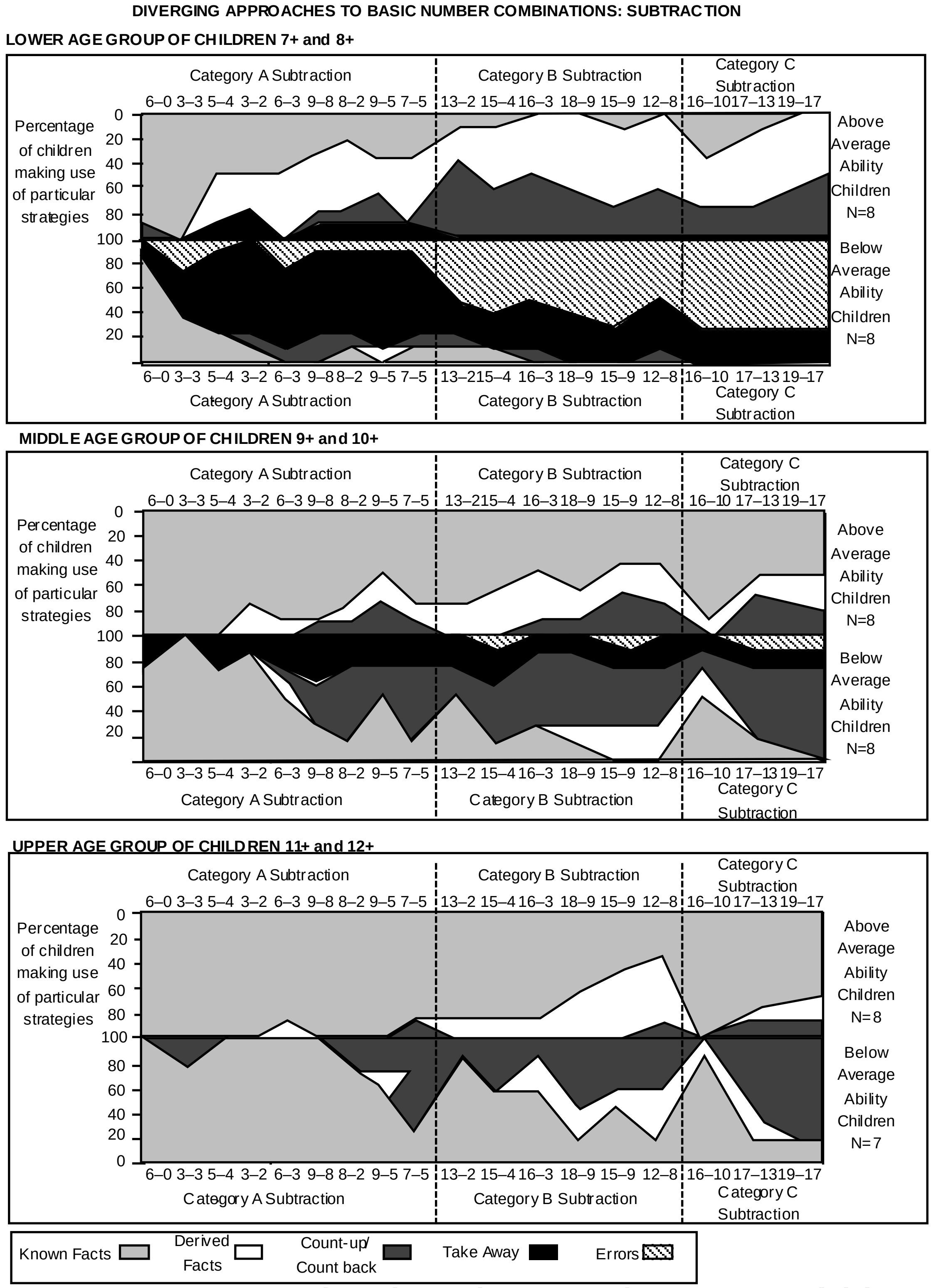


Figure 8: Diverging approaches to basic subtraction combinations. Age and ability comparisons

Combinations are arranged in order of difficulty, through establishing the overall percentage of children within the sample who responded to individual combinations through the use of known facts. The three main groupings correspond to the categories previously considered in figures 4 and 5.

The graphs not only starkly illustrate differences between the below average and the above average children but also how combinations evoke particular responses. Note how combinations involving single digits and a sum between 10 and 20 evokes the use of derived facts by the upper age group of 11 and 12 year olds. Note also how the extensive use of procedural methods amongst the youngest below average group to obtain solutions to number combinations to ten does not provide them with a means of obtaining solutions to harder problems.

