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AN ANALYSIS OF DIVERGING APPROACHES TO SIMPLE ARITHMETIC: PREFERENCE AND ITS CONSEQUENCES

RUNNING HEADER: Approaches to simple arithmetic

ABSTRACT. Earlier research by the author indicated that many below average attainers do not remember number facts and use alternative strategies to obtain solutions to basic arithmetical problems. These alternatives were frequently seen as the 'best way' of finding a solution.

This paper considers the relationship between the various strategies used by mixed ability children aged 7 to 12. An analysis of alternatives suggests that the selection is not underpinned by regression through the learning sequence, but by regression dominated by the child's preference for certain strategies over others. Through the evaluation of a hierarchy of preferences, divergence between the strategies available to the less able and the more able child is revealed. The alternative strategies used are based either on counting - procedural strategies, or on the use of selected known knowledge - deductive strategies. Above average children have <u>both</u> available as alternatives; evidence of deduction is rare amongst below average children. The more able child appears to build up a growing body of known facts from which new known facts are deduced. Less able children – relying mainly on procedural strategies – do not appear have this feedback loop available to them.

This paper contents that, for <u>some</u> children, procedural methods do not encourage the need to remember; the procedure provides security. On the other hand, deductive methods initially enhance the ability to remember other basic facts and eventually help children make extensive use of facts that are known to remove the need to remember new ones. More able children appear to be doing a qualitatively different sort of mathematics than the less able.

THEORETICAL CONSIDERATIONS

The development of numerical concepts and skills within young children has received such attention over the past two decades that the importance of meaningful counting as a basis for arithmetical development would now appear to be beyond question. McEvoy (1989) reviews the evidence of many studies of young children which together trace the sequence of development from counting to the beginnings of formal arithmetic. Knowing how to count is fundamental to the acquisition of early arithmetical skills.

Fuson et al (1982) verified a hierarchy for the development of ability associated with the memorisation of the number word sequence. McEvoy (1989) reminds us that memory plays an important part in the process; children have to have the ability to memorise and recite the number word sequence by rote.

The need to distinguish between the ability to recite the number word sequence and the act of counting, a procedure based on one to one correspondence, was highlighted by Herscovics and Bergeron (1983). They show that any interpretation that treats the Fuson hierarchy as appropriate to counting must be tentative since, although it has been verified for the number word sequence, it has not been verified for the counting procedure.

Herscovics and Bergeron (1983) kept the distinction between recitation and counting in mind when distinguishing six counting procedures that corresponded to the different skills described by Fuson et al. These procedures were linked to specific problems, the solution of which required the use of a corresponding procedure, which ranged from counting from 1, to counting back to a given number. They noted the caution of Steffe et al (1983), who indicated that the final four stages of the Fuson hierarchy, the collective skills of which would enable children to use count-on to solve such problems as 5 + 3 and count back to solve 8 – 3, indicate a very advanced degree of abstraction in that they involve double counting. Herscovics and Bergeron saw understanding based on the procedures as one stage in the process of constructing a conceptual schema. When the procedure is introduced in the context of a certain class of problems the resulting procedural understanding bridges the gap between intuitive understanding and the beginning of abstraction.

Counting is the bed-rock of the procedural approach; it furnishes the process which enables a response to be made. Knowledge is the foundation of the deductive approach; other known facts and the relationship between the facts are used to deduce solutions that are not immediately known. Procedural understanding may well be a significant cog within the arithmetical, or indeed the mathematical entity. However, whilst at one level, and therefore for some children, it can be a link in the conceptual chain which leads towards the growth in relational understanding, at another it may be the reason why some children do not make the links and continue to operate at an instrumental level. For the former, the development of procedural understanding, coupled with the use of selected knowledge, may well lead to the use of deductive strategies to solve numerical problems; eventually the deductive approach may become totally reliant on the use of selected knowledge. For those children who do not make the links, confidence with procedural approaches would appear to prompt an automatic response that may pay little regard to efficiency.

From a pedagogic point of view, a hypothesised hierarchy for counting, together with a classification of the solution strategies used to solve simple arithmetical problems in addition and subtraction, can provide a framework to study the development of arithmetical skills. Within such a framework, the central issue becomes the balance between routine, meaning and the ability of the child to develop efficient strategies to solve problems. Efficiency can be an emotive word when applied to simple arithmetic. One view of efficiency may place the emphasis on the ability to remember; a dominant teaching aim is to establish knowledge of number facts as part of the mental repertoire of the child. An almost contradictory view implies that remembering a fact is of lesser importance than the ability to devise a method for obtaining a solution. However, efficiency is not age independent; what is efficient for the young child may not be regarded as such for the older child. For the latter, the level of sophistication used to obtain solutions to simple arithmetical problems may need to be more in tune with the level of attainment expected in more difficult calculations.

