OBJECTS, ACTIONS AND IMAGES:

A PERSPECTIVE ON EARLY NUMBER DEVELOPMENT

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ABSTRACT

It is the purpose of this paper to present a review of research evidence that indicates the existence of qualitatively different thinking in elementary number development. In doing so the paper summarises empirical evidence obtained over a period of ten years. This evidence first signaled qualitative differences in numerical processing (Gray, 1991), and was seminal in the development of the notion of procept (Gray & Tall, 1994). More recently it examines the role of imagery in elementary number processing (Pitta and Gray, 1997). Its conclusions indicate that in the abstraction of numerical concepts from numerical processes qualitatively different outcomes may arise because children concentrate on different objects or different aspects of the objects, which are components of numerical processing.

INTRODUCTION

The notion that numerical concepts are formed from actions with physical objects underpins the conceived cognitive development of simple arithmetic (see, for example, Piaget, 1965; Steffe, Glaserfeld, Richards, & Cobb, 1983; Kamii, 1985; Gray & Tall, 1994). These conceptions share common ground. The properties by which the physical objects are described and classified need to be ignored; and attention is focused on the *actions* on the objects, which have the potential to create an 'object of the mind'. This can possess new properties associated with new classifications and new relationships. For some, however, meaning remains at an enactive level; elementary arithmetic remains a matter of performing or representing an action. For others the cognitive shift from concrete to abstract involves a qualitative change through which the concept of number can be conceived as a construct that can be manipulated in the mind. Our research shows the broader population filling a spectrum of possible methods of thinking which are typified at each end by these two cognitive styles of mental operation.

The focus of our work has been to consider the thinking that reflects the qualitative differences that emerge in the broad spectrum typified by these distinct styles. Using rich empirical evidence we develop a cognitive theory which may account for what is it that children are doing differently and why? Our research model has focused on extremes of mathematical achievement. It has examined children's interpretations of arithmetical symbolism and imagery associated with these interpretations. Our conclusions suggest that the nature of the object, which is an integral component of children's numerical processing, resonates with different cognitive styles. We hypothesise that these styles influence the quality of the cognitive shift from concrete to abstract thought which contributes to the perceived qualitative difference in children's numerical development.

THE PROCESS/OBJECT TRANSFORMATION

Piaget (1973, p. 80) believed that the growth of numerical knowledge in the child stemmed, "not from the physical properties of particular objects but from the actual actions carried out by the child on the objects". He wrote of how the co-ordination of actions became mental operations—"actions which could be internalised" (Piaget, 1971, p. 21)—and suggested that "actions or operations become thematised objects of thought or assimilation" (Piaget, 1985, p. 49). The formation and meaning of knowledge within the context of learning as well as in mathematics, is rooted within operations on the environment and constructed through active thinking that transforms these dynamic actions into the resulting conceptual entities. Our observations reveal that some children perform these transformations more successfully than others, leading to a divergence in performance in mathematical competence.

Substantial interest in the cognitive development of mathematics has focused on the relationship between actions and entities. For some, grammatical metaphors sharpen the subtle changes that form the basis for numerical constructs. Dienes (1960) described how a predicate (or action) becomes the subject of a further predicate which may in turn become the subject of another and so on. The qualitative benefit from making predicates the servant rather than the master of thought were clear:

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians (Dienes, 1960, p. 21)

Using a similar analogy Davis (1984) signaled the qualitative changes associated with actions becoming objects of thought.

> The procedure, formerly only a thing to be done—a verb—has now become an object of scrutiny and analysis; it is now, in this sense, a noun.
> (Davis, 1984, p. 30)

These distinctions, together with theories accounting for the transformation of processes into concepts, have helped to shift attention from *doing* mathematics to *conceptualizing* mathematics. The way in which dynamic actions become conceptual entities has been variously described as "interiorisation" (Beth & Piaget, 1966), "encapsulation" (Dubinsky, 1991), or "reification" (Sfard, 1991). Dubinsky and his colleagues (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996) formulate the encapsulation as part of the APOS theory (action-process-object-schema), in which actions become repeatable as processes which are then encapsulated into objects to later become part of a mental schema. Sfard indicates a three-phase process: *interiorisation* of the process, *condensation* as a squeezing of the sequence of operations into a whole, then *reification*. She suggests that this is qualitative change manifested by the ontological shift from *operational* thinking (focusing on mathematical processes) to *structural* thinking (focusing on properties of, and relationships between, mathematical objects).