See how the above average make use of very few known facts to establish solutions through the use of derived facts. For instance, "*6 – 3 is 3 because two three's are six*"; "*4 + 7 is 11 because 3 and 7 is 10*"; "*18 – 9 is 9 because 9 x 2 is 18*"; "*8 + 6 is 14 because two sevens are 14*". Simpler facts become "known facts" (or perhaps instantaneous "derived facts"). Harder combinations are less committed to memory, perhaps because the more able realise that it is just as efficient to derive them when required.

Note that even when below average ability children know a substantial number of facts they make very little use of derived solutions. Contrast the efficient solution to 8 + 6 above with a solution derived by a less able children. Stuart (aged 10+) responded to this problem by saying "*I know 8 and 2 is ten, but I have a lot of trouble taking 2 from 6. Now 8 is 4 and 4; 6 and 4 is 10; and another 4 is 14*". We may feel we should congratulate Stuart for the breadth of arithmetical manipulation that he displays but the truth of the matter is that his particular approach indicates not so much what he knows as what he does not know. He knows number combinations that make ten but cannot solve 6–2! His idiosyncratic methods of solution place a severe burden of inventiveness upon him to solve arithmetic problems. It may in the long term prove too great a burden to bear.

The cumulative effect of the proceptual divide

Proceptual encapsulation occurs at various stages throughout mathematics: repeated counting becoming addition, repeated addition becoming multiplication and so on, giving what are usually considered by mathematics educators as a complex hierarchy of relationships (figure 9):

The less able child who is fixed in process can only solve problems at the next level up by coordinating sequential processes. This is, for them, an extremely difficult process. If they are faced with a problem two levels up, then the structure will almost certainly be too burdensome for them to support (see Linchevski & Sfard, 1991). Multiplication facts are almost impossible for them to coordinate whilst they are having difficulty with addition. Even the process of reversing addition to give subtraction is seen by them as a new process ("count-back" instead of "count-up").

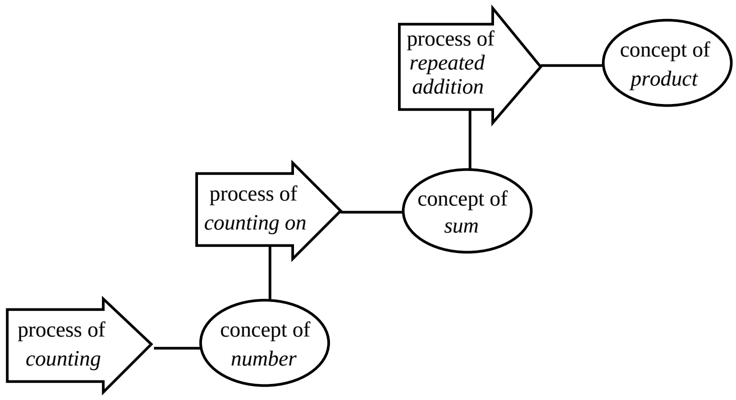


Figure 9 : Higher order encapsulations

The more able, proceptual thinker is faced with an easier task. The symbols for sum and product again represent *numbers*. Thus counting, addition and multiplication are operating on the same procept which can be decomposed into process for calculation purposes whenever desired. A proceptual view which amalgamates process and concept through the use of the same notation therefore *collapses the hierarchy* into a single level in which arithmetic operations (processes) act on numbers (procepts) (figure 10).

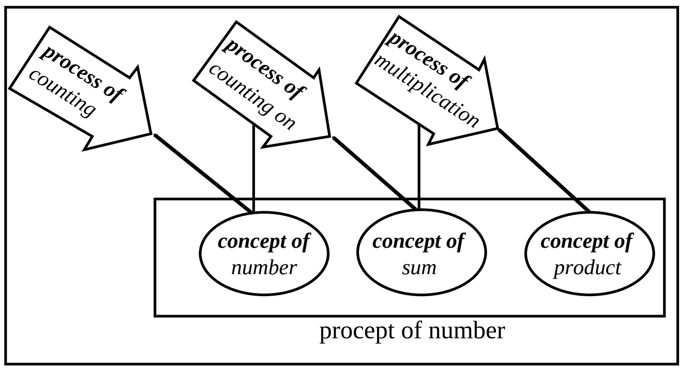


Figure 10: Collapse of hierarchy into operations on numbers

We hypothesise that this is the development by which the more able thinker develops a flexible relational understanding in mathematics, which is seen as a meaningful relationship between notions at the same level, whilst the less able are faced with a hierarchical ladder which is more difficult to climb. It also provides an insight into why the practising expert sees mathematics as such a simple subject and may find it difficult to appreciate the difficulties faced by the novice. As Thurston indicated in our earlier quotation, it is the *compression* of mathematical ideas that makes them so simple. As proceptual thinking grows in conceptual richness, procepts can be manipulated as simple symbols at a higher level or opened up to perform computations, to be decomposed and recomposed at will. Such forms of thinking become entirely unattainable for the procedural thinker who fails to develop a rich proceptual structure.