Gray (1987,1988) indicated that below average children rely extensively on procedural methods to obtain solutions to basic number facts if solutions were not known. The current study was established to confirm the earlier results and to compare the approaches of three identified ability groups of children. Indications from the previous studies were that children used preferred methods as alternatives and that the use of these alternatives indicated disparate outcomes in the use of procedural strategies. To highlight these alternatives, use is made of a preferential hierarchy, that is the child's preferred way of doing things.

PREFERENCES

Solution Strategies

The classification of solution strategies for solving arithmetical word problems made a considerable move forward through the work of Carpenter and Moser (1982). Their refined classification is the natural development of earlier work (Rosenthal and Resnick 1974; Groen and Resnick 1977). Carpenter at al. (1981) draw attention to the care that must be taken when making comparisons between children's solution strategies to verbal problems and their solutions to number calculations. However, for number calculations these categories proved consistent in describing strategies used by children aged 8-12 to solve basic computational problems (Gray, 1988).

To obtain the solution to any one problem based on the addition and subtraction facts to twenty a child may use any one of four basic strategies. The child may

- know how to count: **count-all**
- conceptualise the value of at least one set and use the appropriate counting procedure: **count-on** (or count-up/ count-back)
- use any other known number facts: **derived fact**
- know directly: **known fact**.

Whether these strategies form a conceptual a hierarchy or not can be only partially resolved by documented research. Groen and Resnick (1977) indicated the part that teaching and its consequences had to play in the development of the five year old child's ability to develop counting strategies in addition. Through teaching some children not only modified their addition strategy from count-all to count-on but also adapted it to include the use of commutativity. That count-all is lower in the cognitive hierarchy than count-on were conclusions drawn by Fuson (1982), Carpenter (1982) and Herscovics (1983). The conceptual advance from count-all to count-on was the focus of attention of Secada et al (1983). This view of the relationship between count-all strategies and count-on strategies would seem to have implications for a cognitive hierarchy for subtraction. For a child whose addition strategy is mainly count-all or count-on, the strategy for subtraction can only be seen as a reversal of this processes. Therefore, subtractive strategies which are analogous to count-all would appear to be at a lower level of sophistication than those that use count-back or count-up.

Efficiency and accuracy may eventually be guaranteed using such methods but where does this place the knowledge of facts and the ability to use facts that are known to derive or deduce solutions? The ability to simply recall facts is difficult to established within a hierarchy, because such a strategy can be used without any evidence of meaning. It may only provide evidence of routine and a good memory.

The Preferential Hierarchy

Establishing levels of sophistication now centres on one question; how may we compare the ability to use counting strategies and the ability to recall facts? The short answer is – we cannot compare them using only a cognitive hierarchy, but we can if we look at preferences.

Herscovics and Bergeron (1983) emphasise that any cognitive hierarchy for addition and subtraction can only be assumed for the sake of discussion. In the first instance this is also the case for a preferential hierarchy. If a child's preferred way of solving one of the numerical problems is to remember the answer (known fact) then the preferred, and most efficient alternative if the fact is not known, will be to use other known facts to derive the answer. Should either of these two strategies fail, the child will then need to resort to the next preference which will involve counting.

Logic then, would seem to indicate that the descending order of preference, theoretically available to all of the children, can be indicated and consequently a direction of regression, defined as the move down the preferences, may then be displayed.

The theoretical model in Figure 1 illustrates the preferences available for the separate operations of addition and subtraction. In the former the route of regression is fairly clear cut; failure to solve a problem by the most preferred strategy implies a move down the scale to the next preferred one. Although the model presented in this simplified form does not make distinctions between the different levels of abstraction that can be used in counting, it is clear that the move from count-on to count-all implies that there could be underlying increase in the use of concrete counting aids.

FIG.1 ABOUT HERE

The possible strategies available to solve subtraction problems presents a more complex picture. Reference to previous work (Gray,1988) helps to resolve some of the problems.

First there is the identification of the subtractive process analogous to the addition count-all. In the earlier study, if children operated subtraction in unary form by counting out the minuend and then, from within the set formed, counting out the value of the subtrahend and recounting the remainder, the strategy used was equivalent to the addition strategy of count-all. To make a distinction between the subtractive and additive procedures it was decided to refer to the subtractive strategy as ‘take-away’. This was done for two reasons:

a) it modelled the actual process being undertaken and,

b) no matter what other strategy a child used for subtraction problems, the take-away process was used only when partitioning a counted set and considering the remainder. Other strategies based on counting reflected the complementary nature of addition and subtraction or the use of other known facts.