Gray & Tall (1994) focused on the role of mathematical symbolism within this process. Symbolism represents either a **pro**cess to *do* or a con**cept** to *know*. To emphasise this dual meaning the term **procept** was introduced in elementary arithmetic. Procepts start as simple structures and grow in interiority with the cognitive growth of the child. The word "concept" rather than "object" was used because terms such as "number concept" or "fraction concept" are more common in ordinary language than "number object" or "fraction object". Furthermore, the term is used in a manner related to the "concept image" consisting of "all of the mental pictures and associated properties and processes" related to the concept in the mind of the individual (Tall & Vinner, 1981, p. 152). In this sense there is no claim that there is a "thing" called "a mental object" in the mind. Instead a symbol is used which can be *spoken*, *heard*, *written* and *seen.* It has the capability to evoke appropriate processes to carry out necessary manipulations in the mind of the individual and it can be communicated to share with others.

Theories which refer to the cognitive shift from process to object are process driven, but they form an important backdrop for the theory of procepts. Indeed Anna Sfard's notion of duality (Sfard, 1991) and discussions with her in 1989 were important in its early development. Procepts are dynamic and generic—"things" that are the source of great flexibility and power. The problem in the cognitive context is to identify why some children implicitly seem to recognise this fact but others do not.

'Encapsulation' theories—and here the one word is used as a matter of convenience—have intrinsic differences but also share common ground in attempting to account for process/object links. Notions such as 'interiorisation' or 'repeatable actions' may lead to quantifiable differences in procedure but not qualitative differences in thinking. This distinction is implicit in the finely-grained analysis of counting units of Steffe et al. (1983). Decreasing dependence on *perceptual* material permits children to eventually count *figural* representations of perceptual material; the counting process continues in the absence of the actual items. *Motor acts*, such as pointing, nodding and grasping, that accompany the counting process, can be taken as further substitute units for perceptual items. Dependence on these three forms of unit is further reduced by the realisation that the utterance of a number word, the verbal unit, can be taken as a substitute for countable items that could have been co-ordinated with the uttered number sequence. However, these changes, though quantifiably different, are qualitatively similar—each procedure is an analogue for the process of counting. The concept of unit becomes wholly abstract when the child no longer needs any material to create countable items nor is it necessary to use any counting process.

THE EMPIRICAL EVIDENCE

Theories of encapsulation focus on the manner in which processes are encapsulated as objects. However, the individual's perception of the original objects plays a vital role. Counting starts with objects perceived in the external world which have properties of their own; they may be round or square, red or green or both round and red. However, these properties need to be ignored if the counting process is to be encapsulated into a new entity—a *number* which is *named* and given a *symbol*. It is our contention that *different perceptions of these objects*, whether mental or physical, *are at the heart of different cognitive styles that lead to success and failure in elementary arithmetic*.

Three themes dominate the empirical studies that were used in building the resulting theory:

- diverging approaches displayed when children's attempted to solve elementary number combinations when they could not recall solutions, (Gray, 1991)
- process/concept links identified through the approaches used to carry out elementary computations, (Gray & Tall, 1992
- the nature of any imagery associated with these tactics (Gray & Pitta, 1996 and Pitta & Gray, 1997, Pitta, 1998).

Process Concept Links and the Proceptual Divide.

Qualitative differences in thinking and the consequences of a divergence between the two were revealed by Gray (1991). These results indicated that some children wish to remain at a procedural level which, in terms of information processing, make things very difficult for them, whilst others operated at a conceptual level which was more flexible. The notion of different cognitive styles leading to diverging outcomes came from the observation that the less able, who relied extensively on counting procedures, were "*making things more difficult for themselves and as a consequence become less able*" (Gray, 1991, p. 570). In contrast, the ability to "*compress the long sequences [of procedures] appeared to be almost intuitive to the above-average child"* *(ibid.)*.

