# Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept

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## Introduction

Mathematics is an enigmatic subject in which a few succeed with disarming ease, whilst others seem doomed to failure. After analysing the responses of many children performing simple tasks in arithmetic, we find a phenomenon occurring which suggests a reason for this catastrophic divergence in performance. Quite simply we find that *those who fail are doing a more difficult kind of mathematics compared to those who succeed*.

This difference arises out of the manner in which individuals cope with the progression from procedures of counting to the processes of arithmetic and the concept of number. Whilst the more able progress to use their knowledge in a flexible and powerful way, the less able seek security in counting procedures which work promisingly in simple tasks but fail to generalise when greater sophistication is required (Gray 1991).

A similar phenomenon occurs in many other areas of mathematics from primary to secondary and on to university. Initial experience with procedures may later either develop flexibility and power, or may become fixed in a rigid mode of learning rules. Richard Skemp (1976) described this phenomenon in his much-quoted Mathematics Teaching article on “relational understanding and instrumental understanding”. He described the notion of “faux amis”, where the same word has very different meanings for different individuals. In this article we concentrate on a subtle difference in the meaning of symbols in mathematics. For some a symbol is a mathematical *object*, a *thing* that can be manipulated in the mind. For others it signifies a *procedure* to be carried out. Whilst mental objects are easily manipulated, procedures occur in time and it is extremely difficult to think of two or more procedures simultaneously. It follows that those who concentrate on procedure may very well be good at computations and succeed in the short-term, but long-term they may lack the flexibility that will give them ultimate success.

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We begin by looking at the concept of number to show how it develops from process to concept and report how children in interview show a spectrum of ways of performing simple arithmetic. We then review later stages of the mathematical curriculum and show that the same phenomenon re-occurs in many other mathematical concepts. There are many symbols that evoke either process or concept:

- 3 +2 is either the process of addition of 2 and 3 or the concept of sum,
- 3/4 can mean (amongst other interpretations) the process of division of 3 by 4 or the concept of fraction ¾,
- +2 denotes the process of shifting 2 units to the right, and also the concept of signed number +2.

A symbol which evokes either a process or the product of that process we will call a *procept*. Such a symbol stands dually for both a *pro*cess and a con*cept*. It gives great flexibility in mathematics. It even has an ambiguity which aids flexible thinking. But mathematicians abhor ambiguity. So it has become the common practice to give precise definitions for mathematical concepts which focus on the object at the expense of the inner process. This makes matters particularly difficult for the learner. Those who implicitly sense the flexible power of the symbolism succeed, but the vast majority, who do not, are likely to fail.

## The notion of procept

The idea of a process giving a product, or output, represented by the same symbol is seen to occur at all levels in mathematics. It is therefore worth giving this idea a name:

> We define a *procept* to be a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both.

We do not maintain that all mathematical concepts are procepts. But they do occur widely throughout mathematics, particularly in arithmetic, algebra, calculus and analysis. We may consider number as a procept. The sequence of number names becomes part of a procedure of counting in which the number words are recited in order, and are made to correspond with the elements of a set one at a time, taken once and only once, and the last number word uttered is the number of elements in the set.

In the early stages, number is widely seen as a counting process. It is only when the child realises that the number of elements is independent of the way in which the elements are arranged and of the order in which they are counted that number can begins to take on its own stable existence as a mental object. During Key Stage 1 of the

National Curriculum, most children count at least some of the time, and some children count all the time. Those who count quickly can succeed in the number facts to 10 (level 2) almost as well, and sometimes better, than those who know or can manipulate number facts. But those who achieve higher levels do so because they begin to see numbers as mental objects to be manipulated. The more successful may still count, but they do so less and less, and when they do count they use the technique sparingly in subtle ways which are more likely to succeed than those that continue to count on a regular basis. The latter may develop intricate counting techniques using imaginary fingers, parts of the body, selected objects in the room, and so on, to cope with the number facts to twenty. But in doing so they give themselves a harder job to do than those who use number facts in a flexible way.

The processes of arithmetic depend on whether the child sees number as procedure or procept. The most elementary form of addition is a sequence of counting operations: first count one set, then count the second, then amalgamate the two sets and count them all again (starting from one). This process, called *count-all*, is a co-ordination of three distinct counting procedures.