A second decision limited the range of problems to ones that represented only the unary aspect of subtraction. There was no reference to problems that asked children to "find the difference". Earlier interviews (Gray 1988) had indicated that there was considerable confusion over the word "difference"; very few children linked it to a subtractive process. Although no problem was presented in this form, one or two of the younger children did in fact obtain a solution by finding difference. Since in these cases the procedure involved counting out the value of both subtrahend and minuend, matching the two and then recounting the difference, the decision was taken to subsume any such strategy under the take-away heading.

Thirdly, the essential difference between the addition and subtraction model is that in the first instance, children who revert to counting strategies for subtraction have options available to them. They can either use count-up, count-back or a combination of both. Logically, the latter is of higher order than the single use of one of the two former since it is decision based. This has its parallel in addition when children decide whether or not to use commutativity, particularly when the smaller number is given before the larger and children decide to count on from the larger because "it is nearer the answer". However, the general distinctions are not made within this paper since it is the broader characteristics of the composite use of strategies that are of concern.

The preferential model takes the view that a child seeks alternative strategies to obtain solutions to numerical problems only if the solution to a number fact is not remembered. The alternative is identified as the next best method. Since these children had all been through a pedagogic process which had as its aim the requirement that the number facts should be known, it was hypothesised by the author that failure to remember a fact would trigger a measured descent through the stages of a cognitive hierarchy as outlined in Figure 1. However the strategy combinations observed within the empirical study that is reported do not totally support this view.

THE INVESTIGATION

There is no doubt that children who have difficulty with aspects of computational work use a substantial proportion of procedural methods to obtain solutions to basic number facts. Nor is there a doubt that many children had a preferred way of doing things. Interviews with over 100 children in 12 different English schools between 1986 and 1989 (Gray, in preparation), had indicated this. What was open to question was the relationship between the strategies used by children of different ability and their link to the preferences available. To obtain insight into these relationships a single interviewer used a combination of 'structured' and 'open interviewing' techniques, (Cohen and Manion, 1985), to identify the solution strategies used by children when solving simple arithmetical problems. The interviews were structured in that all of the children were presented with a standard introduction and a common series of problems to solve. The open nature of the interview was apparent when the interviewer discussed the method and preference used to solve particular problems. At such times the content, sequence and wording were entirely in the hands of the interviewer. Interpretation of the outcomes of the open component of the interview were used to categorise a child's solution strategies and give an indication of preferences.

Two schools, considered to represent typical English schools, were approached to take part in the investigation. Although the focus of attention was mainly on the strategies and preferences used by children who had moved beyond the point of pedagogic input in developing knowledge of number facts, a group of children who were still working at this stage, the 7+ children, were included. The class teachers felt that the 8+ and 9+ age groups knew sufficient number facts to move on to the development of computational skills in addition and subtraction with and without exchange, whilst those above these ages were felt to be at least reasonably competent with such problems. Since each school had six classes each class teacher was asked to identify six children. The children were chosen in such a way that in the class teachers opinion they would be representative of above average, average and below average attainers in each class. In this way a total of 72 children were identified who represented the chronological ages 7+ to 12+. Apart from the 12+ group, which through movement and sickness eventually contained only nine children equally spread over the three arithmetical ability levels, each of the other five age groups contained twelve children equally divided over the three teacher–defined ability levels: below average, average, and above average.

Each child was interviewed separately on at least two occasions with a week in between each interview. At the start of the first interview each child was told that the interviewer would present several problems and the child would be asked to find an answer to each problem using the method (s)he thought was best.

During each of the two interviews the children were presented with between 18 and 20 addition and subtraction numerical problems in two stages of difficulty. At the first interview the child was presented with the Stage 1 problems and at the second interview the Stage 2 problems. A third interview was given to those children who had required a substantial amount of time for interviews one and two. Each interview lasted approximately half an hour. The third stage of difficulty, not reported within this paper, was given about two weeks after the child had completed the first two stages.

The problems within each stage were classified into groups:

Stage 1: Addition and subtraction facts to ten. The numerical problems within this stage included:

- the addition and subtraction of zero, and the addition and subtraction of one,
- addition and subtraction involving doubles i.e. 4 + 4, 6 – 3,
- addition and subtraction involving two evens i.e. 6 + 2, 8 – 2; odd and even i.e. 7 + 2, 9 – 4, and two odds i.e. 3 + 5, 7 - 5
- addition and subtraction of a pair of numbers with a difference of one i.e. 4+ 5, 9 – 8

Stage 2: Addition and subtraction facts within the range ten to twenty. Two categories of addition problem were considered:

- The addition of single digit numbers the sum of which was between ten and twenty i.e. 9 + 8 and 4 + 7, and

- a sample of addition problems involving teens where the units to be added included some of those considered in Stage 1 i.e. 12 + 0, 13 + 5, 3 + 16

The subtraction problems also included a sample which involved the use of Stage 1 subtraction facts i.e.15 – 4, 16 – 3 and other subtractions facts to twenty i.e.12 – 8, 18 – 9, 15 – 9.