Drawing upon the children's interpretations of symbolism, the differing cognitive styles evident in this first study were later placed within the context of a 'proceptual divide'. This was hypothesised as a cognitive difference between those children who processed information in a flexible way and those who invoked the use of procedures (Gray & Tall, 1994). Those doing the former have a cognitive advantage. They link procedures to perform arithmetic operations with number concepts through cognitive links relating process and concept. The proceptual divide for us reveals a catastrophic difference between those operating flexibly and those limited to procedures.

The common pedagogical approach to numerical processes builds on the belief that number development should commence with enactive approaches and that, given sufficient time, all children will encapsulate arithmetical processes into numerical concepts.

> When a procedure is first being learned, one experiences it almost one step at a time; the overall pattern and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity—it becomes a thing. It, itself is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. (Davis, 1984, p 29–30)

The existence of a proceptual divide would seem to indicate that some do not perceive the 'wholeness' of an activity. Even when teaching programmes have been designed to shift the lower achievers' focus from processes to thinking strategies (see, for example, Thornton, 1990), lower achievers appear to resist a change from the security offered by their well-known counting procedures. Further, we conjecture that positive efforts to make the relationships implicit in proceptual thinking explicit to those that do not have the associated flexibility run the danger of being seen by some procedural learners as a new set of procedural rules.

So what causes the proceptual divide? We may conjecture that pedagogy may account for it in some degree. There does exist a certain 'conspiracy' between pedagogue and learner which is manifest in the belief that being shown how to do something solves current difficulties (see, for example, Skemp, 1976). We do conjecture that one cause of the proceptual divide is the qualitatively different focus of attention which, on the one hand places the emphasis upon concrete objects and actions upon these objects, and on the other on abstraction and the flexibility intrinsic within the encapsulated object. The fundamental question is *why do some children seem to recognise this power implicitly but others do not?*

IMAGERY AND ELEMENTARY ARITHMETIC

To gain a partial answer to this question our attention turned to imagery. Our fundamental thesis was that different qualities of mathematical abstraction were influenced by the images children associated with mathematical activity and mathematical symbolism. It was hypothesised that the relationship between achievement and the qualitative differences associated with it may be determined by considering:

- the nature of the object that was dominant in a children's imagery,
- the way that imagery is used within elementary arithmetic.

In turning to examine the role of imagery, we had been influenced by psychological research. This had not only identified its importance in cognitive development, but which had also suggested that its role in the child's thought processes may have far-reaching consequences on the development of children's concepts and reasoning (Bruner, Oliver & Greenfield, 1966; Piaget & Inhelder, 1971). Since children appear to use imagery more than adults (Kosslyn, 1980), these consequences may place major constraints on cognitive processes.

The possession of an image of a mathematical idea implies that the individual does not need actions or the specific instances of image making (Pirie & Kieran, 1994). However, in the field of elementary arithmetic, we may not only see highly imaginative and unconventional representations of numbers (Thomas, Mulligan & Goldin, 1995) but many of these representations may be deemed to be 'analogical' in that they are images of quantities that are directly represented by "patterns of dots or other things such as the alignment of apples or a bar of chocolate" (Seron, Pesenti, Noël, Deloche & Cornet,1992, p. 168). In an anological form, an image may be seen to be isomorphic with a concrete object. Our interest evolved around the issue of whether children tended to project different forms of image, and if so, what may be the relationship of these images to notions of numerical achievement.

Research Methodology

An underlying assumption in the work was that an image is mediated by a description (Kosslyn, 1980) and that the representation conforming to an image is more like a description than a picture (Pylyshyn, 1973). The classical notion is usually that of a visual image—though images can be formed from other modalities—which appears to have all of the attributes of actual objects or icons.