When a child begins to realise that it is not necessary to recount the first set starting from the beginning again, a new technique is born – *count-on*. The sum of two numbers is here computed by starting the counting from the first number and counting on. For instance, the procedure to computer 4+3 is to count on three number words past four – "five, six, *seven*". In this way the first number (4) is seen as a mental object – an uttered word in the number sequence – and the second number (3) is seen as a counting process. However, it is a more sophisticated counting process involving a double-count; the numbers "five, six, seven" are counted at the same time as keeping track of the number of words counted "one, two, three".

Concrete materials can be used to support (or rather, avoid) this double counting. For instance, by pointing at a ruler or number line, beginning at the first number (four) and counting on "one, two, three", the fingers alight on the unspoken numbers "five, six, seven" on the physical line – the last of these is the required sum. In this way a number line supports counting-on by replacing it by the easier form of single-counting associated with counting all. It can give the semblance of progress when little progress has actually been achieved and the subtleties of the double-counting of the count-on algorithm have not been sufficiently well apprehended to be carried out without physical supports.

Using a number line may also have a fatal flaw when it comes to subtraction. Those who see counting as process may see subtraction as the reverse process. Often this is performed in the form of "count-back". Thus 19–16 is performed by counting back sixteen numbers down from 19 ("eighteen, seventeen, ..., four, three"). Such a procedure can be carried out on a number line by counting out the second number (16) in the usual order ("one, two, three, ...") and pointing at the unspoken numbers moving backwards on the number line. Counting back on a number line is therefore once more a species of single-counting, hardly more sophisticated than count-all, and it may not generalise into a flexible form of subtraction.

If number is seen as a flexible procept, evoking a mental object, or a counting process, whichever is the more fruitful at the time, then children are likely to build up known facts in a meaningful way. Thus the "fact" that 4+3 is 7 becomes a flexible way of interchanging the notation 4+3 for the number 7. If 4 is taken from 7, then this number triple tells us that the number 3 remains. In this way, seeing addition as a flexible procept leads to subtraction being viewed as another way of formulating addition. Successful children learn how to derive new facts from old in a flexible way. For instance, 19–16 might be given immediately as 3, and seem like a known fact, yet it may be a fact derived instantaneously from knowledge that 9–6 is 3, and that the tens digits cancel out.

In this way procepts become organic. A child who has meaningful known facts, that can be flexibly decomposed and recomposed at will, has generative powers of deducing new facts almost without effort. Meanwhile the less successful child, entrenched in the safety of familiar counting methods, is led down a cul-de-sac in which it is necessary to cope with ever-lengthening sequences of counting to solve more complex problems. The successful child becomes more successful because the mathematics of flexible procepts is easier than the mathematics of inflexible procedures. The gap between success and failure is widened because the less successful are actually doing a qualitatively harder form of mathematics.

## The growing divergence between success and failure

The divergence between those who use interpret processes only as procedures and therefore make mathematics harder for themselves, and those that see them as flexible procepts we call the ***proceptual divide*** (Gray & Tall, 1991). We hypothesise that the difference between success and failure lies in the difference between procept and procedure. Proceptual thinking includes the use of procedures where appropriate and symbols as manipulable objects where appropriate. The flexibility provided by using the ambiguity of notation as process or product gives great mathematical power.

This divide between success and failure is found throughout the mathematics curriculum. At any stage, if the cognitive demands on the individual grow too great, it may be that someone, previously successful, founders and asks “tell me how to do it”, anxiously seeking the security of a procedure rather than the flexibility of procept. From this point on failure is almost inevitable. It is for this reason that mathematics is known chiefly as a subject in which people fail, fail badly, and fail often.

## Examples of procepts in arithmetic

We have already seen that number is a procept that embodies both the process of counting and the concept of number. Once number is encapsulated as a flexible procept it can be manipulated at a higher level in simple arithmetic. *Count-all* is but an extension of the counting process. It happens to be a fairly lengthy process, which occurs in time, so that the two numbers input to be added may be forgotten before the child counts successfully to produce the output. Thus the input and output are less likely to be linked and incidental learning of addition facts is inhibited. *Count-on* sees the first number as a mental object and the second as a (double) counting process. Again, the process element occurs in time, so input and output may not always be linked, but the fact that there is a proceptual number concept developing is more likely to support the development of meaningful known facts. As the latter are developed, the *process* of addition becomes encapsulated as the *concept* of sum and may become a flexible procept.