Stage 3: Addition and subtraction of two and three digit numbers with and without exchange. These problems are not subject to consideration within this paper.

Each problem was presented orally, and on paper in a way appropriate to the usual practice of the school, to each child individually. The solution strategy that each child used was recorded. If this was not completely clear, the child was asked to describe how the answer had been obtained. When children changed strategy they were asked to try and give a reason for the change. The problems within each stage were presented separately until the child had completed a section. If a child was either unable to give an explanation, or began to experience considerable difficulty, as measured by three incorrect solutions or by the length of time involved, the interview was terminated. Structured apparatus i.e. counters, unifix blocks, and colour factor rods, was available and it was suggested that, if they wished, the child may use it. However, the usual practice in both schools was pen and paper so these too were available and children were encouraged to use them.

KNOWN FACTS: AGE AND ABILITY COMPARISONS

A preliminary analysis of the results considered the percentage of solutions within Stages 1 and 2 that appeared to be obtained through immediate recall without evidence of the apparent use of any other strategy. This first analysis, by which an overall percentage of the solutions that were known within each age/ability group was obtained, excluded solutions that were established through the use of alternative knowledge. Using such an approach the initial focus was placed on the category identified by Carpenter and Moser (1982) as ‘Known Fact’.

FIG.2 ABOUT HERE

The evidence from the sample illustrated in Figure 2 indicates that:

(i) as expected, the older children knew more facts than the younger ones.

(ii) the solutions to all of the addition and subtraction facts were only known by the complete group of above average eleven and twelve year olds.

(iii) of the average ability children the twelve year olds were the only group who knew every Stage 1 addition fact.

(iv) no complete below average group knew all of the facts.

(v) there is a two year gap, extending to three years by the age of twelve, between the level of attainment of the below average and the above average children in knowledge of the number facts to ten.

A further feature emerges. Whilst the percentage of facts that are known by the above average children increases steadily as they grow older, the children of average ability appear to have a slight hiccup at the age of 8+. Although there was no difference in the level of ability of above average children and the average ability children at the age of 7+, the average ability eight year old children knew 15% less addition and subtraction facts than their seven year old counterparts. In both schools these children were being taught the addition and subtraction algorithms with exchange. One wonders if the complexity of the broader issues of exchange may be responsible for this phenomenon. It may be that the children have temporarily lost confidence in their own ability to remember and this may have had some effect on their preferences.

The attainment ‘age gap’ identified in the analysis of the known facts to ten becomes far more apparent when the addition and subtraction facts to twenty are considered (Figure 3).

FIG. 3 ABOUT HERE

(i) No one complete group of children knew all the facts relating to the Stage 2 numerical problems.

(ii) The overall percentage of known facts increased in all groups up to the age of 11+ but then the above average children at 12+ appear to know fewer than those at 10+ or 11+, whilst the average ability children continue to show a slight increase.

(iii) None of the below average ability seven year old children knew any of the Stage 2 addition or subtraction facts. Neither could any of the above average children of 7+ recall any of the subtraction facts between ten and twenty. They used derived facts, count-back or count-up.

(iv) There is confirmation of at least a two year attainment gap between the above average and the below average children, and evidence that for number facts within the teens, the attainment gap is more normally three years.

The analysis of the known facts amongst the three groups of children, and the somewhat surprising limitations in the children's ability to remember the facts, leads to some interesting questions. A central feature of the study was to consider the strategies that the children used to obtain the correct solutions to the variety of numerical problems that were presented to them. If they cannot remember facts, what are the alternative strategies available to them and how are these alternatives used?

COMPOSITE STRATEGY USE - THE USE OF ALTERNATIVE STRATEGIES

Individual children did illustrate the use of a single of strategy. At the least sophisticated level, one of the below average children solved every addition and subtraction problem based on the facts to ten by using count-all for addition and take away for subtraction. The only fact that this child could immediately recall was 6 – 0, a fact that only one child in the whole sample did not know.

At the other extreme, an above average nine year old knew all of the presented addition and subtraction facts to ten. This child indicated the start of a pattern that was to culminate with all of the average and above average twelve year olds recalling every addition fact to ten.

The above average ten year olds were the only complete group to display this level of attainment with the subtraction facts.

The network in Figure 4 (below) is a composite picture of the routes used by all of the children to solve the addition and subtraction problems based on facts to ten (the Stage 1 problems). The percentages are rounded to the nearest whole and enable comparisons to be made between the proportions of children using a range of particular strategies. Through following particular routes we are able to identify that 97% of the children use known facts (KF) but 72% supplement its use with alternative strategies when dealing with a range of addition problems. The comparable figures for subtraction are 97% and 73%. From these figures it can be established that 25% of the children knew all of the presented addition facts whilst 24% knew the subtraction facts. Only 3% of the children in each case had to totally resort to the use of count-all or take-away.