Examples from other areas of mathematics

Our empirical evidence in this paper has concentrated on simple arithmetic. However other research can also be re-interpreted in proceptual terms. We have evidence that the lack of formation of the procept for an algebraic expression causes difficulties for pupils who see the symbolism representing only a general procedure for computation: an expression such as $2+3x$ may be conceived as a process which cannot be carried out because the value of x is not known (Tall & Thomas, 1991). We have evidence that the conception of a trigonometric ratio only as a process of calculation (opposite over hypotenuse) and not a flexible procept causes difficulties in trigonometry (Blackett 1990, Blackett & Tall 1991). In both of these cases we have evidence that the use of the computer to carry out the process, and so enable the learner to concentrate on the product, significantly improves the learning experience. The difference between ratio and rate also has an obvious interpretation in terms of procept where ratio is a process and rate the mathematical object produced by that process.

The case of the function concept, where f(x) in traditional mathematics represents both the process of calculating the value for a specific value of x and the concept of function for general x, is another example where the modern method of conceiving a function as an encapsulated object causes great difficulty (Sfard, 1989). There is evidence (Schwingendorf et al, in press) that the programming of the function as a procedure whose name may also be used as an object, significantly improves understanding of function as a procept.

The limit concept is also a procept, but of a subtly different kind. The symbolism for limit represents both the *process* of tending to a limit, “as $n \to \infty$ so $s_n \to s$” or “$\lim_{n\to\infty} s_n = s$”, and the *value* of the limit “$s = \lim_{n\to\infty} s_n$”. As Cornu (1981, 1983) showed, this causes a problem for students because there is no explicit procedure to calculate the limit, instead it has to be computed by indirect means using general theorems on limits which may not be adequate to compute the precise value. Thus the notion of a procept for which the process has no explicit procedure causes difficulties for students because it seems to violate their intuitions (which have been built up from previous experience, including arithmetic where processes *do* have explicit procedures of calculation).

We therefore are confident that the notion of procept allows a more insightful analysis of the process of learning mathematics, in which the precision of definition of modern mathematics (“a function is a set of ordered pairs such that ...") causes great difficulties for students. The ambiguity of process and product represented by the notion of a procept provides a more natural cognitive development which gives enormous power to the more able. It exhibits the proceptual chasm faced by the less able in attempting to grasp what is – for them – the spiralling complexity of the subject.

References

Baroody, A. J., & Ginsburg, H. P. (1986). The Relationship between Initial Meaningful and Mechanical Knowledge of Arithmetic. In Hiebert (Ed.) *Conceptual and Procedural Knowledge: The Case for Mathematics*, Hillsdale. N.J.: Erlbaum.

Beth, E. W., & Piaget J. (1966). *Mathematical Epistemology and Psychology* (W. Mays, trans.), Dordrecht: Reidel.

Blackett, N. (1990). *Developing Understanding of Trigonometry in Boys and Girls using a Computer to Link Numerical and Visual Representations*, unpublished Ph.D. Thesis, University of Warwick.

Blackett, N., & Tall D. O. (1991). Gender and the Versatile Learning of Trigonometry Using Computer Software, *PME15*, Assisi, **1,** 144–151.

Carpenter, T. P., (1986). Conceptual Knowledge as a Foundation for Procedural Knowledge. In Hiebert (Ed.) *Conceptual and procedural Knowledge: The Case for Mathematics*, Hillsdale. N.J.: Erlbaum.

Carpenter, T. P., Hiebert, J., & Moser, J. M., (1981). Problem Structure and First Grade Children's Initial Solution Processes for Simple Addition and Subtraction Problems, *Journal for Research into Mathematical Education* 12 (1), 27-39.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1982). Cognitive Development and Children's Solutions to Verbal Arithmetic Problems, *Journal for Research in Mathematics Education*, *13* (2), 83-98.

Cockcroft, W. H. (Chairman) (1982). *Mathematics Counts* (Report of the Committee of Inquiry into the Teaching of Mathematics in Schools) H.M.S.O.: London.