Paivio (1991) suggested that the generation of an image promotes the development of a trace in the brain that integrates the separate components of the item in question. Accessing a part of the information encoded in memory prompts the retrieval of all other pieces of information contained in the image (Woloshyn, Wood & Pressley, 1990, cited in Drake, 1996). To gain a sense of the nature of children's imagery associated with both concrete and abstract objects and the relationship this may have with mentally processing elementary number combinations, 24 children were selected to represent the extremes of ability, 'low achievers' and 'high achievers', across four age groups, 9+ to 12+. The levels of achievement was largely identified through performance on the two mathematical components of the Richmond Tests in Basic Skills (France, Hieronymus and Lindquist, 1974) which was used by the school to identify teaching groups. The average test scores of the 'high achievers' was 130, that of the 'low achievers' 83 (The standardised mean of the test is 100, and the standard deviateion 15).

The children were first asked to respond to

- ***auditory items*** such as 'ball', 'car', 'triangle', 'five', 'fraction' and 'number', and
- ***visual items*** which included *icons* named, through most frequent responses by the children as 'windows', 'marbles' and 'dancing man', and *geometric shapes* forming a 'house', and *symbols* such as '5', and '3÷4'.

The children were later asked to provide mental solutions to a series of elementary arithmetic combination in addition and subtraction.

The research methodology used semi-structured clinical interviews (see Gray & Pitta, 1996; Pitta & Gray, 1996, Pitta 1998). Items that prompted discussion were presented in a way that gave the interviewees the freedom to follow their own inclinations. Data from each individual was collected in a variety of ways including records of achievement and teacher assessment. The initial selection of children was made from class records of achievement. Each individual interview was audio and video taped and subsequent transcriptions formed the basis for response classification. When responding to each item within the auditory and visual sections, children were asked to provide a first notion of ‘what came to mind’ when they first heard or saw the item. They were then given 30 seconds to talk aloud about the item in question. Children were also asked to provide an explanation of the auditory items so that an extra-terrestrial (ET) may understand what it was.

Qualitative differences in interpretation

Though there are a wide variety of conclusions that may be drawn from each item, the analysis of the 30 second results indicates that similarities in the children’s descriptions of imagery are remarkable both for their consistency across the range of items, and for the differences they displayed between the ‘high achievers’ and the ‘low achievers’.

When responding to the auditory items, the ‘low achievers’ tended to highlight the descriptive qualities of items:

> *“[A fraction is] two numbers with a line in the middle”.*

> *“Some [numbers] are straight [1. 4], some are bent and lines [5] and some are circles [8,6].”*

Often description was associated with personalising the items.

> *“Very big. It’s got different shapes on it. It’s stripy, small or big – red and green and pink and orange. My name in black (I see my ball with my name on it).”*

> *“A blue car–one wheel cap missing. My mum’s car”.*

Though these qualities were also evident when the children responded to the visual items, (“A green card, black writing 3,4 and something else but can’t remember. Writing black - 3 and division sign and 4” [3÷4]), there was a tendency to associate these with a story. For example a picture of small circles (which for the purpose of analysis had been termed ‘marbles’ since so many children had suggested this on the first response) was described as:

> *“A game of marbles, some in a circle and boys are using others to hit them”.*

Whilst the outline of the ‘dancing man’ was said to be:

> *“Jack Frost. But he would be transparent. It is a crook running away from a robbery”.*

In a sense, for the low achievers, many of the visually presented items were seen as pictures that required colour, detail and a realistic content.

Though it was tendency of the “low achievers” to concentrate on the descriptive and concretise the items, ‘high achievers’ concentrated on the more abstract qualities within both series of items. For example 3÷4 evoked the response that it

> *“Was a sum. We may need to do it by division but its… 0.75, 3/4. 75%…”*

Though they initially focused on core concepts, the “high achiever’ could traverse, at will, a hierarchical network of knowledge from which they abstracted these notions or representational features. The visual item ‘windows’ was identified as”

> *“A shape in four quarters, half shaded, a picture of a window” or*

"Two out of four, half, cupboard, windows where they don't use shutters".

One of the geometric shapes, 'hexagons' was suggestive of:

"Hexagons, symmetry, four light, twelve dark, one quarter and three quarters".

Many items seemed to trigger 'low achievers' to provide descriptions which were *qualitatively similar* whereas 'high achievers' used each comment to trigger a *qualitatively different* comment.