FIG.4 ABOUT HERE

The almost equal proportion of similar strategies that the children used to solve both the addition and subtraction problems is striking. The one exception to this appears to be the greater proportion of subtraction facts that were solved by take-away. 11% of the children used no other alternative strategy if the facts were not known. Indeed, the percentages that included some form of counting as an alternative strategy were otherwise almost equal; 60% for addition and 62% for subtraction. The balance, by implication, used only derived facts together with known facts; 12% addition and 11% for subtraction.

FIG.5 ABOUT HERE

At Stage 2 (Figure 5) a slightly different picture emerges:

(i) there was a considerable decline in the percentage of children who knew all of the facts.

(ii) there is a considerable increase in the percentage of children who use derived facts.

(iii) a greater percentage of children, through their exclusive use of counting strategies, indicate that they could not recall at least one fact or use a fact they did know to establish a different fact.

(iv) some children were unable to start the problems, hence the difference in totals.

Although such composite pictures do give an indication of what is happening across the spectrum of the age range considered they do not enable us to begin to identify the contrast in strategy use by children of different abilities and ages. To do this we need to consider the variety of strategies used by separate groups.

THE DIVERGING USE OF STRATEGIES

The strategy combinations that are outlined above begin to present some very clear pictures if they are related to preferences, age and the teacher identified level of ability of the children.

One of the stated objectives of teachers of the younger children was that these children should know the facts to ten. It became clear that this aim was not achieved overall and that proficiency in the knowledge was a function not only of the age of the children, but also of their ability. Further, if the facts were not known, the use of particular supportive strategies and the preferences available was also a function of the children’s age and ability.

91% of the whole sample indicated that knowing the answer was the best way of getting a solution to the addition problems within Stage 1. Three-quarters of the below average seven and eight year olds indicated that counting was the best way. One eight year old stated that the "best way to get the answer was to know them but I usually have to count because I don’t know many". An alternative view was expressed by an eight year old above average child who said "I usually count, but some I know". In contrast 84% of the sample claimed that knowing the solution was the best way of obtaining the answer for a subtraction facts. All of the eight year old below average, three quarters of the eight year average and three-quarters of the seven year below average indicated that counting was the best way to deal with subtraction.

Almost all of the average and below children who claimed that knowing the fact was the best way to obtain a solution indicated that counting was the best if the solution was not known. Individual problems e.g. 9–5 did indicate that there were exceptions to this general rule, but the exceptions were not common enough for these children to make a definitive statement which related to the selection of other known facts. Even the eight year above average children, who extensively used derived facts, were not able to indicate that the use of selected facts was an option for them. Their general approach to counting however was that they used that method if they couldn’t "do it" (find a solution to the problem) any other way.

Even though derived facts were extensively used when children were dealing with the Stage 2 problems, the ability to state that deduction was an option was only expressed by the older average ability children e.g. 11 and 12 year olds, and the older half of the above average children e.g. 10 to 12 year old olds. Below average children only saw the method as applicable to particular problems. The implication is that this method for most children of middle school age is intuitive and signalled by the problem they have to deal with. It is suggested that only through continued successful use does it become an approach which can take its place as a preference.

The representations (Figure 6 and Figure 7) are area graphs which illustrate the cumulative percentage of the range of strategies used by identified groups of children to obtain solutions to Stage 1 and Stage 2 problems. The area graphs indicate the trends between age groups and show the proportions of the strategies that the children of the three ability ranges, and the six age groups, used to solve the addition and subtraction problems.

FIG. 6 ABOUT HERE

The analysis of the strategies used by the below average group of children at Stage 1 (Figure 6) shows that the percentage of known facts increases with a corresponding decline in the percentage of counting. However, there is also a change in the pattern of counting within for addition. The less sophisticated strategy of count-all decreases at the expense of an increase in count-on. The proportional use of count-on peaks at the age of 10+ but remains high for 11+ and 12+ children.

For the above average children there is a distinct absence of count-all strategies. Even the use of count-on is only clearly evident amongst the seven year olds and, to a considerably lesser extent, the nine year olds.

FIG. 7 ABOUT HERE

The broad distinctions apparent in the Stage 1 solutions strategies are again evident with the Stage two problems (Figure 7). The extensive use of counting by the below average group is clearly identified. Within the other two ability groups the decline in counting with an accompanying growth in recall methods, a combination of derived facts and known facts, is clearly identifiable.

THE USE OF DEDUCTIVE APPROACHES

The most striking feature of the graphs is the extent to which different groups make use of derived facts.

It can be clearly seen that the above average children, when dealing with the Stage 1 problems (Figure 6) made considerable and more extensive use of other known knowledge to derive solutions at a younger age than the average ability children. In contrast, it appears that this latter strategy is not readily available to the below average children when dealing with the number facts to ten.