Cornu, B. (1981). Apprentissage de la notion de limite : modèles spontanés et modèles propres, *Actes due Cinquième Colloque du Groupe International P.M.E.*, Grenoble 322-326.

Cornu, B. (1983). *'Apprentissage de la notion de limite: conceptions et obstacles'*, Thèse de doctorat, Grenoble.

Davis, R. B. (1983). Complex Mathematical Cognition. In H. P. Ginsburg (Ed.) *The Development of Mathematical Thinking*, Academic Press, New York, 254–290.

Dienes, Z. P. (1960). *Building up Mathematics*, Hutchinson Educational: London.

Fuson, K. C., (1982). An Analysis of the Counting-on Solution Procedure in Addition. In T. Carpenter *et al.* (Eds.) *Addition and Subtraction: A Cognitive Perspective*, Hillsdale. N.J.: Erlbaum.

Fuson, K. & Fuson, A. M. (1992). Instruction Supporting Children's Counting On for Addition and Counting Up for Subtraction, *Journal for Research in Mathematics Education*, *23* (1), 52–78.

Fuson, K. C., Richards, J., Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In Brainerd, C. (Ed.), *Progress in Cognitive Development, Vol. 1. Children's Logical and Mathematical Cognition*, New York: Springer Verlag.

Gelman, R., & Meck, E. (1986). The Notion of Principle: The Case of Counting. In Hiebert (Ed.), *Conceptual and procedural Knowledge: The Case for Mathematics,* Hillsdale. N.J.,: Erlbaum.

Gray, E. M., & Tall D. O. (1991). Duality, Ambiguity and Flexibility in Successful Mathematical Thinking, *Proceedings of PME XIII, Assisi* Vol. II 72-79.
Gray, E. M. (1991). An Analysis of Diverging Approaches to Simple Arithmetic: Preference and its Consequences*, Educational Studies in Mathematics*, ***22***, 551-574.
Greeno, J. (1983). Conceptual Entities. In D. Genter & A. L. Stevens (Eds.), *Mental Models*, 227-252.
Herscovics, N., & Bergeron, J. C. (1983). Models of Understanding, *Zentralblatt fur Didaktik der Mathematik*, *83*, 75-83.
Hiebert, J., & Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In Hiebert (Ed.) *Conceptual and procedural Knowledge: The Case for Mathematics*, Hillsdale. N.J.: Erlbaum.
Kaput, J. (1982). Differential effects of the symbol systems of arithmetic and geometry on the interpretation of algebraic symbols. Paper presented at the meeting of the *American Educational Research Association*, New York.
Linchevski, L., & Sfard, A. (1991). Rules without reasons as processes without objects: the case of equations and inequalities, *Proceedings of PME XIII,* Assisi, II, 317–324.
(The) National Curriculum (1989). *The National Curriculum - From Policy to Practice*, London: H.M.S.O.
Piaget, J. (1972). *The Principles of Genetic Epistemology* (W. Mays trans.) London: Routledge & Kegan Paul.
Schwingendorf *et al* (in press). Horizontal and Vertical Development of the Function Concept. In E. Dubinsky & G. Harel, (Eds.) *Learning and teaching the concept of function.*
Secada, W. G., Fuson, K. C., & Hall, J. W. (1983). The Transition from Counting All to Counting On in Addition", *Journal for Research in Mathematics Education*, ***14*** (1), 47-57.
Sfard, A. (1989). Transition from Operational to Structural Conception: The notion of function revisited, *Proceedings of PME XIII*, Paris, 151–158.
Skemp, R. R. (1979). *Intelligence, Learning and Action*, London: Wiley.
Sinclair, H., & Sinclair, A. (1986). Children's Mastery of Written numerals and the Construction of Basic Number Concepts. In Hiebert (Ed.) *Conceptual and procedural Knowledge: The Case for Mathematics*, Hillsdale. N.J: Erlbaum.
Steffe, L. P., Richard, J., & von Glaserfeld, E. (1981). Children's counting types: Philosophy, Theory and Case Studies, Monograph presented at the Conference on Interdisciplinary Research on Number, Athens, Ga. April 1981.
Tall D. O. & Thomas M. O. J. (1991), Encouraging Versatile Thinking in Algebra using the Computer, *Educational Studies in Mathematics*, ***22*** (2), 125–147.
Thurston, W. P. (1990). Mathematical Education, *Notices of the American Mathematical Society*, ***37*** (7), 844–850.

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