Responses reflected the degree within which the children were involved with the abstract qualities of the objects. The higher the involvement, the more the child was able to talk about the items at an impersonal level. 'High achievers' often referred to the symbol using phrases such as "it is" to illustrate semantic aspects of the object. For example, the word 'five' drew responses such as "it is two plus three, one hundred take away ninety five", or "it is prime because it is only divisible by one and five". This does not mean to say that they did not attach qualities arising from episodic memory, such as "I had five candles on my cake for my fifth birthday"; high achievers were able to do both. On the other hand 'low achievers' almost always displayed examples of episodic memory, concretised the item, "I have five fingers", or associated its use with some arithmetical action such as counting.

The similarities within groups and differences between groups may be summed up by concluding that images of the low achievers are more often episodic, specific and action-based, whilst those of the high achievers are more often semantic, generic and concept-based. We use the terms 'episodic' and 'semantic' to draw a distinction between images arising from memory associated with the recollection of personal happenings and events, and images associated with organised knowledge, having meaning and relationships. The former is based upon access to previous experience, the latter no longer depend on learning episodes that provided the original basis for knowledge (see Tulving, 1985; Davis, 1996)).

The qualitatively different responses to the words, icons and symbols suggests that the 'low achievers' were reluctant to reject information. If there was little to describe, they created description by building stories around the items using images from their known physical world. Often they were participants in the image, elaborating the detail whenever it seemed that such embellishment was required. In some instances they drew upon one image which acted as a symbol, for example, "my football", "my mother's car". The objects referred to were invariably real, quantifiably different, but qualitatively the same. In contrast, 'high achievers' filtered out the superficial to concentrate on the more abstract qualities of the items. Though they focused on real world concepts, they were also able to relate to a hierarchy of ideas which allowed them to refer to objects in the abstract by using qualitatively different notions or representational features.

Images in Elementary Arithmetic

Such differences became marked when images associated with children's responses to the range of elementary number problems were considered (see, for example, Pitta & Gray, 1997). Again 'low achievers' tended to concretise and focus on all of the information. Symbols were translated into numerical processes supported by the use of imaginistic objects that possess shape and in many instances colour.

Frequently 'low achievers' reported imagery strongly associated with the notion of number track although the common object which formed the basis of each 'unit' of the track was derived from fingers. In some instances children reported seeing full picture images of fingers, in others it was 'finger like'. The essential thing is that the object of thought was 'finger' and the mental use of finger invoked a double counting procedure.

The objects of thought of the ‘low achievers’ were analogues of perceptual items that seemed to force them to carry out procedures in the mind, as if they were carrying out the procedures with perceptual items on the desk in front of them. Their images were essential to the action; they maintained the focus of attention. When the image failed they used the real items. For these children mathematics involved action and to carry out the action they used ‘real’ things.

Symbolism enables us to utilise short term memory to better effect but the differences between the ‘low achievers’ imagery associated with symbolism and that described by the ‘high achievers’ was stark. It is here that we may see clearly the ‘low achiever’s’ inability to filter out information thus providing the contrast between their uneconomical use of memory and the ‘high achievers’ economic use. Here, we should explain that we use the word ‘economic’ not simply to illustrate differences in the detail but also in arrangement as well as quality.

Symbolic images played considerably less part in processing for ‘low achievers’ than they did for ‘high achievers’. They were also reported far less than analogical images. The notion of “spinning” seemed to be a common feature of the ‘low achievers’ descriptions, implying that images remained for some time and possessed movement. Even when adding 2+1 a nine year old reported seeing all of the operation symbols “spinning around on one side and a big black 3 on the other”. In some instances images were associated with approximation. When adding 6+3 another nine-year-old reported seeing “a jumble of numbers with 8 and 9 standing out because they are near the answer.” This was a similar response to that given by a twelve-year-old who, when doing the same combination reported an image that consisted of 3, 6, 9, 12, 15, and 18. “All the numbers were in the three times table”. Whilst the “three and the six stayed there because they were part of the nine, the twelve, fifteen and the eighteen just fall away.”