If the younger below average child does not know a solution the evidence is that (s)he will use counting. Such a strategy remains a dominant alternative for each group of below average children.The above average children on the other hand, by the age of eight, were able to recall, either through knowing or through using other known knowledge, over 80% of the answers.

It is the tendency, or otherwise, to use other known knowledge that amplifies distinctions between the below average children and the other two groups. The use of derived facts may hold the key to providing insight into the relationship between the individual differences identified within the preferential hierarchy. The deductive approach, based as it is on the use of known knowledge, carries with it the implication that children cannot use the derived fact strategy until such time as something is known.

The above average children demonstrated that if they did not recall solutions they could bring to the unknown solution a variety of other known facts to enable them to derive solutions. The use of the known pairs of numbers that make ten enabled the seven and eight year olds to obtain solutions to such Stage 1 numerical problems as 4 + 5, 7 + 2 and 6 + 3. The facts to ten and the use of known addition facts also enabled them to derive solutions to 5 – 4, 6 – 3, 9 – 5 and 9 – 8. In some instances multiplication table facts were used to obtain solutions i.e. 6 + 3; "six has two three’s and three three’s make nine". The average ability children also made similar use of such knowledge but the essential difference between the two groups was that the range of solutions used by the average ability children was limited and the children were older when they started using them. The evidence of the use of derived facts amongst below average children was much more restricted, confined solely to deriving a solution to 4 + 5 and/or 9 – 5.

Although the use of doubles was very limited amongst the above average children when dealing with Stage 1 problems evidence of such strategies was considerable when they sought solutions to Stage 2 problems. Such problems as 9 + 8, 8 + 6 and 18 – 9 were solved extensively by the use of doubles, whilst understanding of the structure of the number system helped the above average child to use a combination of facts to solve such problems as 13 + 5, 15 + 4, 4 + 7, 15 – 9 and 12 – 8. The use of such alternative knowledge amongst the average ability children was to a large extent restricted to age 9+ and above.There were of course some exceptions to this generalisation; the ability to remember and use the fact that ‘two nines are eighteen’ extended across the age range so that the solution to 18 – 9 could be found by such an approach. The solution to 14 + 4 could be found by some of the younger children through a combination of known knowledge i.e. 4 + 4 = 8, and an understanding of the number structure viz. "ten and eight is eighteen".

The ability to deduce a solution is dependant on something being known. The counting methods used by the below average children do not appear to reinforce their knowledge of number facts. The children did not appear to be making the links between the problem, the procedure and the solution. After using counting to obtain solutions to such numerical problems as 5 + 3, 9 + 8 and 15 – 9 many of the below average children gave the solutions with relief. More importantly however, many of the younger children within this group could not remember the problem that had triggered their procedure. The link between the numerical problem and its solution had been obscured by the lengthy counting routine that had been used to obtain the solution. It appears that the younger below average child does not receive any feedback from the counting procedure; the process is not being encapsulated into a known concept.

CONCLUSION

Given any one of the numerical problems an individual child took one of three general approaches to obtain a solution:

(i) immediately recalled the fact; e.g. 2 + 3 is almost immediately seen as five,

(ii) deduced the solution from alternative known facts,

(iii) used a procedure which they felt confident with.

It was the child's failure to solve the problems by immediate recall that triggered the use of other approaches. The evidence from this study is that the child reverts back, not through the cognitive hierarchy, but through the preferential hierarchy and that there are two distinct approaches to the regression. The first makes use of other known knowledge, the deductive approach. The second is dominated by the use of counting, the procedural approach. The former is clearly displayed by the above average children and the average ability children who use it extensively when dealing with the Stage 2 problems. Evidence of the deductive approach is very limited amongst the below average ability children who make substantial use of procedural approaches.

What has become fairly clear through this study, since it confirms earlier work (Gray,1988), is the fact that the below average ability child is neither successful at learning the number bonds nor in making use of the ones that they do know. However, during the middle years of schooling there appears to be a subtle change in the use that such children make of procedural methods. If the younger children do not remember the solution to a problem they search for an alternative route and consequently, placing the emphasis on instrumental rather than relational approaches, take a route that involves a radical change in strategy: memory is abandoned for a procedure that involves the use of physical or quasi-physical objects. The bits they do know do not appear to be held together, with the result that this change in strategy may involve the child in long sequences of counting to arrive at solutions. In one sense they make things more difficult for themselves and as a consequence become less able. However, by the end of middle years of schooling such children feel secure, even confident, in their procedure. It is successful, may well have been refined and it leads to solutions. Why change? What need is there to look for alternatives?