Level 2 of the National Curriculum specifies “knowing and using addition and subtraction facts to ten” and level 3 states “learning and using multiplication facts up to 5×5”, showing the perceived difference in difficulty between addition and multiplication. Multiplication is the process of *repeated addition*. If addition is seen only as a counting procedure, then repeating this procedure several times to get the procedure for multiplication is excessively complex. It needs addition to take on a flexible proceptual quality, so that the addition facts can be reassembled flexibly to obtain the multiplication facts.

Children who remember tables only as lists of multiples (“three, six, nine, twelve, …”) have a process of working out successive multiples without necessarily linking them to the precise product. To compute six threes using such a table requires an extended double-counting process – saying the table at the same time keeping count of the number of multiples. Again the procedural child will find such a process more difficult than the proceptual thinker.

There is also the question of whether the symbol $3 \times 4$ means “3 lots of 4” or “3 multiplied by 4”, where the first is 3 fours and the second is 4 threes. The processes are different, but the result of the two processes are the same. Invariably the first step is to give meaning to each process in its own way; but the child who goes on to realise the proceptual idea that the mathematics is essentially the same has the advantage over the child who focuses only on the fact that the procedures are totally different.

The notion of fraction is a procept, and an important one at that. Children are known to have great difficulty, particularly with the arithmetic of fractions. This involves the use of whole number arithmetic. We have seen that if addition of whole numbers is not proceptual, then multiplication becomes hard. At a higher level, if whole number arithmetic is not proceptual then fractional arithmetic becomes even more difficult. In other words, we hypothesise that children who see whole number arithmetic as counting procedure will find far greater difficulty with fractional arithmetic than those who see whole number arithmetic as flexible procept. Indeed, we would go further and suggest that, whilst children may appreciate the concept of fraction in a practical sense, those who are procedural will have little hope of giving flexible meaning to equivalent fractions and arithmetic of fractions.

Ratio is a further proceptual extension. In the secondary curriculum it is one of the hardest of all – involving the comparison of comparisons between quantities in the form “a is to b as c is to d”. It becomes manageable when ratios are encapsulated as fractions to be compared through simple arithmetic. In other words it becomes manageable when the process of comparing ratios is seen flexibly as the concept of equal fractions.

Signed numbers are procepts. The number “+2” means both “shift two units on the number line” and the signed number +2 represented as a point on the number line. The proceptual thinker who conveniently moves between the process of “shifting by 2” or “adding two” and the number concept “+2” is the one who will succeed in the future. Of course, in the early stages there may be a need to distinguish between process and concept. But the fetish for *permanent* distinction between “add two” as a process and “+2” as a concept is an artefact of the development of the concept which we contend is counterproductive to long-term success.

Likewise, the “definition” of negative numbers as “shifts left” on the number line. or “equivalence classes of ordered pairs of whole numbers related by difference” which was one of the worst excesses of sixties new mathematics, really overcomplicates a simple issue. Negative numbers can be seen as a simple extension of the number line, either horizontally to the left, or vertically down below zero, like temperature on a thermometer.

## Procepts in algebra

An algebraic expression such as "3*a*+4*b*" is a procept which stands dually for the process "add three times *a* to four times *b*" and the algebraic expression which can be manipulated mentally as an object. Clearly children with a procedural view of notation will be confused by an expression involving letters, for it cannot be processed until the values of the letters are known and, if they are known, then they are redundant and the whole thing could have been done as arithmetic using the number values. Thus the procedural child is likely to see algebra as an unnecessary complication. And if the child does not have a flexible view of arithmetic allowing decomposing and recomposing of arithmetic expressions, it will be highly improbable for the child to use such notions at the level of generalised arithmetic. This suggests that it is fatuous to teach paper and pencil algebra manipulation to a child who has not got a proceptual view of arithmetic. Elsewhere (Tall & Thomas 1991) the suggestion has been made to give meaning to algebraic symbolism using simple programming. Thus, in BASIC, if the command a=3 is typed followed by PRINT a+2, then the response 5 offers a result whose patterns can soon be predicted. In this way meaning can be given to expressions using letters as variable numbers. The fact that expressions such as 4**a*+4**b* always evaluate to the same value as 4*(*a*+*b*) focuses on the interpretation that the processes of calculation may be different, but the output values are always the same. The symbols represent different processes, but the same flexible procept.