Where it was used the use of symbolic imagery amongst ‘high achievers’ was far more economical. The word “flashing” dominated their descriptions. Images came and went very quickly. “I saw ‘3+4’ flash through my mind and I told you the answer”, “I saw a flash of answer and told you.” It was not unusual for the children to note that they saw both question and answer “in a flash”, sometimes the numerical symbol denoting the answer “rising out of” the symbols representing the question. In instances where children reported the use of derived facts it was frequently the numerical transformation that ‘flashed’. For instance when given 9 + 7 one eleven year old produced the answer 16 accompanied by the statement. “10 and 6 flashed through my mind.”

DISCUSSION

The quality of imagery generated by the two groups differed considerably. On the one hand we see the dominant objects being either physical, such as fingers and counters, or figural representations of physical items. On the other we see it as an object of thought. ‘Low achievers’ appeared to concentrate upon analogues of physical actions. Where they use symbolism they continue to carry out actions associated with such analogues. Their images are not so much associated with conceptually “knowing” mathematics but with actively “doing” mathematics. In contrast the symbolic images of ‘high achievers’, appear to act as thought generators. They appear to flash as memory reminders, momentarily coming to the fore so that new actions or transformations may take place.

Distinct trends indicated that ‘low achievers’ had a tendency to concretise and focus on virtually *all* of the information. In the numerical context their imagery was strongly associated with procedural aspects of numerical processes. ‘High achievers’ appeared to focus on those abstractions that enable them to make choices—they display the ability to reject information. We suggest that such differences have overriding consequences for children’s mathematical achievement. The one conclusion that may be drawn from the use of analogical images is that it would seem to place a tremendous strain on working memory. Geary (1994) has suggested that a component of developmental difficulties in mathematics is a working-memory deficit. We would suggest that on the contrary these low achievers show an *extraordinary* use of working memory. Their problem is one associated with its *use* and not its *capacity*. Not only is the child focusing on the representation but also on discrete numbers in that representation.

The ability to filter out information and see the strength of such a simple device as a mathematical symbol works to the advantage of the 'high achievers'. In contrast the evidence suggests that children who are 'low achievers' in mathematics appear unable to detach themselves from the search for substance and meaning — almost no information is rejected, no surface feature filtered out.

The notion of procedural compression and the interiorisation of mathematical processes is strongly embedded in the literature. Interpretations of Piagetian notions that enactive approaches will form a foundation for procedural encapsulation may be associated with Bruner's (1968) view that past experience may be conserved through such enactive approaches. Of course, whilst the latter must also be seen within the context of iconic and symbolic conservation, it would seem that far from 'encapsulating' enactive interpretations of arithmetical processes, the 'low achievers' are mentally imitating them and dependent upon them.

We suggest that the quality of image formed from enactive approaches is dependent upon what it is that the child chooses to create an image of. In turn we suggest that this will influence the use to which the image is put. We conjecture that this will not only have consequences for the quality of the action that is taken into consideration but it will also affect the quality of the object which dominates the child's imagery. It would seem reasonable that if some children concentrate on actions with physical objects and work hard to develop competence with these actions the more they are likely to use them.

Such considerations add a new quality to the notion of proceptual divide, one that is so strongly associated with image formation that it is possible that children's interpretations of mathematical actions may be strongly influenced by their interpretations of their real world. In early mathematics children are faced with not one but *two* interpretations of their interaction with externally perceived objects. On the one hand it is the identification of the qualities of objects that arise from manipulation and perception which lead eventually to the development of geometrical concepts. On the other, though perception and manipulation are the dominant actions, it is the cognitive shift associated with the result of these actions that brings about the development of numerical concepts. The objects that are the catalysts for both strands of development are the same but the conceptual development is different. We believe that this has serious implications for pedagogy.

Early years within school are dominated by enactive methods in the belief that given the appropriate experience all children will "encapsulate" arithmetical processes to form arithmetical concepts. *Observation within any classroom shows that this is not the case*. Children may be focusing on *different* aspects of their experience. For some the dominant focus is on objects and the actions *on those objects*, others are able to focus more flexibly on the results of those actions expressed as number concepts. The former may seek the security of counting procedures on objects rather than the longer-term development of flexible arithmetic. We need to determine which, so that we may provide the necessary support both to those who develop flexibly and also to those who, at the very start of their mathematical development, appear to traverse a cognitively different route.

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