In contrast, condensing the long sequences appears to be almost intuitive to the above average child. This eventually becomes the cornerstone to their higher level of attainment; they can take short cuts and operate with increasing levels of abstraction. All the disparate aspects of the number system are brought together to establish solutions. Again there appears to be two outcomes. Up to the age of about eleven the ability to condense procedures through the use of alternative known facts appears to strengthen the bonds between deriving and knowing. After eleven there is a hint that the above average child is increasingly content to use a base of knowledge to derive other knowledge. An above average child may well have to learn that addition and subtraction are different but the method of handling 3 + 5 may well be the same as that used to handle 8 – 3. It is suggested that such a recognition is initially intuitive. Eventually, because a generalisation can be made from it, the children recognise that they have an appropriate alternative to knowing and they can express their alternative in words; the approach can now be identified as a preferred alternative. The scope of the options available to the above average and, with a small time gap, the average child is broadened. The below average child on the other hand, is more likely to handle 5 + 3 by counting on and 8 – 3 by count-back or count-up. Such procedures involve the co-ordination of sequential processes which imply a greater use of memory. Thus, compared to their more able counterparts the below average child has more to remember but in fact the ability to remember less.

The divergence between procedural and deductive approaches becomes a reality and indicates that the above average and, to a slightly lesser extent, the average are indeed doing a different form of mathematics than the below average.

The enhancement of procedural understanding to provide a foundation for encapsulation would appear to be taking place within those children who are using an approach with a combination of deductive and procedural techniques. Such children know some facts but not enough; it would appear that they use the procedural approach to bridge the gap between their knowledge and the use of that knowledge. The younger above average children illustrate this trend.

Counting, as used by the below average child, does not appear to enhance the ability to learn and then use facts. The learning comes purely from familiarity but the relationship between knowledge and procedure remains weak; counting does not enable the child to encapsulate the knowledge, indeed the indications are that it in fact does the reverse.

For the above average child relational learning is more likely to have taken place through the use of deduction to arrive at solutions. The use of derived facts strengthens the bonds between the known knowledge and the relationships and structures that are inherent in the number system. With Stage 1 problems using derived facts to obtain solutions is an auxiliary approach which enhances the ability to remember. As the problems became harder and the children older, this approach removed the need to remember. Thus it is suggested that the use of derived facts for the younger children is an indispensable stage in developing knowledge of the number bonds but in the older children it becomes a expedient which removes the necessity to remember them all.

The procedural understanding that dominates the alternatives available to the below average child does not seem to be a step towards the beginning of abstraction. Instead it appears to becomes an end in its own right and as such demonstrates the child’s inability to make the generalisations that will ease the load on the quantity that is to be remembered. We will not help such children by continuing to accept this situation and at the same time continuing to push forward attempting to teach them more complex algorithms. We have to help them overcome the hard work that is necessary to remember some facts and reinforce the relationships between the facts that they do know to establish other facts.

From the teaching point of view two issues arise that have implications for classroom practice. The first embraces issue of the relationship between procedural and deductive methods and children's solution strategies for the number facts, and the second, far less easy to respond to, the qualitatively different ways that less able children appear to be doing mathematics compared to their more able peers.

Teachers of young children must make a conscious effort to teach count-on. Secada et al (1983) provide an indication of how this may be done. However, because count-on is a process completed over time, the links between the problem, the process and the product of the process are less tenuous for the below average child. The focus of attention for the below average child is the process which concentrates action by the child. The product, frequently arrived at with a sense of relief, is marginalised so that longer term benefits of a procedure which would enable the child to develop and internalise an expanded number schema are limited. The author has now developed a degree a scepticism in claiming that benefits arise from the use of structural apparatus with these children. Such aids alter the procedural focus and usually involve a ‘translational’ stage, for example the selection of a colour factor rod to represent a number, which can expand rather than contract the procedural process. Below average ability children must be given an opportunity to expand their experience of seeing problem and product without the clutter of a procedure. Graphic calculators, which enable a simultaneous display of a problem and its product, may provide such an opportunity.

An emphasis on count-on as a procedure for addition will also have implications for teaching. Many children will view subtraction as a reversal of this process. The evidence from this study, though not made a substantive part of the paper, is that a substantial number of children use count-back to solve subtraction problems. This can be an horrendous process, particularly for the below average child. Indeed, watching such a child attempting to cope with 12 – 8 or 15 – 9 using such an approach, is a savage indictment of teaching methods that condone it . It is also an indictmentof the belief that children should be allowed to develop their own personal modes of performing arithmetic. Count back may work well with 8 – 2 but it does not generalise easily to 15 –9. Teachers therefore, must look once again at standard practices which make use of such aids as the number track or the number line. Within many English schools current use of such pedagogic aids tends to reinforce the reversal of count-on to obtain solutions for subtraction problems. An appreciation of the binary or difference perspective of subtraction, would enable children to establish count-up procedures and thus give them greater flexibility.