## Procepts in higher mathematics

The analysis of procepts does not stop with simple arithmetic and algebra, but goes on to higher levels. For instance, a limit concept, such as

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

uses the notation to represent both the *process* of tending to a limit and the *value* of the limit. Again the symbolism dually represents process and product and is yet another instance of procept. But here there is a subtle difference. The procepts in arithmetic (addition, multiplication, etc.) have clearly defined procedures to compute the result (counting, repeated addition, etc.), and the procepts in algebra can at least *potentially* be computed if the values of the variables were known. But limit procepts often do *not* have a guaranteed procedure of computation. For instance, what is the value of the following sum?:

$$\lim_{n\to\infty}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)$$

The value is now known to be $\pi^2/6$. But this was first computed using methods of complex analysis rather than any obvious arithmetical method. In this way procepts in higher mathematics may operate in ways that are not expected by students based on previous experience and may lead to further cognitive conflict.

## Reflections

At all levels we can see that the notion of symbolism dually evoking either process or concept has powerful applications for the flexible thinker yet creates great difficulty for the inflexible person who accumulates procedures to solve each new problem. The proceptual divide between those who succeed easily and those who fail catastrophically is perpetuated and grows ever wider.

Such a divide is embodied in Ausubel's (1968) differentiation between *meaningful* and *rote* learning, or Skemp's distinction between *relational* and *instrumental* understanding (1976). However, the theory we give here has an extra ingredient. It is not just the *relating* of one idea to another, or the giving of a *meaning* to a process or concept. It is the ability to give meaning to the process *in a flexible way that allows process and concept to be interchanged at will*, often without any distinction being made between the two.

This suggests a rethink of the way in which we teach mathematics at all levels. For instance, in whole number arithmetic, children are often encouraged to develop their own methods of computation. Such a policy is fine if the methods are flexible and generalisable. But if the child obtains success through idiosyncratic counting methods, then this can lead to the development of a procedural approach that gives short-term success but possible long-term failure. In the day-to-day running of a classroom short-term success may be more immediate and instantly rewarding, but if it is at the cost of eventual failure, it is a devastatingly bad strategy. It also bodes ill for payment by results in the National Curriculum, for if a teacher uses short-term procedural methods to get quick and easy solutions to pass this years' test, it may store up future problems for the child and for those who teach that child later.

The analysis also suggests that the current method of allowing children to work at their own pace from work cards can actually disguise the symptoms of eventual failure. The child may succeed at addition sums more slowly through counting procedures, yet may be developing the very strategies that lead down a cul-de-sac. Only through discussion and *listening* to a child talking through the processes being used can one hope to diagnose the possible development of inappropriate strategies.

This burden imposed on the less successful, who are constrained to perform harder (procedural) mathematics rather than the more powerful and easier proceptual mathematics, provides a challenge that seems to have little hope of resolution by traditional means. Simply giving children more practice at the procedures they cannot do only serves to reinforce inflexible procedural methods.

Clearly the attempt to impose flexible methods on children who are already struggling with the procedures of mathematics may only serve to increase the burden. Perhaps the attempt to teach them to be proceptual may only serve to teach them new procedures that have a semblance of flexibility.

A hopeful strategy would involve giving children access to more powerful methods without increasing the cognitive strain. This may involve using the right tools for the job. For instance, it was assumed by many that the calculator would interfere with the child's ability to perform mental arithmetic. Yet experience shows this to be false. Using a calculator enables children to deal with arithmetic involving larger numbers where patterns become more obvious. They need not be constrained in their earlier years to pass from number bonds to ten into number bonds to twenty, where the irregularity of language often obscures the pattern. Instead of handling sums like "thirteen plus four" where the link to "three plus four is seven" may not be apparent, they may be dealing with more euphonious instances such as "twenty-three plus four" whose solution is "twenty - seven". Another factor which is often overlooked is that the use of the calculator suppresses the need to count. Thus it replaces the procedural use of counting by the procedure of typing the sum into the calculator. But at least in the latter case the sum may be recorded in its entirety and used in the search for pattern that may lead to flexible use of mathematical notation.

Likewise the computer can be used to give meaning to algebraic notation, to draw graphs and visualise concepts in the calculus, and to carry out procedural computations so that the learner may concentrate on the meaning of the results of those computations. The use of the computer challenges our traditional perceptions of mathematics for it can perform all the traditional procedures that are at the root of a procedural approach to mathematics. By looking more at the proceptual nature of mathematical meaning, rather than focusing solely on the procedural method of mathematical computation, we may be entering a new era of mathematical understanding.

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