Flexibility, particularly amongst the older below average children, can often mean the development of personalised counting methods. Again the evidence from the interviews within this study indicates that if children develop highly personalised methods of coping with a limited range of arithmetical problems (often by counting using various parts of the body or various finger configurations to represent different numbers) then they may achieve short term success with small numbers but considerable difficulty,and even failure, with larger numbers.

Within the classroom a greater degree of urgency must be applied to remembering number facts but this is not to be seen as an end in itself. To understand the concept of number implies its assimilation into the number-schema of natural numbers. What for a young child may be activities based on the use of structured apparatus, such as colour factor or unifix, to write 'the story of eight' must be extended with older children into oral practice and games to indicate what is known about 8; "Tell me all you know about eight". Responses must be encouraged that not only emphasise the numerosity of eight but an expansion of the schema of which eight is already a part. "Eight is 4 + 4, eight is 10 - 2, eight is 5 × 2 – 2".

This basic numerical knowledge can then be associated with place value numeration to provide greater power through which extrapolation of knowledge increases; 3 + 4 = 7, 13 + 4 = 17, 23 + 4 = 27, eventually going as far as 103 + 4 = 107, 203 + 4 = 207 etc.

Re-emphasising the need to provide continued opportunities for children to expand their number schema may, in the end, only serve to increase the distinctions between the less able and other children within the class. What appears to be clear, is that to keep accepting without question procedural methods used by lower ability children is doing them an injustice – it only serves to reinforce their status and will ensure that the divide between them and their more able peers will grow ever wider. However, reactions to their difficulties which involve the continued recycling of procedural methods, because of a belief that they promote understanding, can only serve to place the emphasis, not on mathematcs, but on the use of memory. The hierarchial ladder grows ever longer and these children, faced with the problem of co-ordinating sequential processes, "try to remember by brute force a multitude of rules, facts and procedures"(Byers and Erlwanger, 1985, p.277). Graphic calculators may nurture the problem - product linkages, but the arena for further consideration must focus more sharply on the different types of thought that are brought to bear on mathematical activity by the more able and the less able.

REFERENCES

Byers, V., Erlwanger, S.: 1985, 'Memory in mathematical understanding', Educational Studies in Mathematics, 16, 259-281

Carpenter, T.P., Hiebert, J., & Moser. J.M.: 1981, 'Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems', Journal for Research in Mathematics Education 12, 27-39.

Carpenter, T.P., Hiebert, J., & Moser. J.M.: 1982, 'Cognitive development and children's solutions to verbal arithmetic problems', Journal for Research in Mathematics Education, 13, 83-98.

Cohen, L., Manion, L.: 1985, Research Methods in Education, 2nd Ed., Croom Helm Ltd., London.

Fuson, K.C., Richards, J., & Briars, D.J.: 1982, 'The acquisition and elaboration of the number word sequence'. In Brainerd, C.(Ed.) Progress in Cognitive Development: Vol. 1, Children's logical and mathematical cognition, Springer-Verlag, New York

Fuson, K.C.: 1982, 'An analysis of the counting-on solution procedure in addition'. In T. Carpenter et al. (Eds.) Addition and Subtraction: A Cognitive Perspective. Laurence Earlbaum, New Jersey.

Gray, E.M.: 'Barriers to numerical competence created by efforts to understand', in preparation.

Gray, E.M.: 1987, 'Slow learning children and their range of risk in in basic addition calculations', Proceedings of BSRLM Day Conference, University of London Institute of Education, November 7th 1987, 8.

Gray, E.M.: 1988, ' "O God - It's Take Away": An examination of the strategies that below average children use to solve basic subtraction problems', Proceedings of BSRLM Conference, University of Warwick, May 1988, 27-29.

Groen, G.J., Resnick, L.B.: 1977, 'Can preschool children invent addition algorithms?', Journal of Educational Psychology, 69, 645-652.

Herscovics, N., Bergeron, J.C.: 1983, 'Models of Understanding', Zentralblatt fur Didaktik der Mathematik, 83, 75-83.

McEvoy, J.: 1989, 'From counting to arithmetic: The development of early number skills', British Journal of Special Education, 16, 107-110.

Rosenthal, D.J., Resnick L.B.: 1974, 'Children's solution processes in arithmetic word problems', Journal of Educational Psychology, 66, 817-825

Secada, G.W., Fuson, C.K., Hall, J.W.: 1983, 'The transition from counting-all to counting-on in addition', Journal for Research in Mathematics Education, 14, 47-57

Skemp, R.R.: 1976, 'Relational understanding and instrumental understanding', Mathematics Teaching, 77, 20-26

Steffe, L., von Glaserfeld, E., Richards, J., Cobb, P.: 1983, Children's Counting Types: Philosophy, Theory and Applications, Preagar, New York

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